Julius-Maximilians-
UNIVERSITÄT WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Computational Geometry

Winter semester 2016/17

Convex Hulls in 3D

Lecture #9
(Chapter 11)

Joachim Spoerhase

Chair for Computer Science I

Complexity of the Convex Hull

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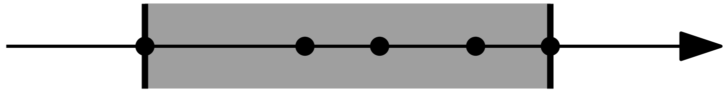

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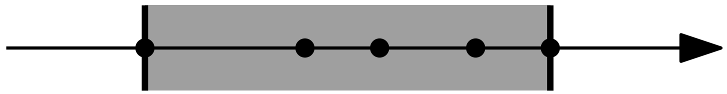

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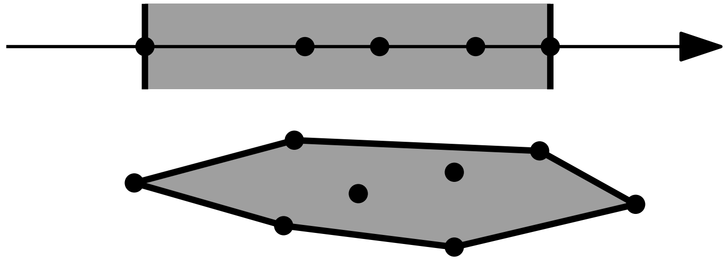

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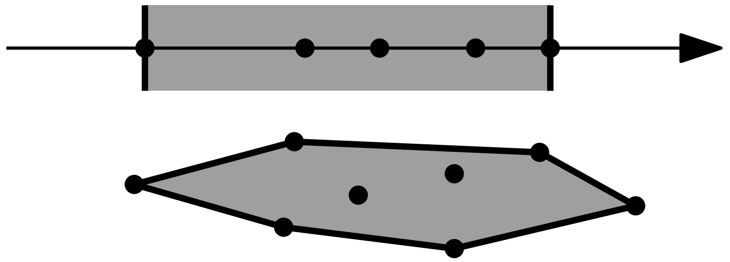

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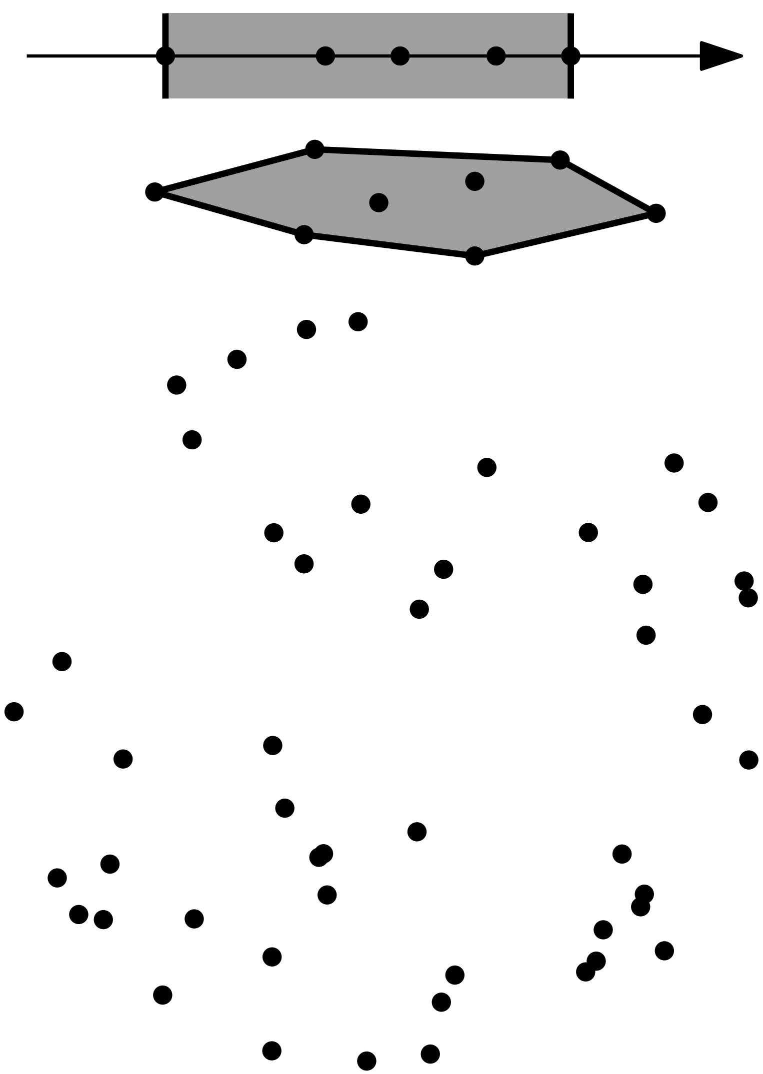

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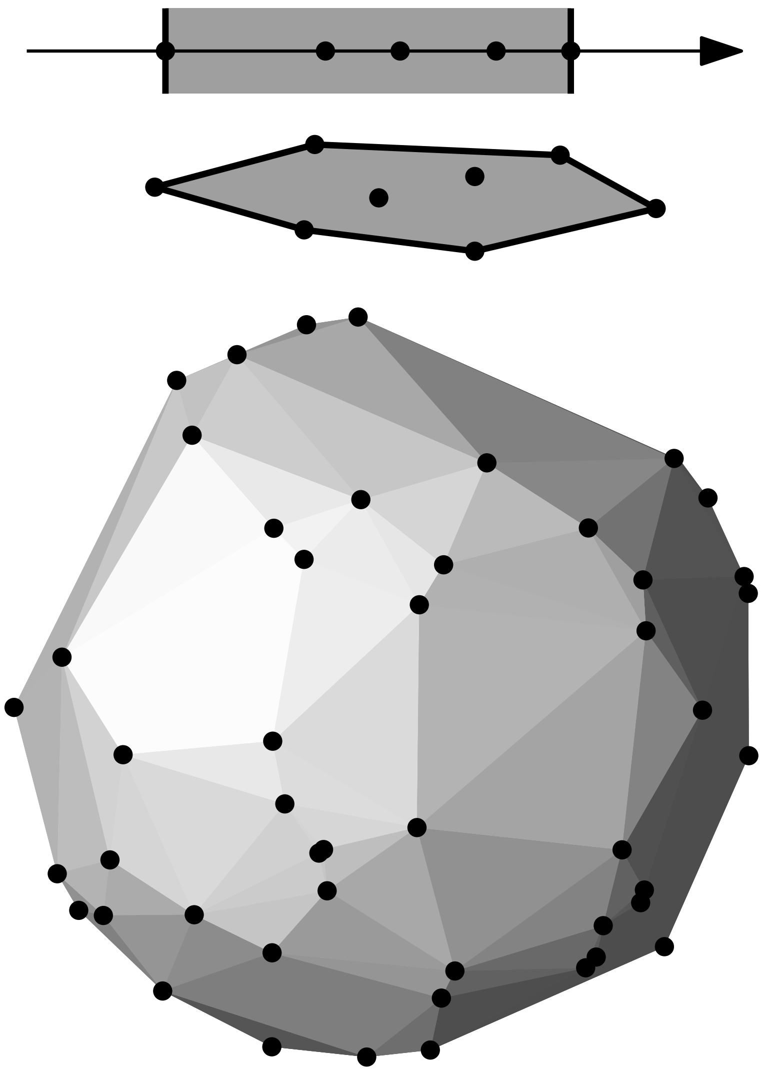

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3	**Your task!**
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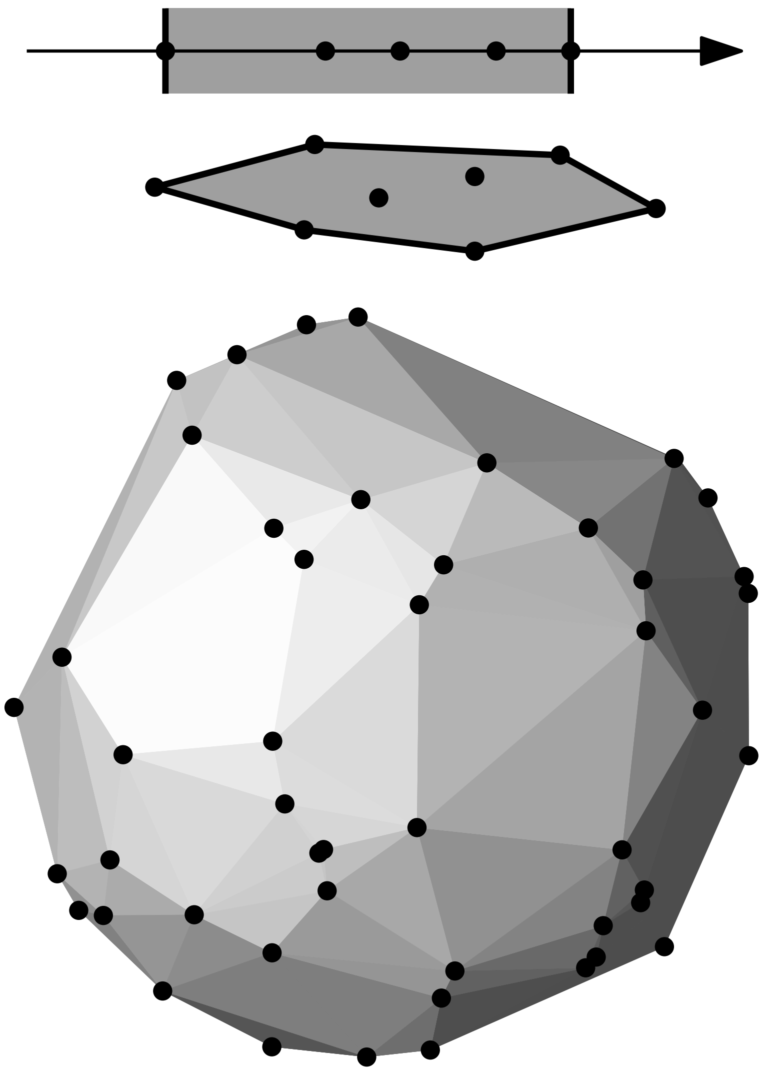

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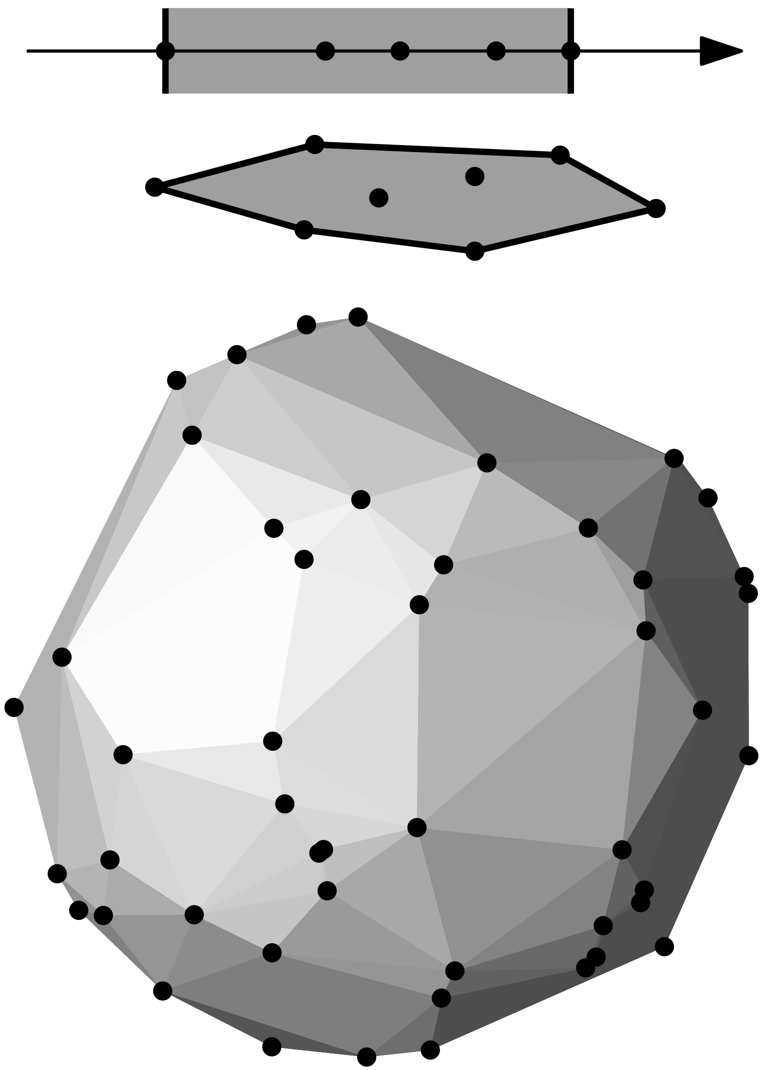

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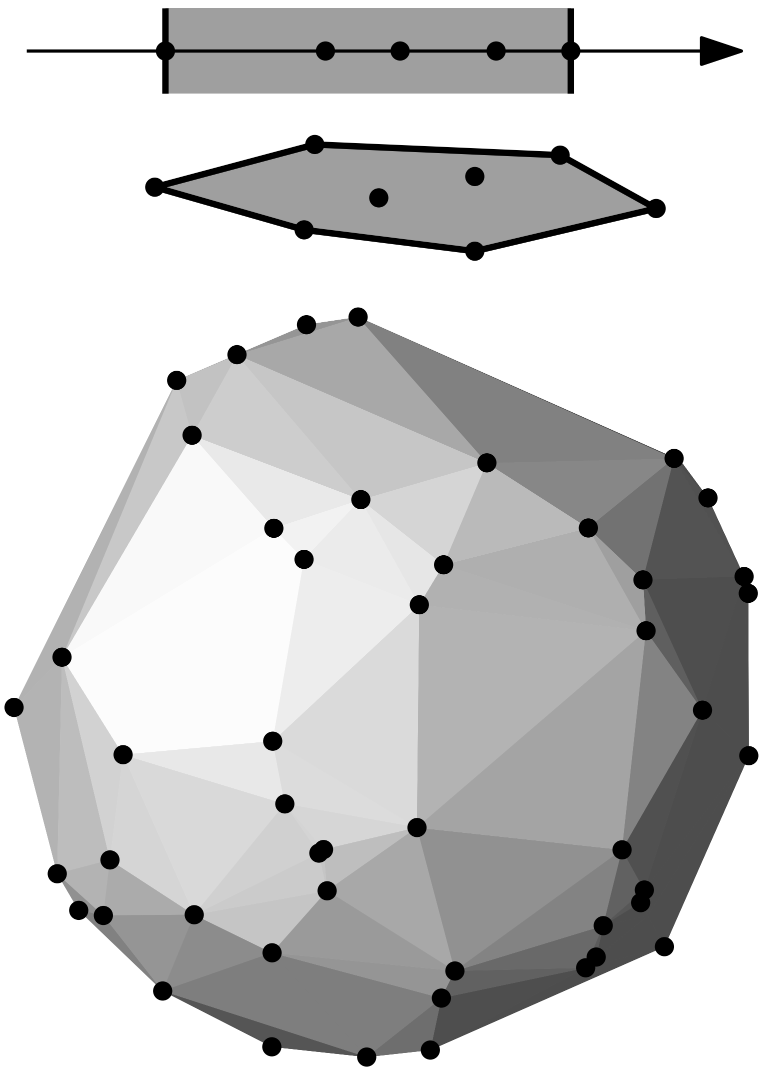

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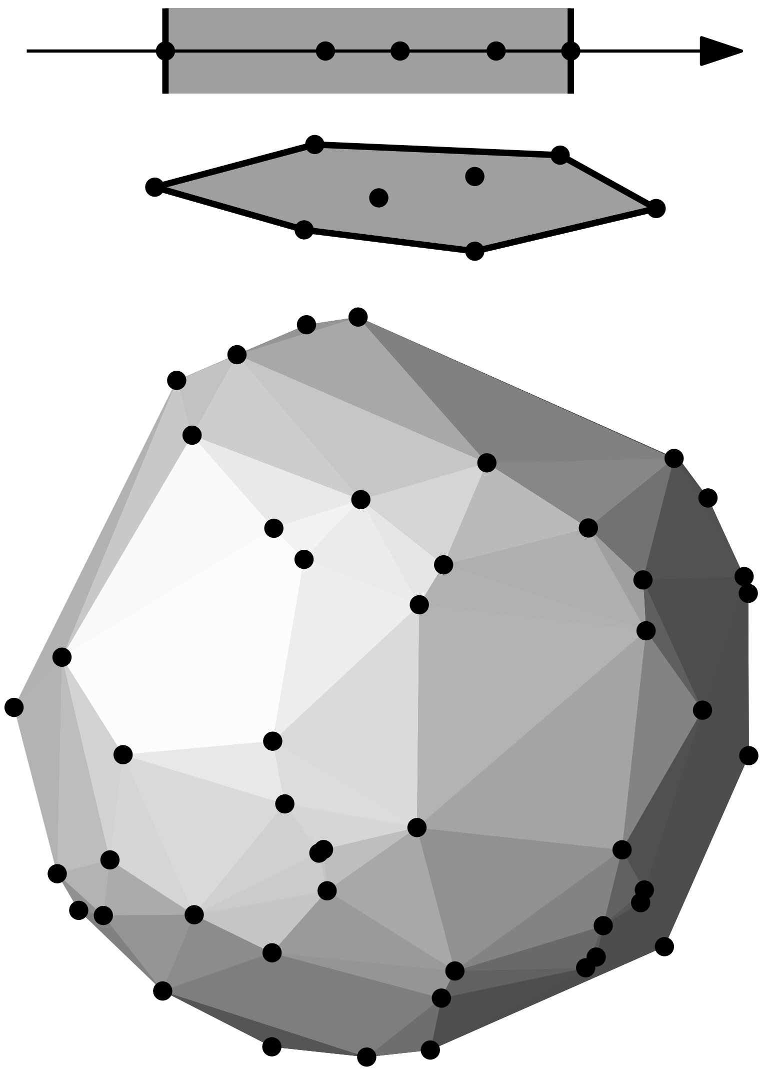
Upper Bound Theorem



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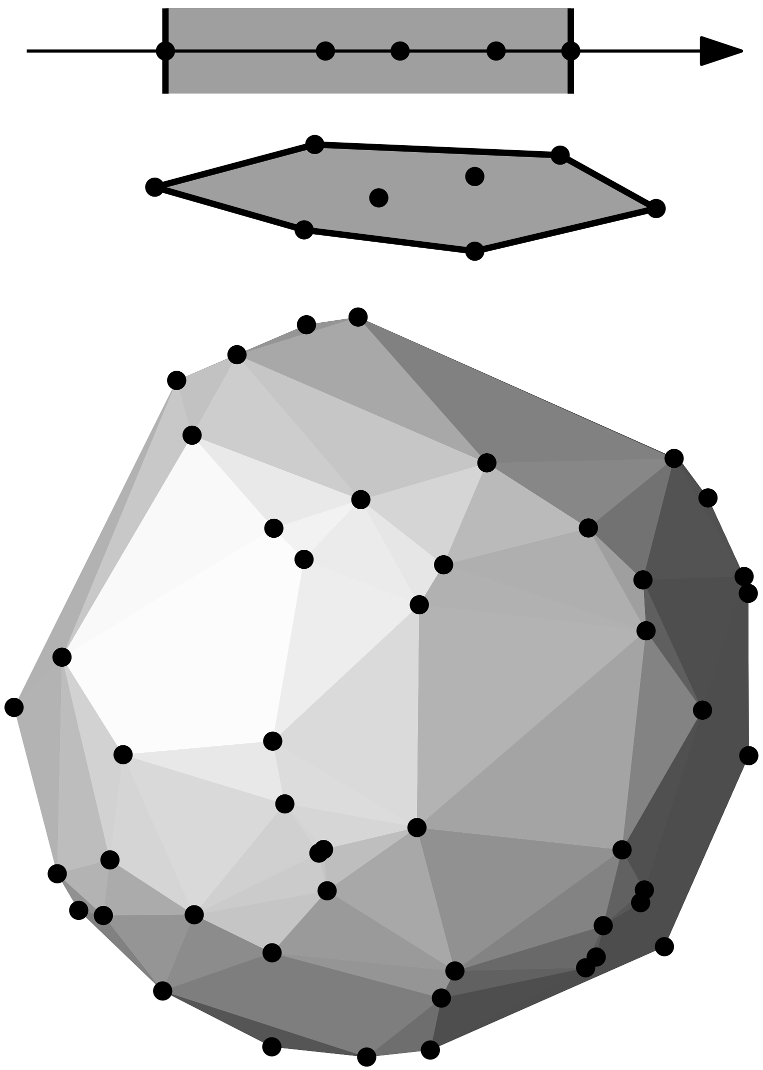

Construction?

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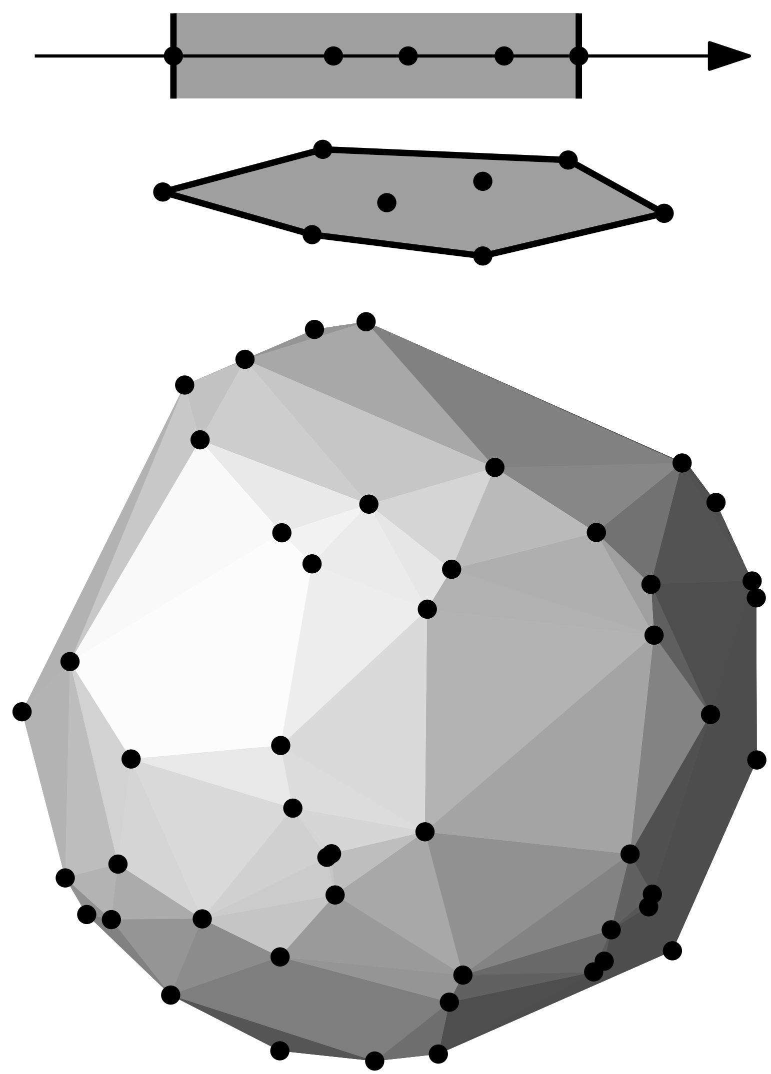
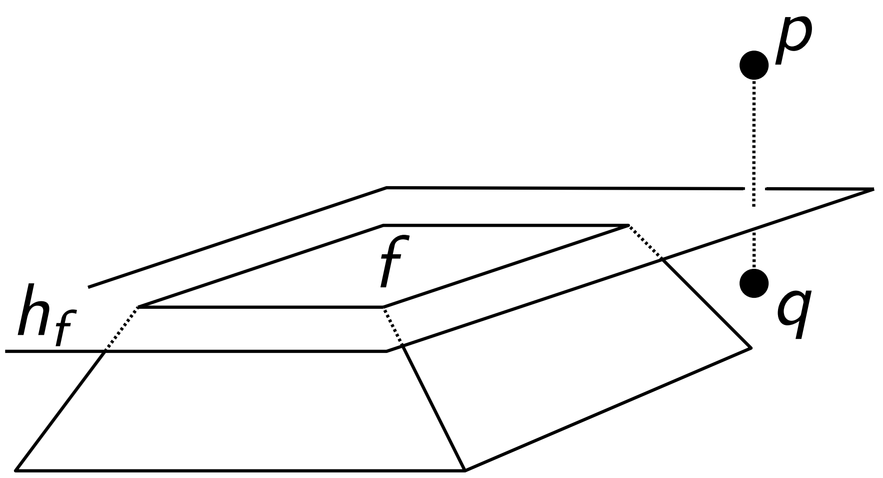


Construction

randomized-incremental!

Visibility



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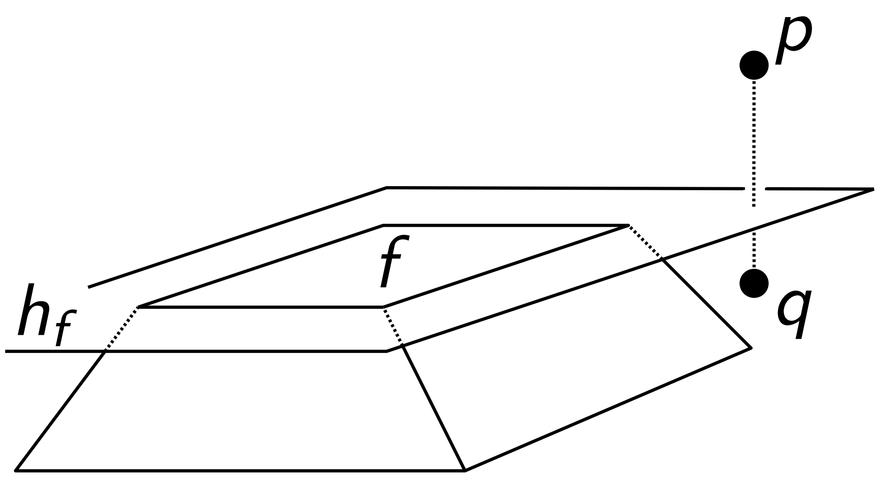

Face f is *visible* from p but not from q.

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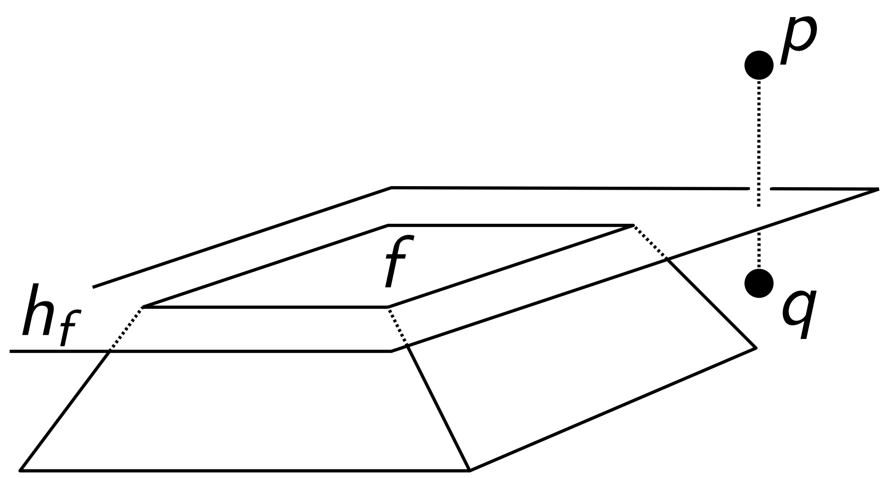

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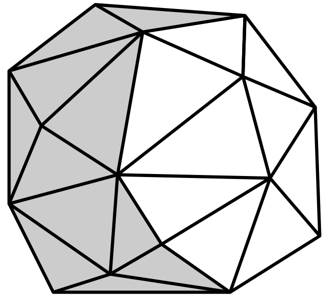

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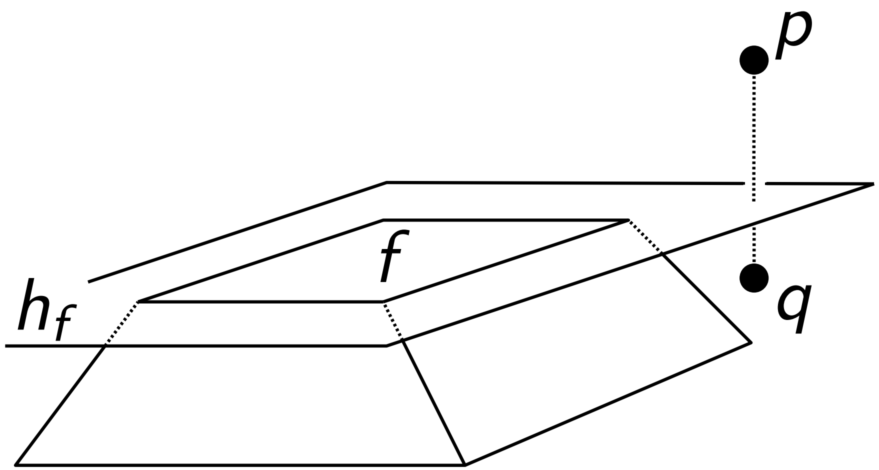

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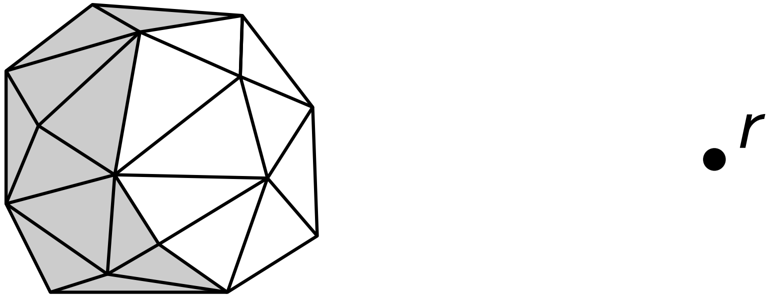

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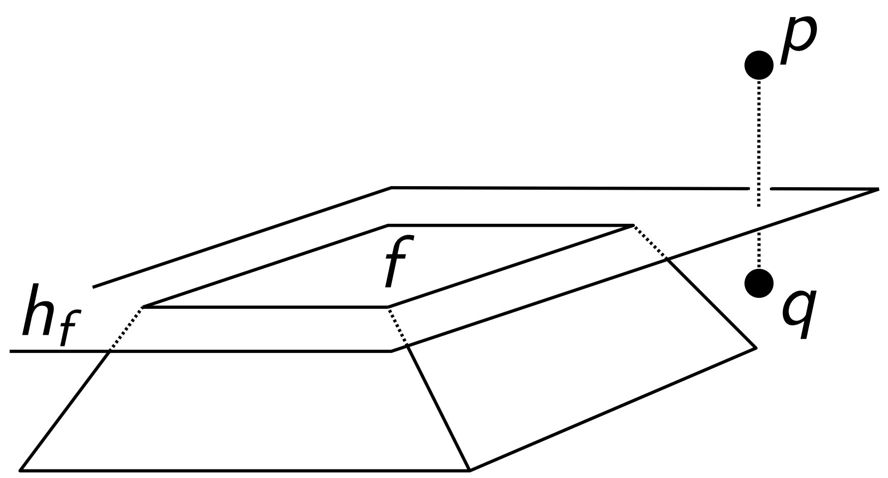

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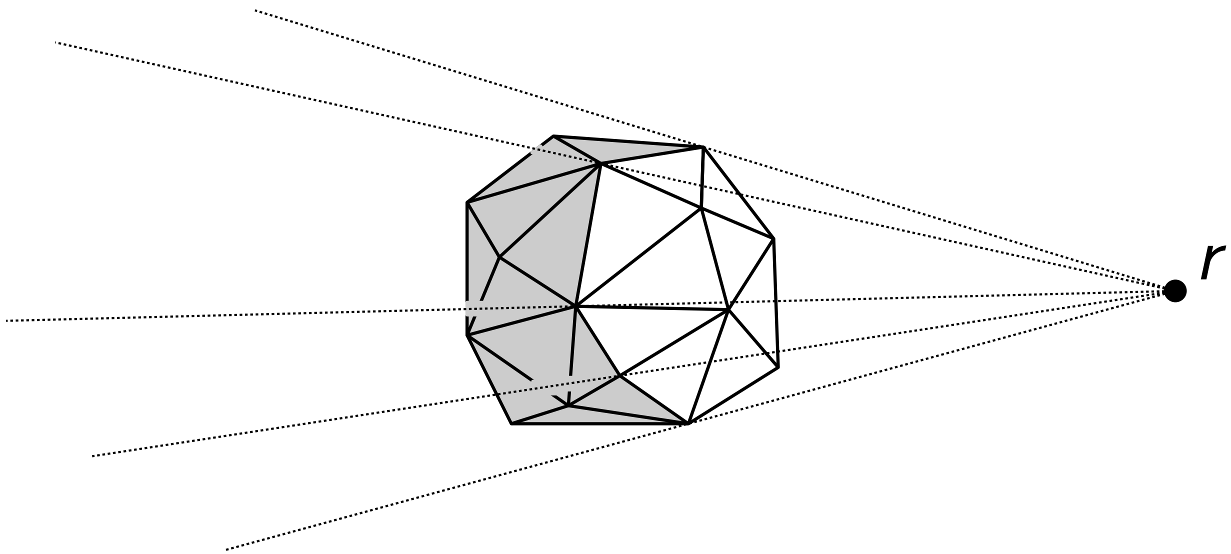

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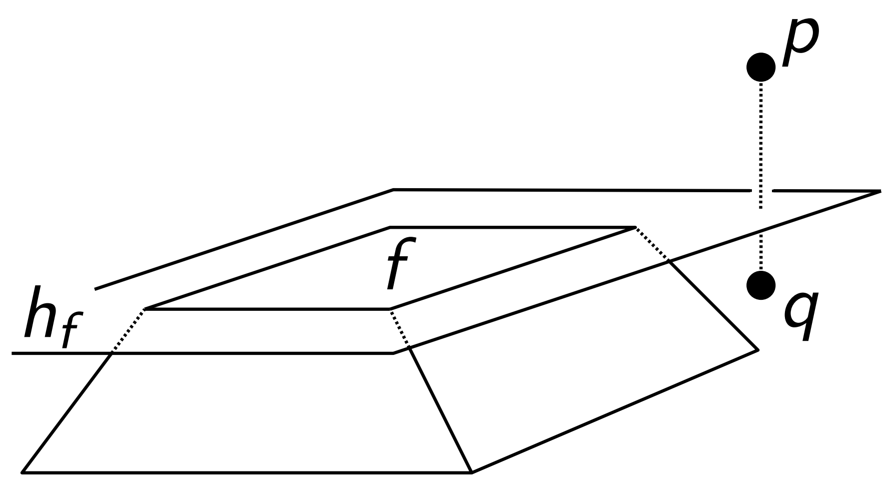

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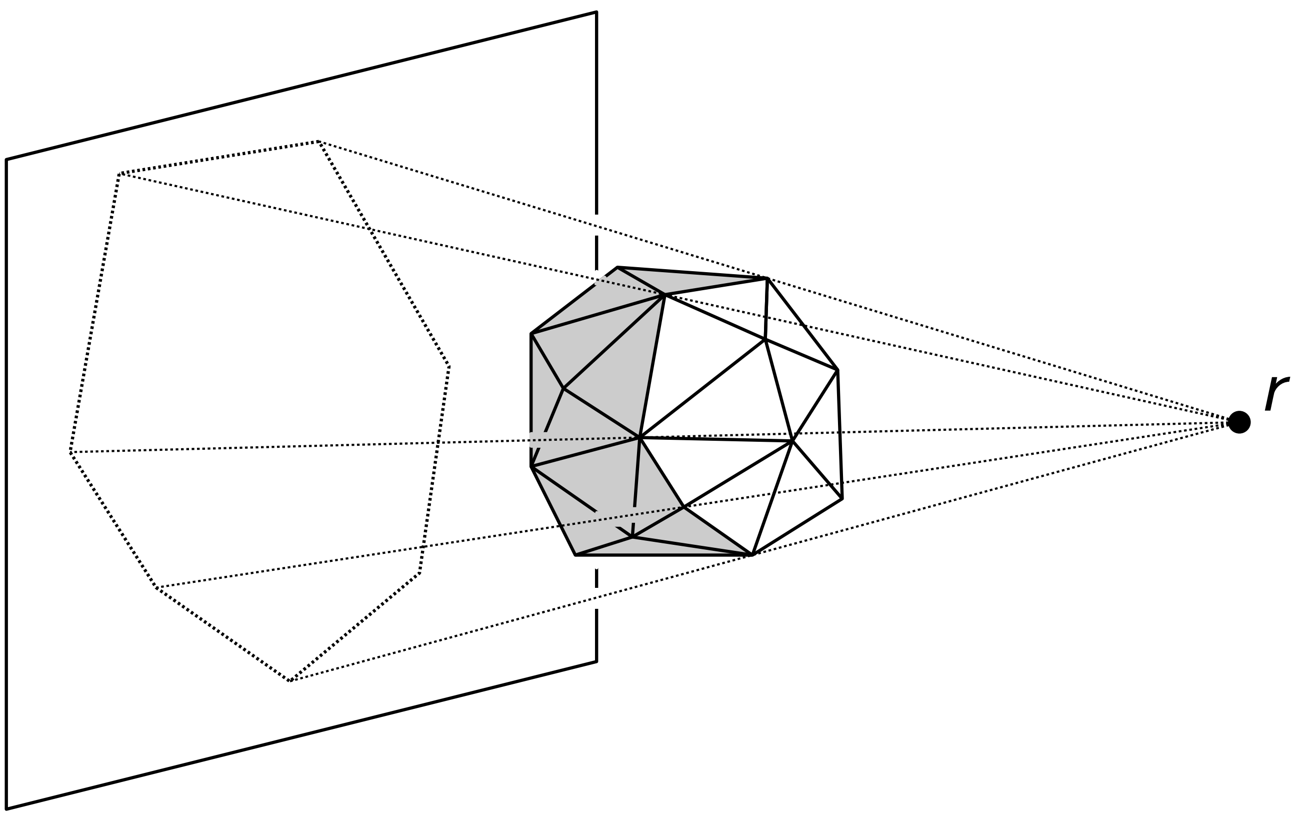

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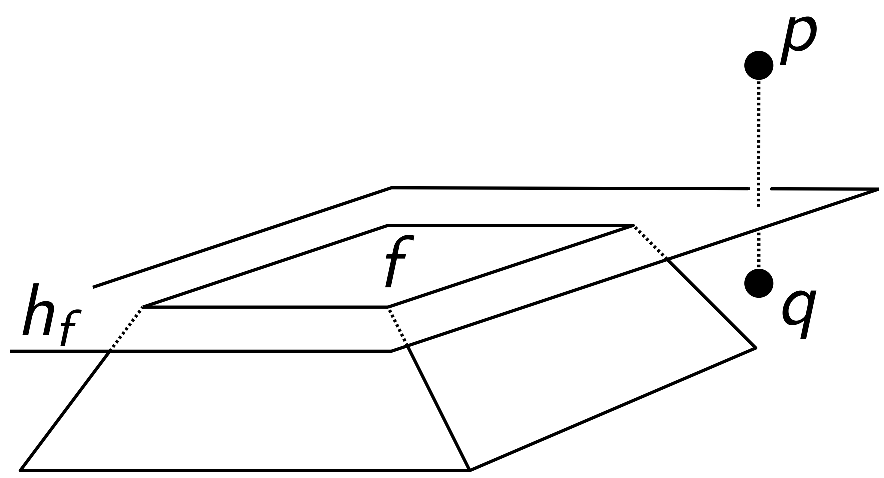

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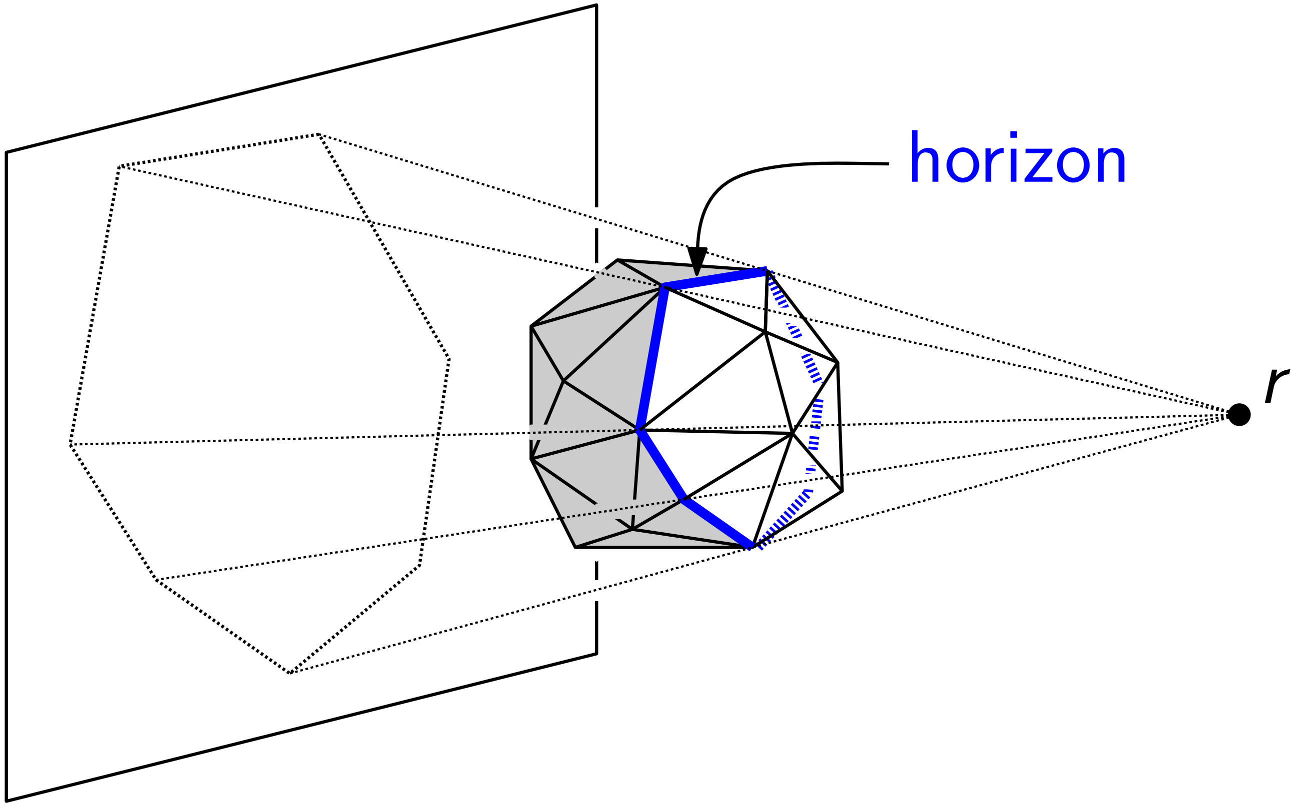

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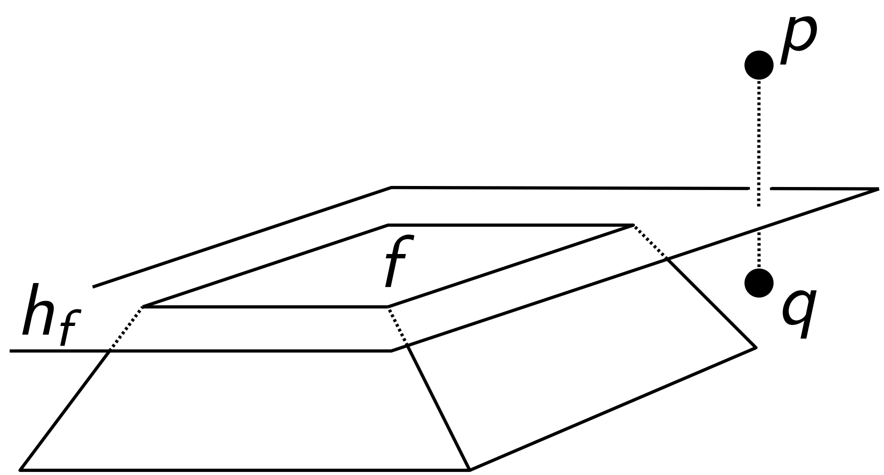

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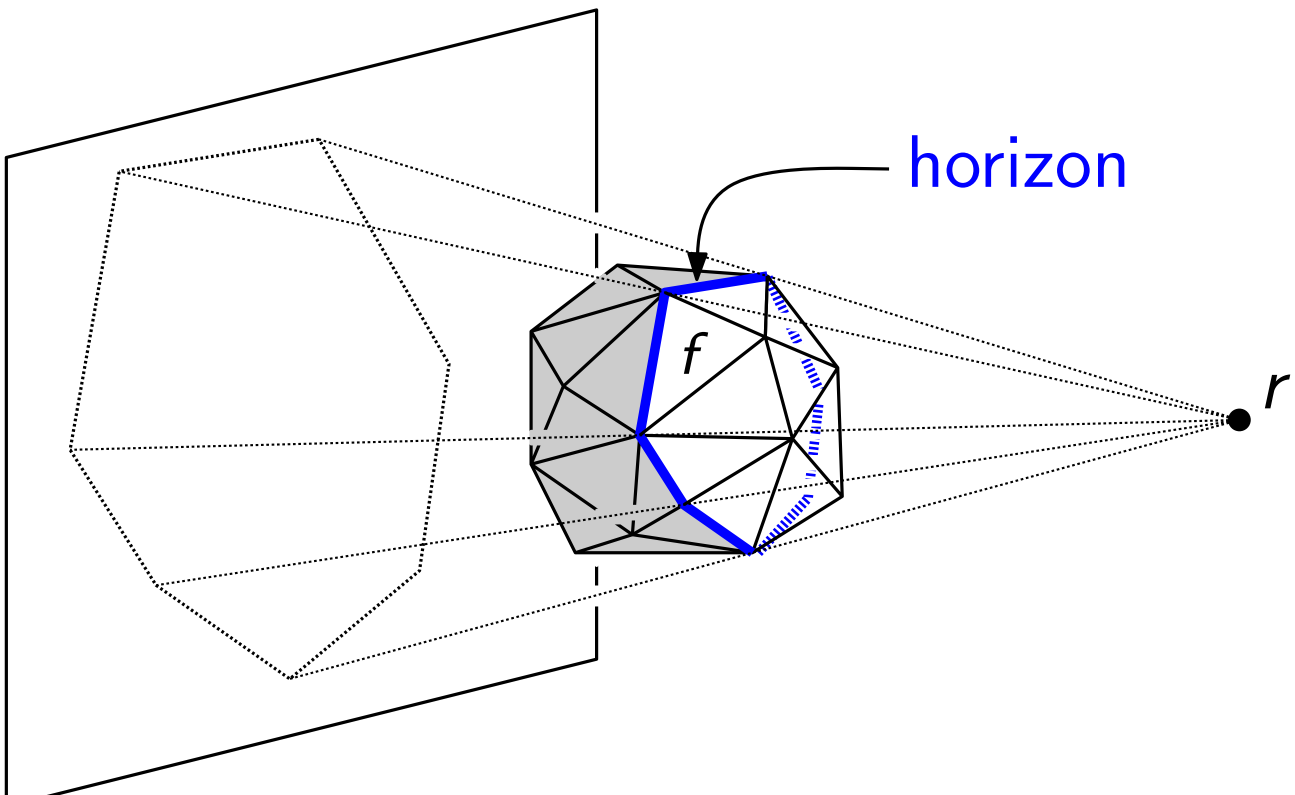
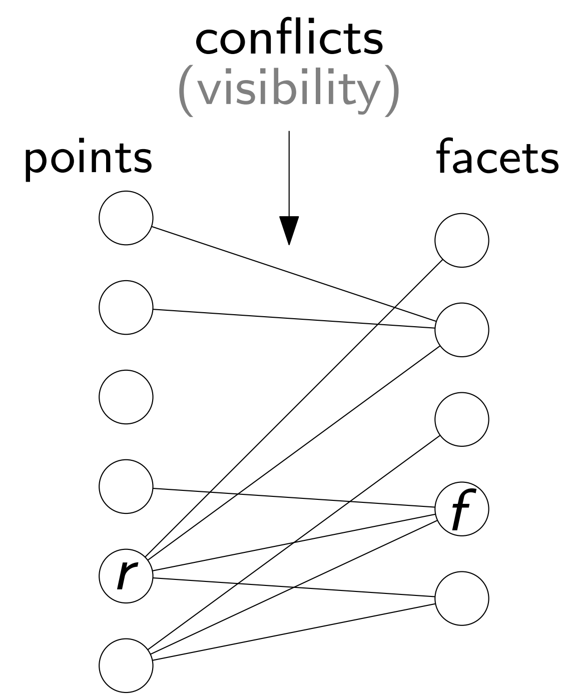


Define conflict graph G:



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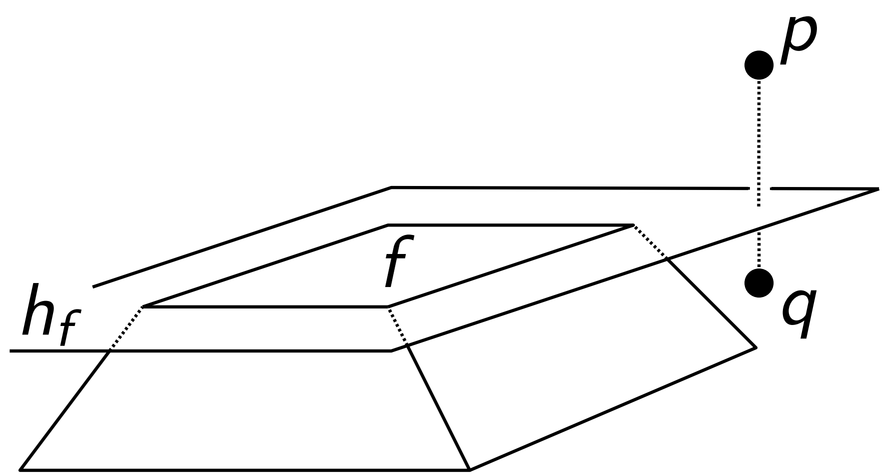

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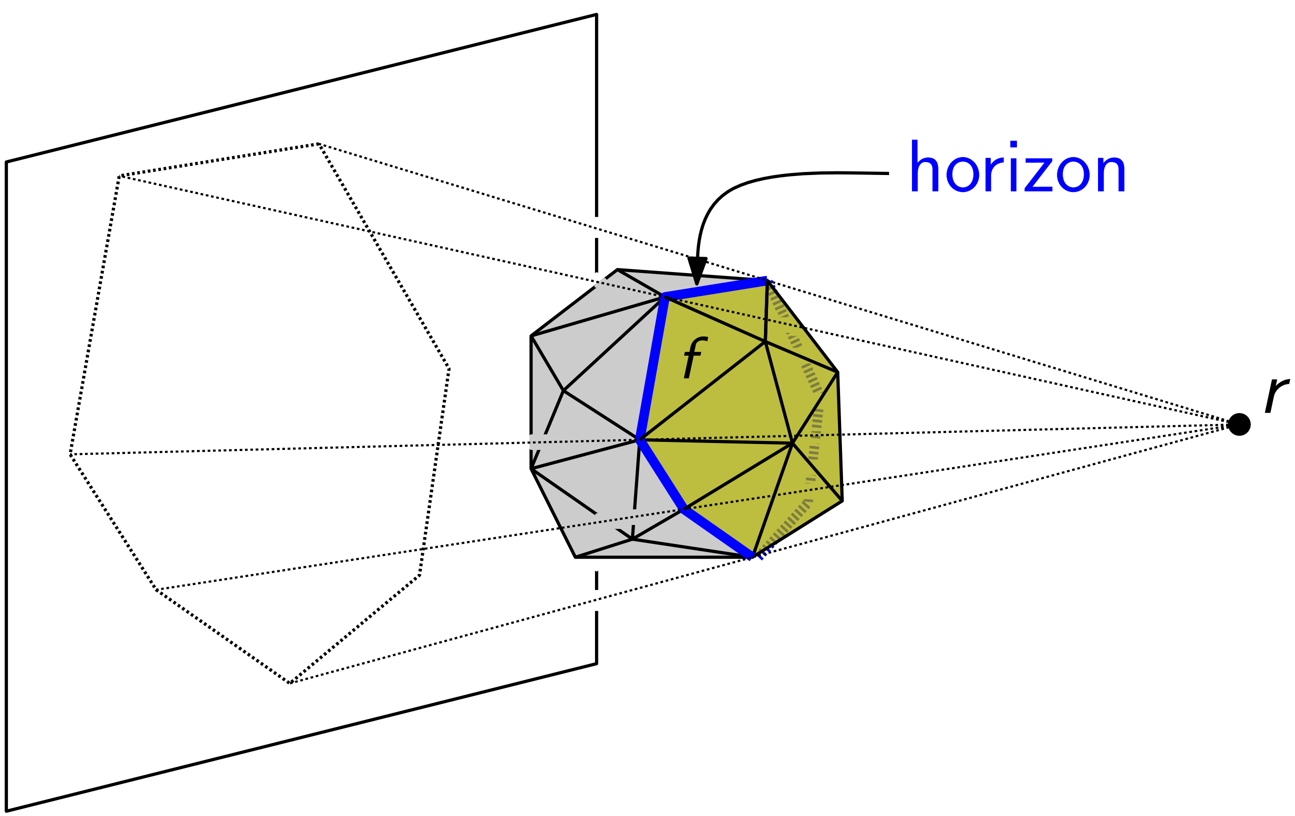

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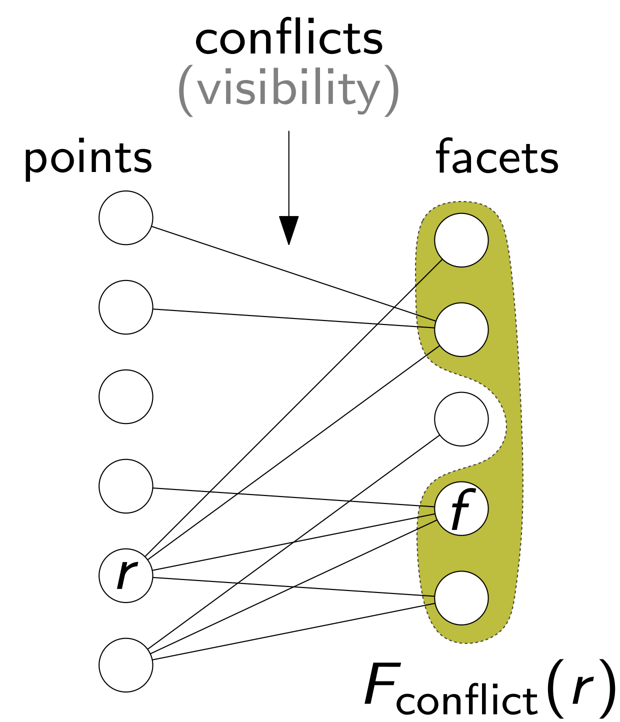

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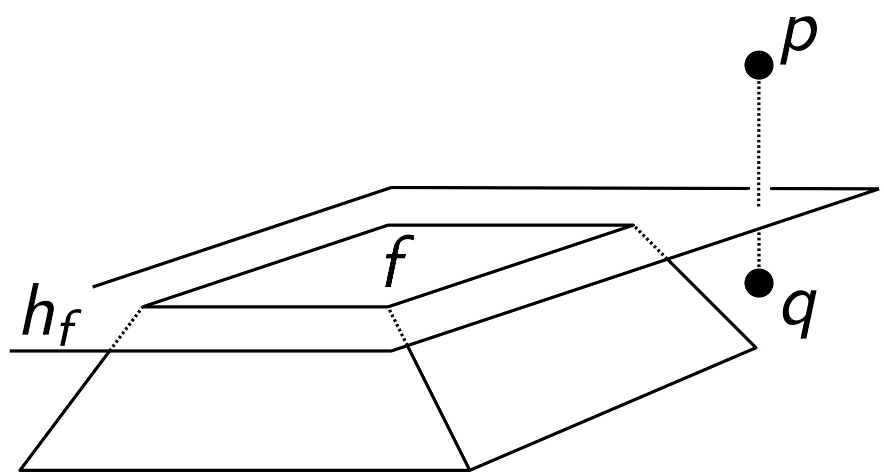

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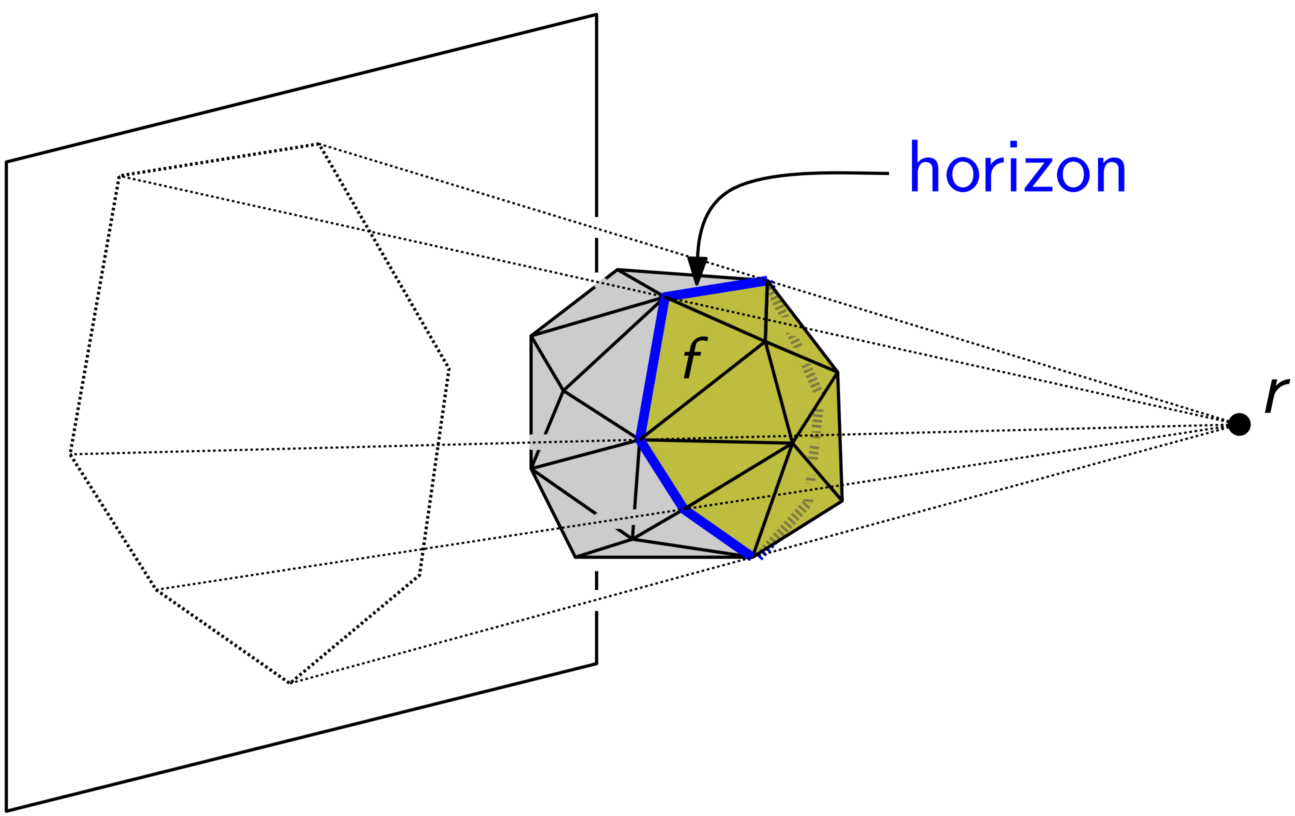

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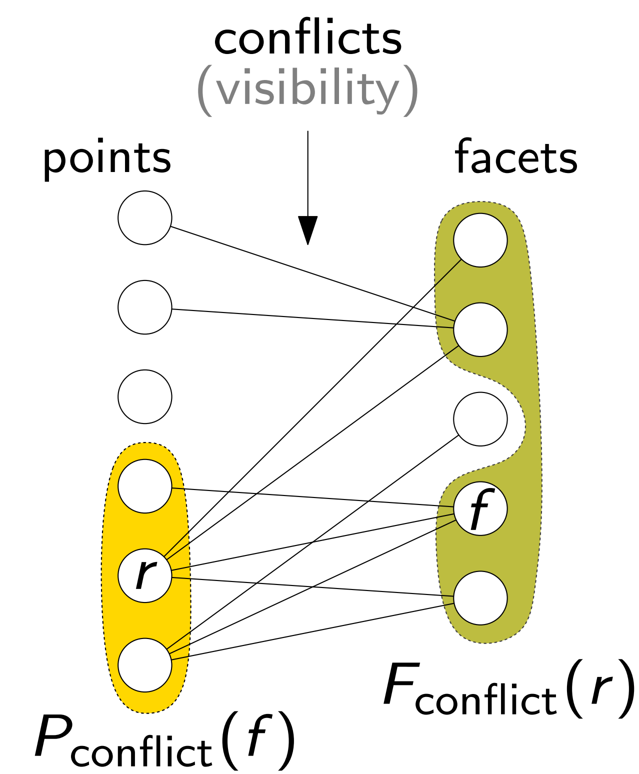
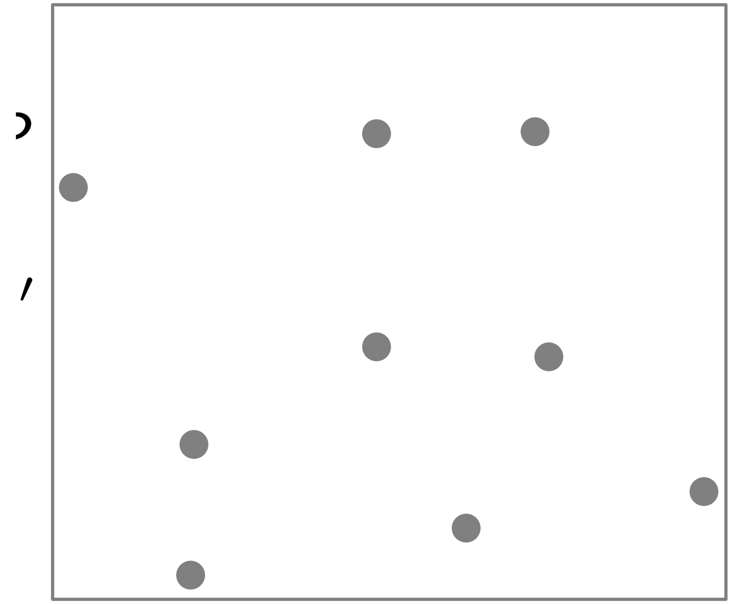


Rand3dConvexHull($P \subset \mathbb{R}^3$)



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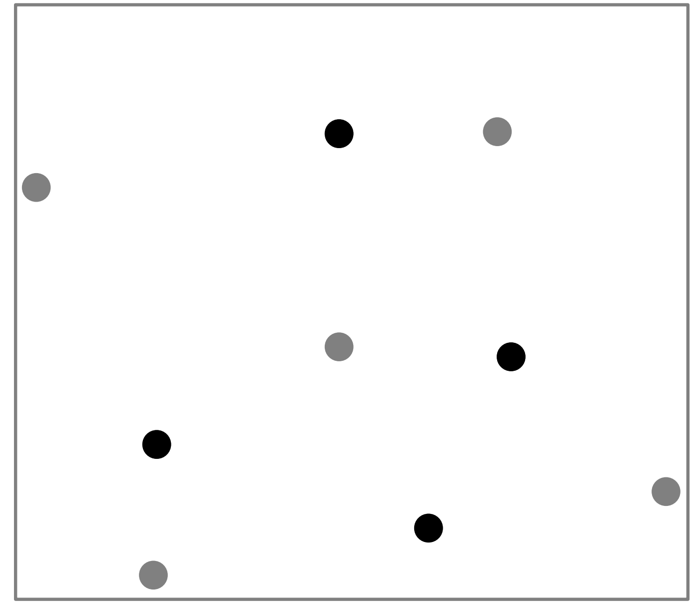
pick non-coplanar set $P' = \{p_1, \ldots, p_4\} \subseteq P$



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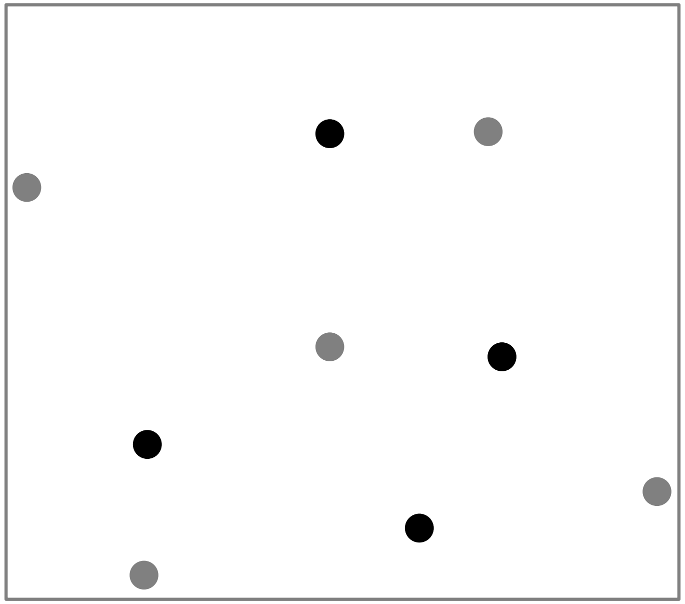

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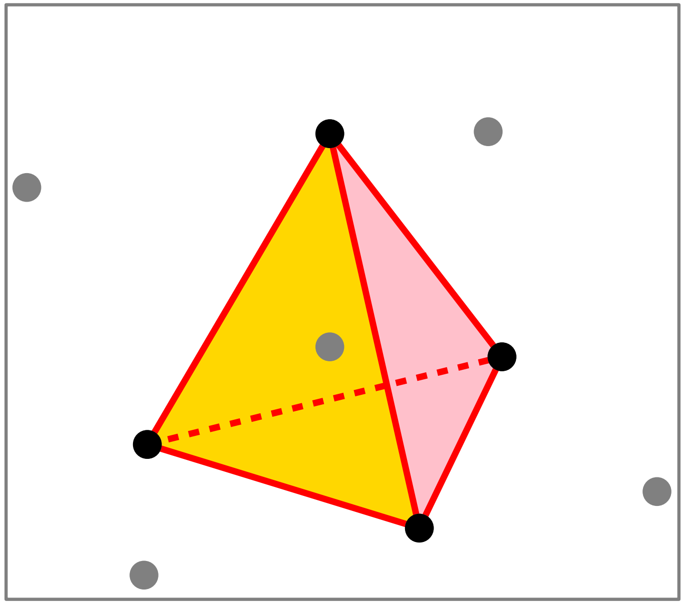

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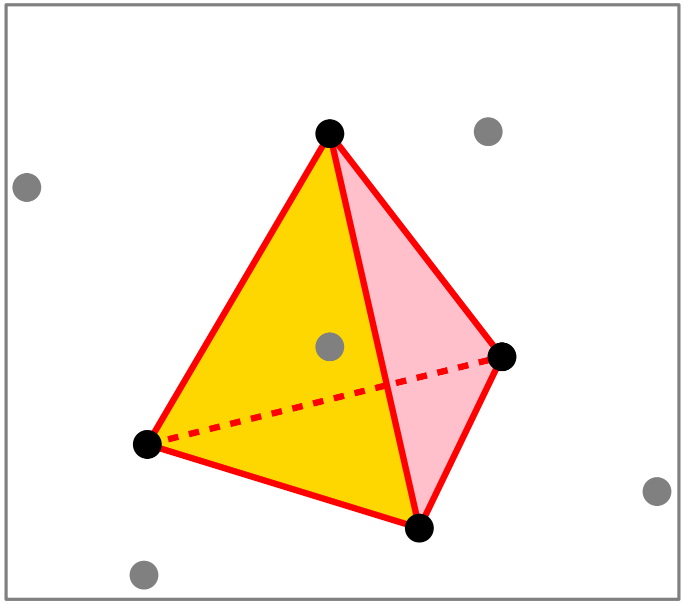

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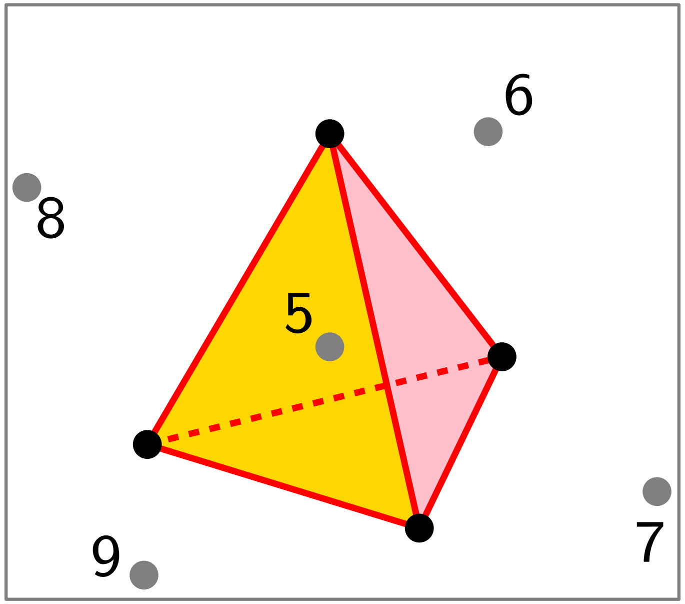

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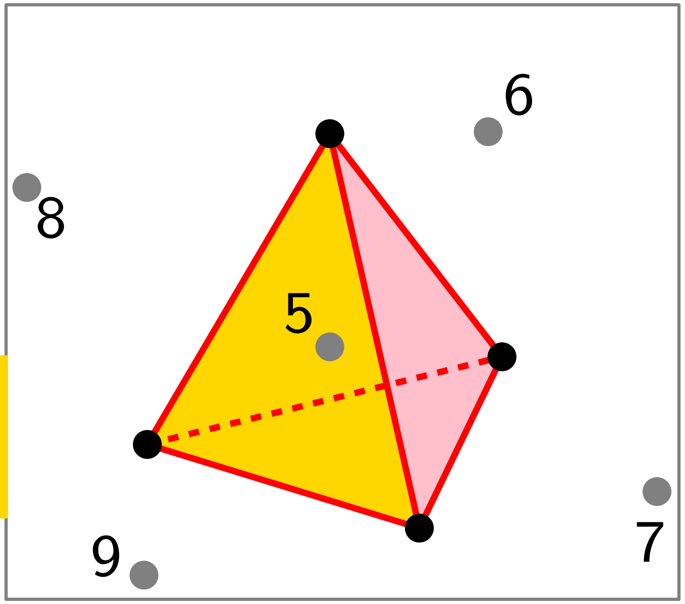

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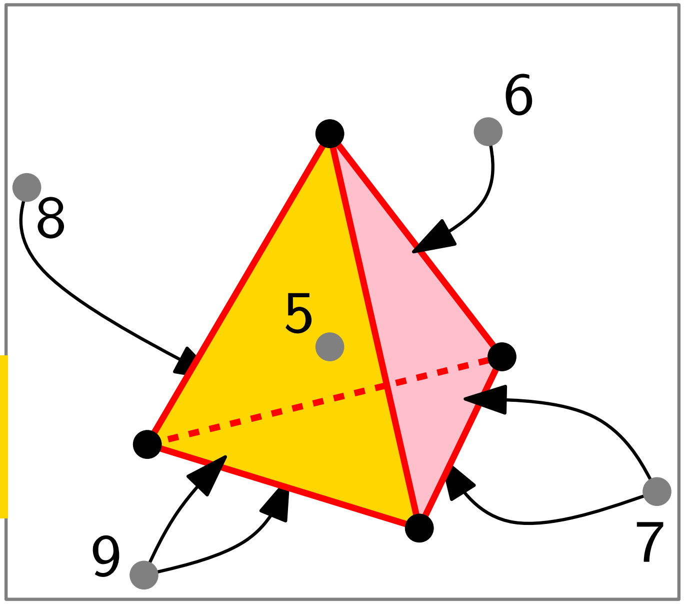

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for $r = 5$ **to** n **do**

  if $F_{\mathrm{conflict}}(p_r) \neq \emptyset$ **then** $\{\ p_r \notin C\ \}$

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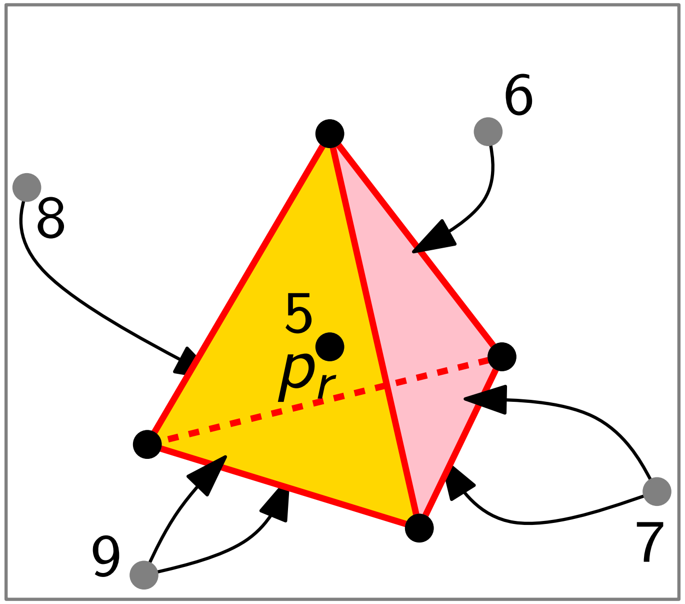

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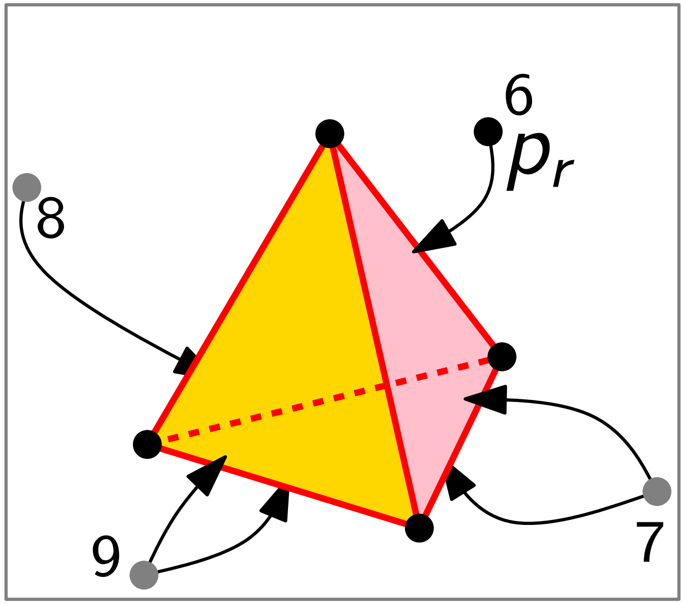

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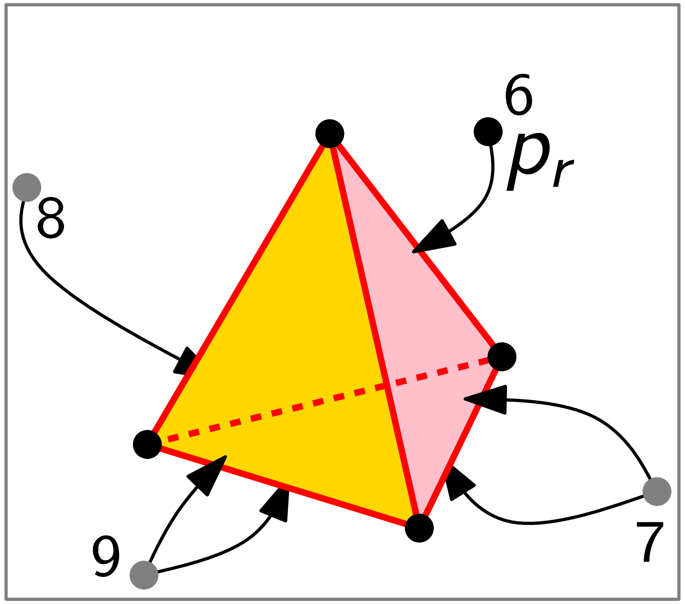

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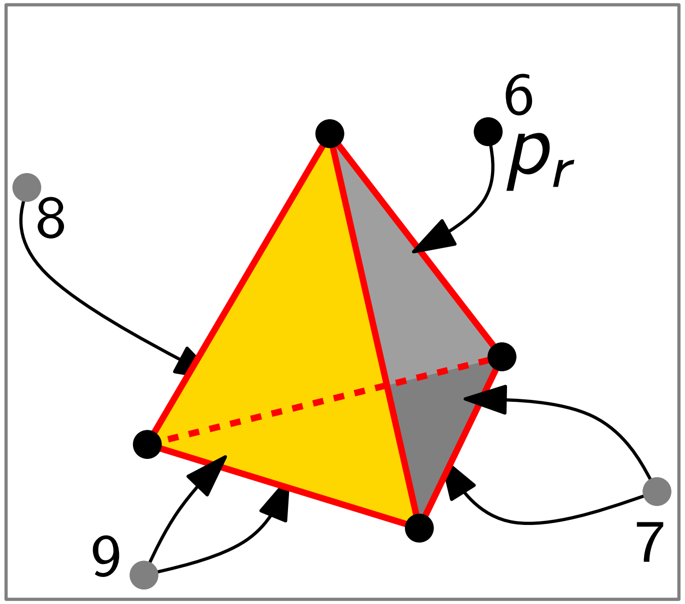

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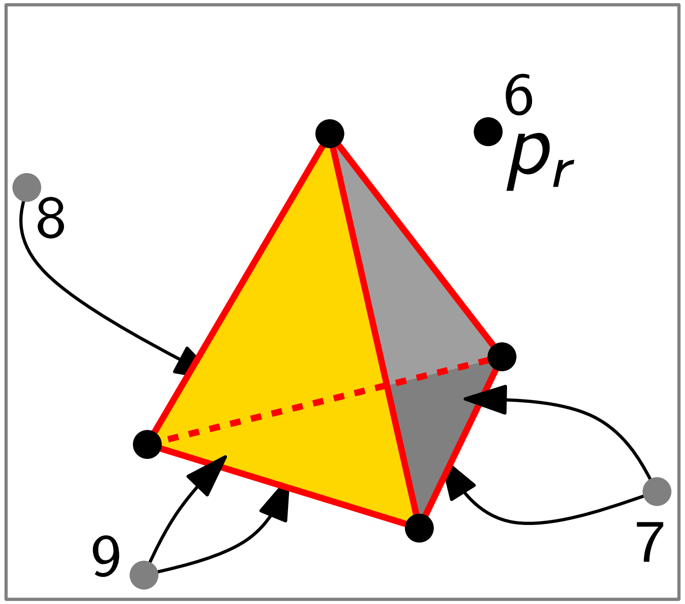

Rand3dConvexHull($P \subset \mathbb{R}^3$)

- pick non-coplanar set $P' = \{p_1, \ldots, p_4\} \subseteq P$
- $C \leftarrow \mathrm{CH}(P')$
- compute rand. perm. $(p_5, \ldots, p_n)$ of $P \setminus P'$
- initialize conflict graph G
- **for** $r = 5$ **to** n **do**
  - **if** $F_{\text{conflict}}(p_r) \neq \emptyset$ **then** $\{\ p_r \notin C\ \}$
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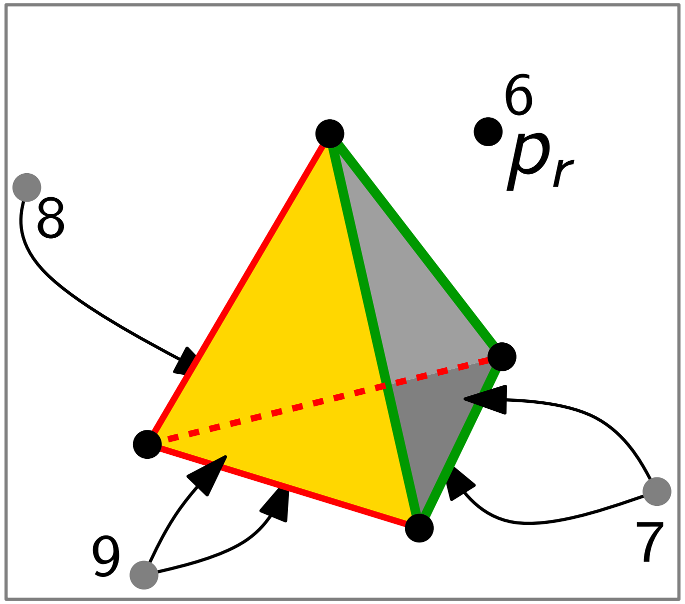
    - $\mathcal{L} \leftarrow$ list of horizon edges visible from p_r
- **return** C



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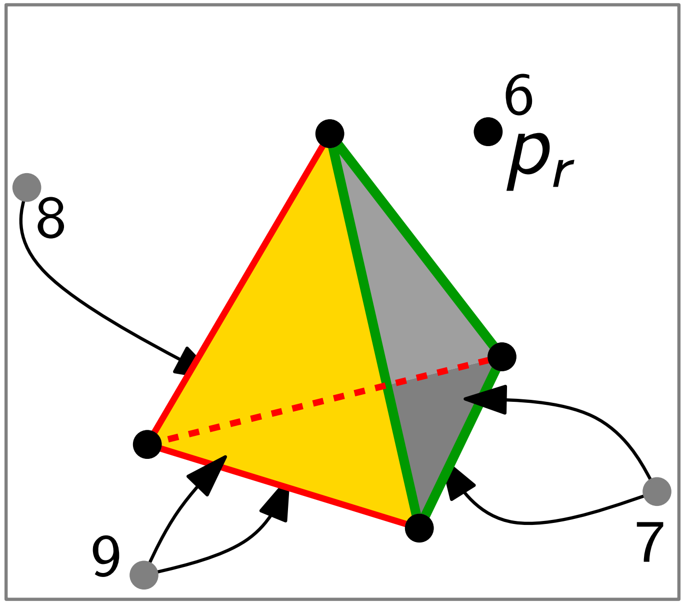

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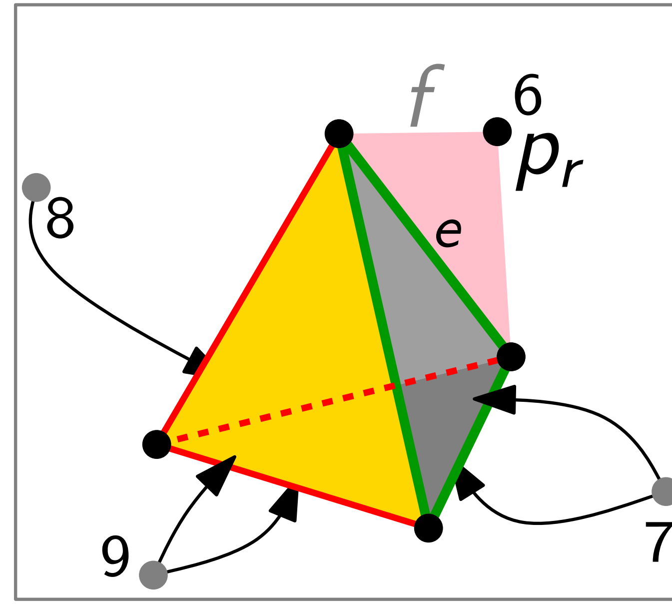

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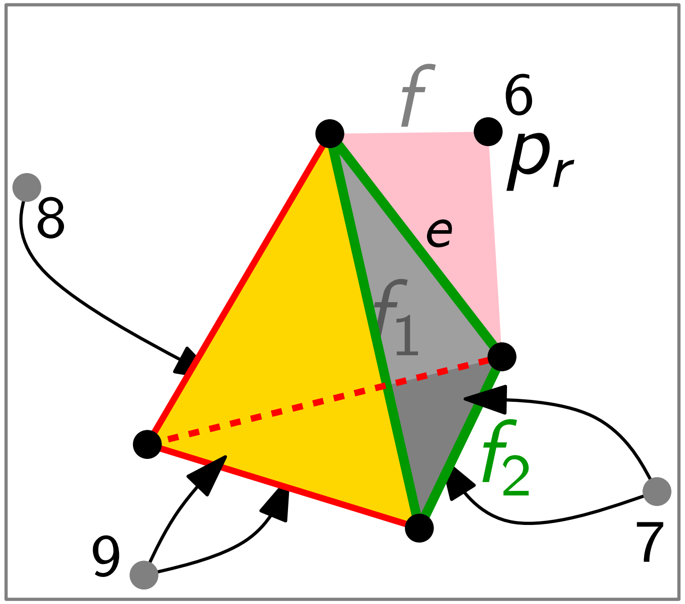

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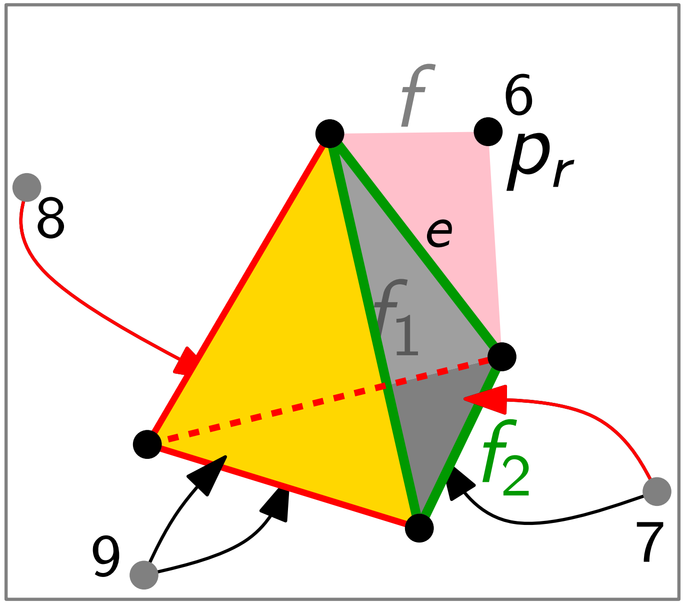

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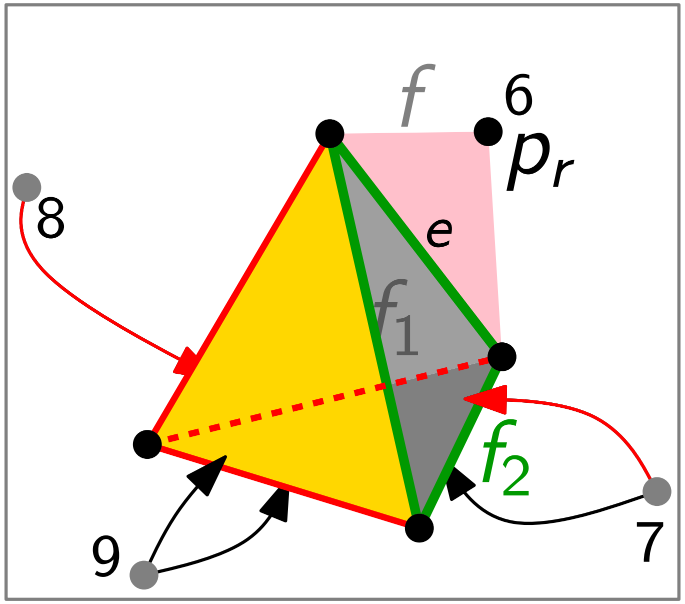

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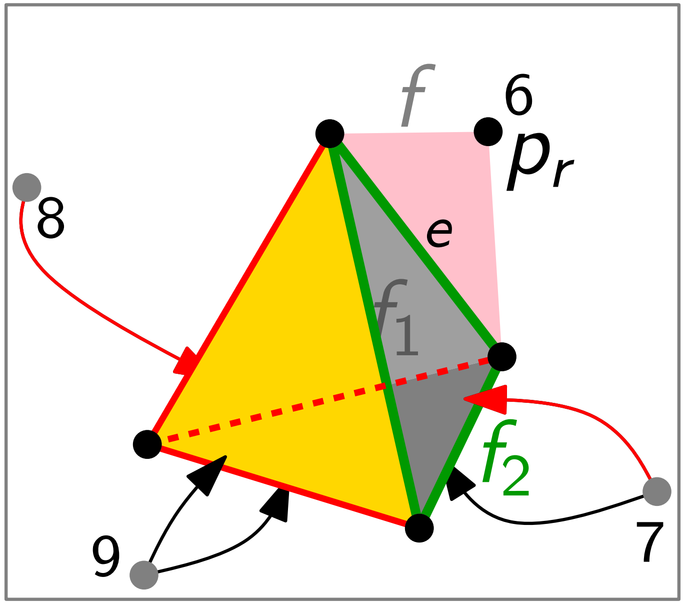

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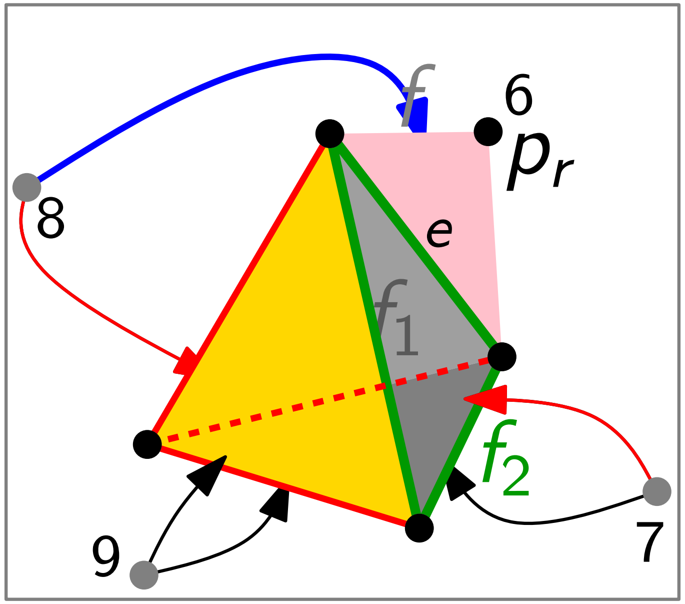

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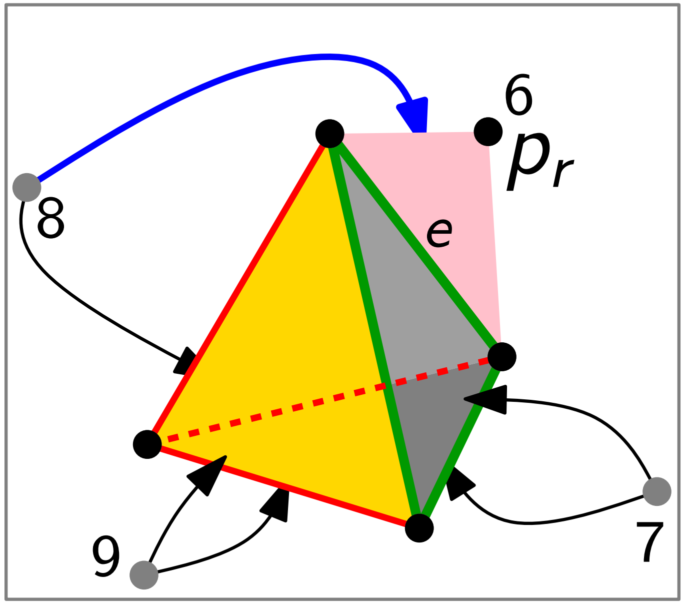

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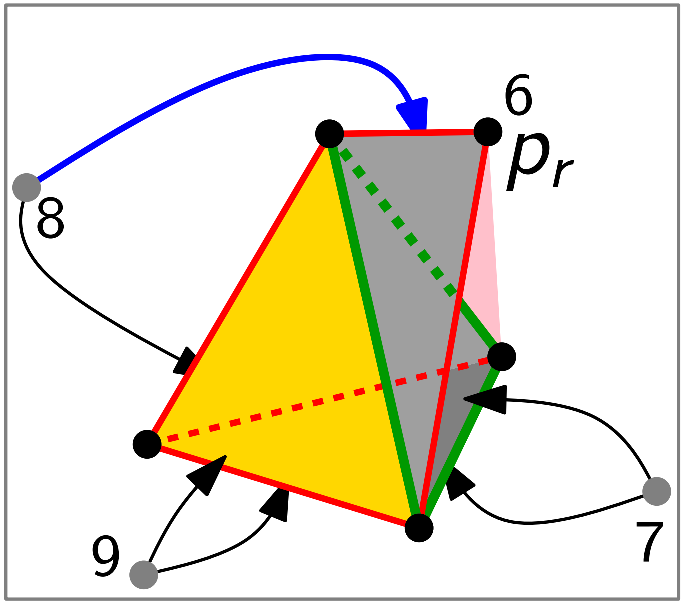

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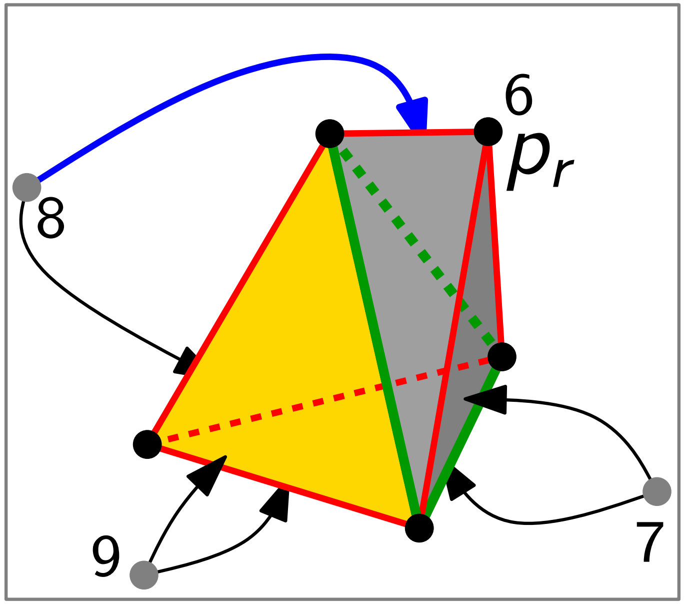

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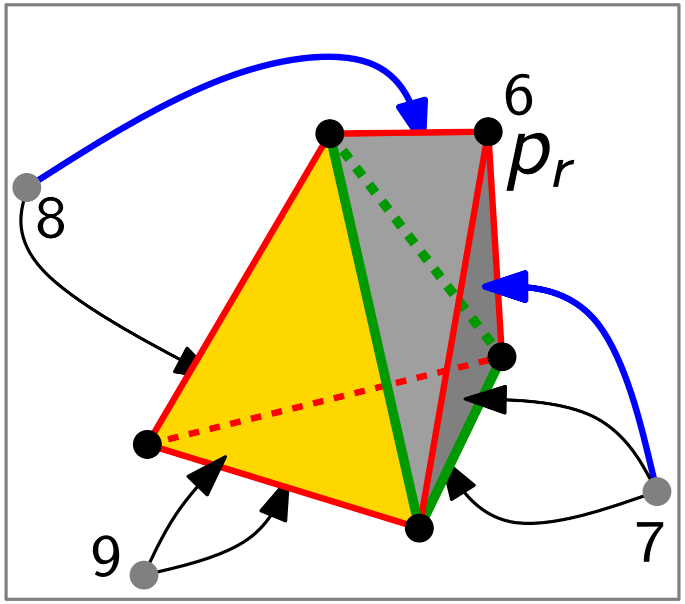

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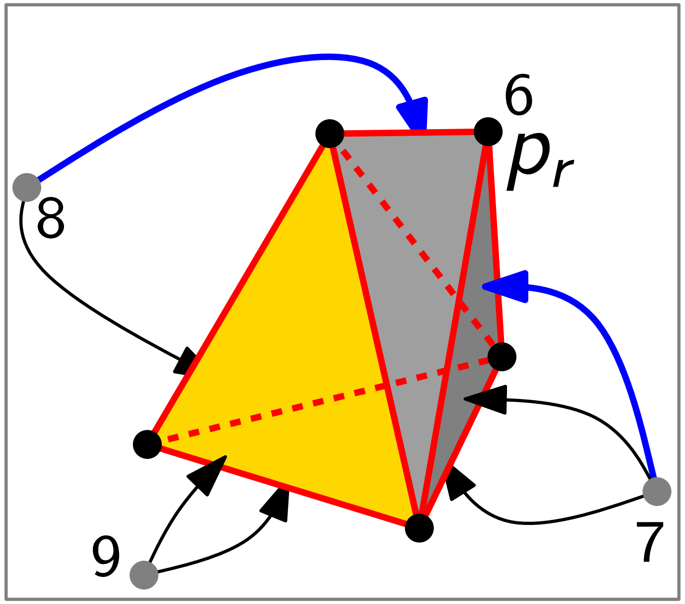

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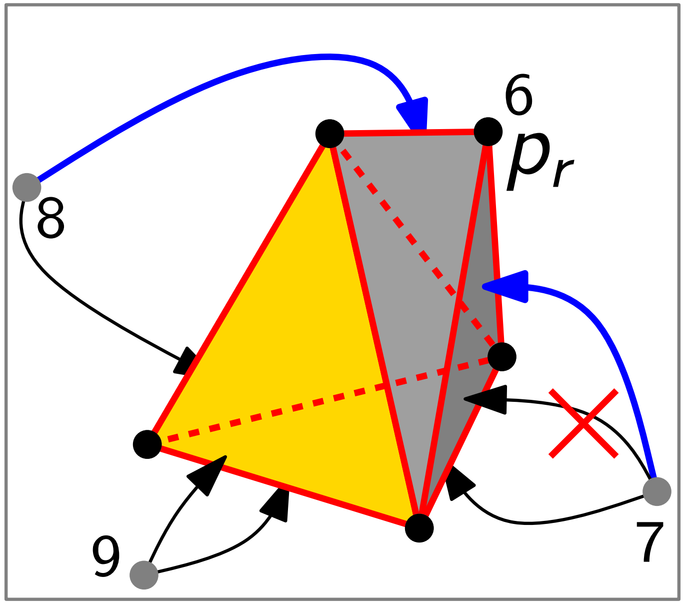

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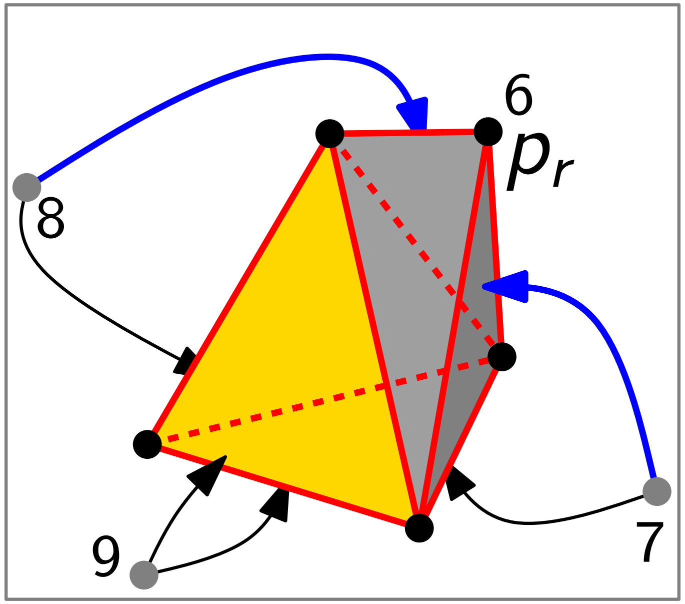

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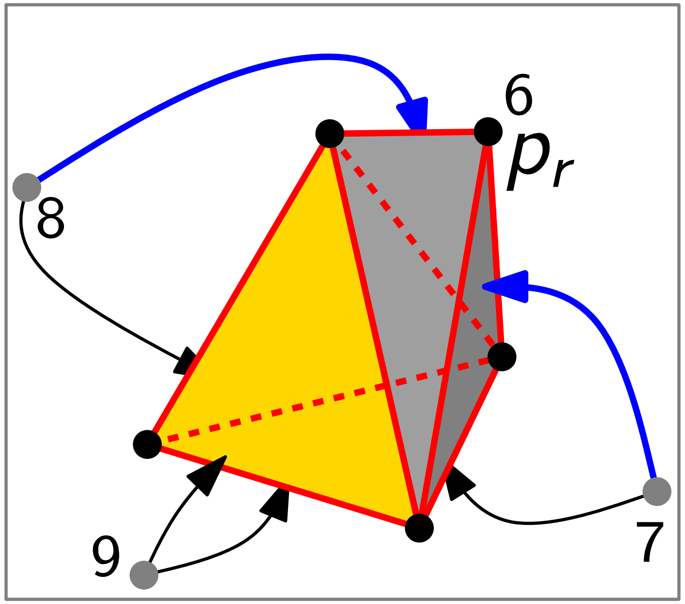

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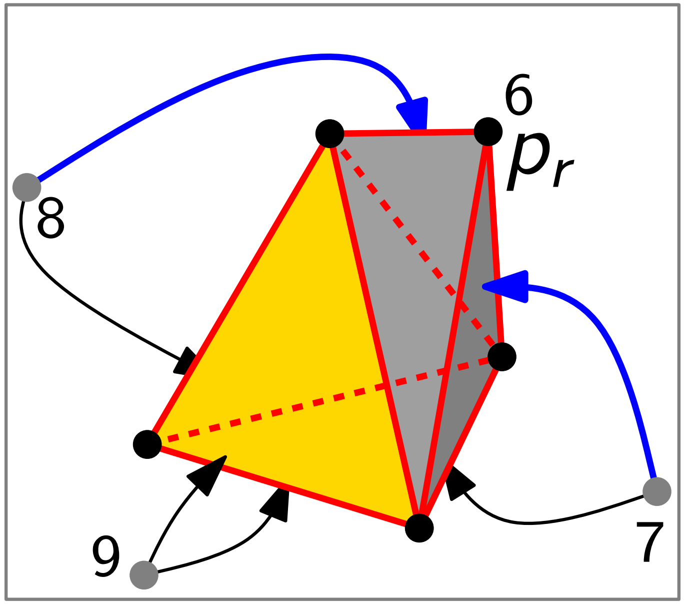

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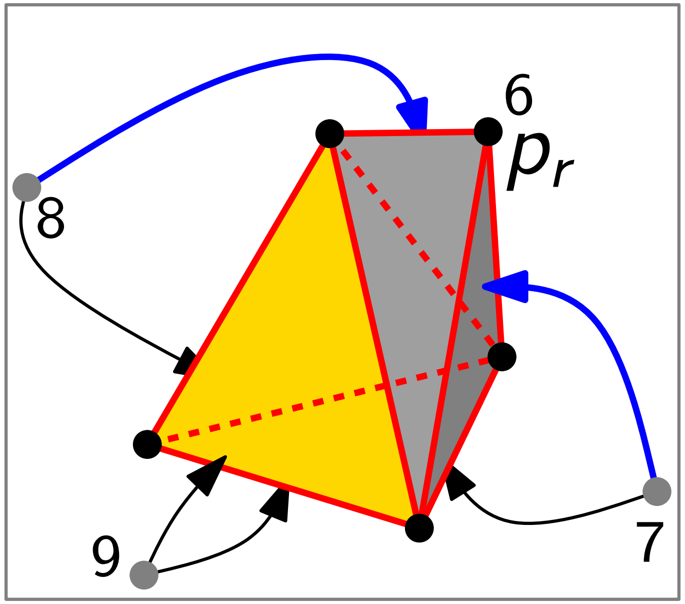

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Worst-case running time $= O(n^3)$

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(#facets = #edges)

For $r > 4$:

$$E[\deg(p_r, \mathrm{CH}(P_r))] = \frac{1}{r-4} \sum_{i=5}^{r} \deg(p_i, \mathrm{CH}(P_r))$$

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$E[\deg(p_r, \mathrm{CH}(P_r)] = \frac{1}{r-4} \sum_{i=5}^{r} \deg(p_i, \mathrm{CH}(P_r))$

$\leq \frac{1}{r-4} \left[\left(\sum_{i=1}^{r} \deg(p_i) \right) - 12 \right]$

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$$= 4 + \sum_{r=5}^{n} E[\underbrace{\#\text{facets incident to } p_r \text{ in } \mathrm{CH}(P_r)}_{\deg(p_r, \mathrm{CH}(P_r))}]$$

#edges

For $r > 4$:

$$E[\deg(p_r, \mathrm{CH}(P_r))] = \frac{1}{r-4} \sum_{i=5}^{r} \deg(p_i, \mathrm{CH}(P_r))$$

$$\leq \frac{1}{r-4} \left[\underbrace{\left(\sum_{i=1}^{r} \deg(p_i) \right)} - 12 \right]$$

Analysis

Idea: Bound expected *structural change*, that is, the total #facets created by the algorithm.

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Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

```
Rand3dConvexHull(P ⊂ ℝ³)
  pick set P' = {p_1, ..., p_4} ⊆ P of 4 non-coplanar pts
  C ← CH(P')
  compute a random permutation (p_5, ..., p_n) of P \ P'
  initialize conflict graph G
  for r = 5 to n do
      if F_conflict(p_r) ≠ ∅ then
          delete all facets in F_conflict(p_r) from C
          ℒ ← list of horizon edges visible from p_r
          foreach e ∈ ℒ do
              f ← C.create_facet(e, p_r); create vtx for f in G
              (f_1, f_2) ← previously_incident_C(e)
              P(e) ← P_conflict(f_1) ∪ P_conflict(f_2)
              foreach p ∈ P(e) do
                  if f visible from p then add edge (p, f) to G
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Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

$O(n)$ time

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  for r = 5 to n do
      if F_conflict(p_r) ≠ ∅ then
          delete all facets in F_conflict(p_r) from C
          L ← list of horizon edges visible from p_r
          foreach e ∈ L do
              f ← C.create_facet(e, p_r); create vtx for f in G
              (f_1, f_2) ← previously_incident_C(e)
              P(e) ← P_conflict(f_1) ∪ P_conflict(f_2)
              foreach p ∈ P(e) do
                  if f visible from p then add edge (p, f) to G
          delete vtc {p_r} ∪ F_conflict(p_r) from G
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```

Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

Stage r of for-loop (w/o outer foreach loop)

$O(n)$ time

```
Rand3dConvexHull(P ⊂ ℝ³)
  pick set P' = {p_1, ..., p_4} ⊆ P of 4 non-coplanar pts
  C ← CH(P')
  compute a random permutation (p_5, ..., p_n) of P \ P'
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  for r = 5 to n do
    if F_conflict(p_r) ≠ ∅ then
      delete all facets in F_conflict(p_r) from C
      ℒ ← list of horizon edges visible from p_r
      foreach e ∈ ℒ do
        f ← C.create_facet(e, p_r); create vtx for f in G
        (f_1, f_2) ← previously_incident_C(e)
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        foreach p ∈ P(e) do
          if f visible from p then add edge (p, f) to G
      delete vtc {p_r} ∪ F_conflict(p_r) from G
  return C
```

Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

Stage r of for-loop (w/o outer foreach loop) takes time $O(|F_{\text{conflict}}(p_r)|) =$

$O(n)$ time

```
Rand3dConvexHull(P ⊂ ℝ³)
  pick set P′ = {p1, ..., p4} ⊆ P of 4 non-coplanar pts
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        (f1, f2) ← previously_incident_C(e)
        P(e) ← P_conflict(f1) ∪ P_conflict(f2)
        foreach p ∈ P(e) do
          if f visible from p then add edge (p, f) to G
      delete vtc {p_r} ∪ F_conflict(p_r) from G
  return C
```

Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

$O(n)$ time (for the initialization steps, marked by a brace):

```
Rand3dConvexHull(P ⊂ ℝ³)
  pick set P' = {p_1, ..., p_4} ⊆ P of 4 non-coplanar pts
  C ← CH(P')
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  for r = 5 to n do
      if F_conflict(p_r) ≠ ∅ then
          delete all facets in F_conflict(p_r) from C
          ℒ ← list of horizon edges visible from p_r
          foreach e ∈ ℒ do
              f ← C.create_facet(e, p_r); create vtx for f in G
              (f_1, f_2) ← previously_incident_C(e)
              P(e) ← P_conflict(f_1) ∪ P_conflict(f_2)
              foreach p ∈ P(e) do
                  if f visible from p then add edge (p, f) to G
          delete vtc {p_r} ∪ F_conflict(p_r) from G
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```

Stage r of for-loop (w/o outer foreach loop) takes time $O(|F_{\text{conflict}}(p_r)|) = O(\#\text{facets deleted when adding } p_r)$

Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

$O(n)$ time

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Rand3dConvexHull(P ⊂ ℝ³)
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        f ← C.create_facet(e, p_r); create vtx for f in G
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        P(e) ← P_conflict(f_1) ∪ P_conflict(f_2)
        foreach p ∈ P(e) do
          if f visible from p then add edge (p, f) to G
      delete vtc {p_r} ∪ F_conflict(p_r) from G
  return C
```

Stage r of for-loop (w/o outer foreach loop) takes time $O(|F_{\text{conflict}}(p_r)|) =$ $O(\#\text{facets deleted when adding } p_r)$

This part of for-loop in total:

Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

$O(n)$ time (applies to the first four lines of the algorithm below):

```
Rand3dConvexHull(P ⊂ ℝ³)
  pick set P' = {p_1, ..., p_4} ⊆ P of 4 non-coplanar pts
  C ← CH(P')
  compute a random permutation (p_5, ..., p_n) of P \ P'
  initialize conflict graph G
  for r = 5 to n do
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```

Stage r of for-loop (w/o outer foreach loop) takes time $O(|F_{\text{conflict}}(p_r)|) =$ $O(\#\text{facets deleted when adding } p_r)$

This part of for-loop in total: $E[\#\text{facets deleted}] =$

Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

$O(n)$ time

```
Rand3dConvexHull(P ⊂ ℝ^3)
  pick set P' = {p_1, ..., p_4} ⊆ P of 4 non-coplanar pts
  C ← CH(P')
  compute a random permutation (p_5, ..., p_n) of P \ P'
  initialize conflict graph G
  for r = 5 to n do
    if F_conflict(p_r) ≠ ∅ then
      delete all facets in F_conflict(p_r) from C
      ℒ ← list of horizon edges visible from p_r
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```

Stage r of for-loop (w/o outer foreach loop) takes time $O(|F_{\text{conflict}}(p_r)|) =$ $O(\#\text{facets deleted when adding } p_r)$

This part of for-loop in total:

$E[\#\text{facets deleted}] =$

$\leq E[\#\text{facets created}] =$

Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

$O(n)$ time

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$E[\#\text{facets deleted}] =$
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Lemma

Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

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Stage r of for-loop (w/o outer foreach loop) takes time $O(|F_{\text{conflict}}(p_r)|) =$ $O(\#\text{facets deleted when adding } p_r)$

This part of for-loop in total:
$E[\#\text{facets deleted}] =$
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Lemma

Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

$O(n)$ time

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Lemma

Outer foreach-loop:

Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

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Outer foreach-loop:
– in stage r: $O(\sum_{e \in \mathcal{L}} |P(e)|)$

Running Time

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              P(e) ← P_conflict(f_1) ∪ P_conflict(f_2)
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Stage r of for-loop (w/o outer foreach loop) takes time $O(|F_{\text{conflict}}(p_r)|) = O(\#\text{facets deleted when adding } p_r)$

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Lemma

Outer foreach-loop:
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– in total:

Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

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This part of for-loop in total:
$E[\#\text{facets deleted}] =$
$\leq E[\#\text{facets created}] = O(n).$
Lemma

Outer foreach-loop:
– in stage r: $O(\sum_{e \in \mathcal{L}} |P(e)|)$
– in total:

$$O\left(\sum_{e \text{ on horizon at some moment}} |P(e)|\right)$$

Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

$O(n)$ time

```
Rand3dConvexHull(P ⊂ ℝ³)
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          foreach e ∈ L do
              f ← C.create_facet(e, p_r); create vtx for f in G
              (f_1, f_2) ← previously_incident_C(e)
              P(e) ← P_conflict(f_1) ∪ P_conflict(f_2)
              foreach p ∈ P(e) do
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Stage r of for-loop (w/o outer foreach loop) takes time $O(|F_{\text{conflict}}(p_r)|) = O(\#\text{facets deleted when adding } p_r)$

This part of for-loop in total:
$E[\#\text{facets deleted}] =$
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Lemma

Outer foreach-loop:
– in stage r: $O(\sum_{e \in \mathcal{L}} |P(e)|)$
– in total:

$$O\left(\sum_{e \text{ on horizon at some moment}} |P(e)|\right)$$

$$= O(n \log n)$$

Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

$O(n)$ time

```
Rand3dConvexHull(P ⊂ ℝ³)
  pick set P' = {p_1, ..., p_4} ⊆ P of 4 non-coplanar pts
  C ← CH(P')
  compute a random permutation (p_5, ..., p_n) of P \ P'
  initialize conflict graph G
  for r = 5 to n do
    if F_conflict(p_r) ≠ ∅ then
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      L ← list of horizon edges visible from p_r
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using *configuration spaces*, Section 9.5 [De Berg et al.]

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Exercise: Give a simple deterministic algorithm that computes the convex hull in $O(n^2)$ (worst-case) time.

Convex Hulls and Half-Space Intersections

Convex Hulls and Half-~~Space~~ Plane Intersections

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Define dualtity $\star$ between pts and (non-vertical) lines:

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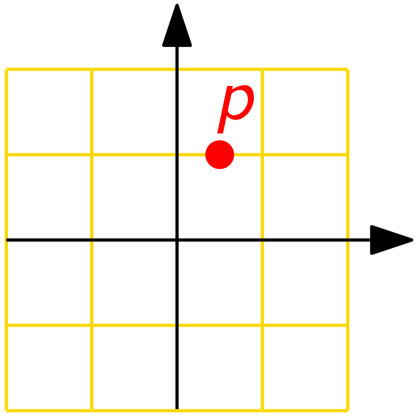

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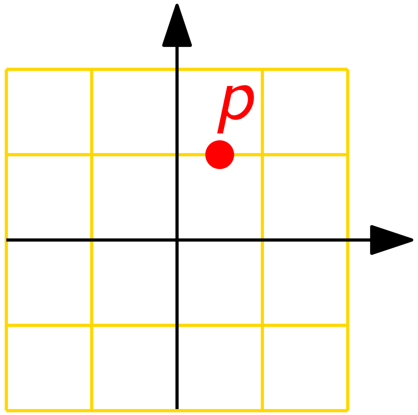

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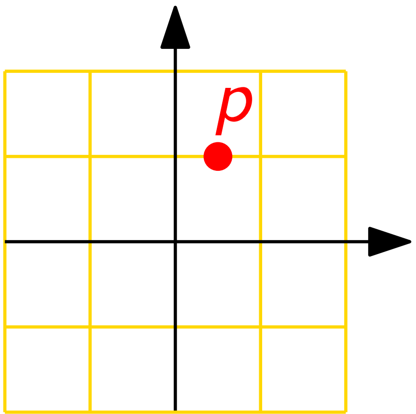

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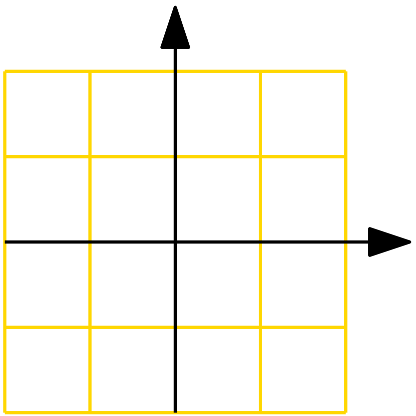

Convex Hulls and Half-~~Space~~ Intersections
Plane

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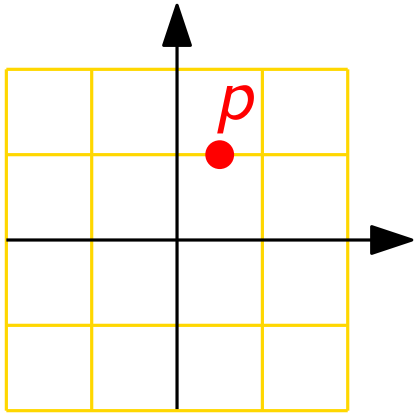

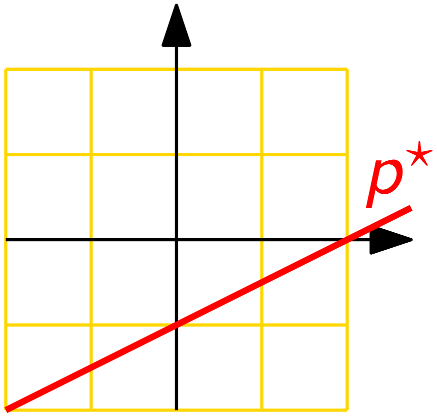

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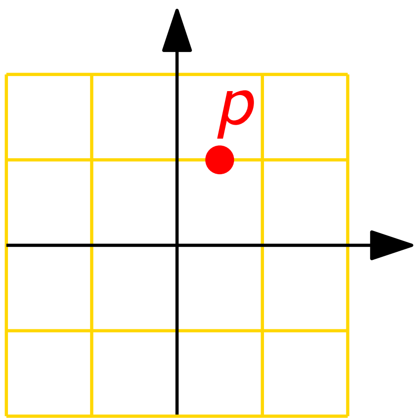

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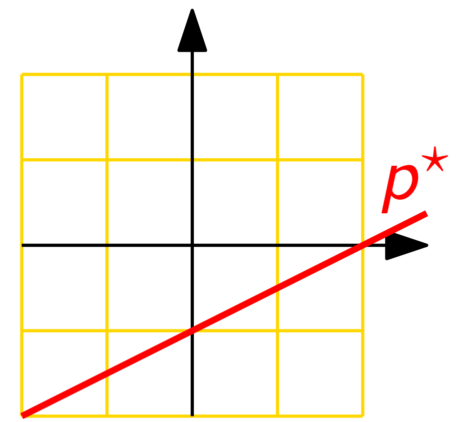

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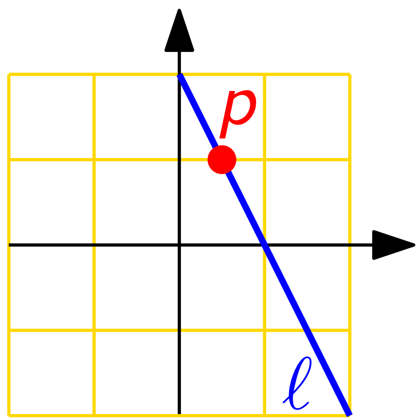

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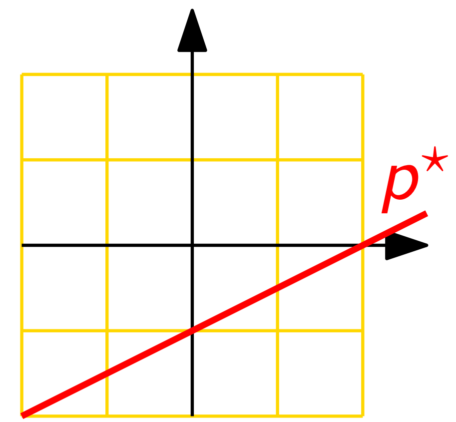

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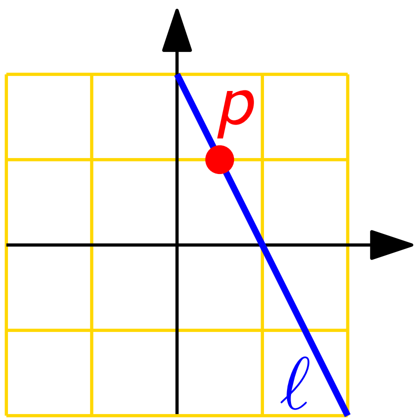

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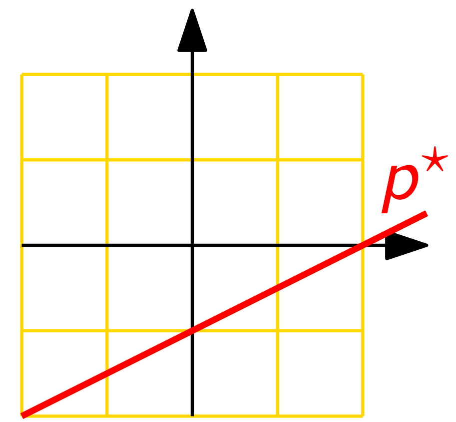

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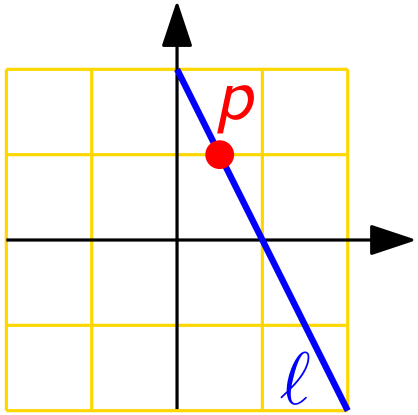

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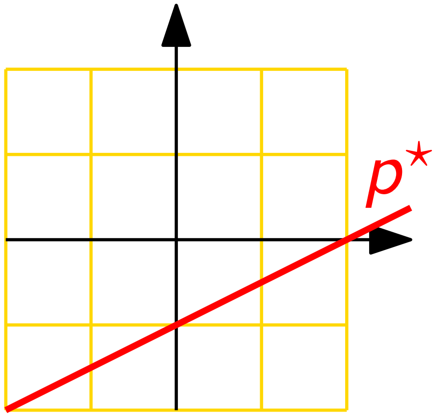

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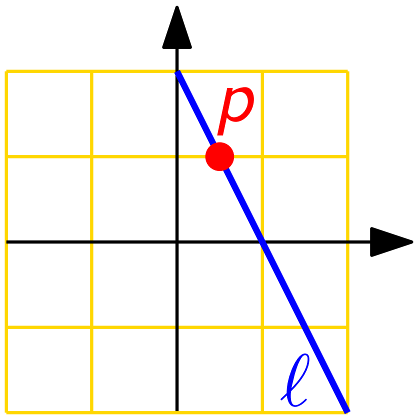

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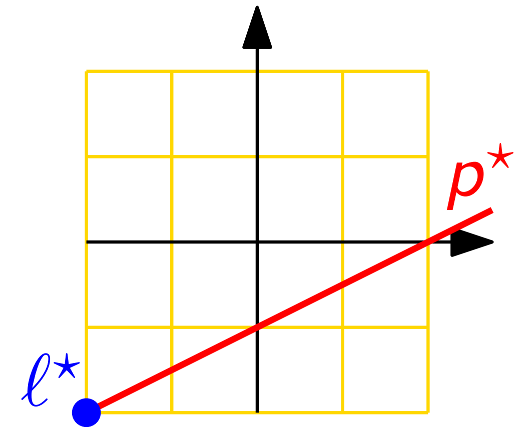

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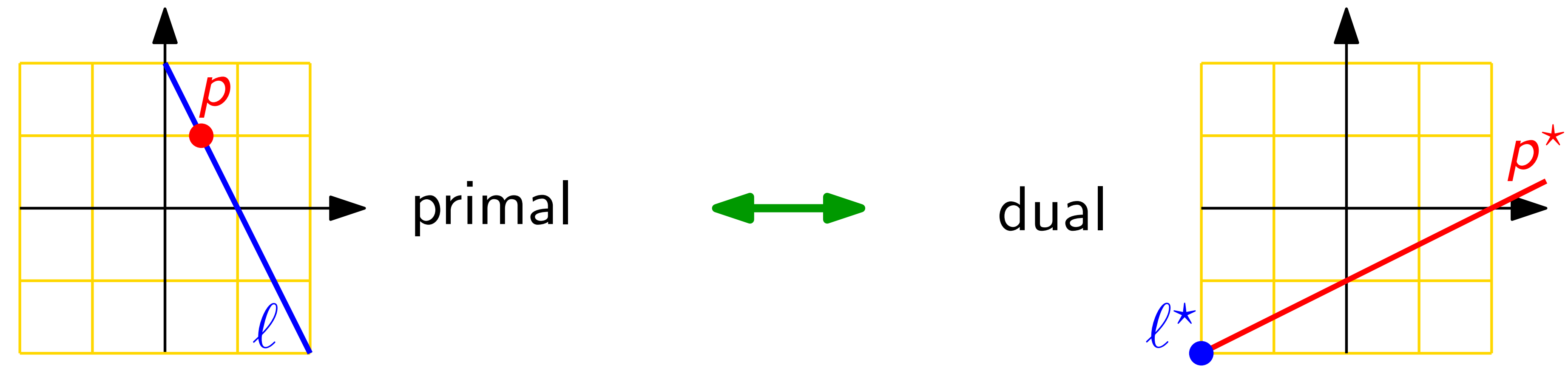

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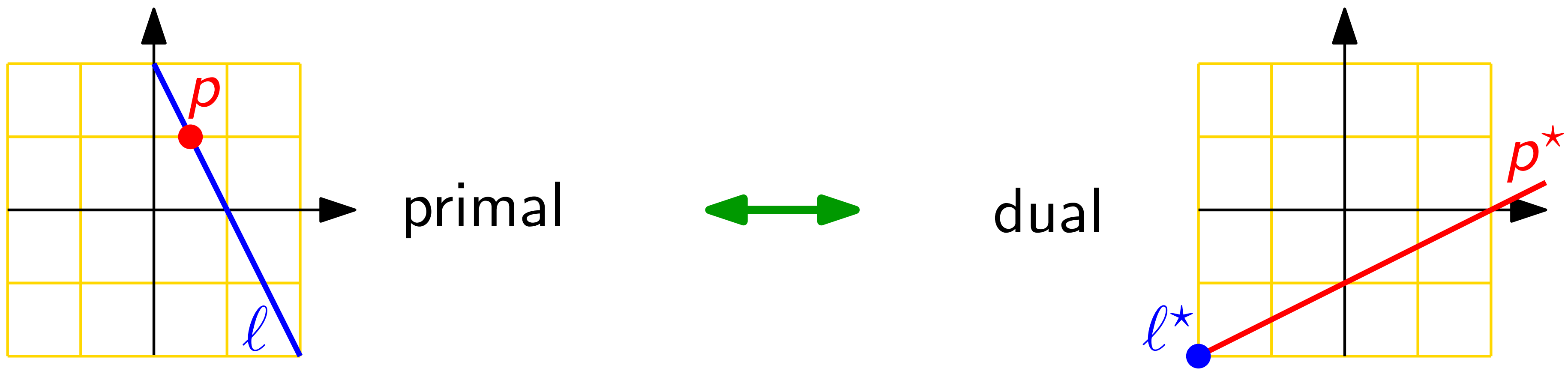

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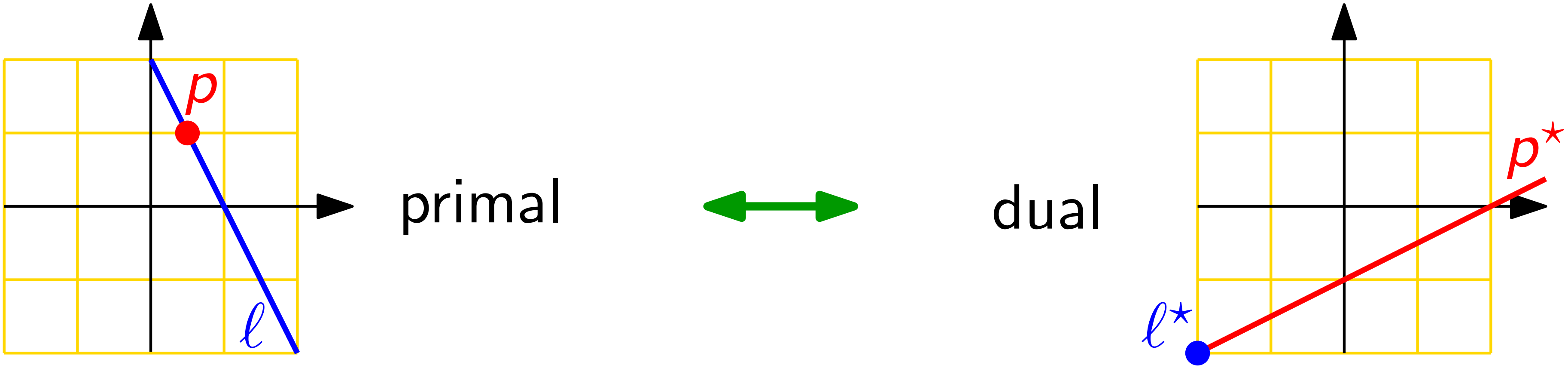

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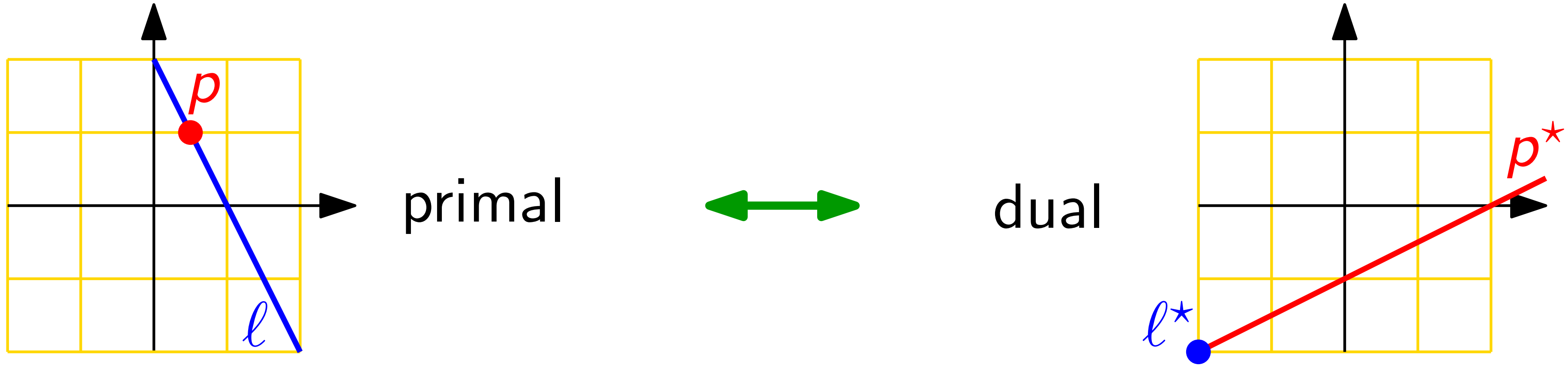

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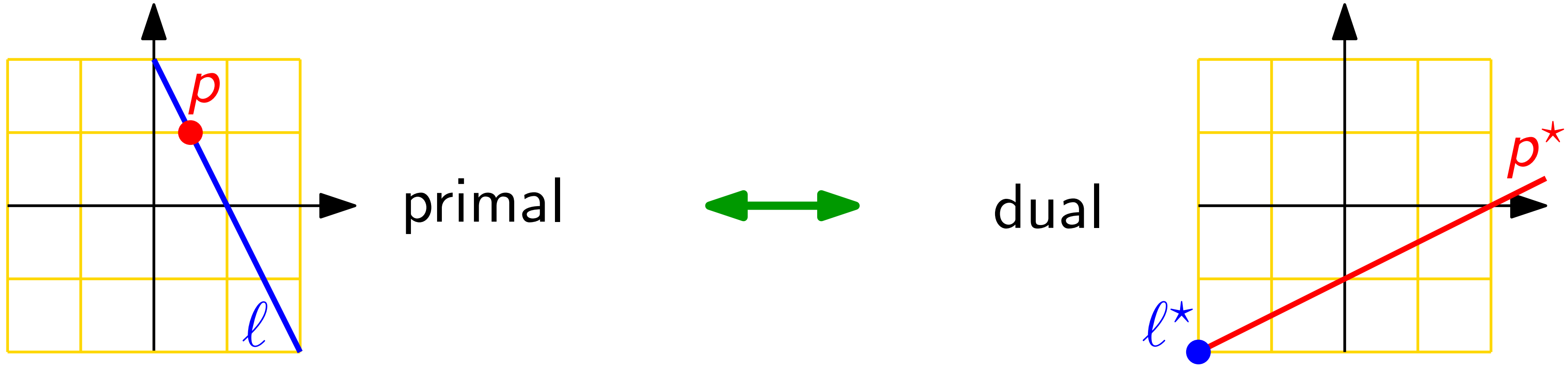

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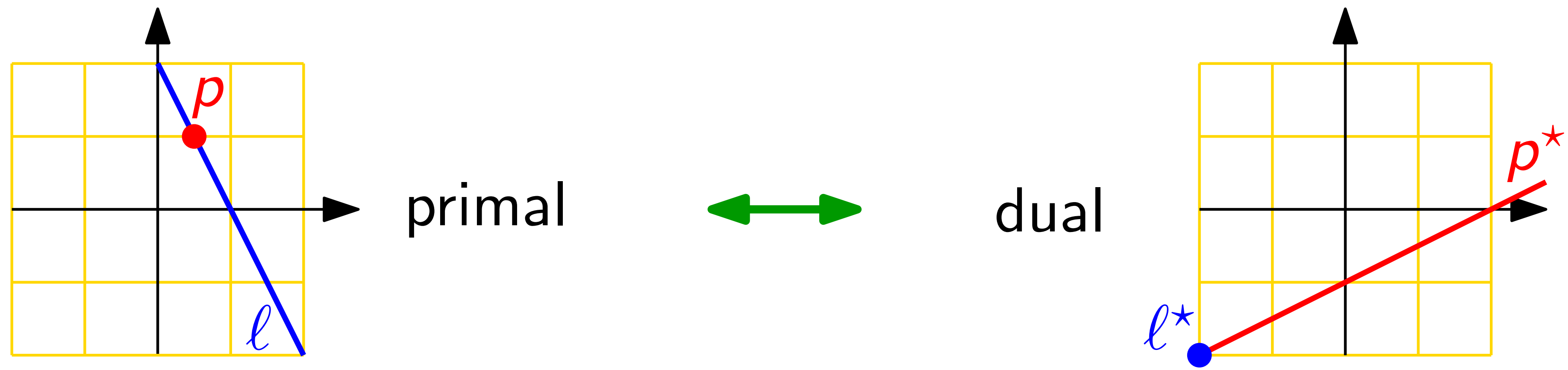

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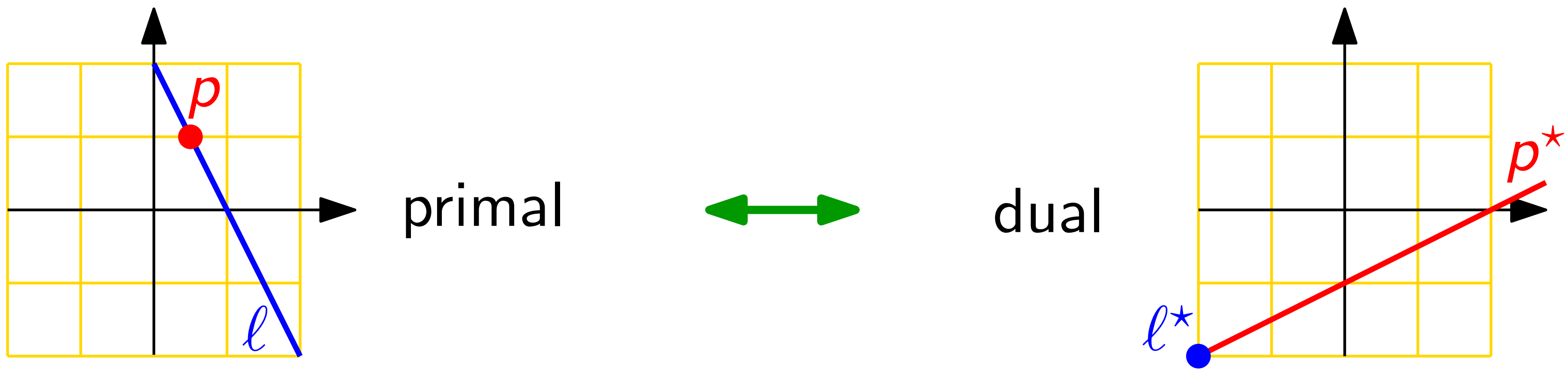

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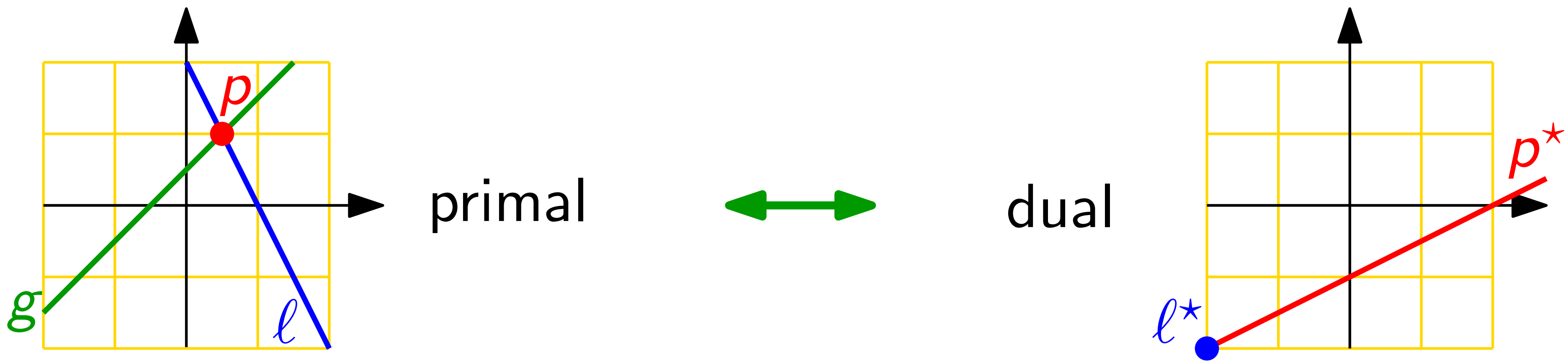

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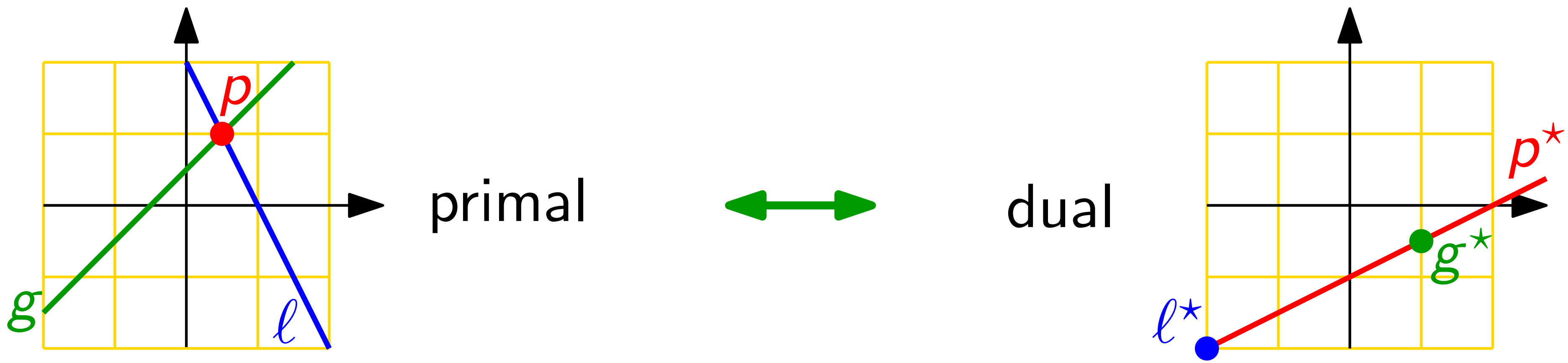

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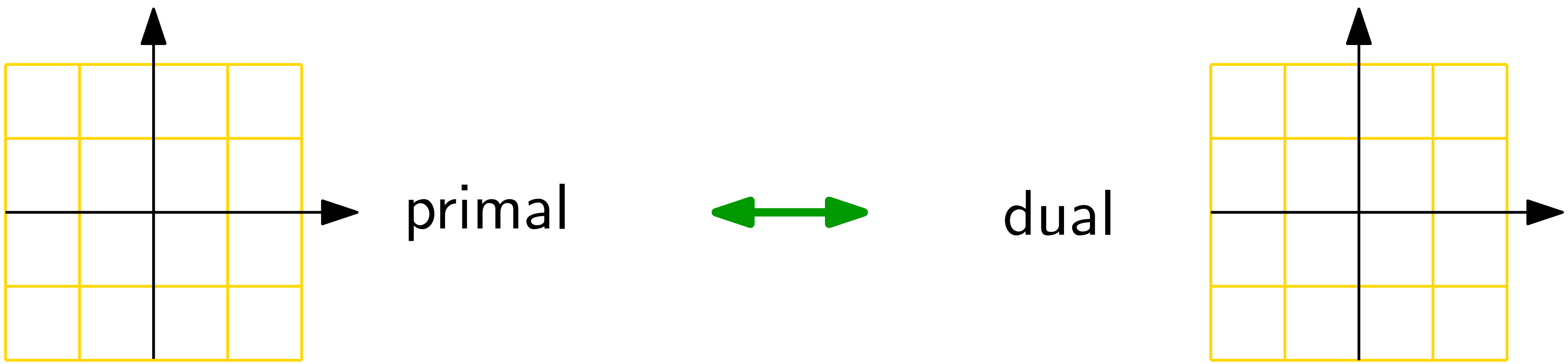

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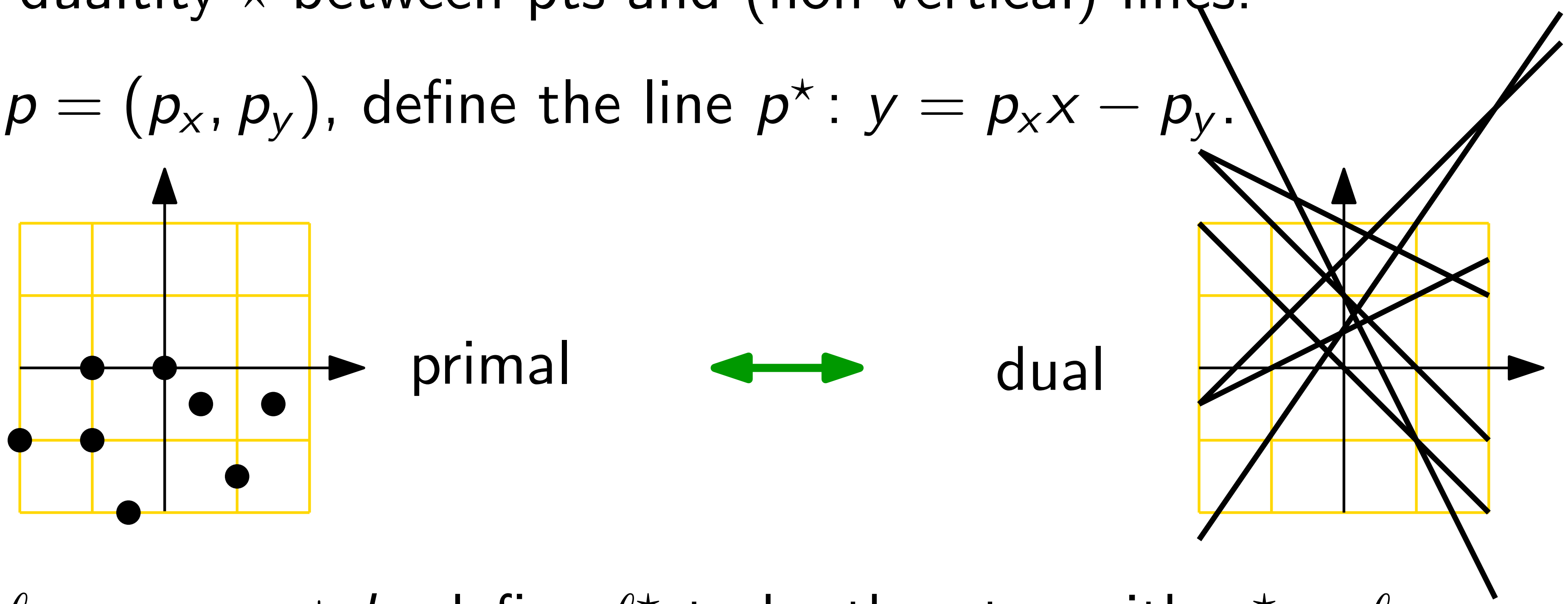

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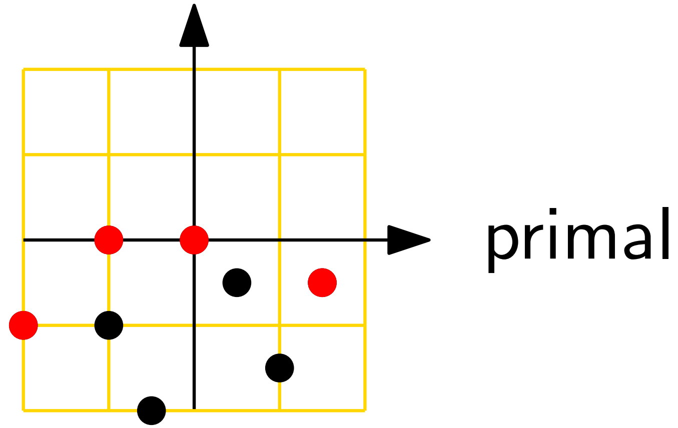

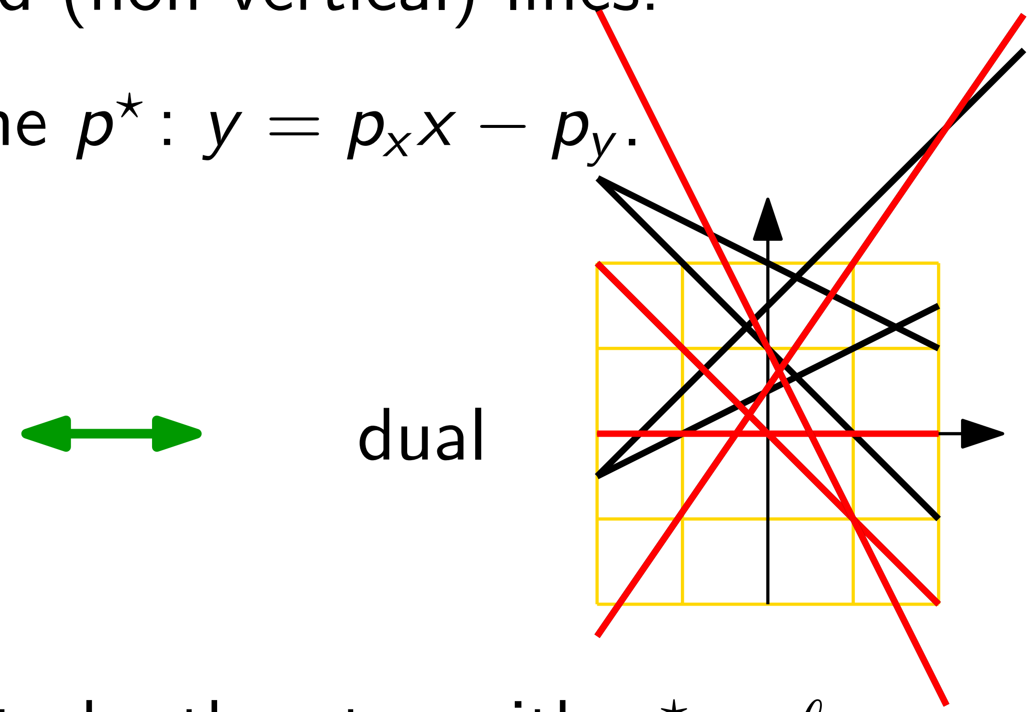

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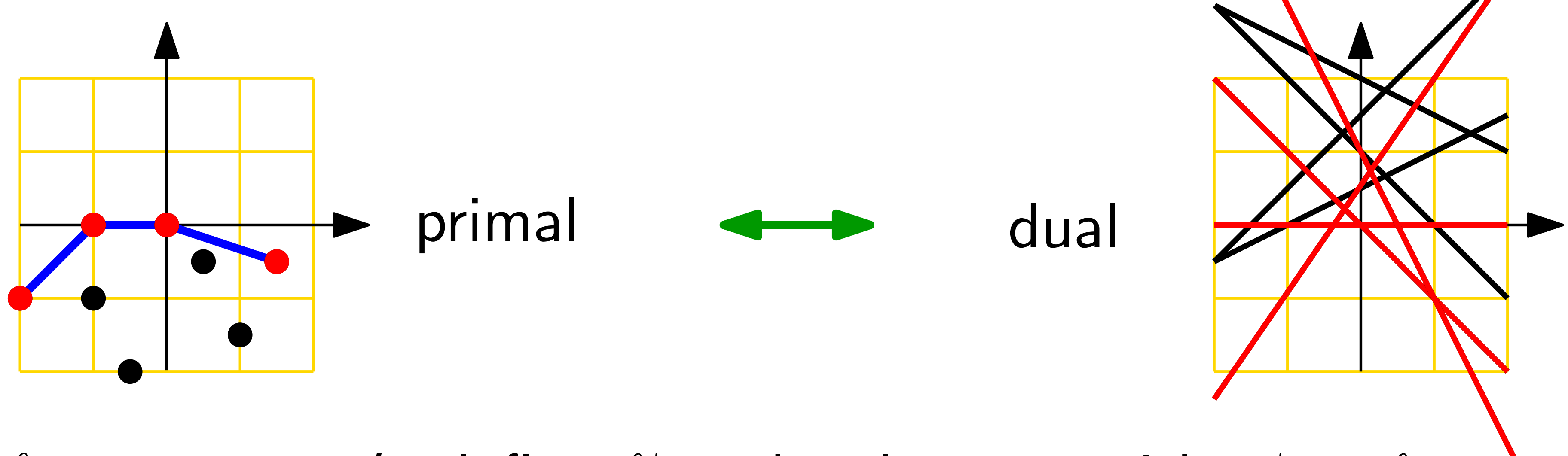

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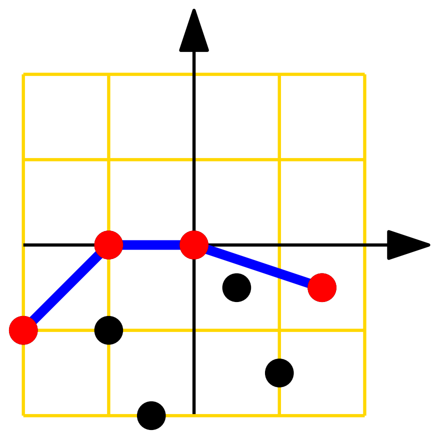

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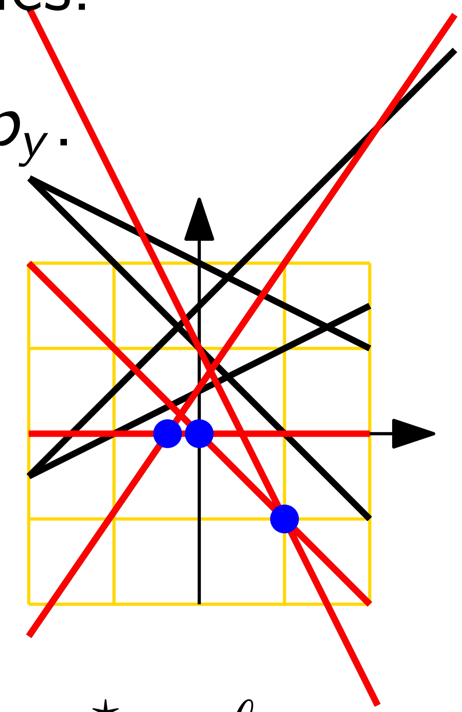

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Convex Hulls and Half-~~Space~~ Plane Intersections

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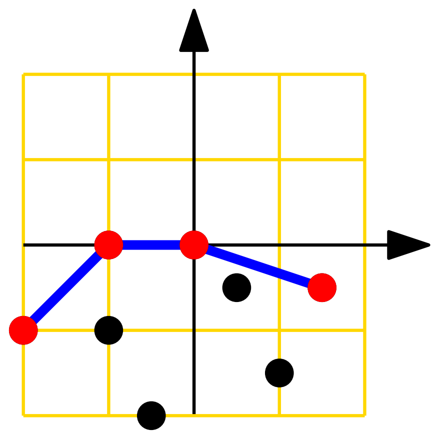

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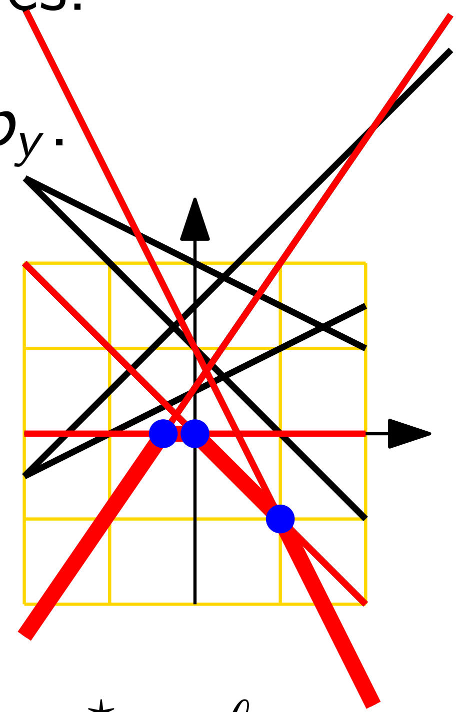

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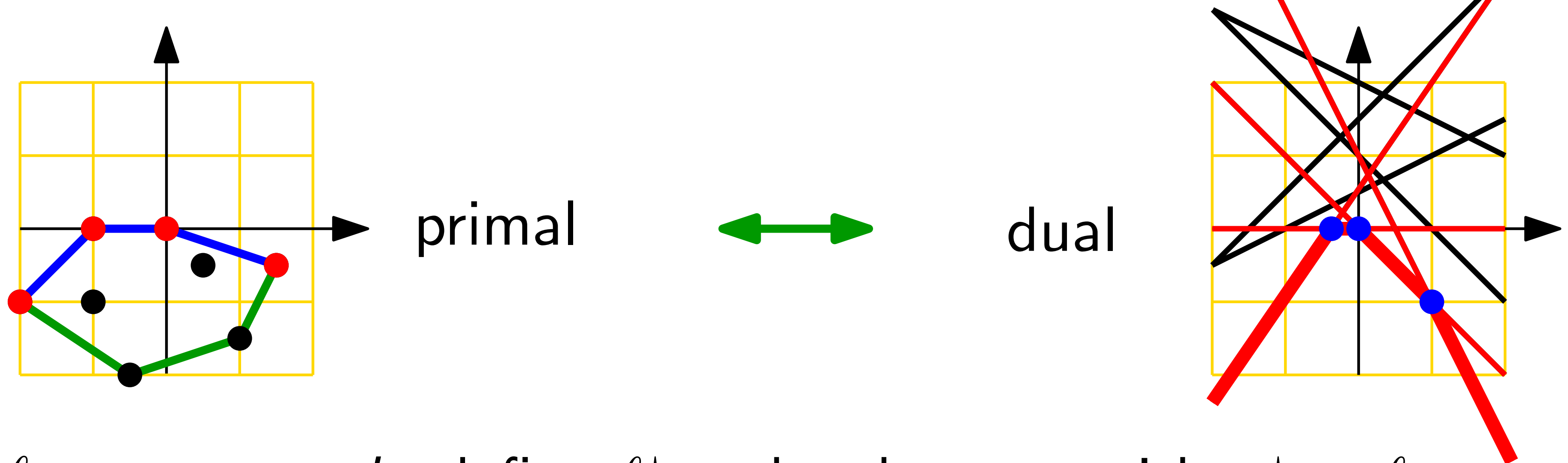

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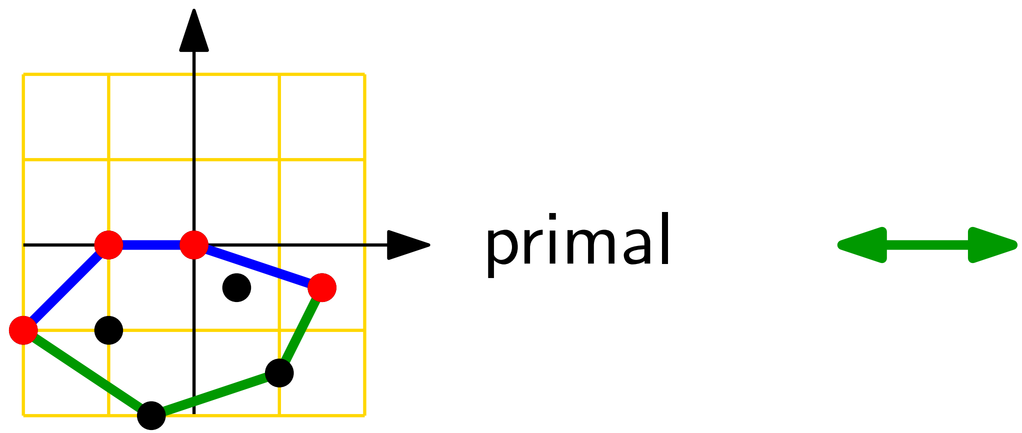

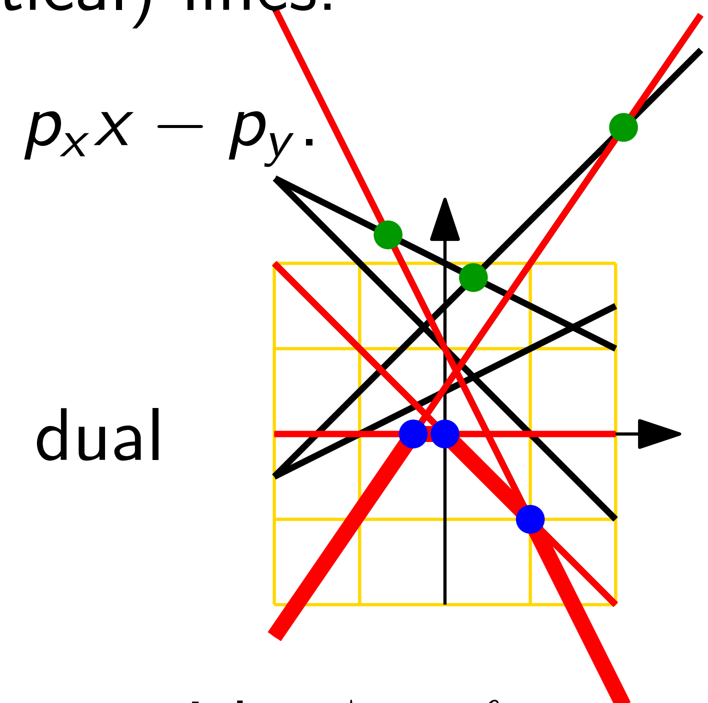

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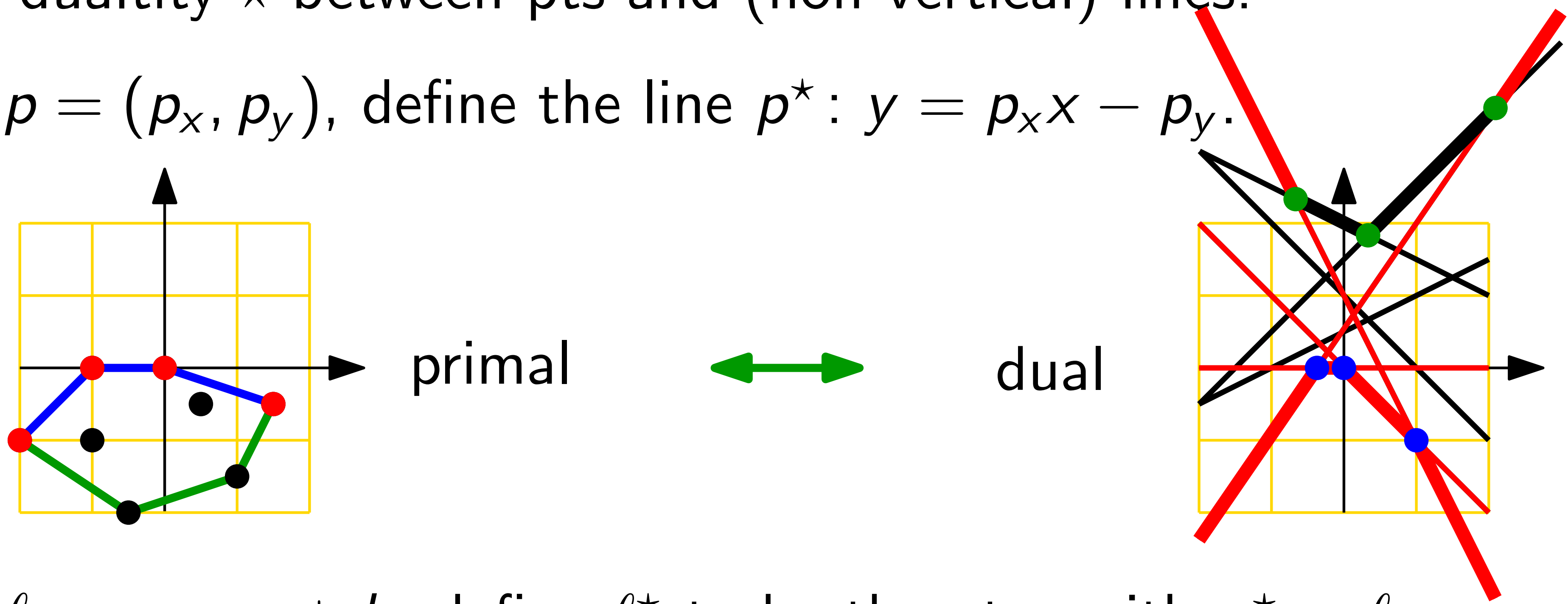

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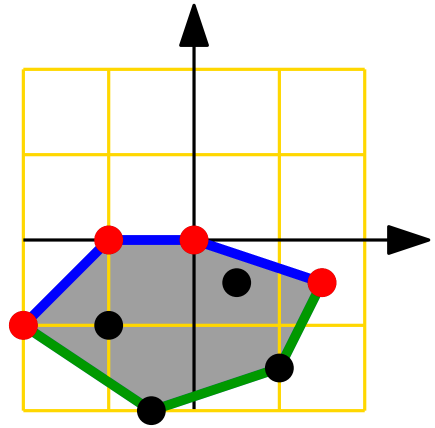

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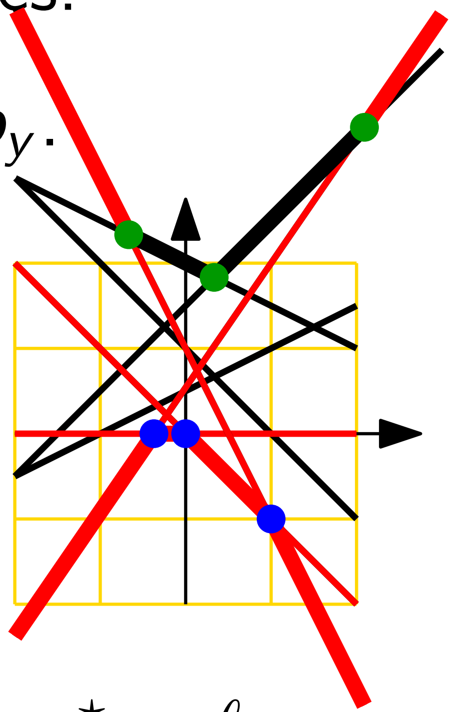

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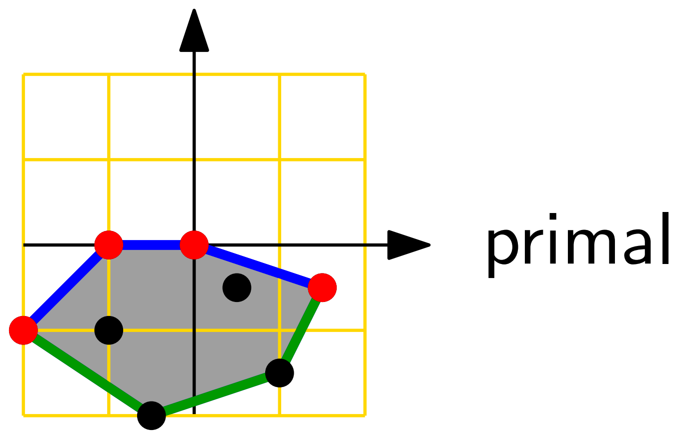

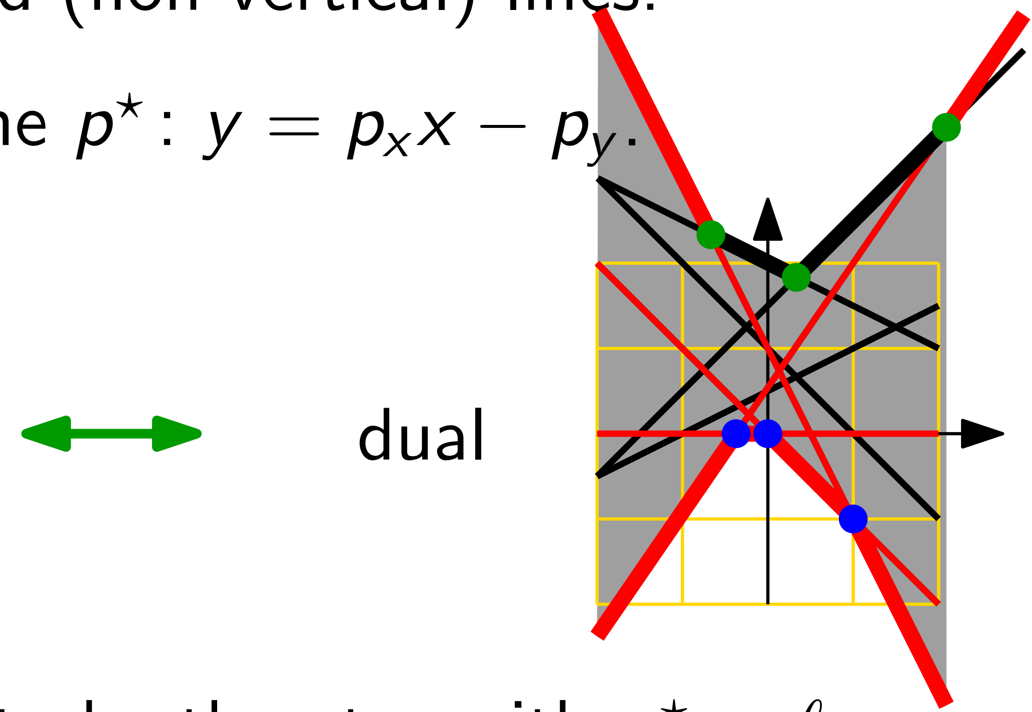

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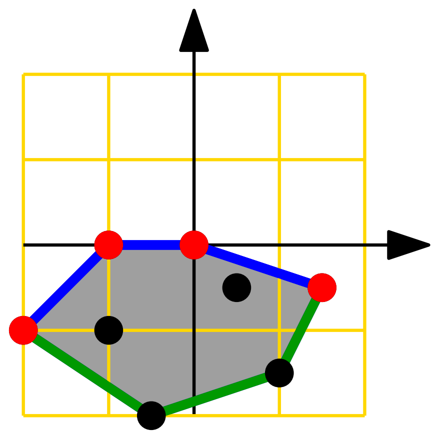

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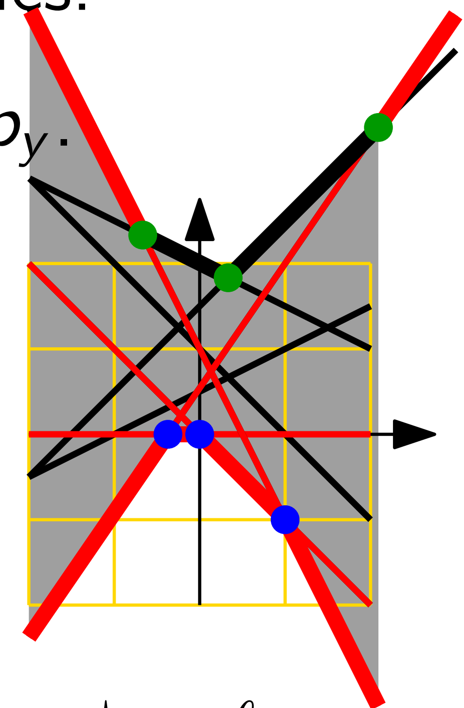

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Observe: upper convex hulls of pts $\leftrightarrow$ lower envelopes of lines

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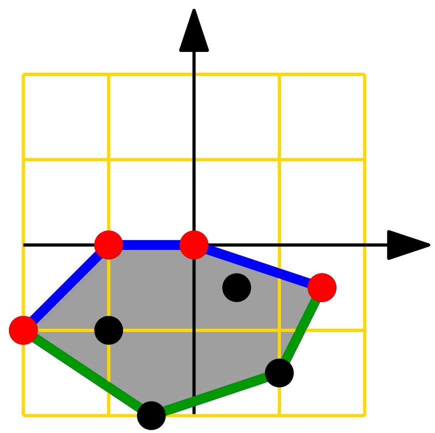

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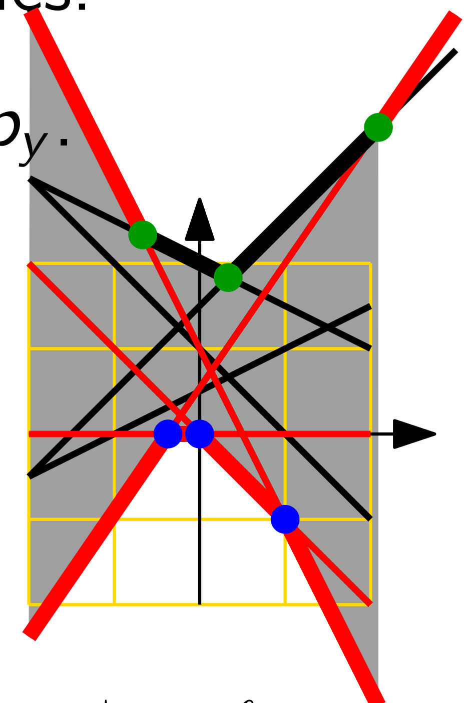

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Observe:
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- can compute intersections of “lower/upper” half planes (spaces) via upper/lower convex hulls

Voronoi Diagrams Revisited

Let $U\colon z = x^2 + y^2$ be the *unit paraboloid* in $\mathbb{R}^3$.

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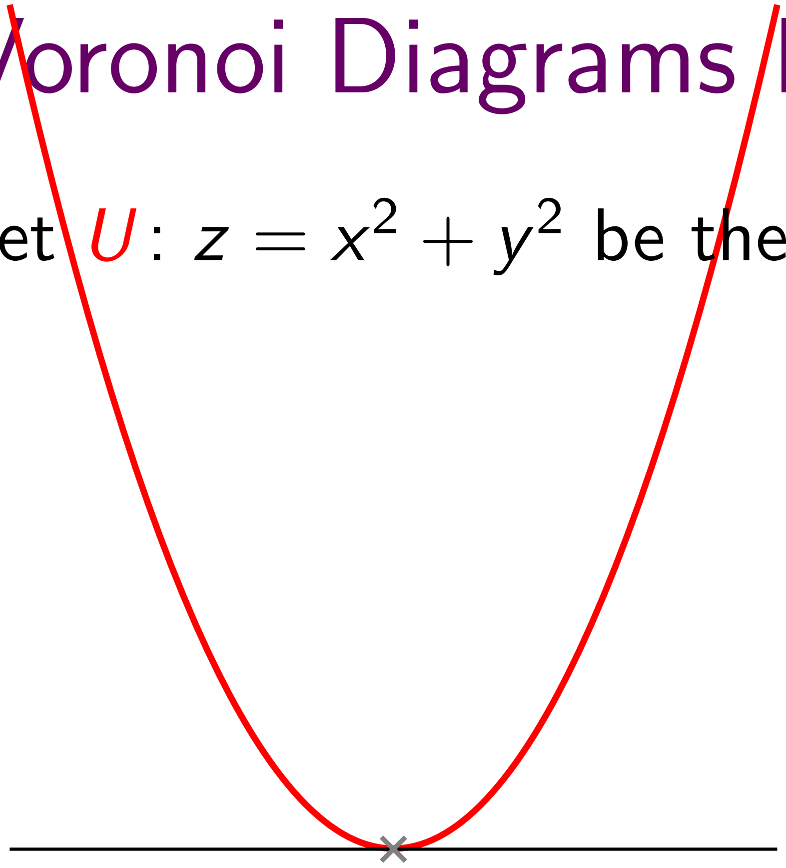

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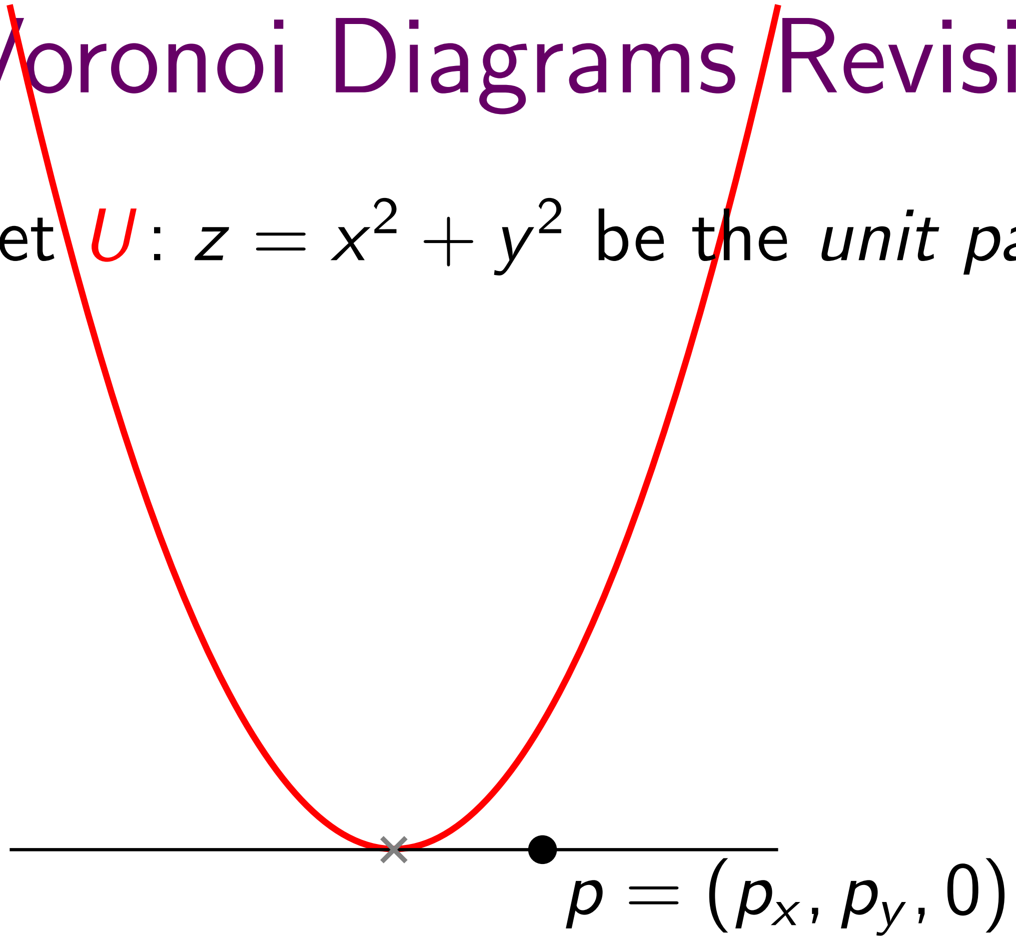

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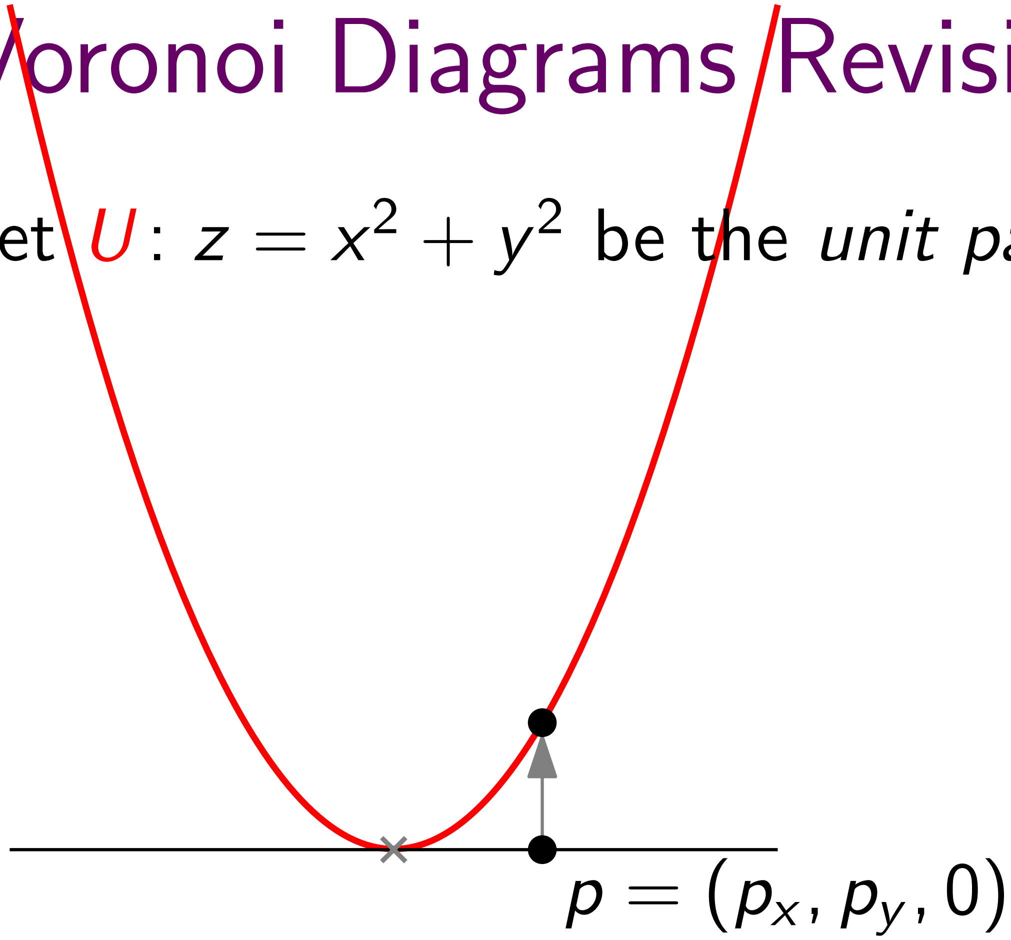

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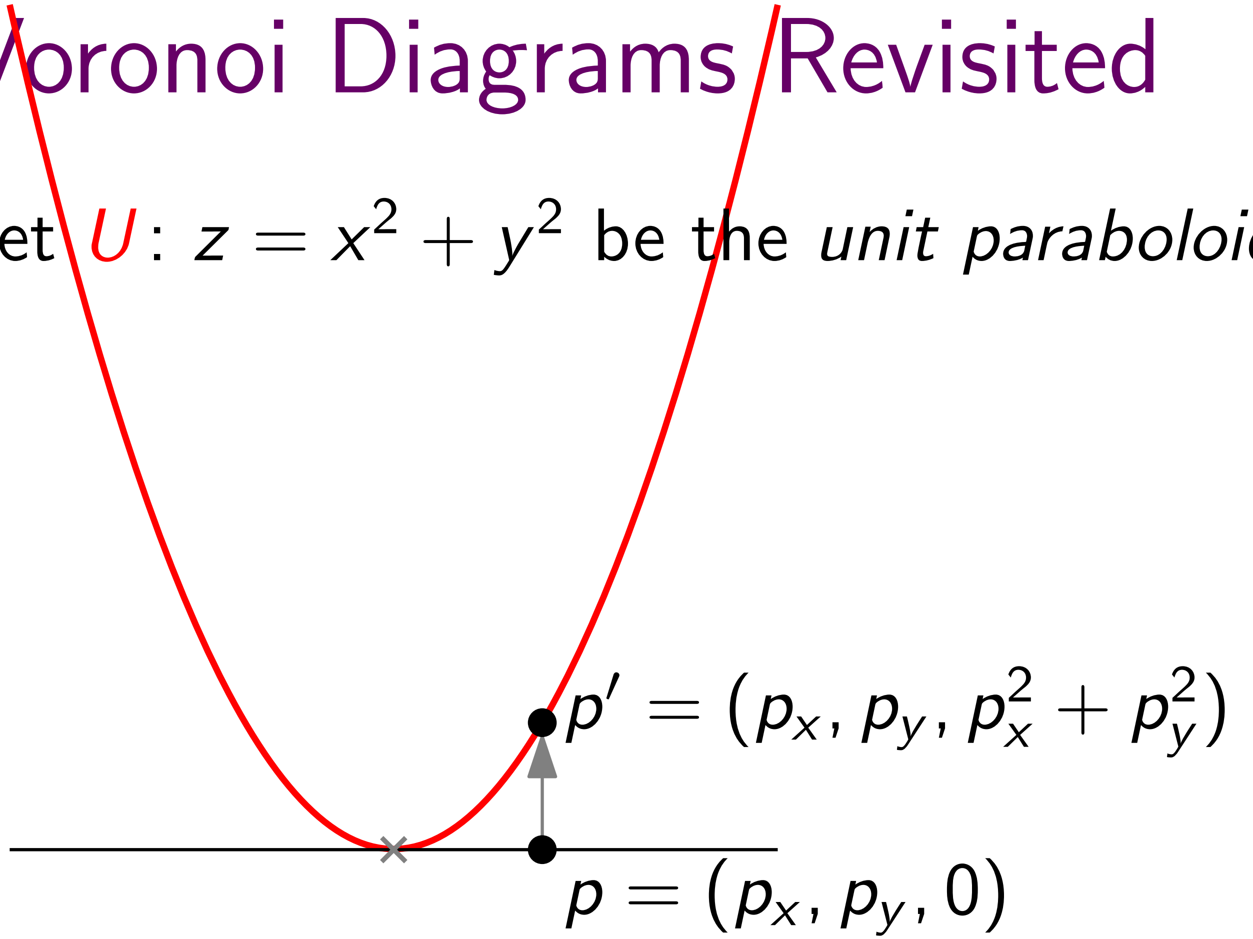

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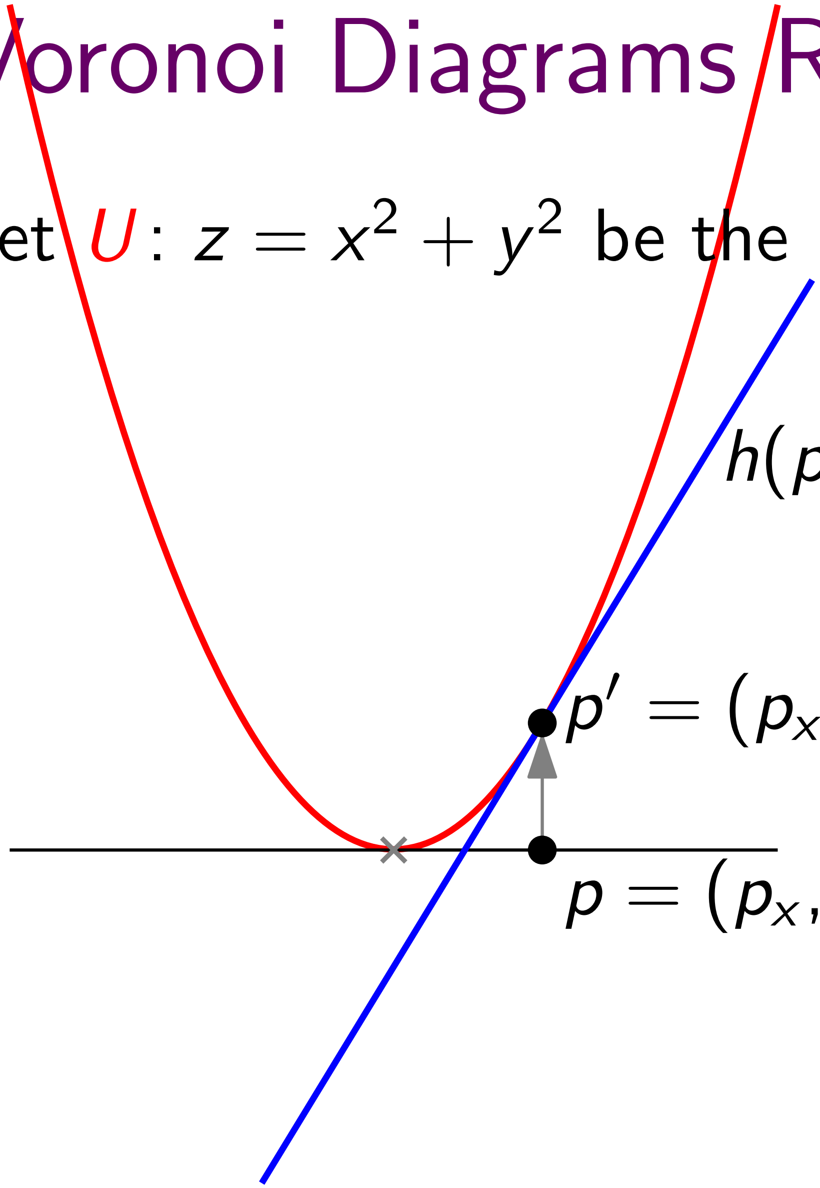

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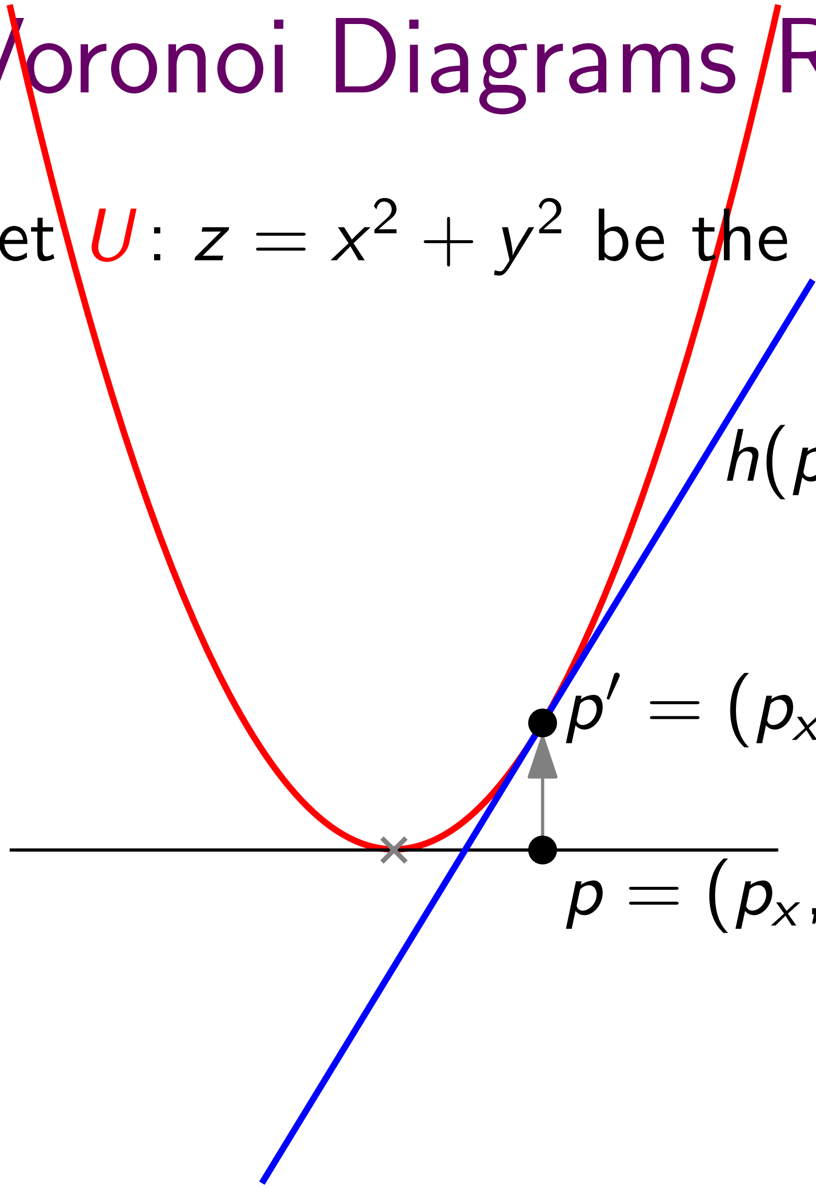

$h(p)$: $z = (2p_x)x + (2p_y)y - (p_x^2 + p_y^2)$

Note that $p' \in h(p)$.

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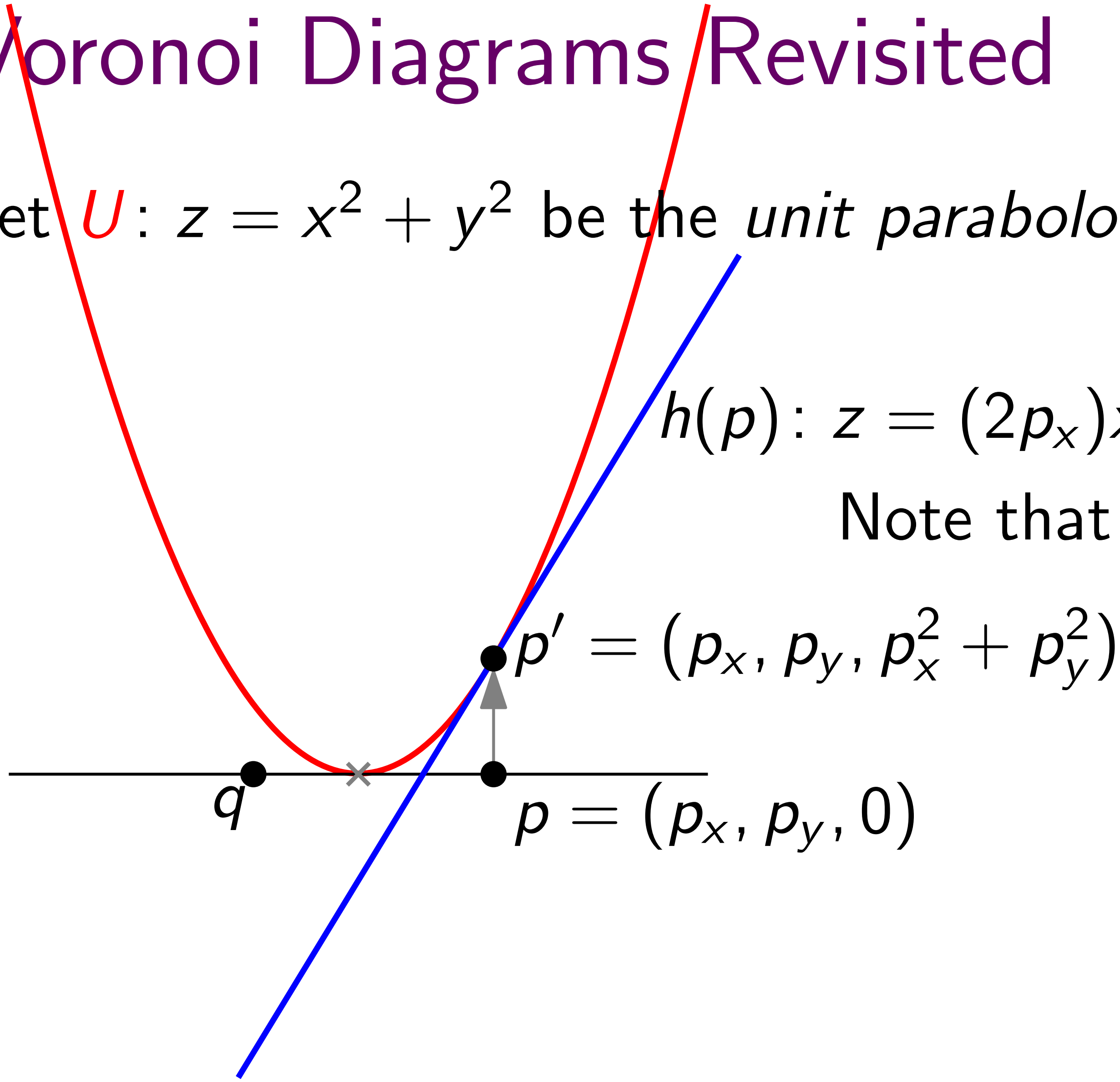

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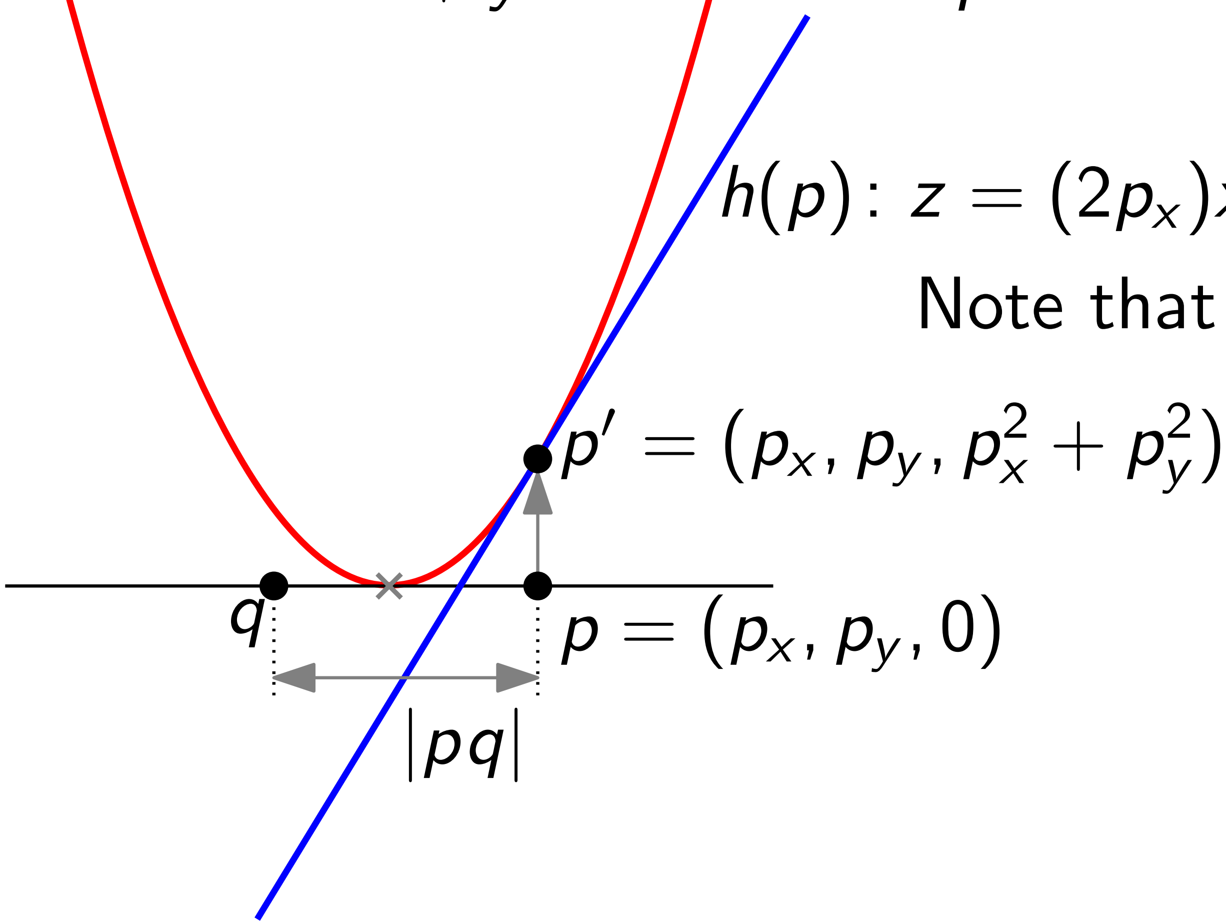

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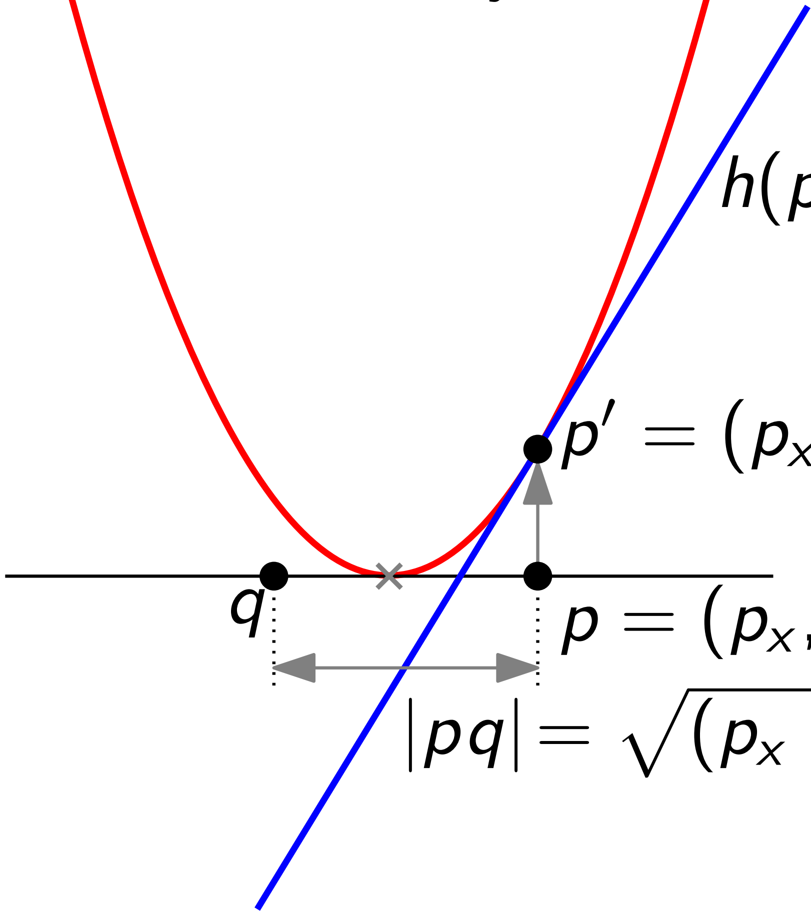

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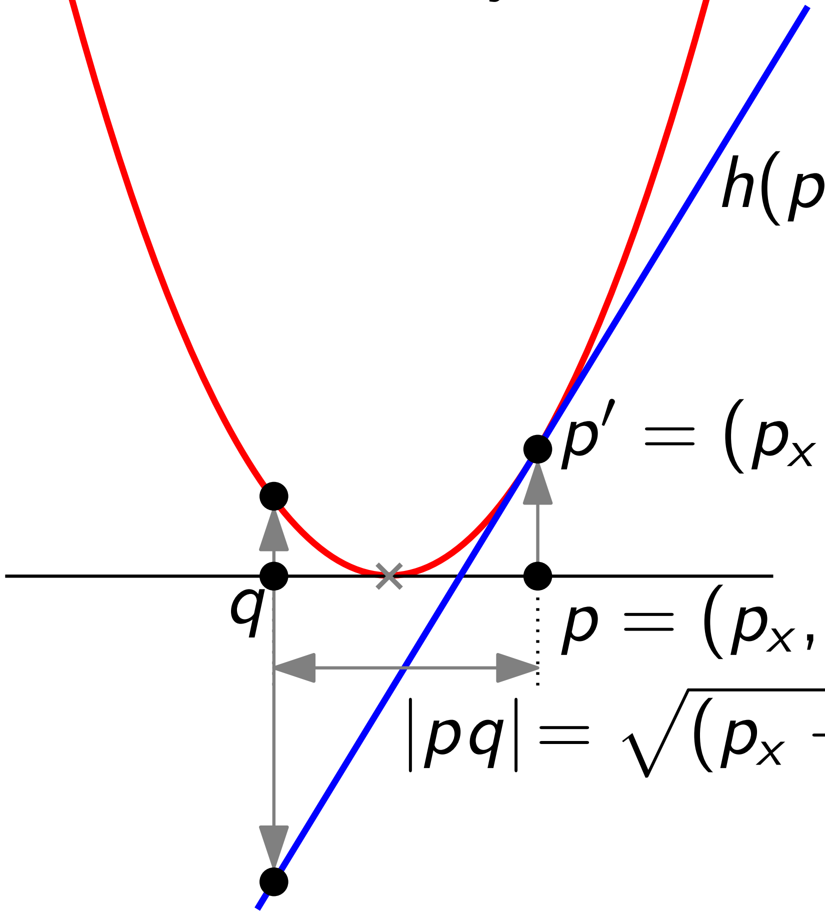

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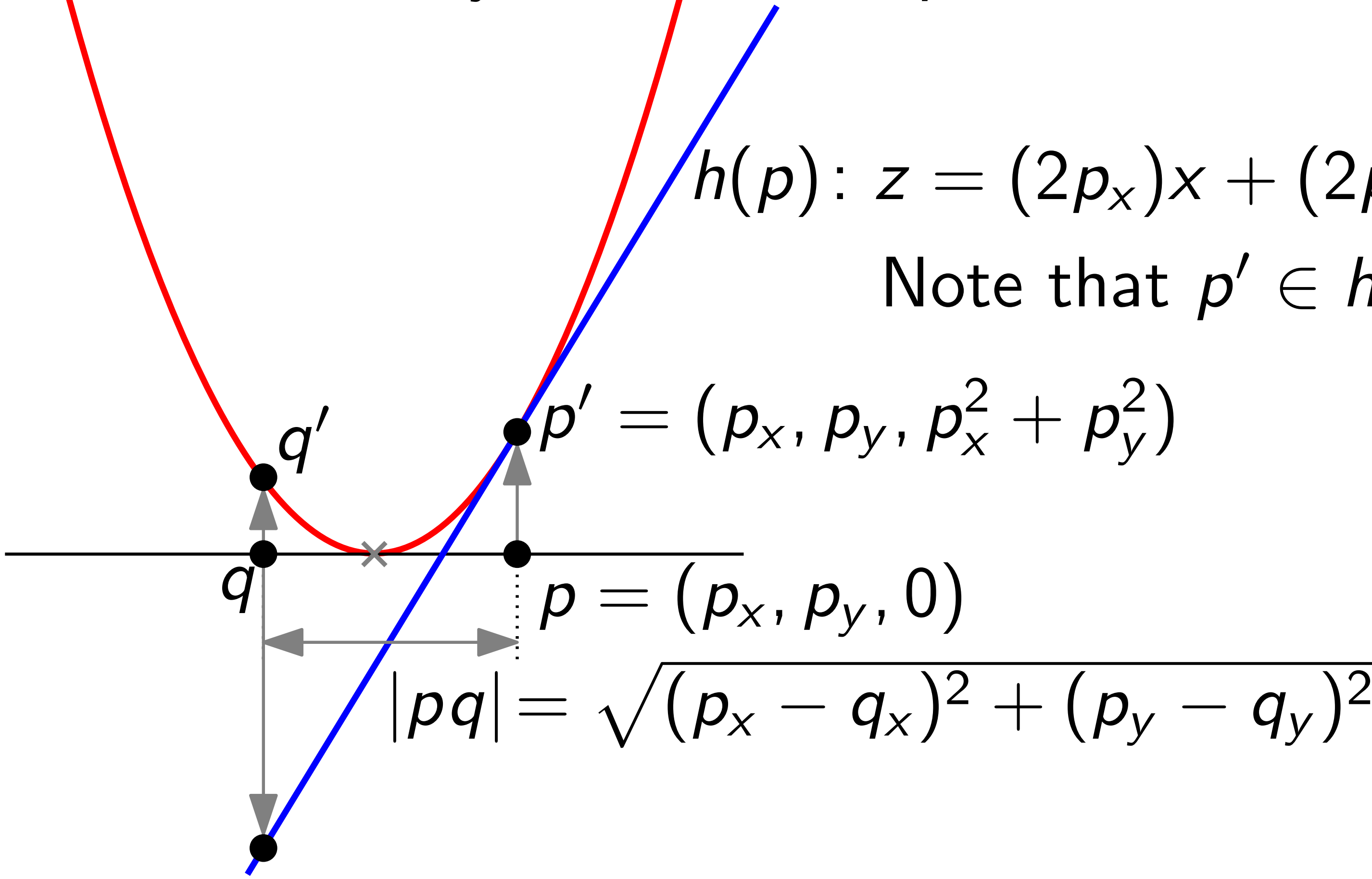

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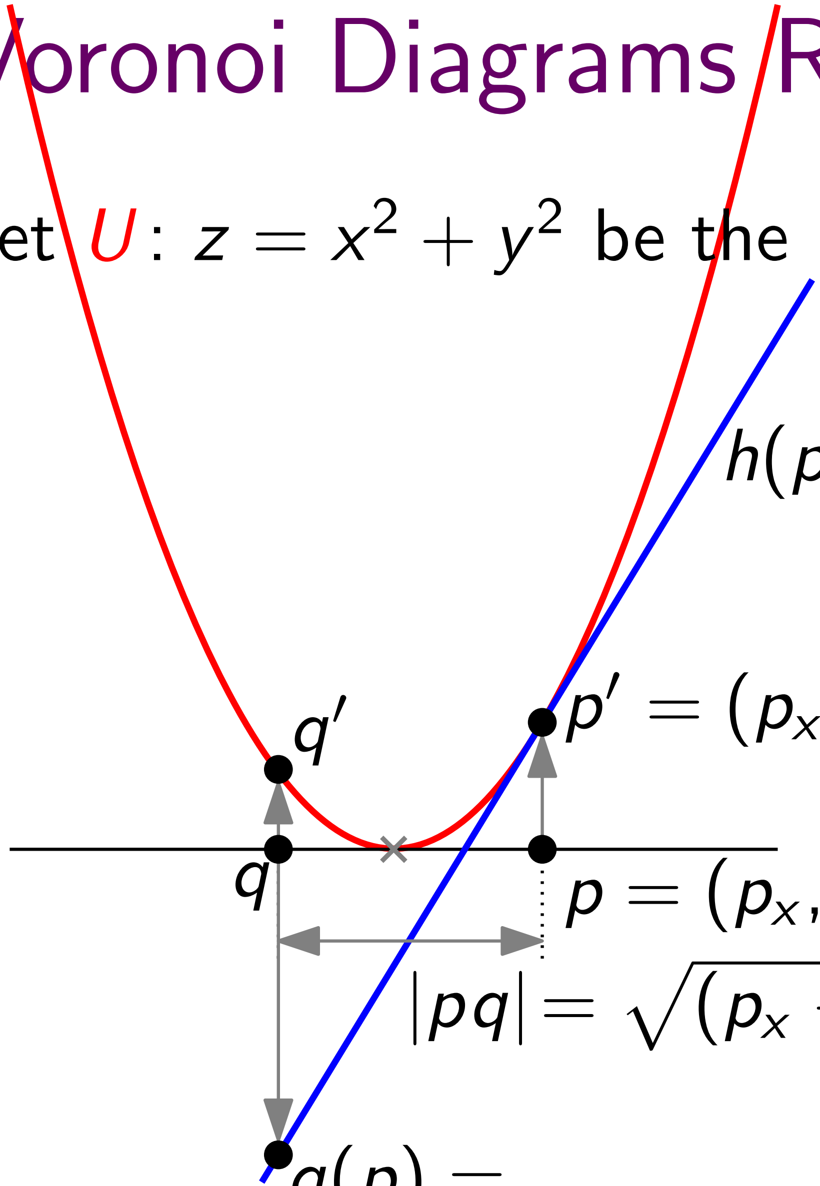

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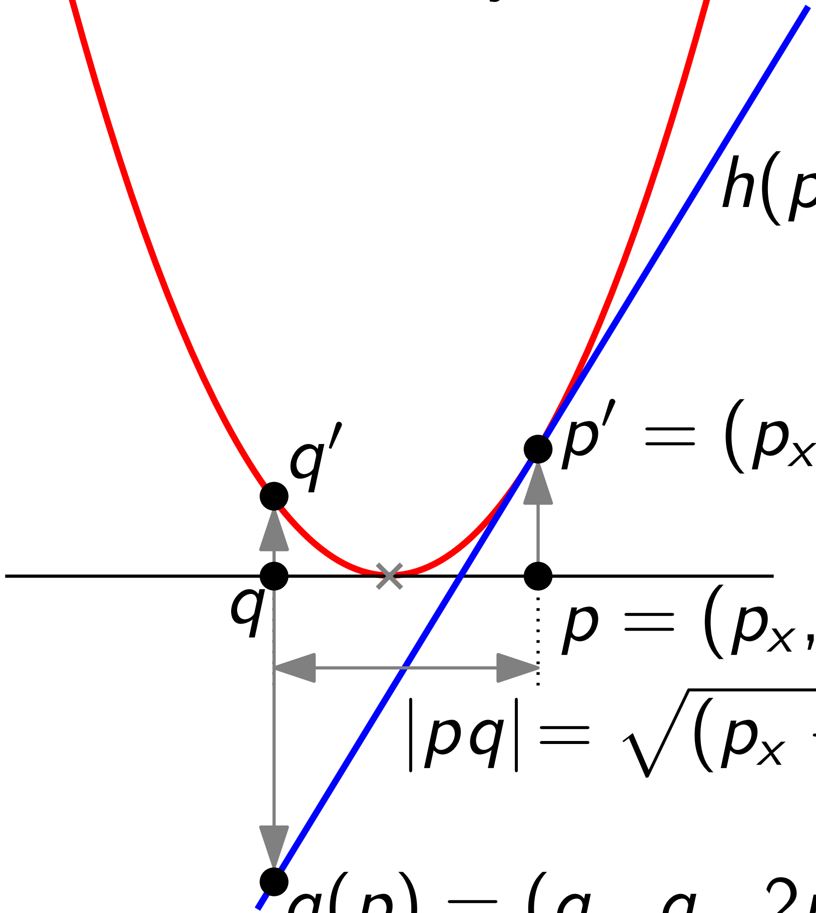

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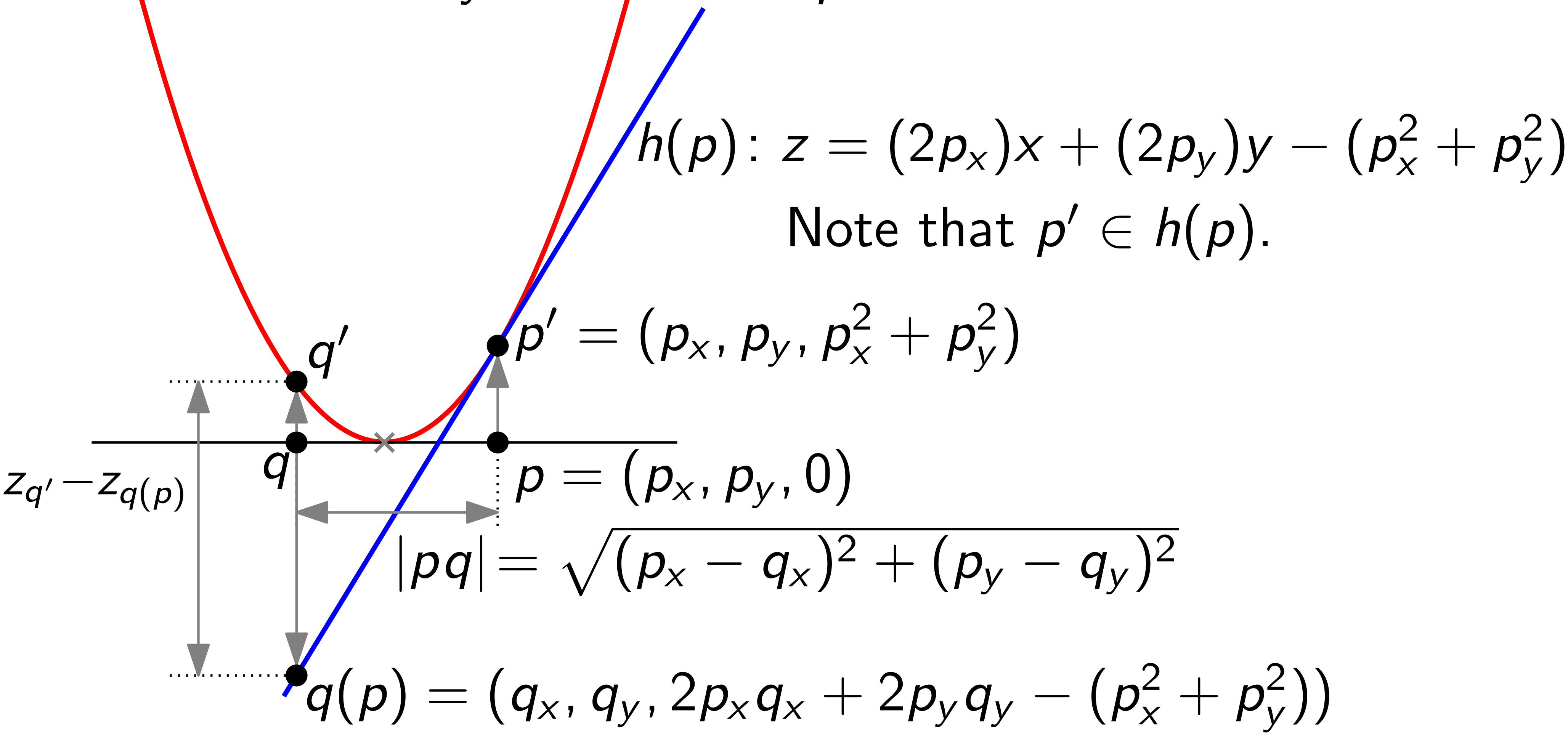

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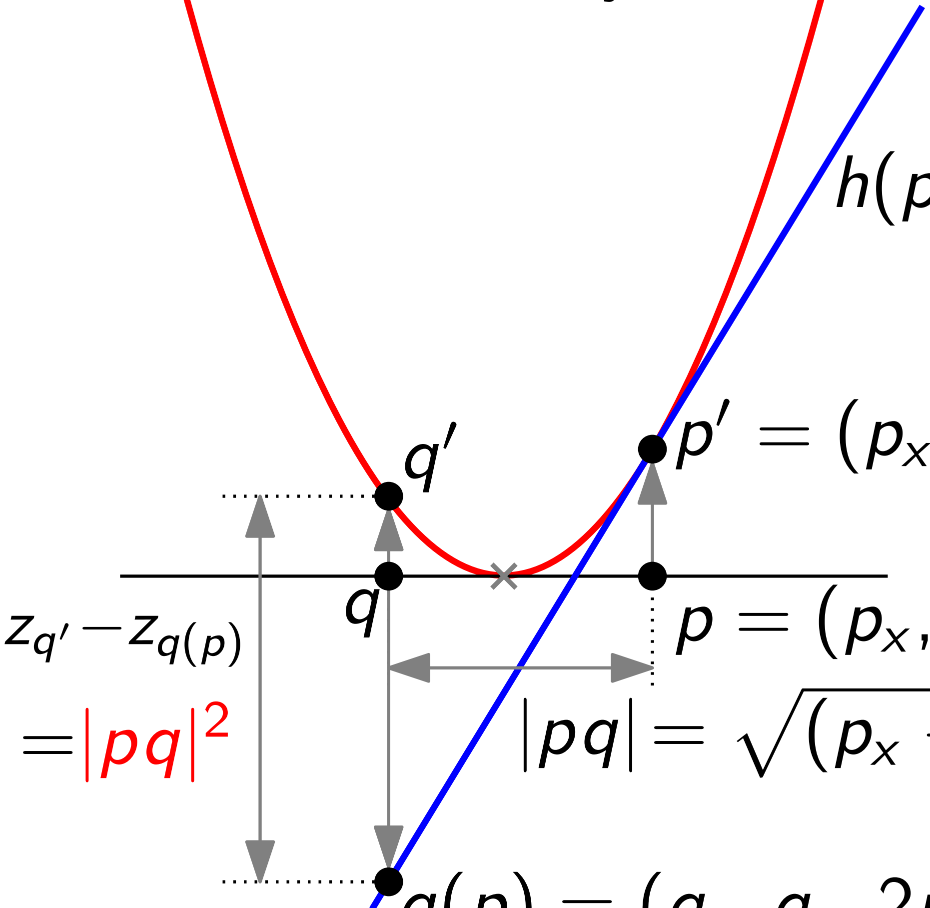

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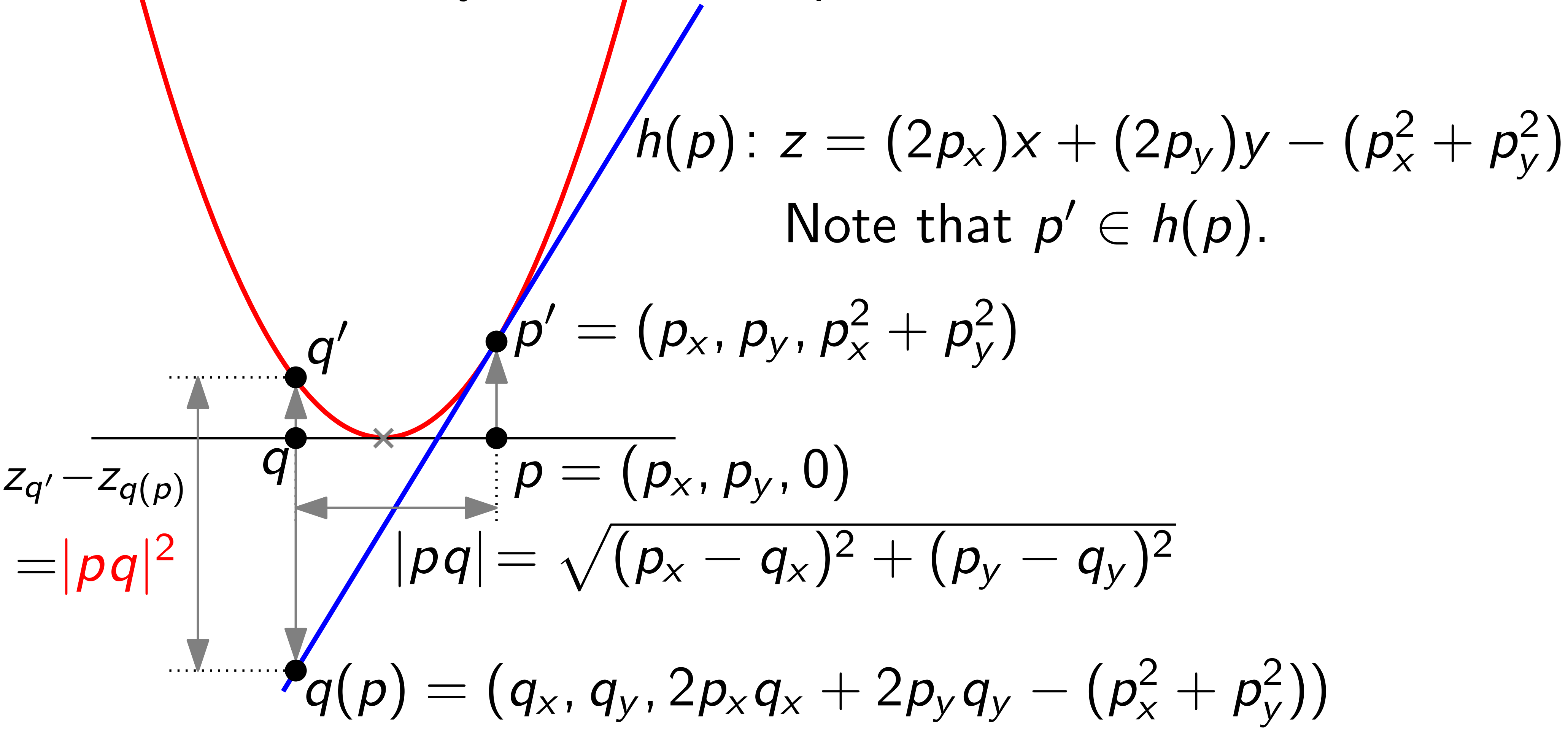

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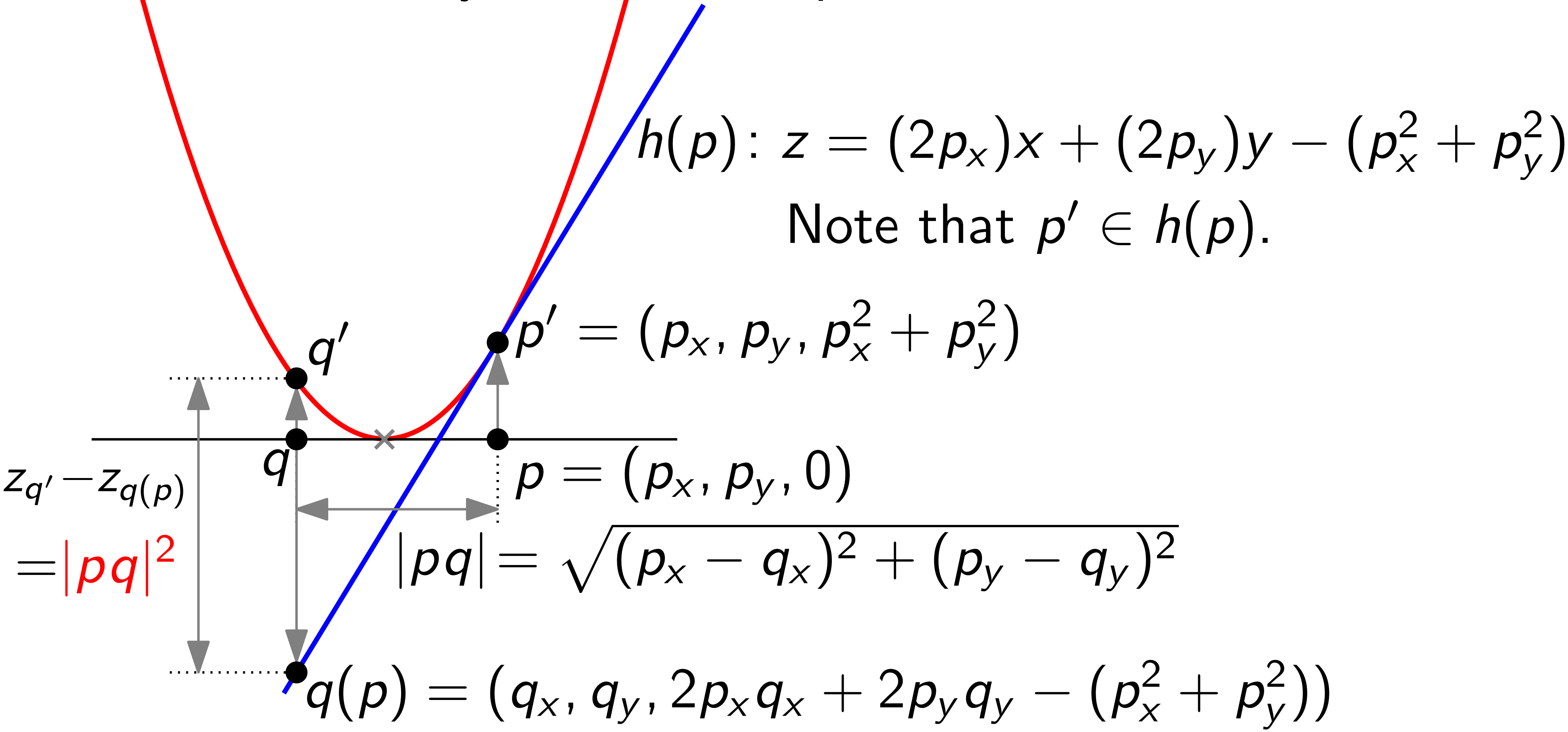

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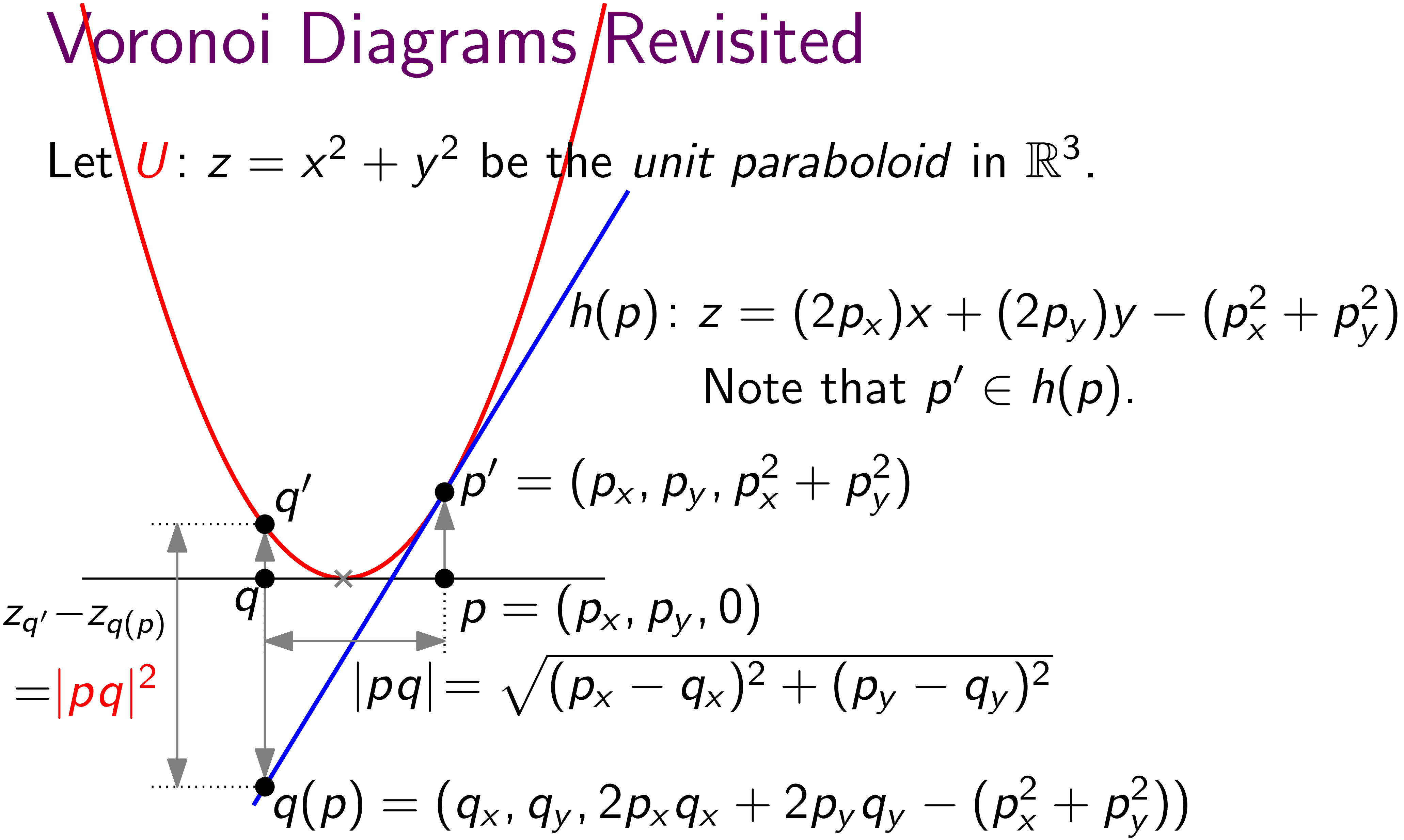

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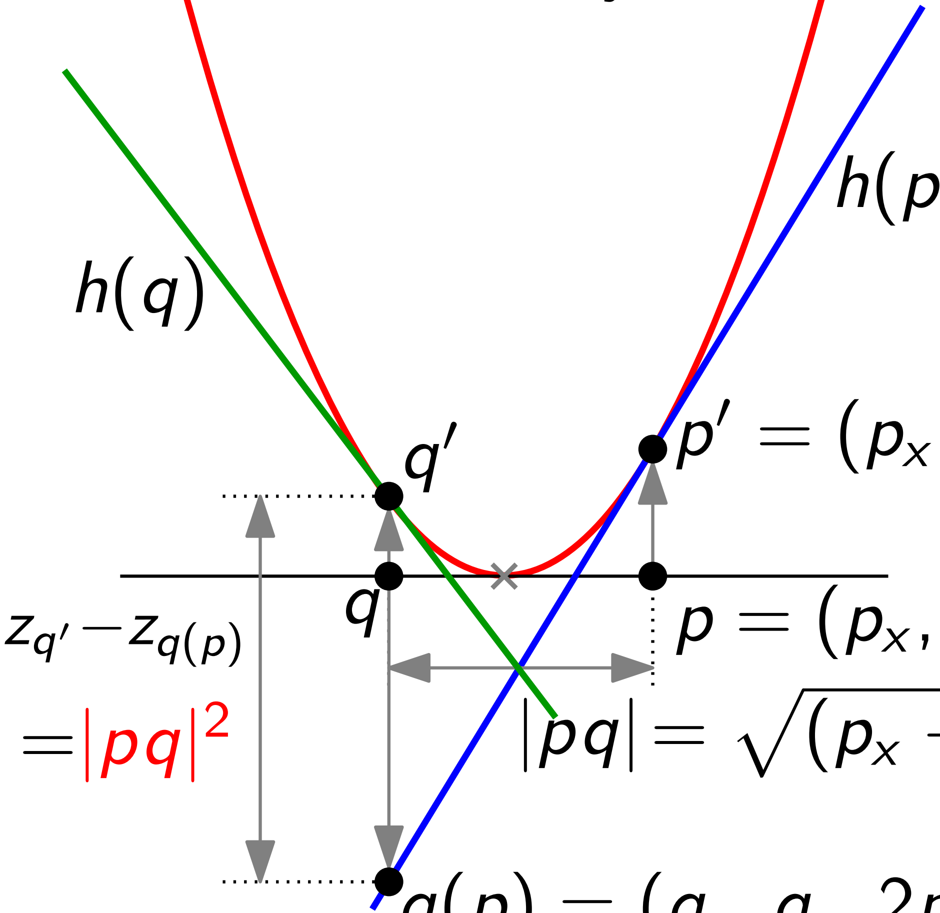

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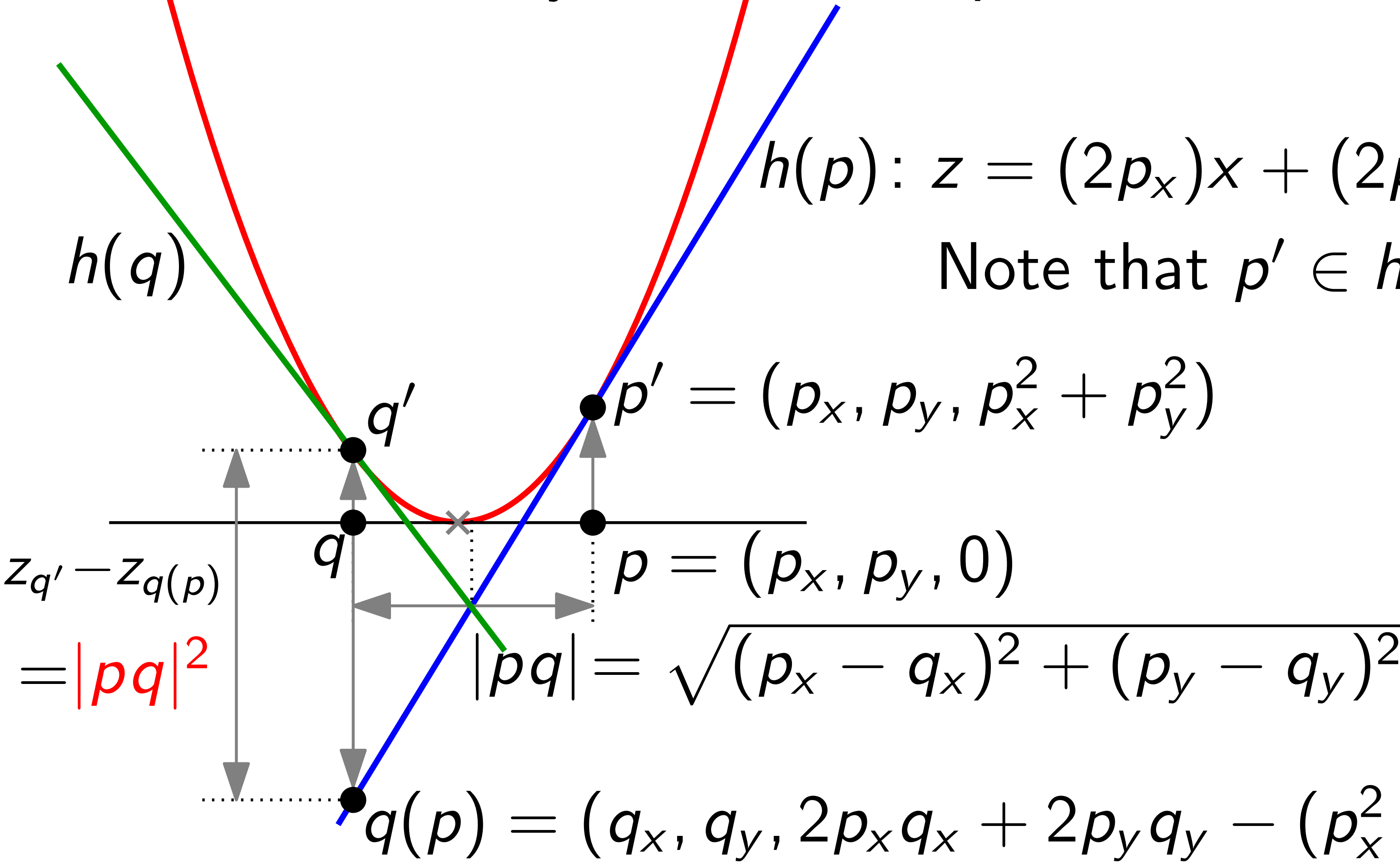

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V Revisited

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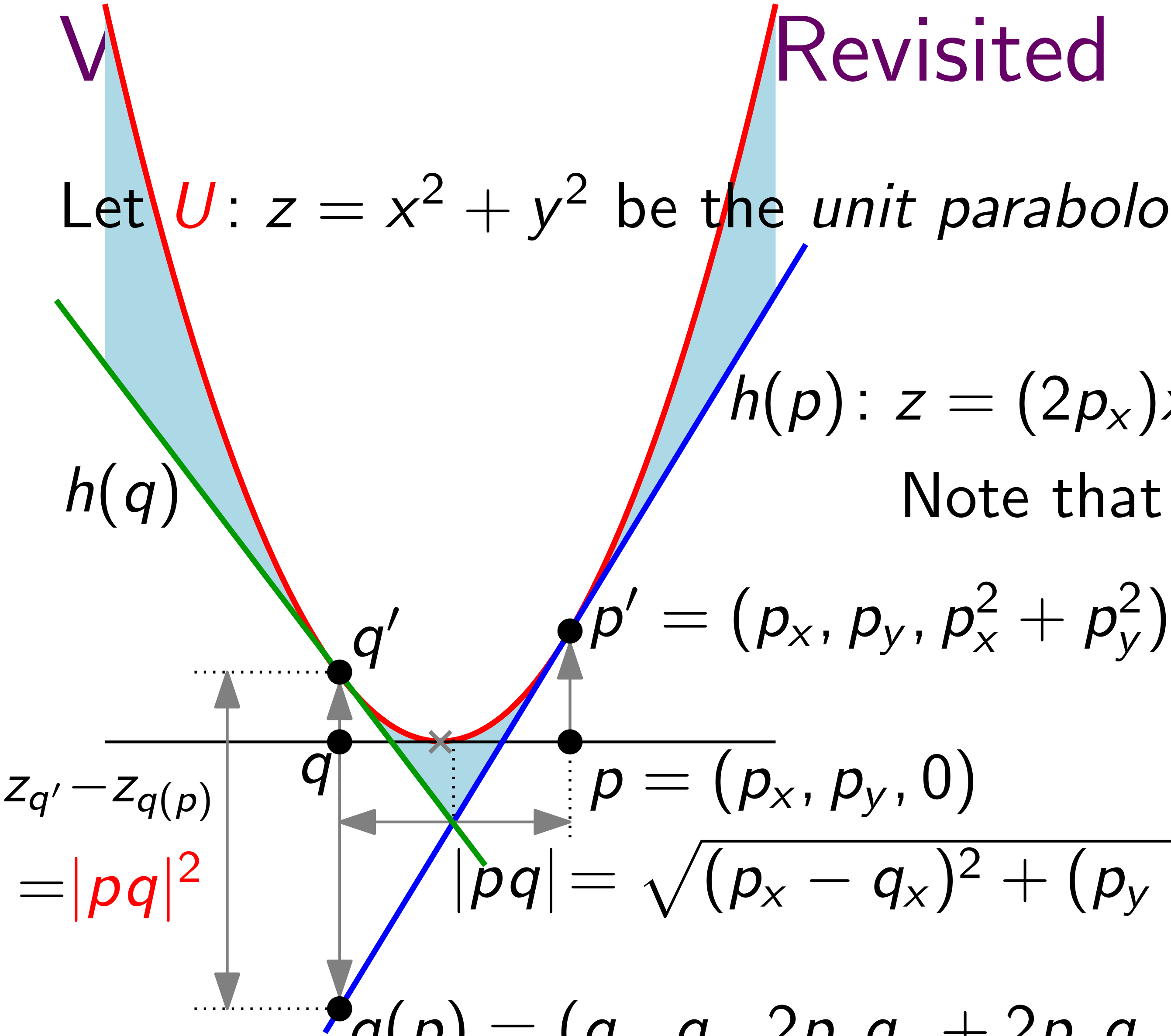

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The Upper Envelope Strikes Back

Theorem: Let $P \subset \mathbb{R}^2 \times \{0\}$

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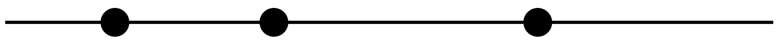
Theorem: Let $P \subset \mathbb{R}^2 \times \{0\}$ and $\mathcal{H} = \{h(p) \mid p \in P\}$.



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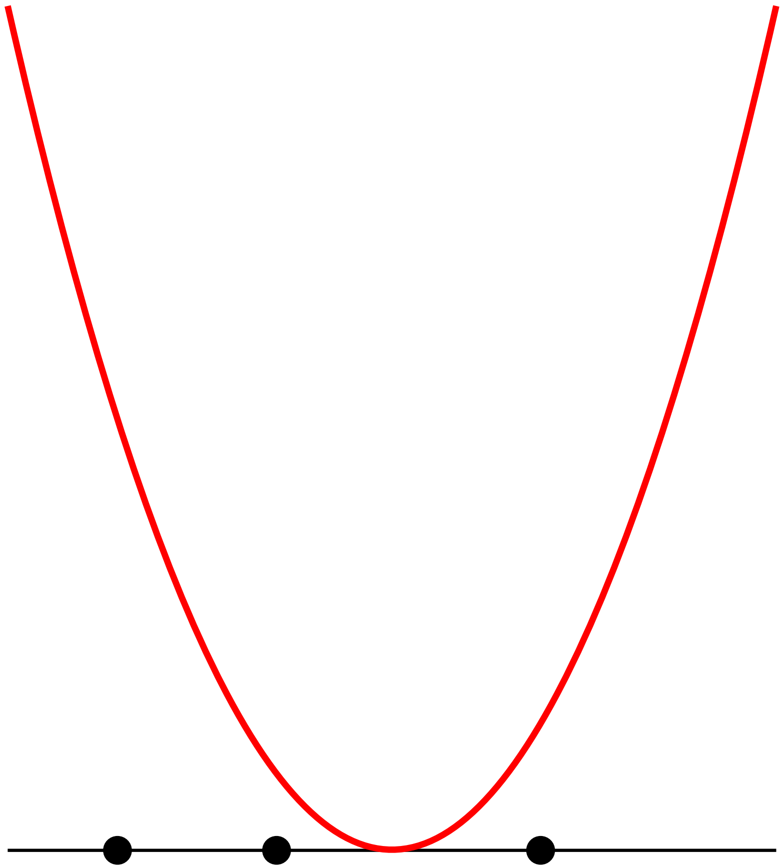

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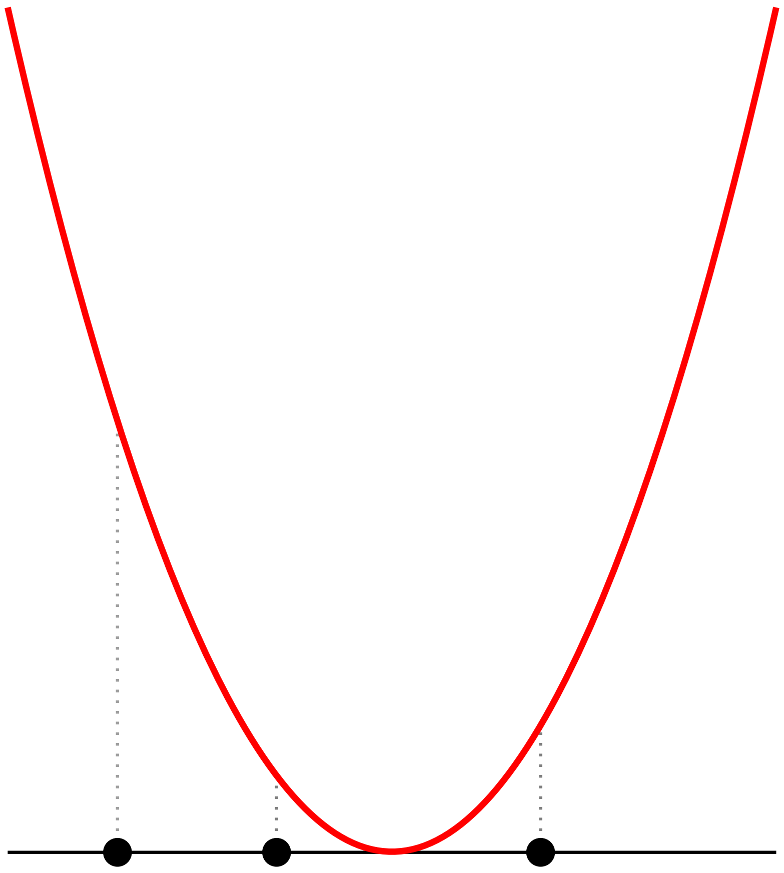

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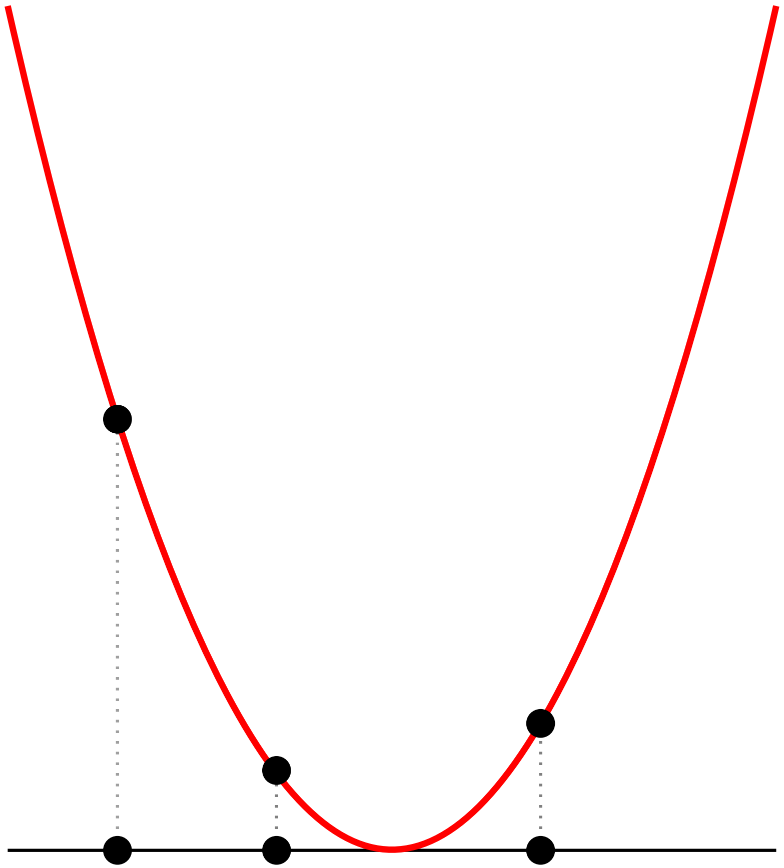

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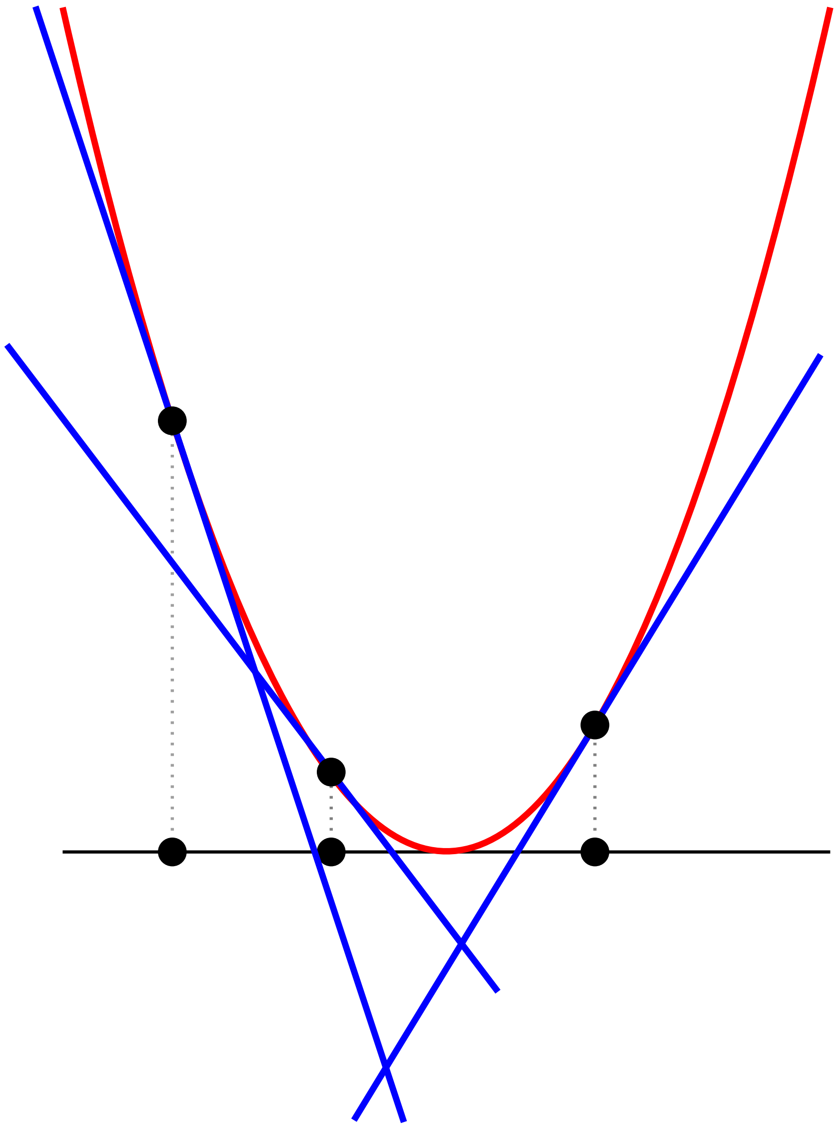

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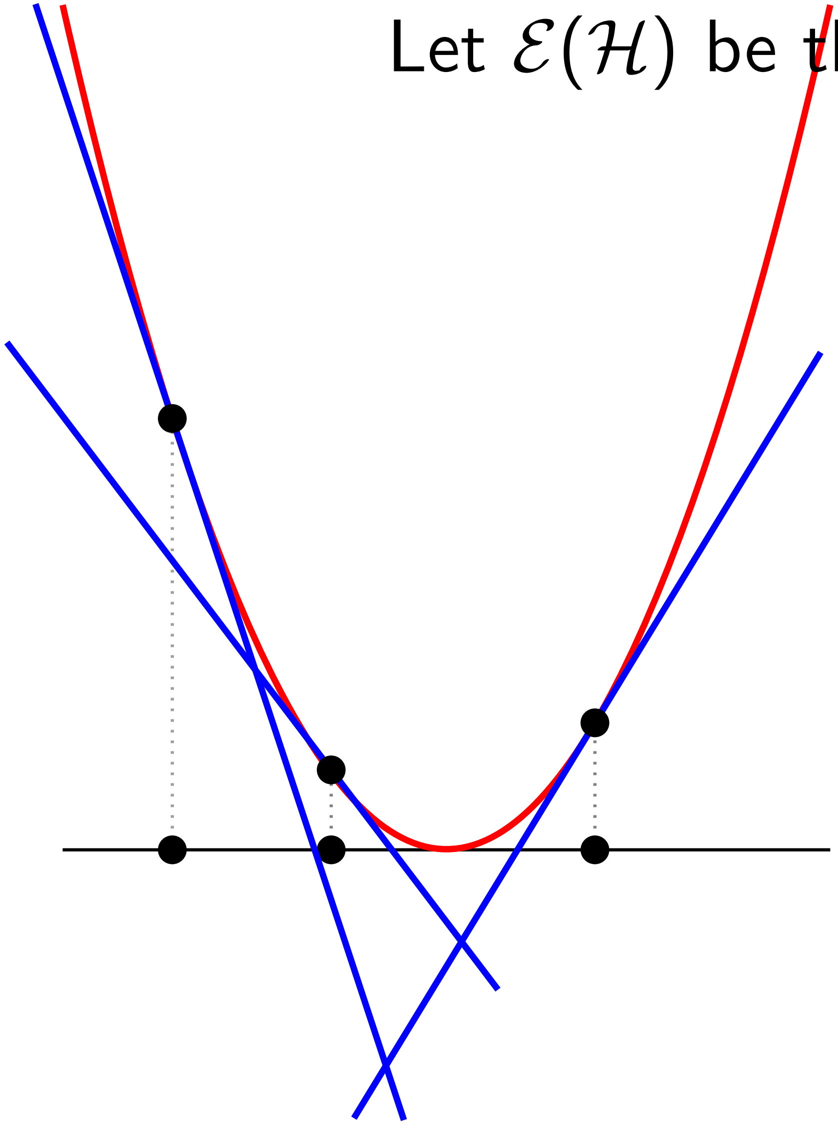

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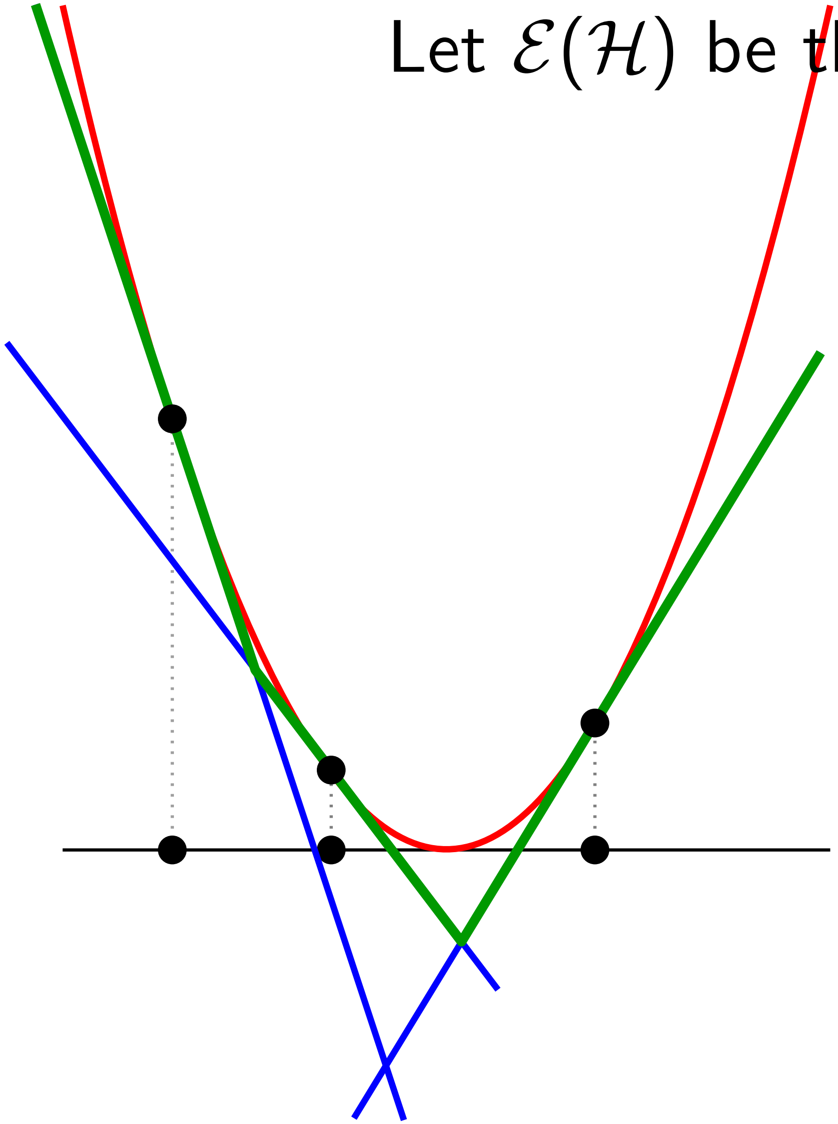

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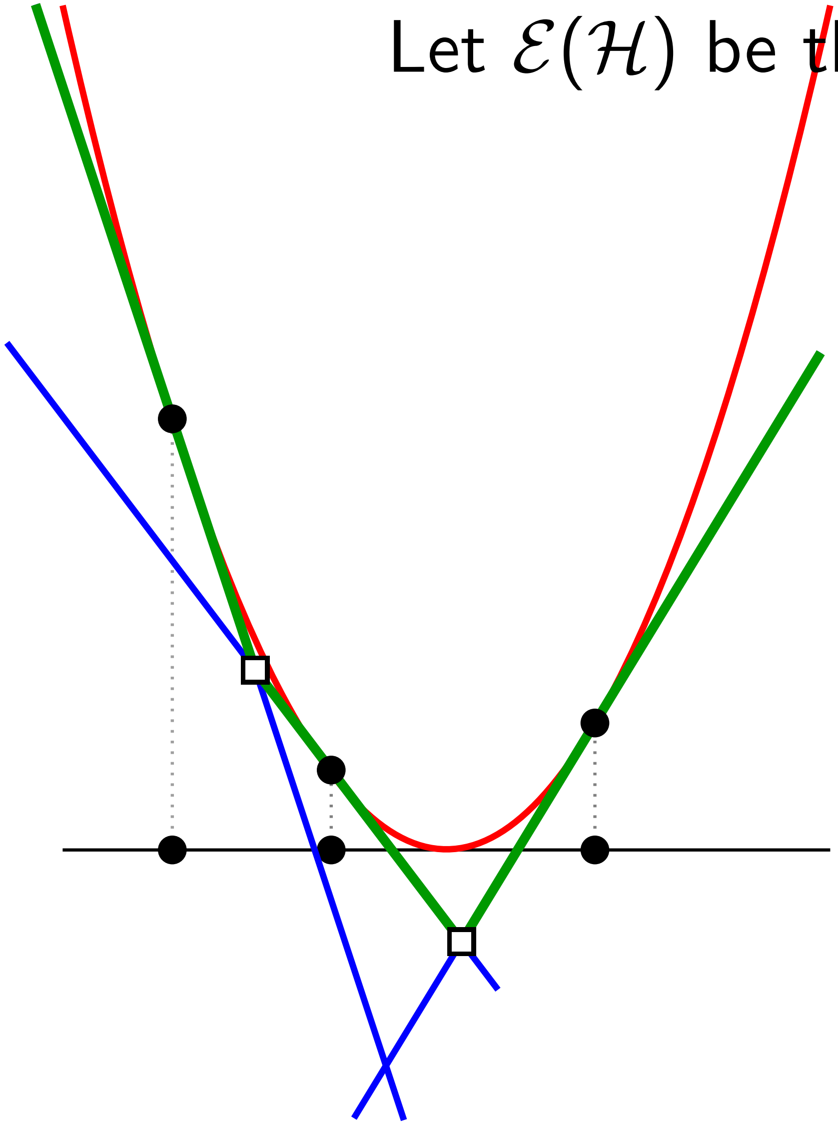

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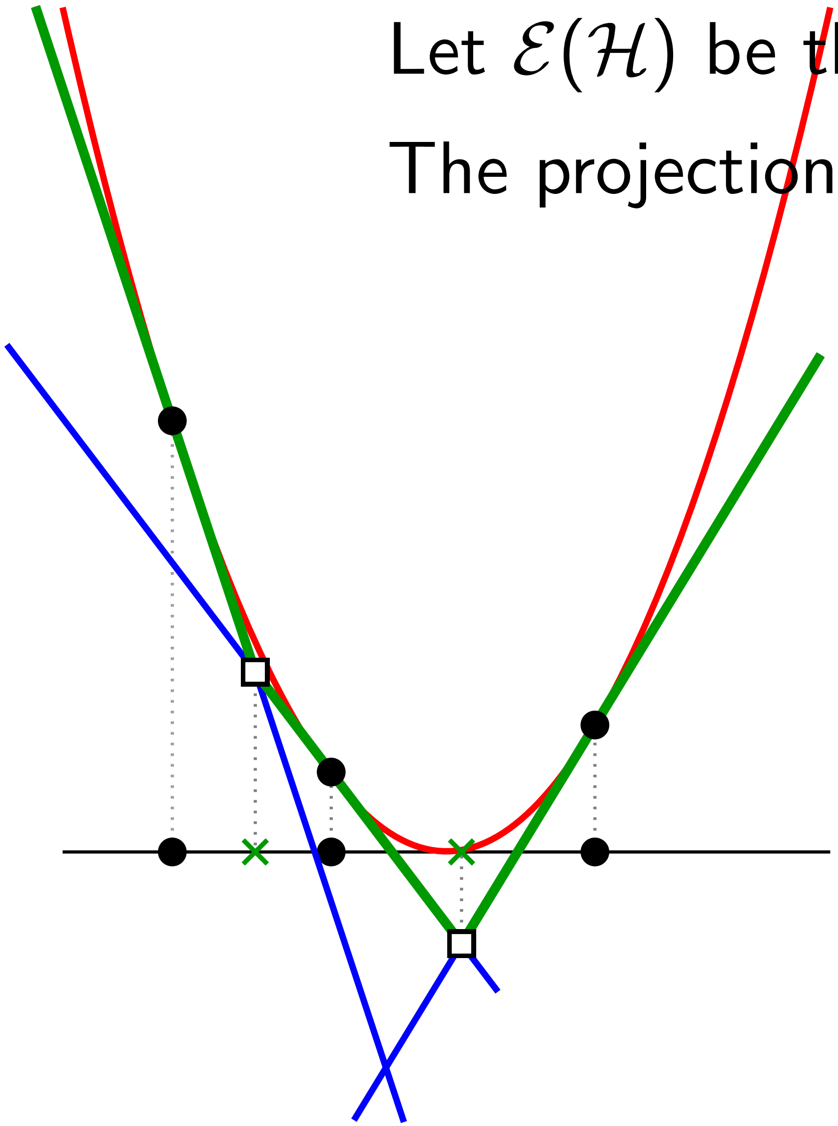

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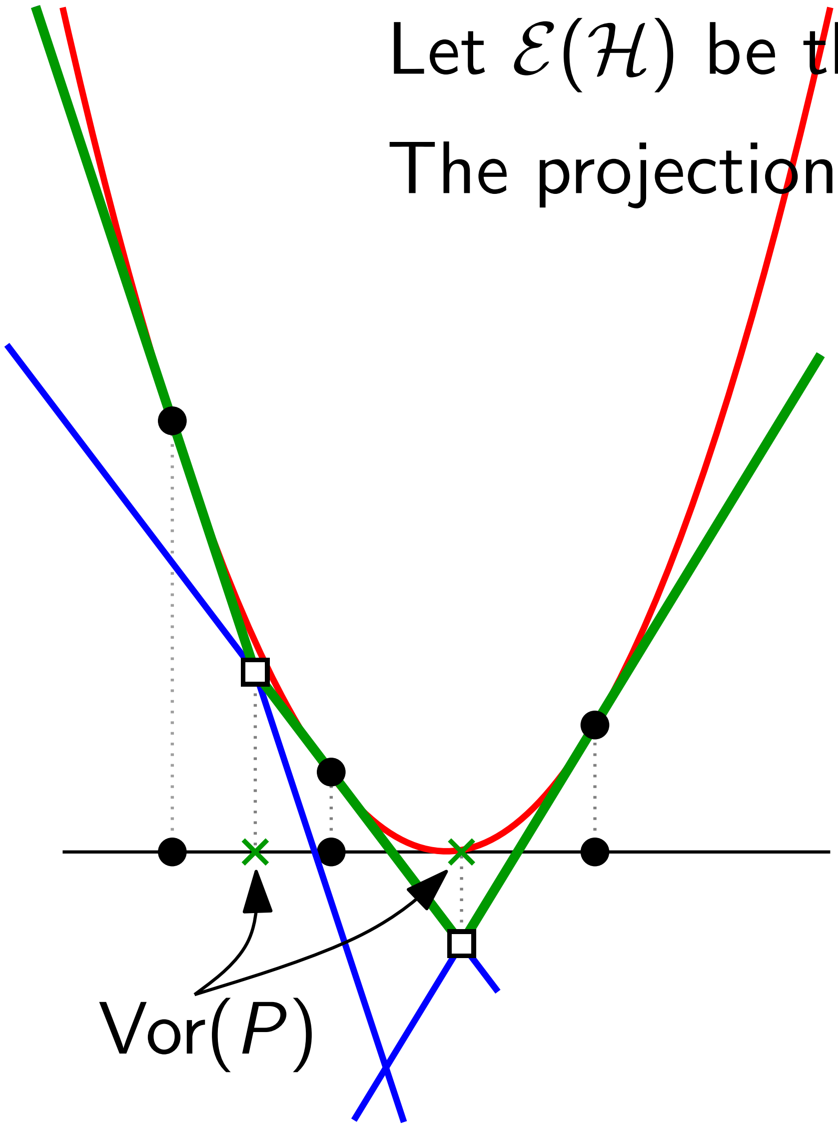

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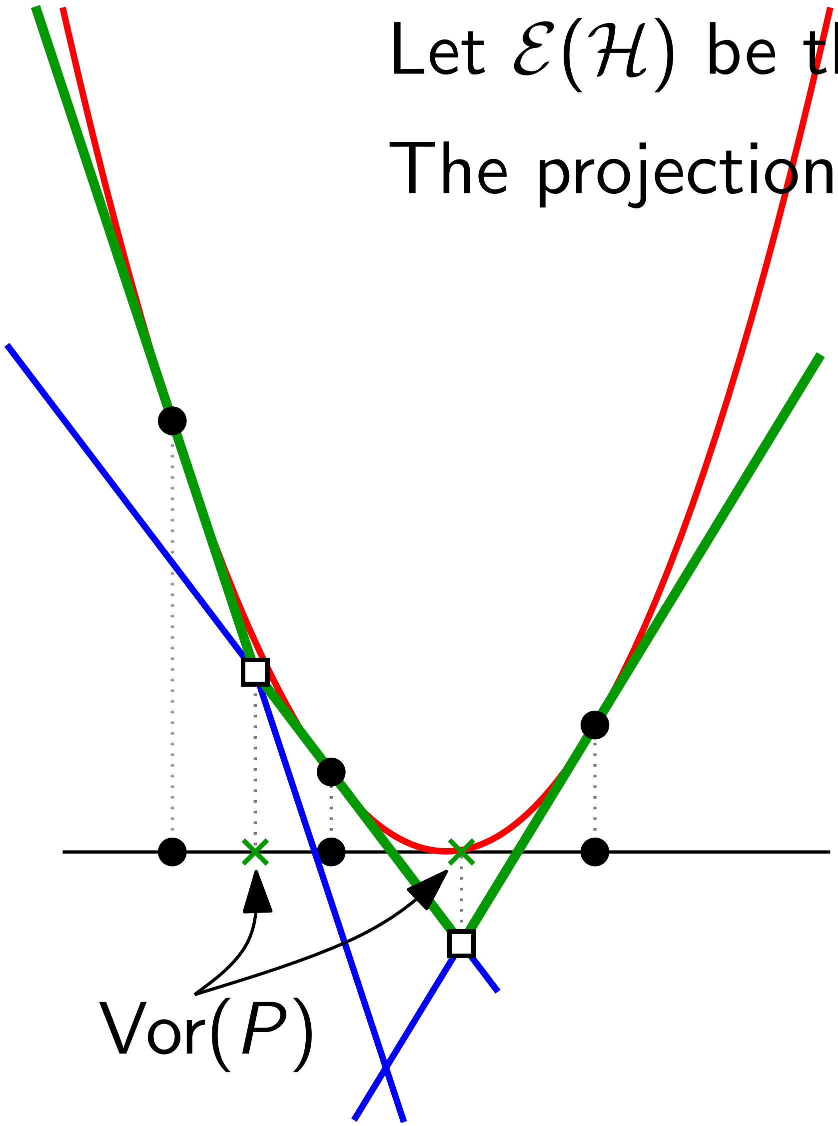

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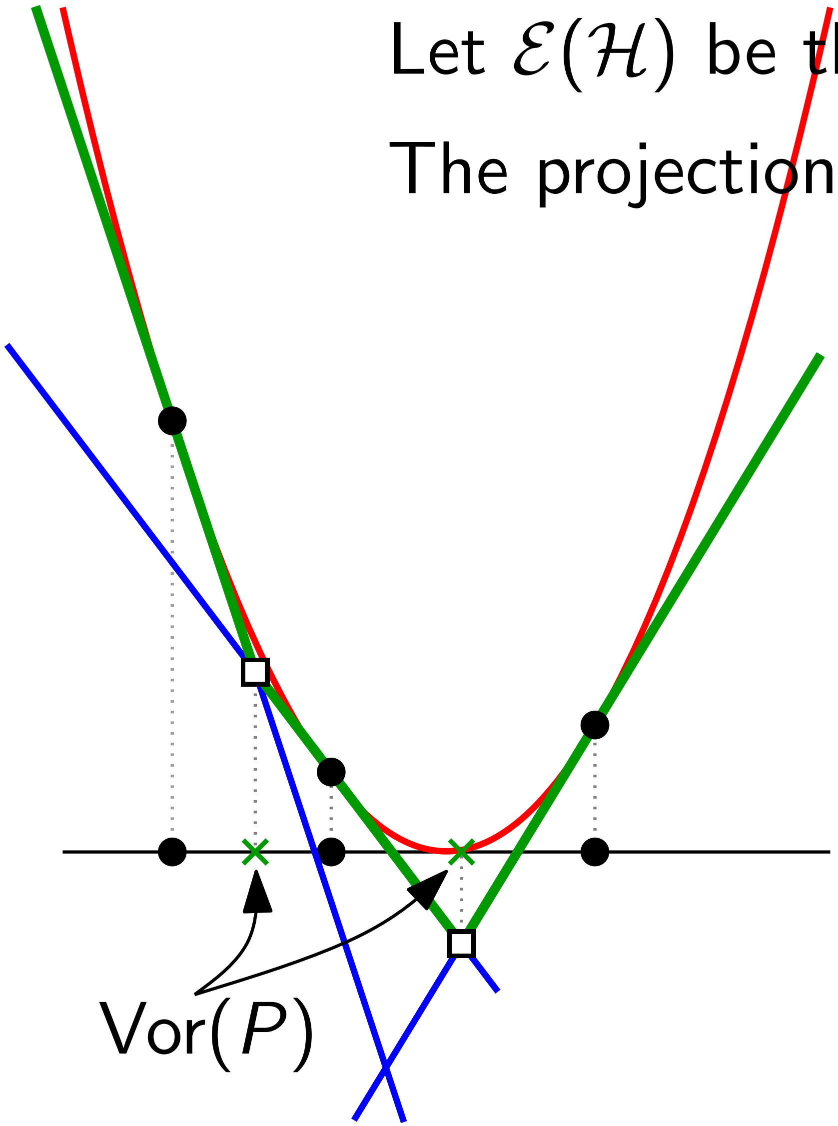

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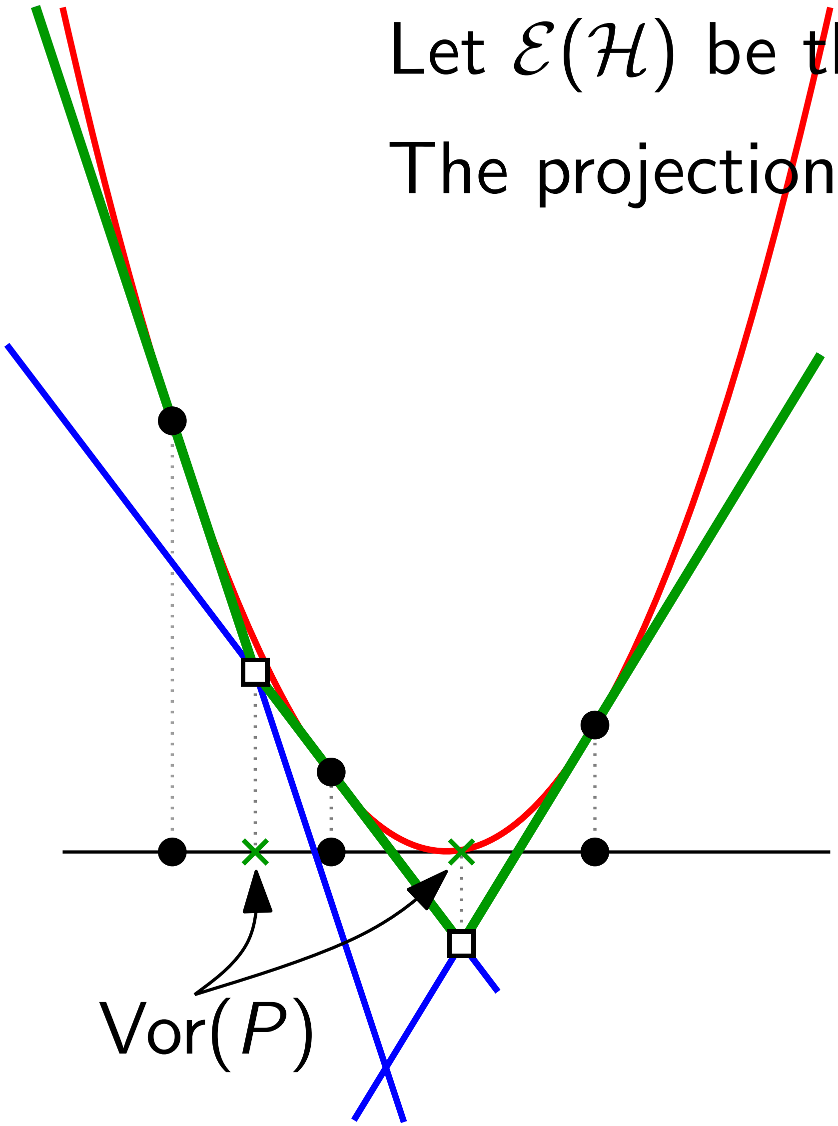

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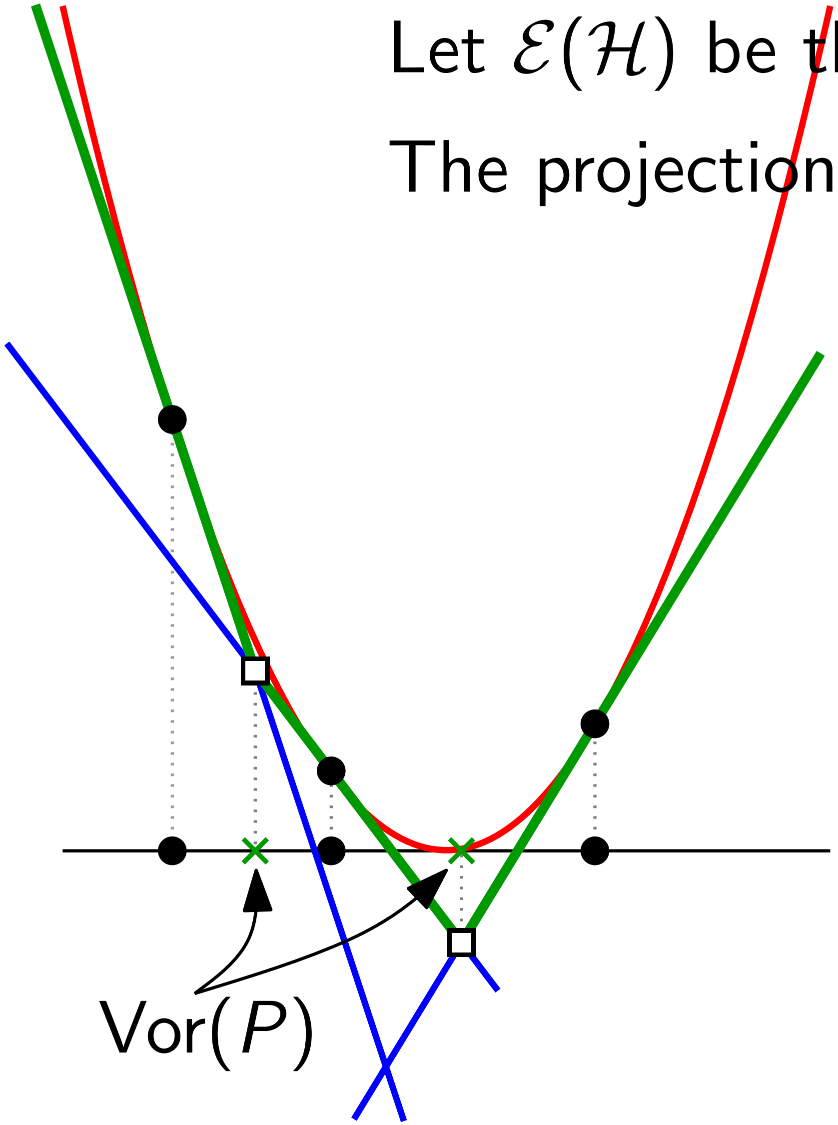

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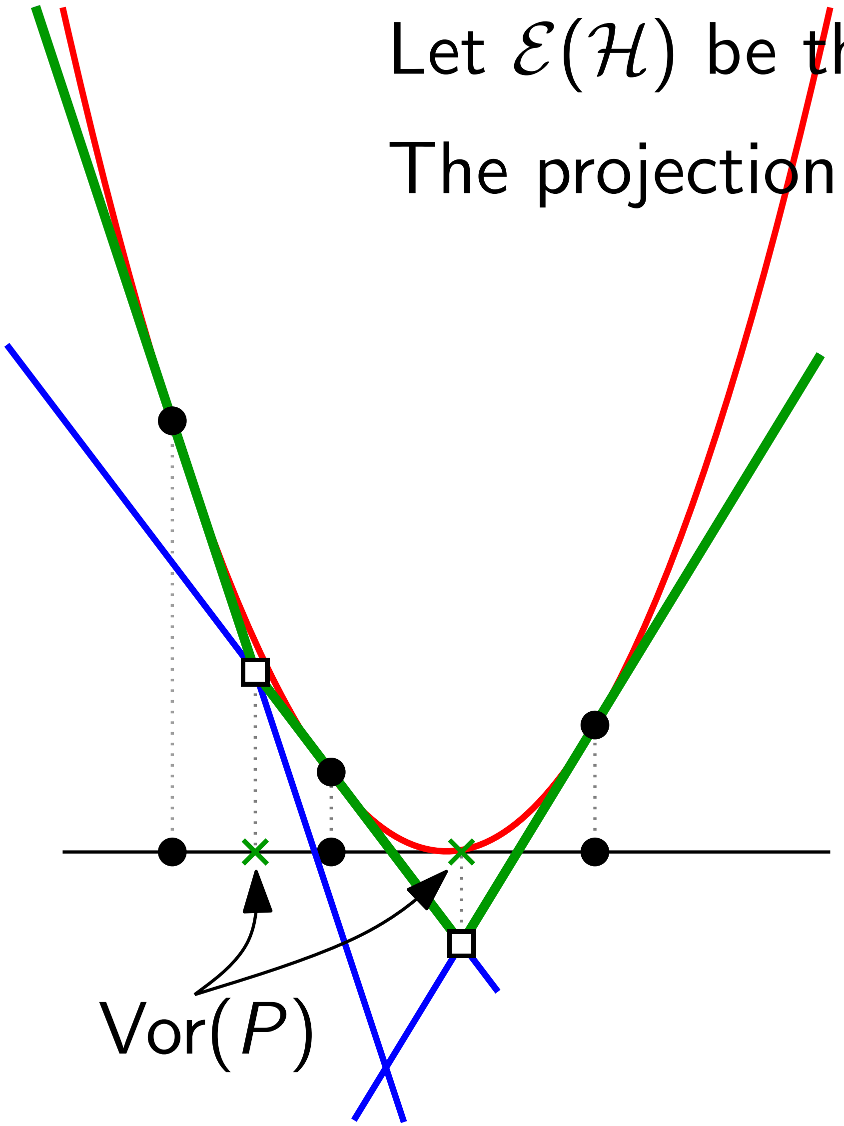

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use algorithm Rand3dConvexHull!