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Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Computational Geometry

Winter semester 2016/17

Convex Hulls in 3D

Lecture #9
(Chapter 11)

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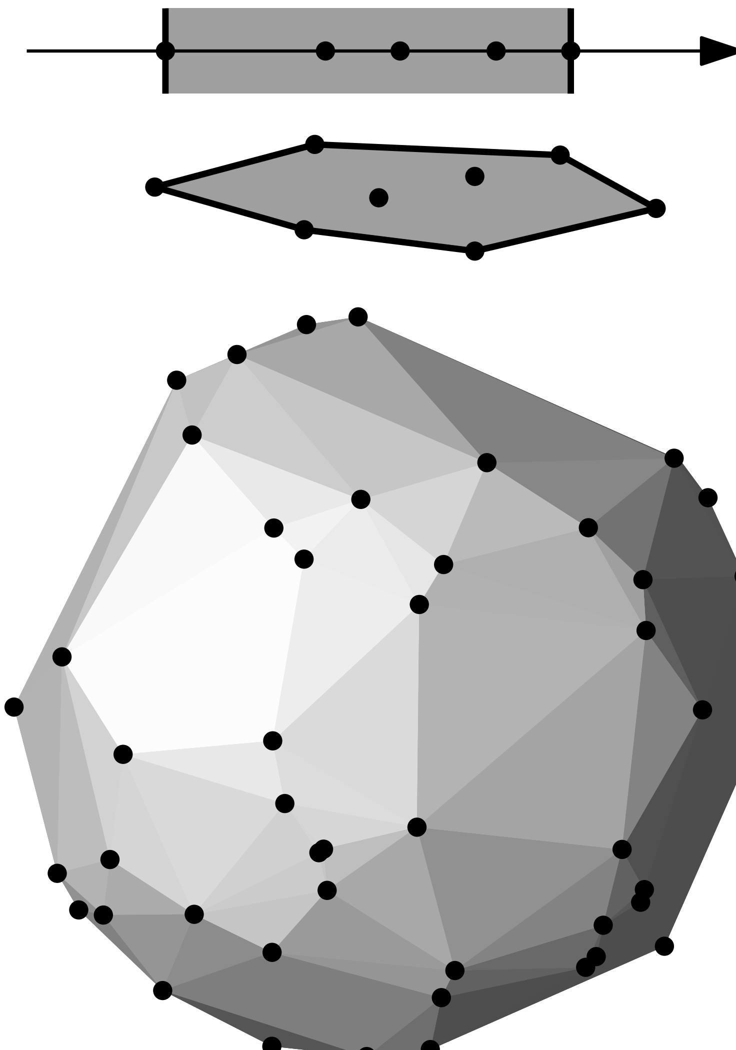
Complexity of the Convex Hull

Given set S of n points in $\mathbb{R}^d$, what is max. #edges on $\partial\mathrm{CH}(S)$?

dim	w-c complexity of CH(S)
1	$2 \in \Theta(1)$
2	$n \in \Theta(n)$
3	$3n - 6 \in \Theta(n)$
d	$\Theta(n^{\lfloor d/2 \rfloor})$

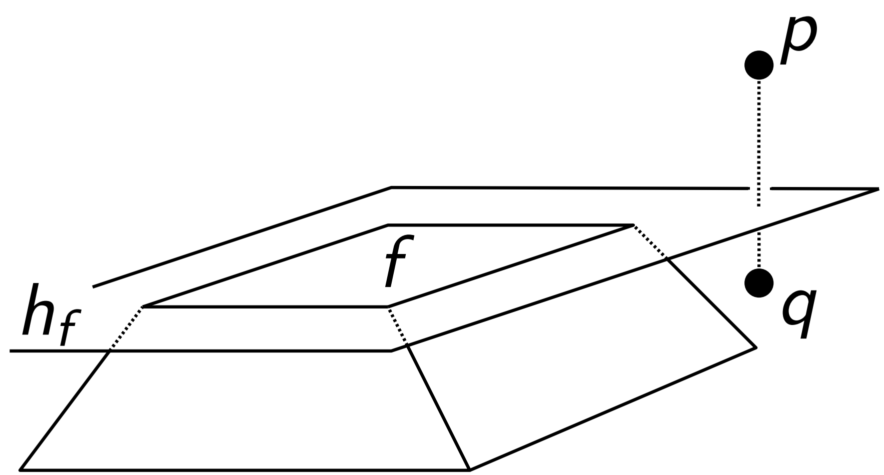
Upper Bound Theorem



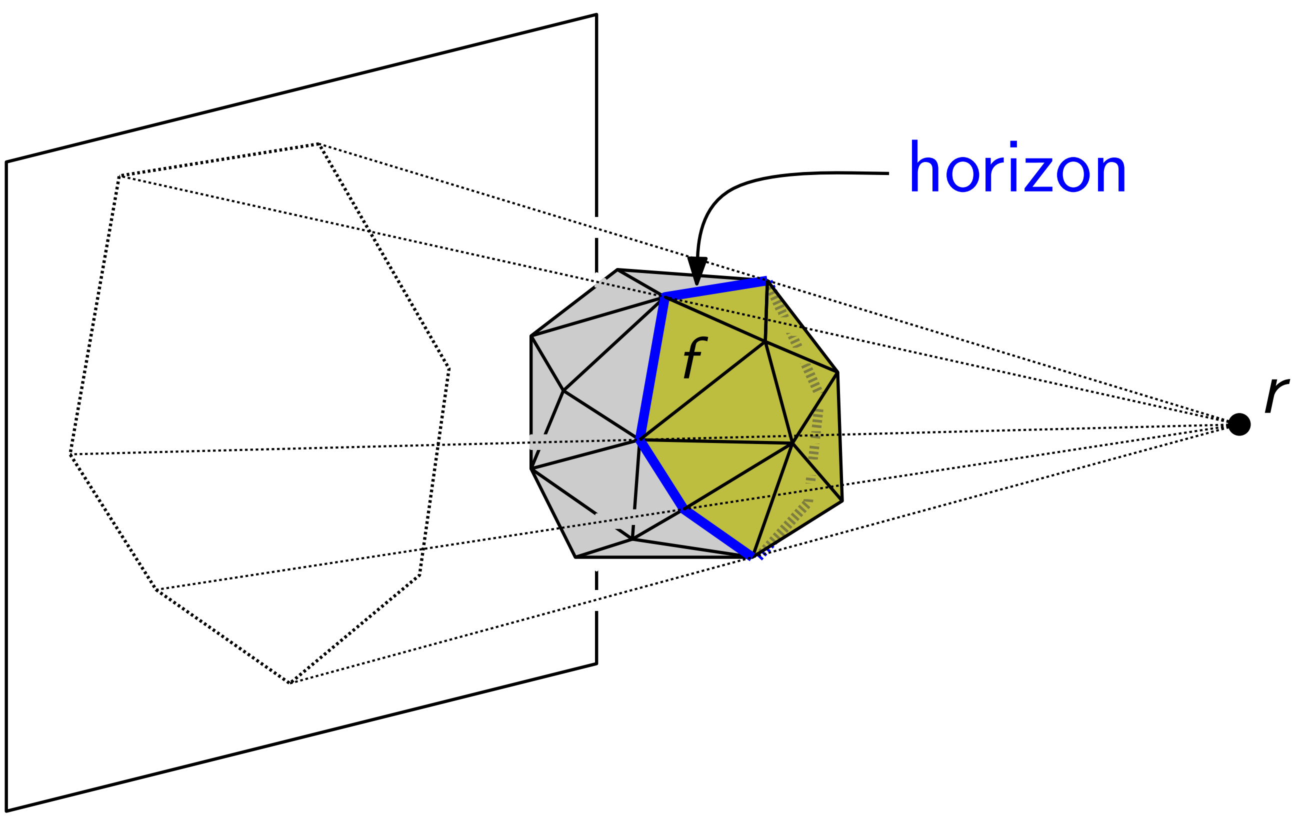
Construction

randomized-incremental!

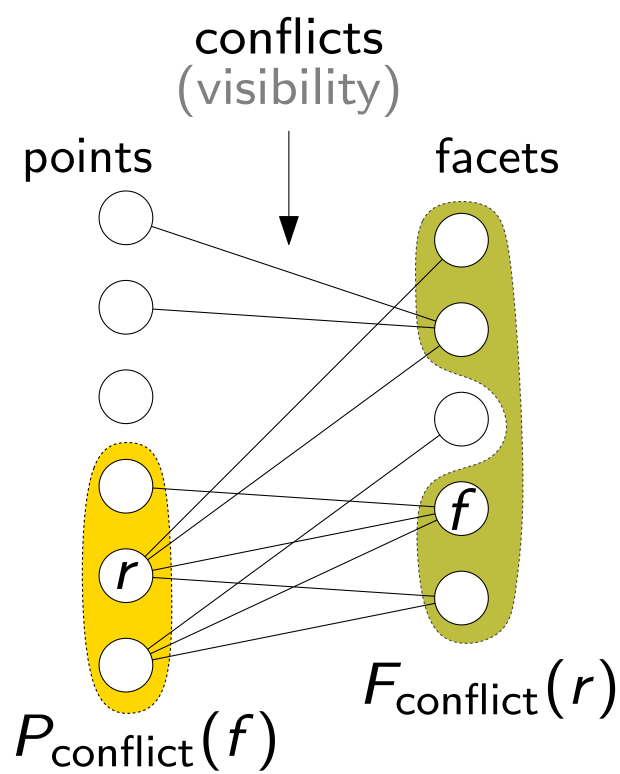
Visibility



Face f is *visible* from p but not from q.



Define conflict graph G:



Rand3dConvexHull($P \subset \mathbb{R}^3$)

pick non-coplanar set $P' = \{p_1, \ldots, p_4\} \subseteq P$

$C \leftarrow \mathrm{CH}(P')$

compute rand. perm. $(p_5, \ldots, p_n)$ of $P \setminus P'$

initialize conflict graph G

for $r = 5$ **to** n **do**

  if $F_{\text{conflict}}(p_r) \neq \emptyset$ **then** $\{ p_r \notin C \}$

    delete all facets in $F_{\text{conflict}}(p_r)$ from C

    $\mathcal{L} \leftarrow$ list of horizon edges visible from p_r

    foreach $e \in \mathcal{L}$ **do**

      $f \leftarrow C$.create_facet(e, p_r); create vtx for f in G

      $(f_1, f_2) \leftarrow$ previously_incident$_C(e)$

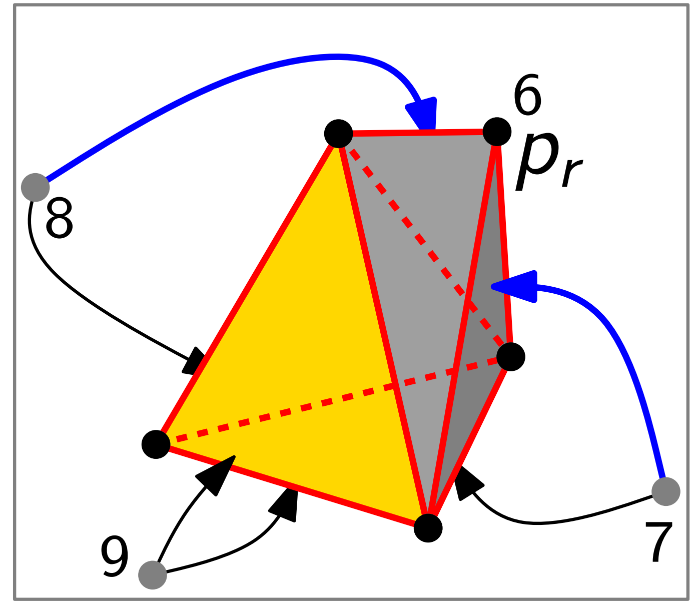
      $P(e) \leftarrow P_{\text{conflict}}(f_1) \cup P_{\text{conflict}}(f_2)$

      foreach $p \in P(e)$ **do**

        if f is visible from p **then** add edge (p, f) to G

    delete vtc $\{p_r\} \cup F_{\text{conflict}}(p_r)$ from G

return C



Worst-case running time $= O(n^3)$

Analysis

Idea: Bound expected *structural change*,
that is, the total #facets created by the algorithm.

Lemma. The expected #facets created is at most $6n - 20$.

Proof. $E[\#\text{facets created}] =$

$$= 4 + \sum_{r=5}^{n} E[\underbrace{\#\text{facets incident to } p_r \text{ in } \mathrm{CH}(P_r)}_{\deg(p_r,\, \mathrm{CH}(P_r))}] \le 6n - 20$$

(#edges)

For $r > 4$:

$$E[\deg(p_r, \mathrm{CH}(P_r))] = \frac{1}{r-4} \sum_{i=5}^{r} \deg(p_i, \mathrm{CH}(P_r))$$

$$\le \frac{1}{r-4} \left[\underbrace{\left(\sum_{i=1}^{r} \deg(p_i) \right)}_{2 \cdot \#\text{ edges of } \mathrm{CH}(P_r)} - 12 \right]$$

$$\le \frac{1}{r-4} \left[2 \cdot (3r - 6) - 12 \right] \le 6$$

Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

$O(n)$ time

```
Rand3dConvexHull(P ⊂ ℝ³)
  pick set P' = {p_1, ..., p_4} ⊆ P of 4 non-coplanar pts
  C ← CH(P')
  compute a random permutation (p_5, ..., p_n) of P \ P'
  initialize conflict graph G
  for r = 5 to n do
    if F_conflict(p_r) ≠ ∅ then
      delete all facets in F_conflict(p_r) from C
      ℒ ← list of horizon edges visible from p_r
      foreach e ∈ ℒ do
        f ← C.create_facet(e, p_r); create vtx for f in G
        (f_1, f_2) ← previously_incident_C(e)
        P(e) ← P_conflict(f_1) ∪ P_conflict(f_2)
        foreach p ∈ P(e) do
          if f visible from p then add edge (p, f) to G
      delete vtc {p_r} ∪ F_conflict(p_r) from G
  return C
```

Stage r of for-loop (w/o outer foreach loop) takes time $O(|F_{\text{conflict}}(p_r)|) =$ $O(\#\text{facets deleted when adding } p_r)$

This part of for-loop in total:
$E[\#\text{facets deleted}] =$
$\leq E[\#\text{facets created}] = O(n)$.
Lemma

Outer foreach-loop:
– in stage r: $O(\sum_{e \in \mathcal{L}} |P(e)|)$
– in total:

$$O\left(\sum_{e \text{ on horizon at some moment}} |P(e)|\right)$$

using *configuration spaces*, Section 9.5 [De Berg et al.] → $= O(n \log n)$

Running times – expected vs. worst case

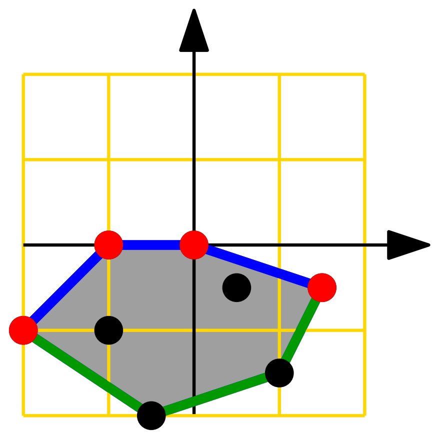
Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

Exercise: Give a simple deterministic algorithm that computes the convex hull in $O(n^2)$ (worst-case) time.

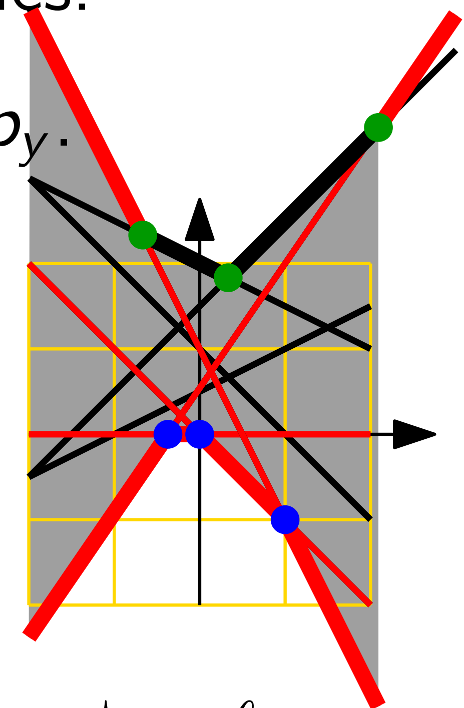
Convex Hulls and Half-~~Space~~ Plane Intersections

Define dualtity $\star$ between pts and (non-vertical) lines:

For $p = (p_x, p_y)$, define the line $p^\star\colon\ y = p_x x - p_y$.



primal ⟷ dual



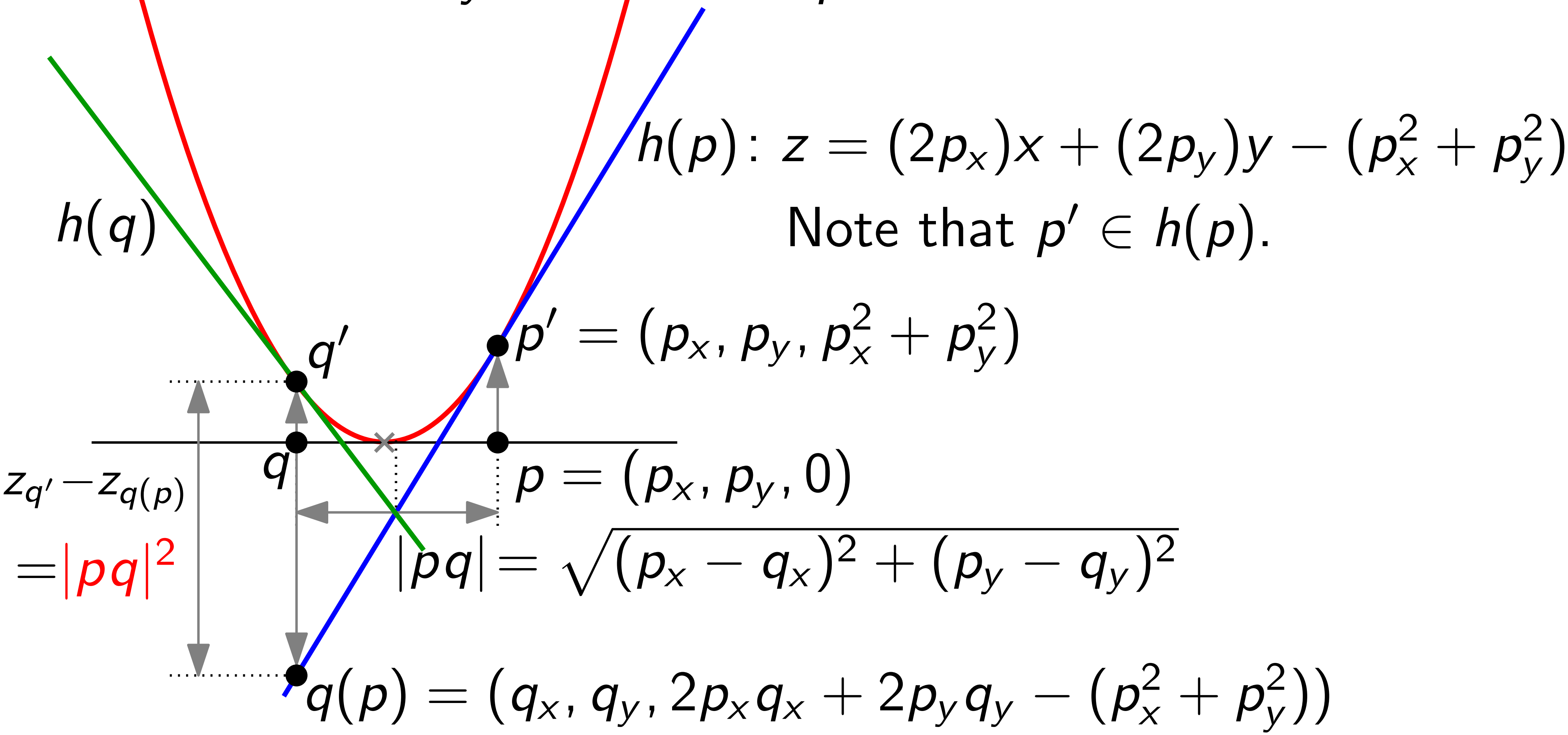
For $\ell\colon\ y = mx + b$, define $\ell^\star$ to be the pt q with $q^\star = \ell$,

that is, $\ell^\star = (m, -b)$.

Observe:
- upper convex hulls of pts $\leftrightarrow$ lower envelopes of lines
- can compute intersections of "lower/upper" half planes (spaces) via upper/lower convex hulls

Voronoi Diagrams Revisited

Let $U\colon z = x^2 + y^2$ be the *unit paraboloid* in $\mathbb{R}^3$.



$\Rightarrow$ $h(p)$ and U encode dist. betw. p and any other pt in $z = 0$.

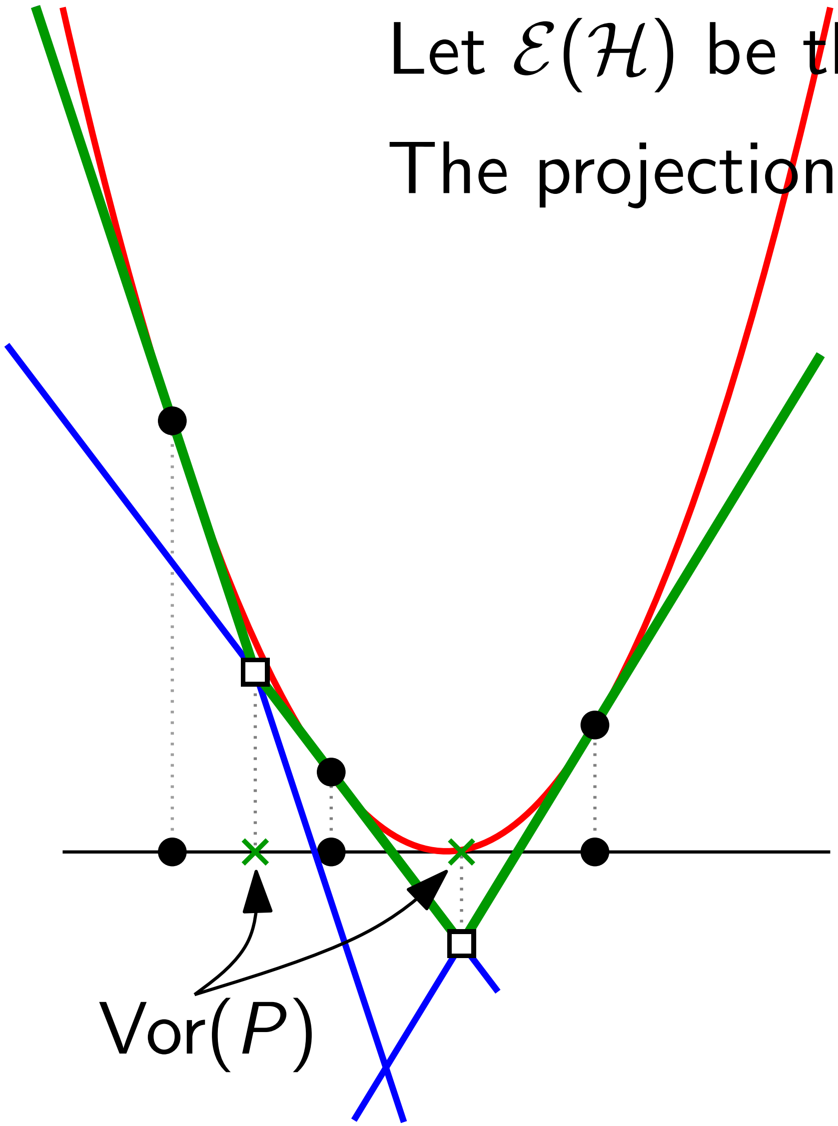
$\Rightarrow$ $h(p) \cap U = \{p'\}$ $\Rightarrow$ $h(p)$ is tangent to U (in p')

The Upper Envelope Strikes Back

Theorem: Let $P \subset \mathbb{R}^2 \times \{0\}$ and $\mathcal{H} = \{h(p) \mid p \in P\}$.

Let $\mathcal{E}(\mathcal{H})$ be the upper envelope of $\mathcal{H}$.

The projection of $\mathcal{E}(\mathcal{H})$ on $z = 0$ is $\mathrm{Vor}(P)$.



can compute $\mathrm{Vor}(P)$ in $\mathbb{R}^2$ via upper envelope in $\mathbb{R}^3$

↓ exercise 11.10

upper envelope in $\mathbb{R}^3$ is in one-to-one correspondence to lower convex hull of the pt set $\mathcal{H}^\star$

↓

use algorithm Rand3dConvexHull!