Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Lehrstuhl für
INFORMATIK I
Effiziente Algorithmen und
wissensbasierte Systeme

Institut für Informatik

Computational Geometry

Winter semester 2016/17

Arrangements and Duality

Lecture #12
(Chapter 8)

Joachim Spoerhase *Chair for Computer Science I*

Rendering und Ray-Tracing

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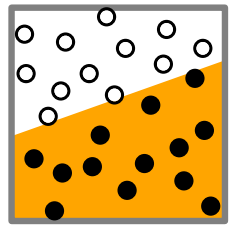

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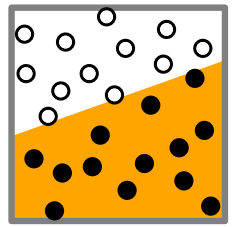

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Thm. The halfplane discrepancy of a set of n pts in the unit square can be computed in $O(n^2)$ time.

TODO!

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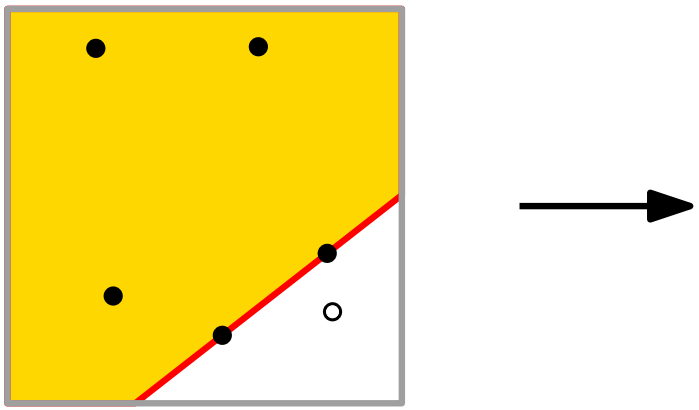

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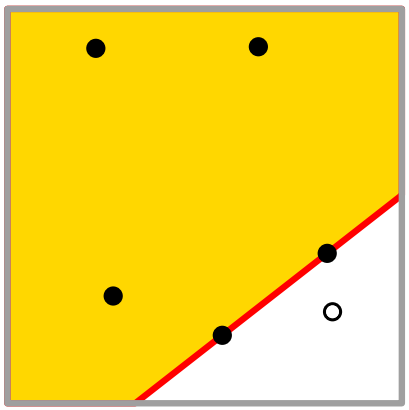

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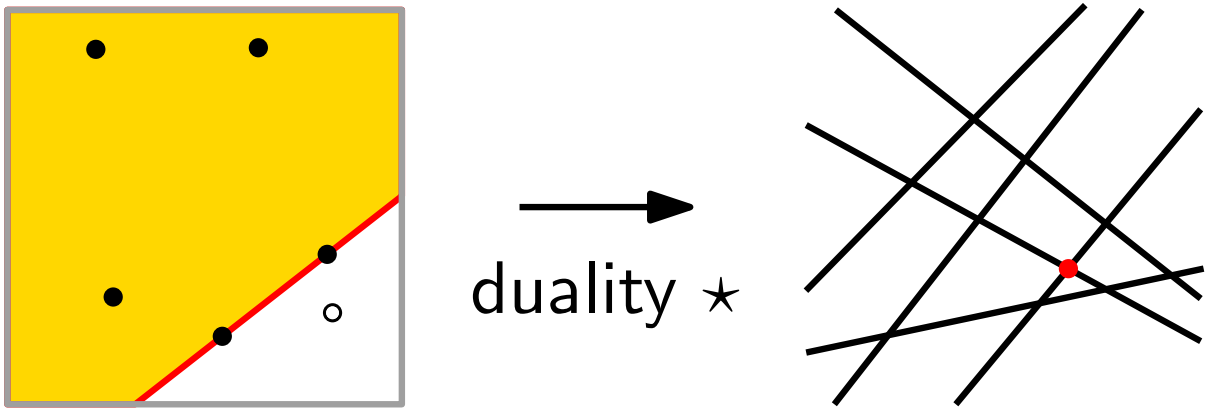

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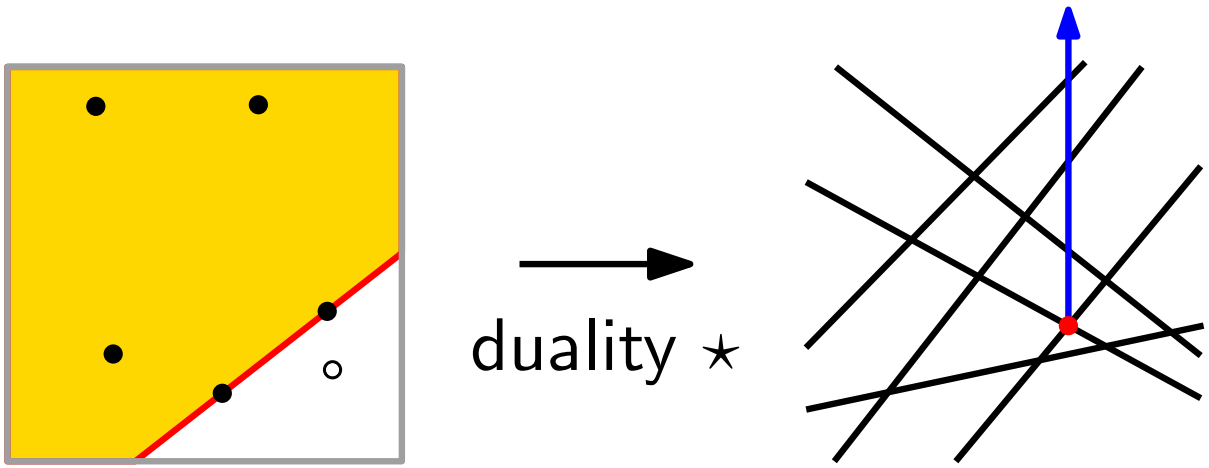

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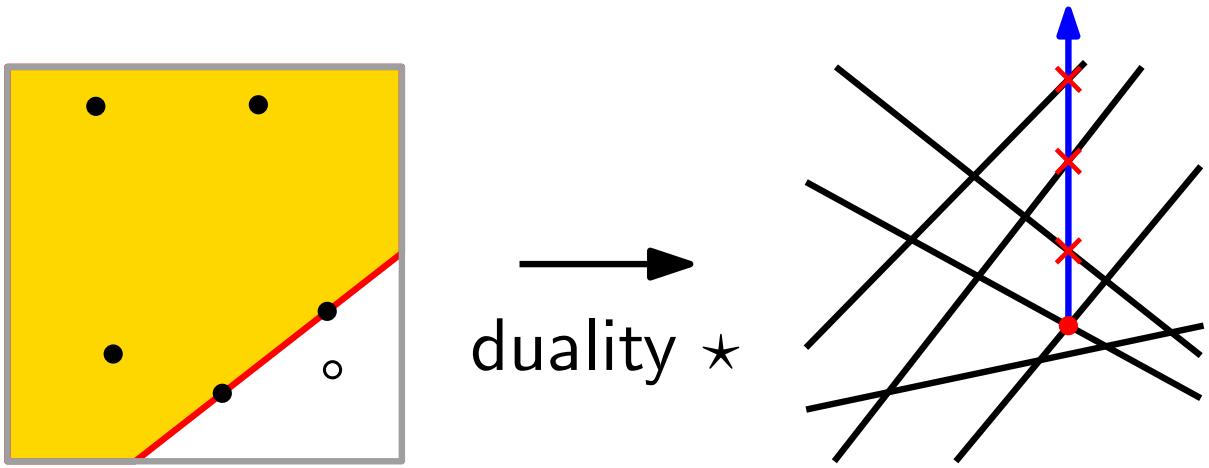

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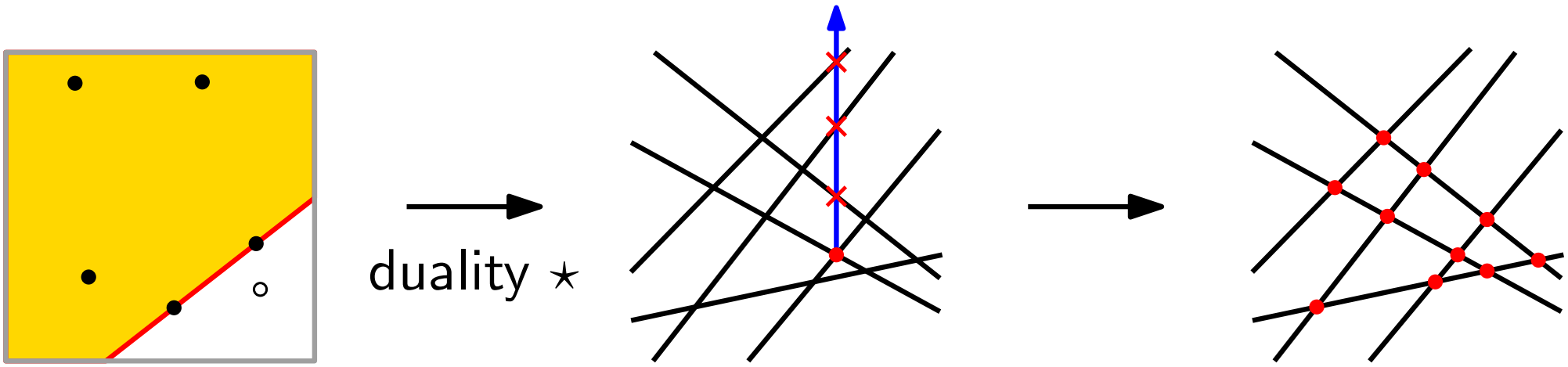

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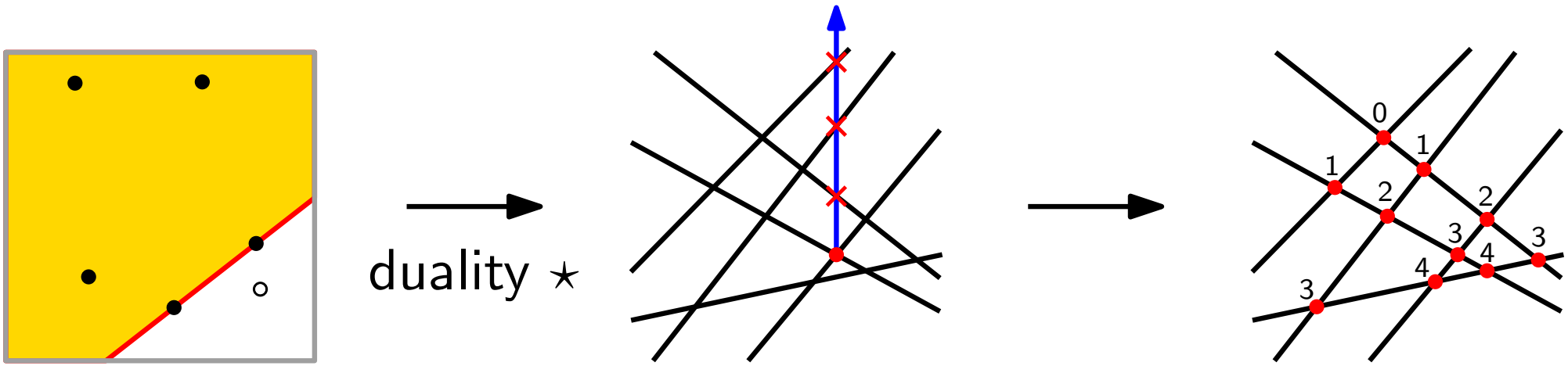

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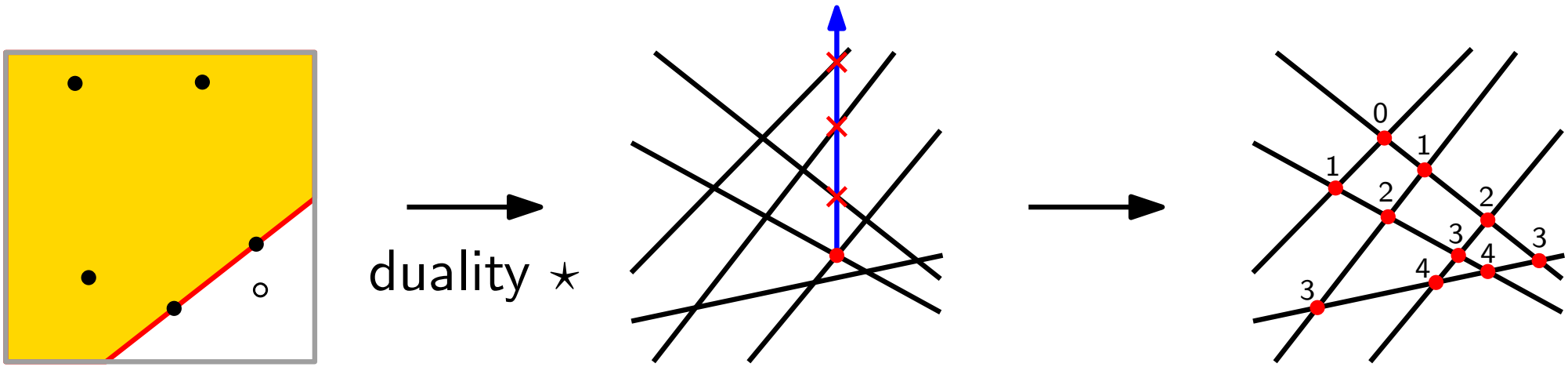

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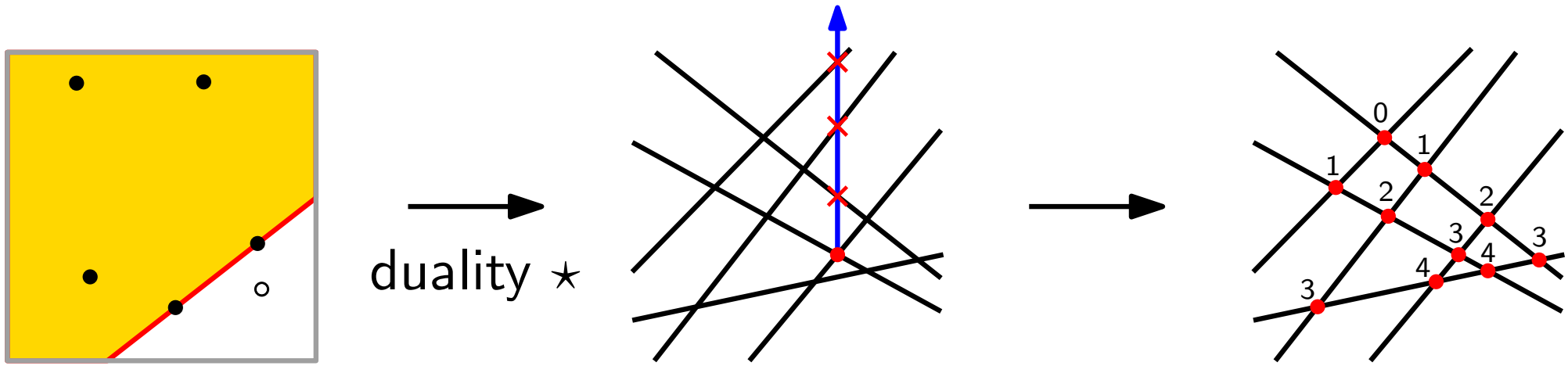

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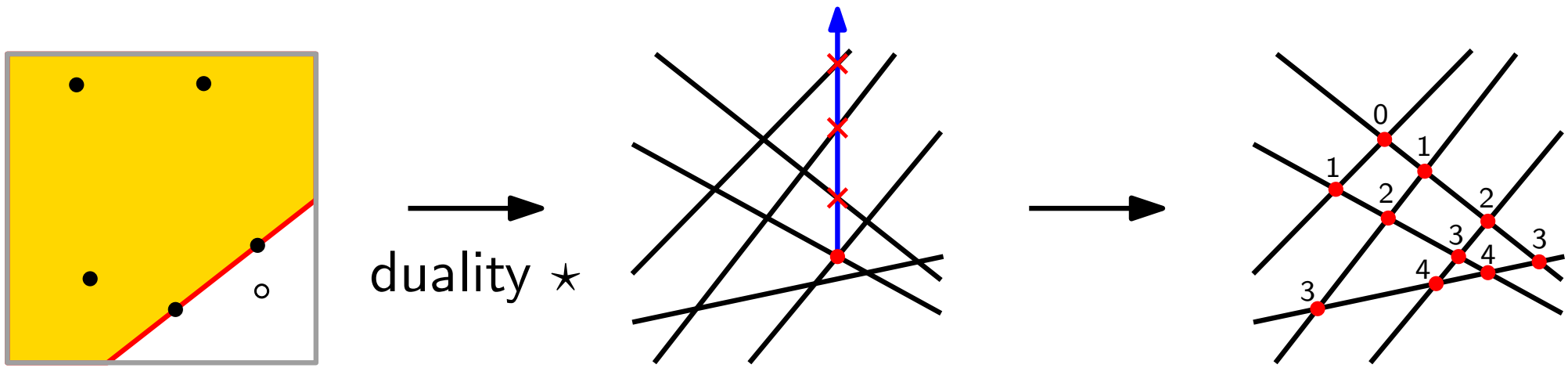

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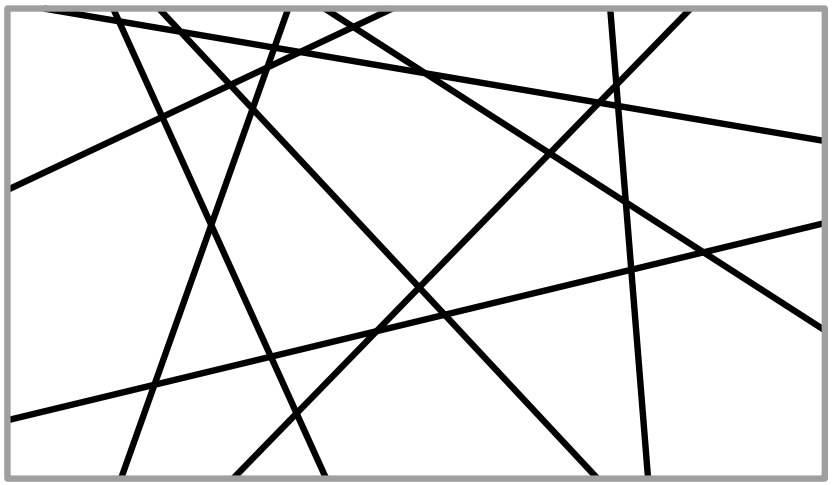

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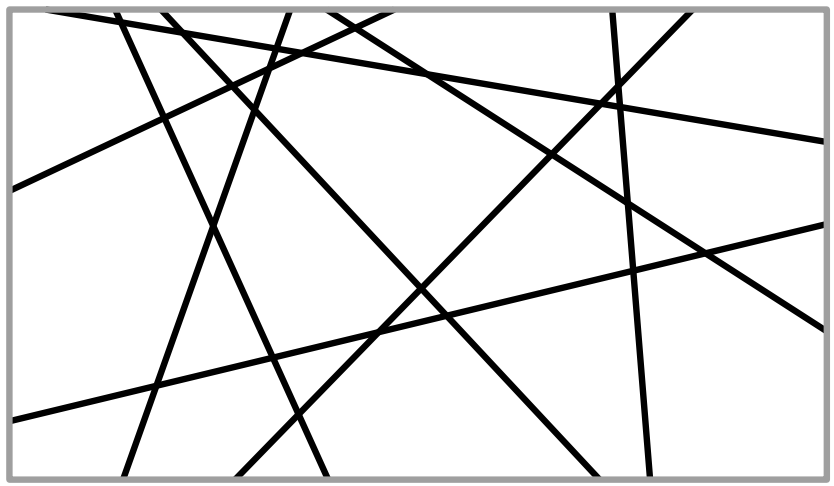

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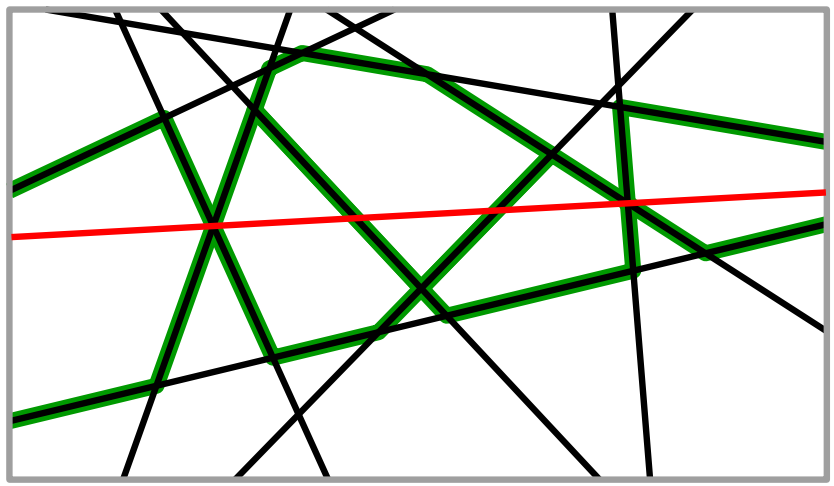

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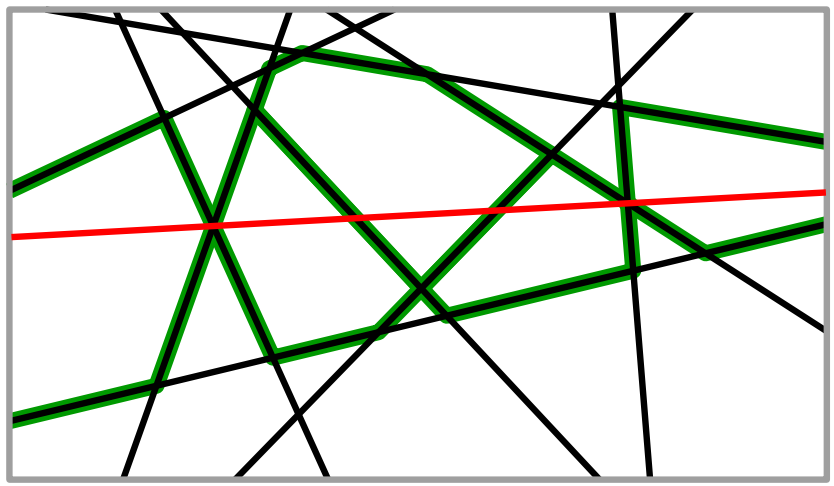

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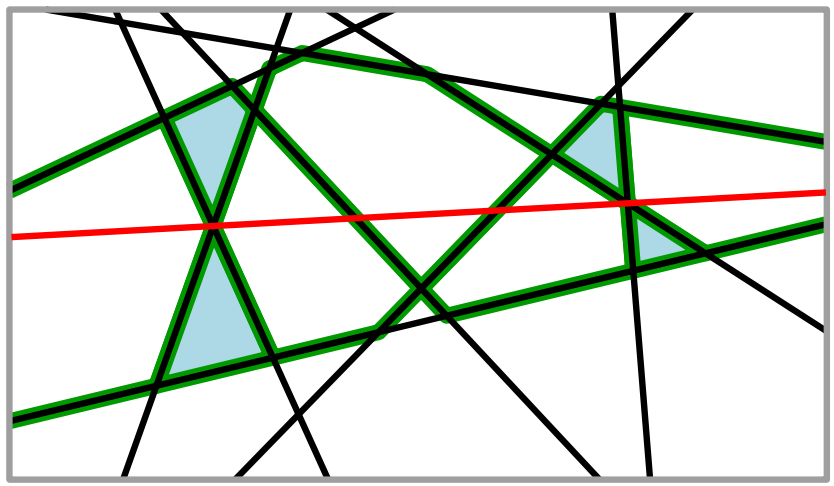

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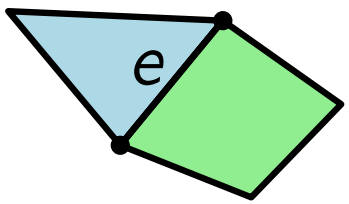

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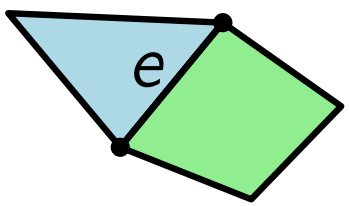

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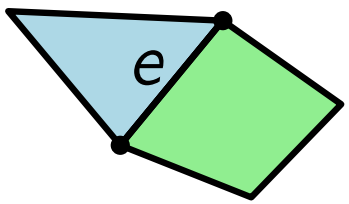

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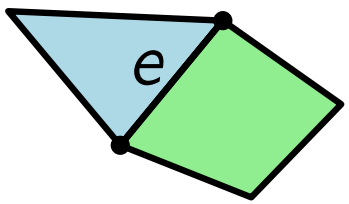

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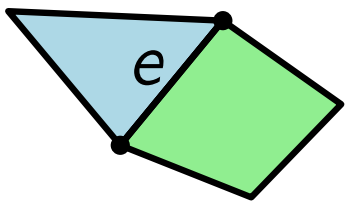

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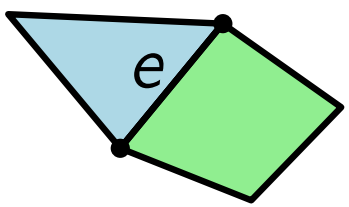

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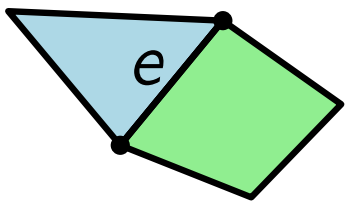

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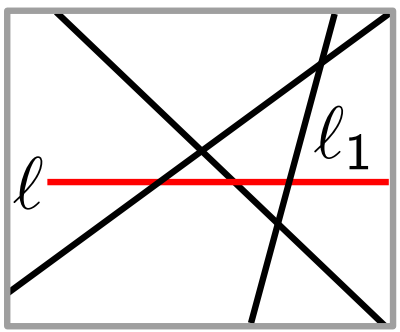

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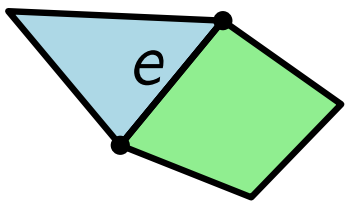

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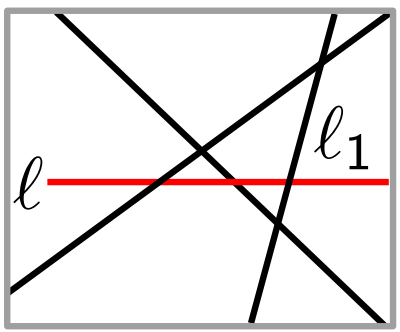

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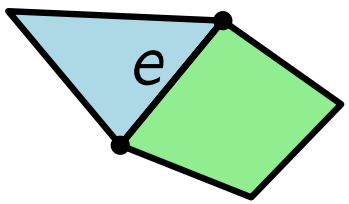

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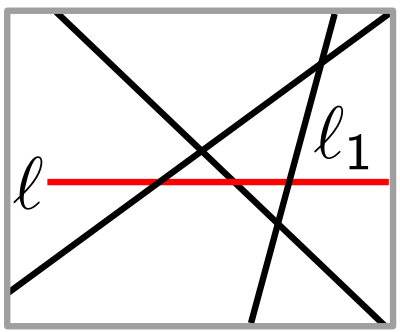

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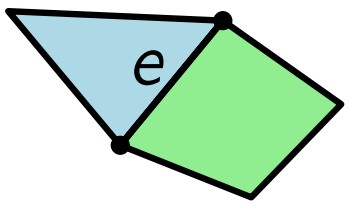

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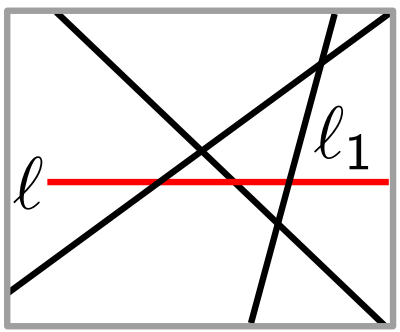

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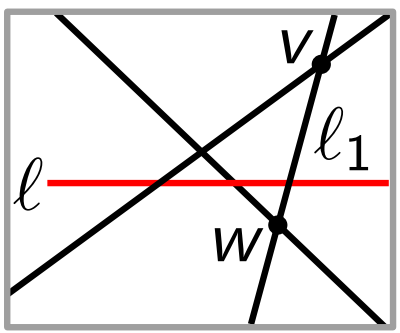
Now we re-insert ℓ_1 into $\mathcal{A}(\mathcal{L} \setminus \{\ell_1\})$.

Proof cont'd

Let v be the first intersection pt on ℓ_1 above ℓ. (if such pts exist)
w below



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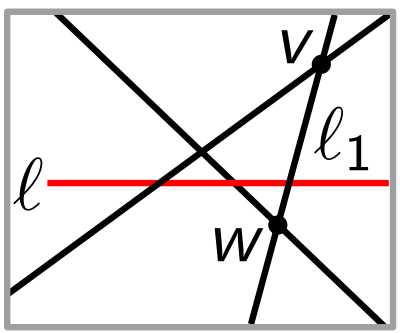

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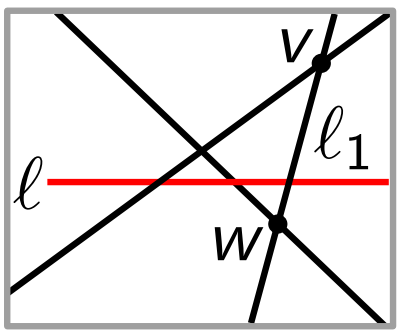

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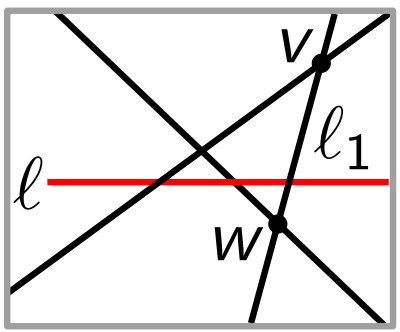

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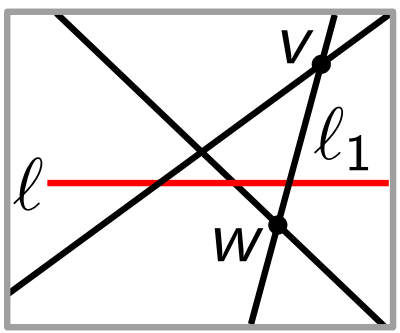

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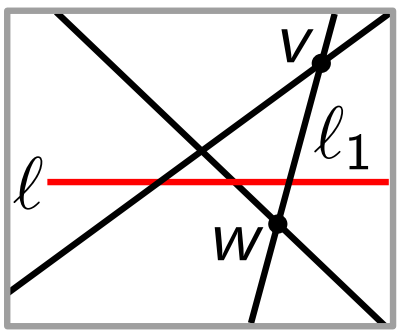

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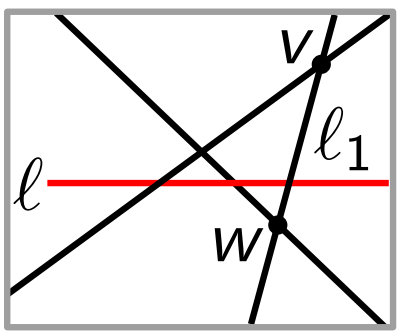

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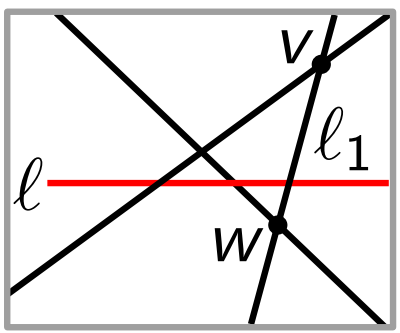

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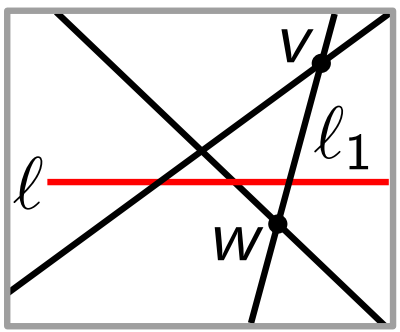

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$\Rightarrow$ Our above bound holds in this case, too.

□

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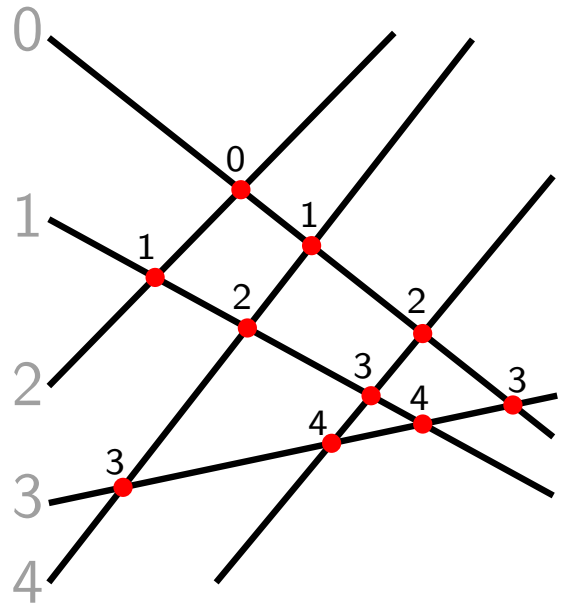

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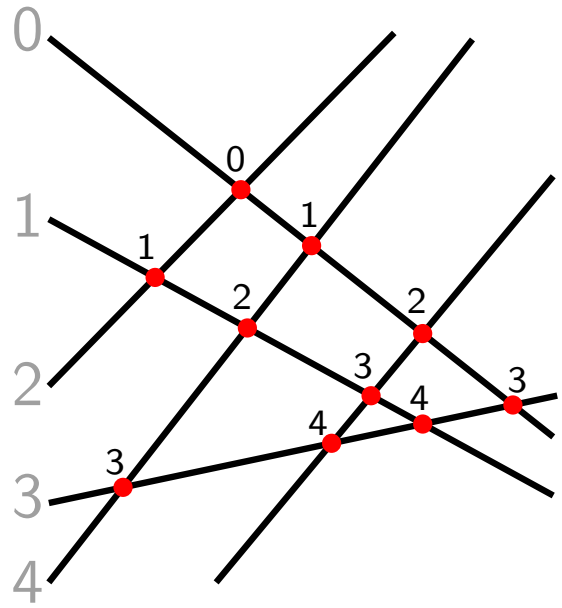

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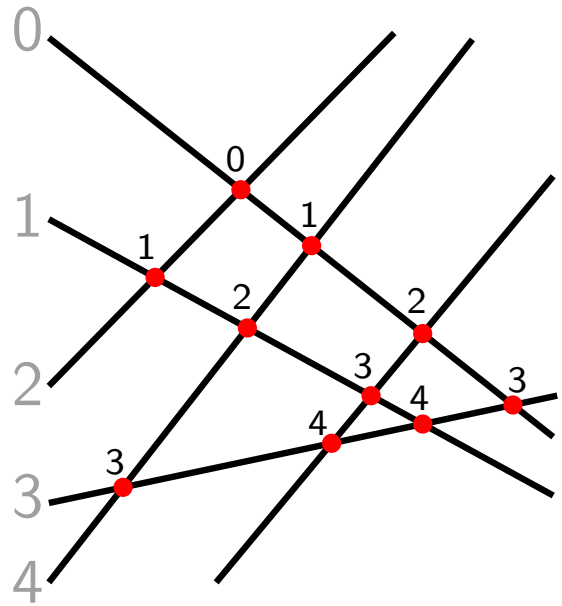

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Back to Discrepancy

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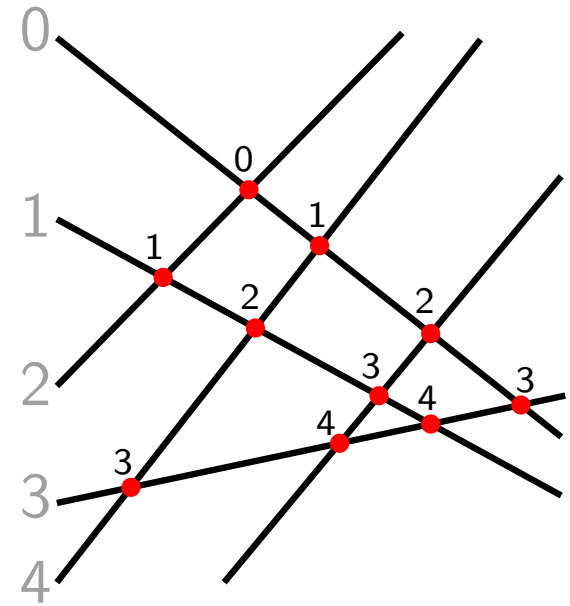

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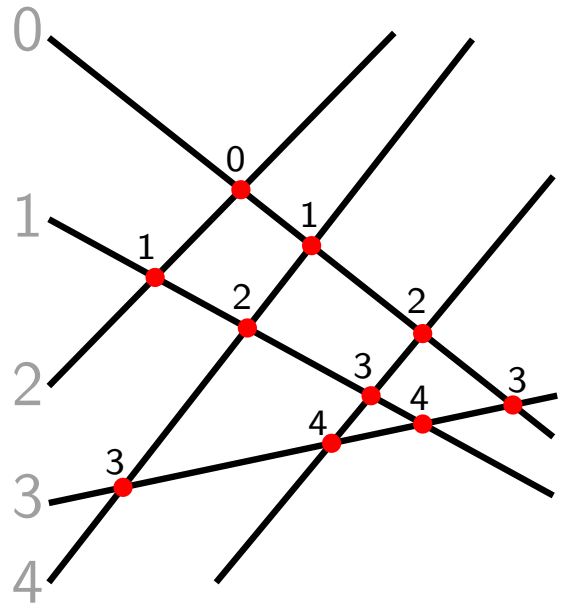

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  $\Rightarrow$ depth at $x = -\infty$ corresponds to rank in this order (minus 1)
- Traverse each line from left to right.
  Adjust its depth by $+1$ (-1) if the intersecting line comes from below (above).