

Computational Geometry

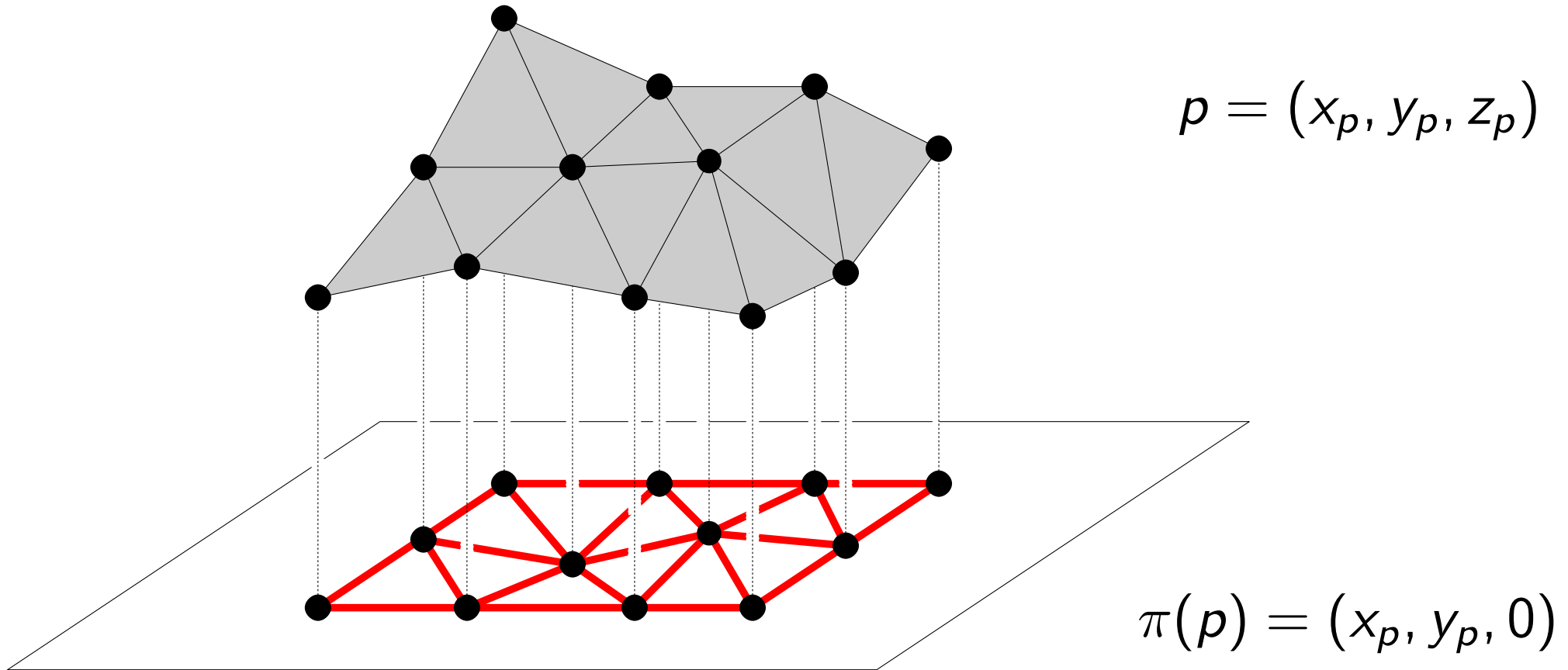
Winter semester 2016/17

Height Interpolation

Lecture #8

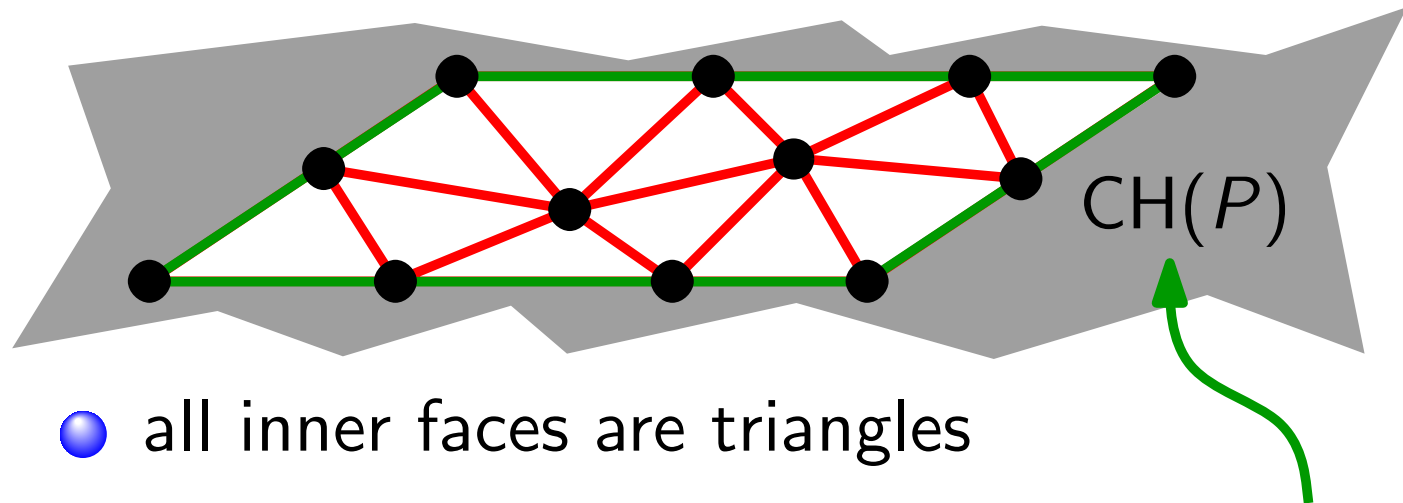
(Chapter 9 in the textbook)

Height Interpolation



Triangulation of Planar Point Sets

Definition: Given $P \subset \mathbb{R}^2$, a *triangulation* of P is a maximal planar subdivision with vtx set P , that is, no edge can be added without crossing.

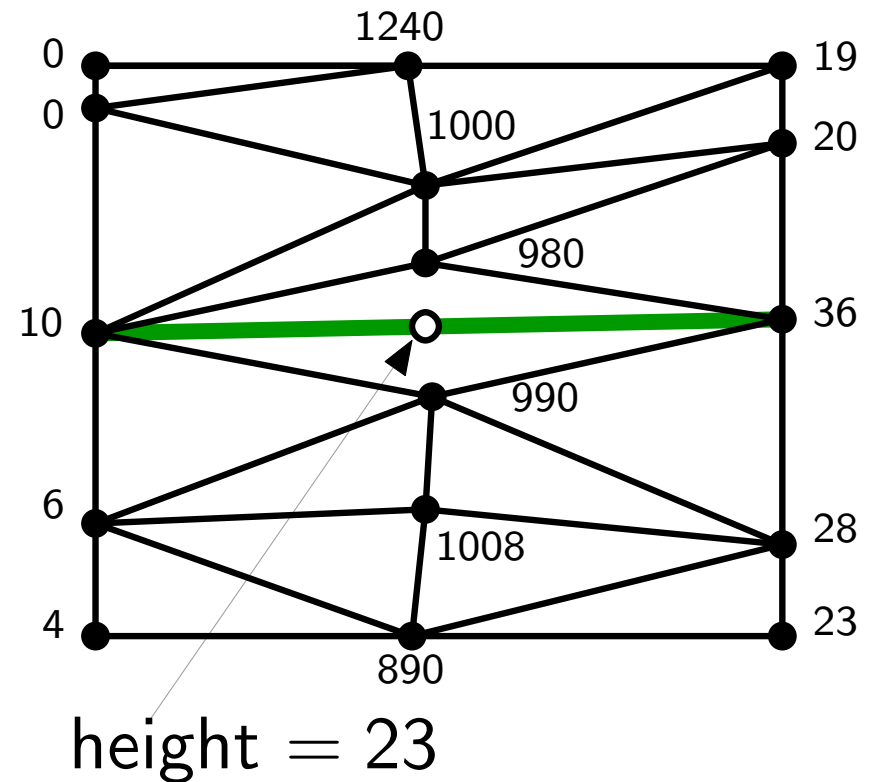
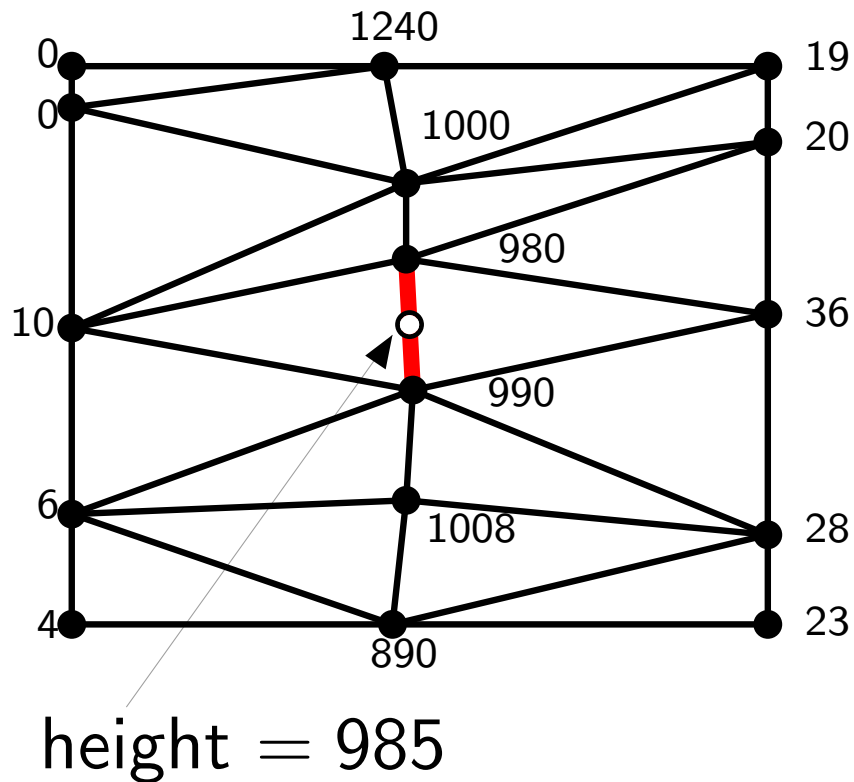


Observe:

- all inner faces are triangles
- outer face is complement of a convex polygon

Theorem: Let $P \subset \mathbb{R}^2$ be a set of n sites, not all collinear, and let h be the number of sites on $\partial\text{CH}(P)$. Then *any* triangulation of P has $t(n, h)$ triangles and $e(n, h)$ edges. **Task:** Compute t and e !

Back to Height Interpolation



Intuition: Avoid “skinny” triangles!

In other words: avoid small angles!

Angle-Optimal Triangulations

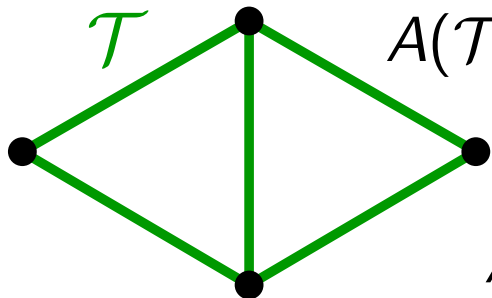
Definition: Given a set $P \subset \mathbb{R}^2$ and a triangulation \mathcal{T} of P , let m be the number of triangles in \mathcal{T} and let $A(\mathcal{T}) = (\alpha_1, \dots, \alpha_{3m})$ be the *angle vector* of \mathcal{T} , where $\alpha_1 \leq \dots \leq \alpha_{3m}$ are the angles in the triangles of \mathcal{T} .

We say $A(\mathcal{T}) > A(\mathcal{T}')$

if $\exists i \in \{1, \dots, 3m\} : \alpha_i > \alpha'_i$ and $\forall j < i : \alpha_j = \alpha'_j$.

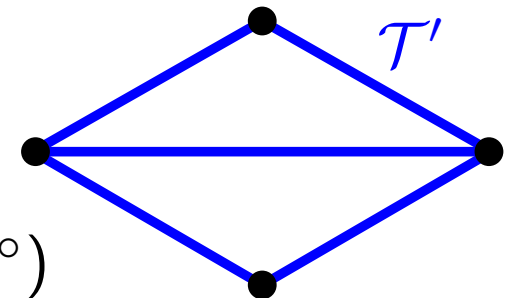
\mathcal{T} is *angle-optimal* if

$A(\mathcal{T}) \geq A(\mathcal{T}')$ for all triangulations \mathcal{T}' of P .



$$A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ)$$

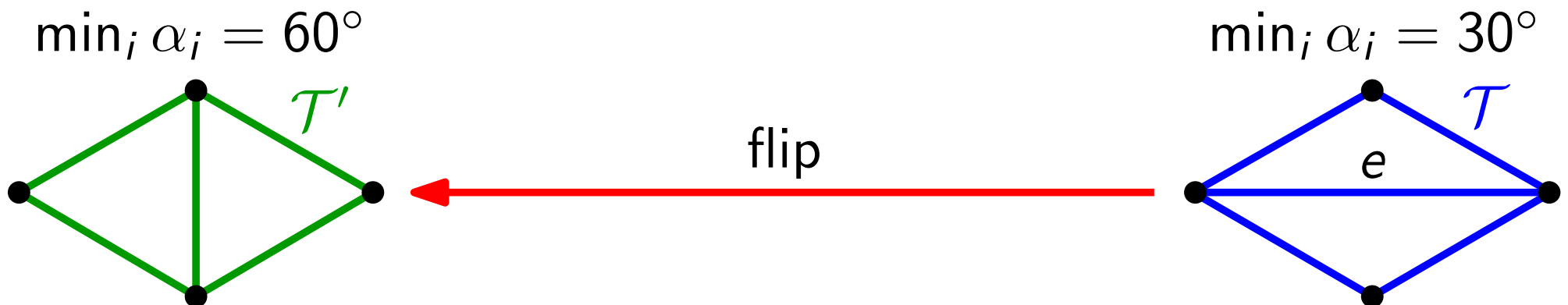
$$A(\mathcal{T}') = (30^\circ, 30^\circ, 30^\circ, 30^\circ, 120^\circ, 120^\circ)$$



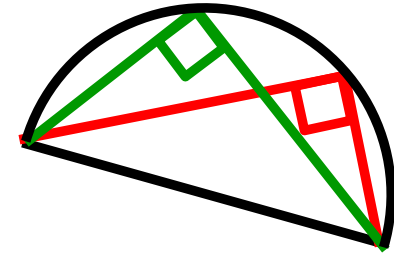
Edge Flips

Definition: \mathcal{T} a triangulation. An edge e of \mathcal{T} is **illegal** if the minimum angle in the two triangles adjacent to e increases when flipping.

Observe: Let e be an illegal edge of \mathcal{T} , and $\mathcal{T}' = \text{flip}(\mathcal{T}, e)$. Then $A(\mathcal{T}') > A(\mathcal{T})$.



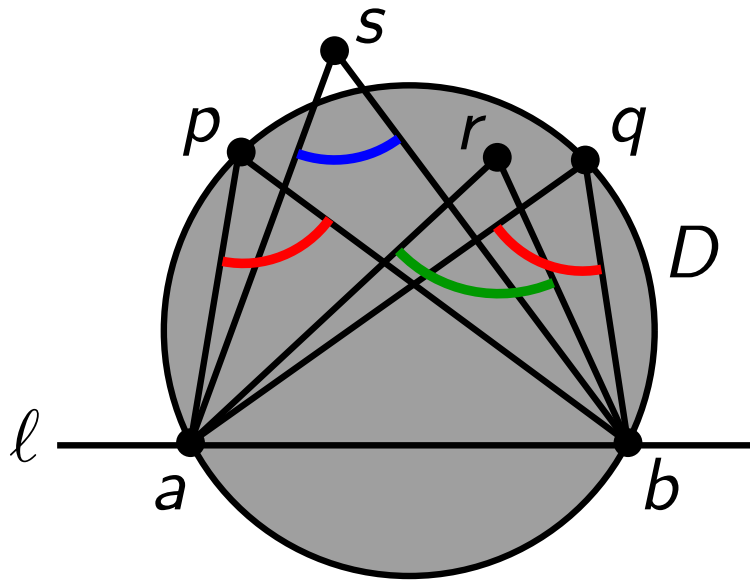
This is all Greek to me...



Theorem: (Thales)

The diameter of a circle always subtends a right angle to any point on the circle.

Theorem: (Thales++)



$$\{a, b\} := \ell \cap \partial D \quad (a \neq b)$$

$$p, q \in \partial D$$

$$r \in \text{int}(D)$$

$$s \notin D$$

$$\angle asb < \angle apb = \angle aqb < \angle arb$$

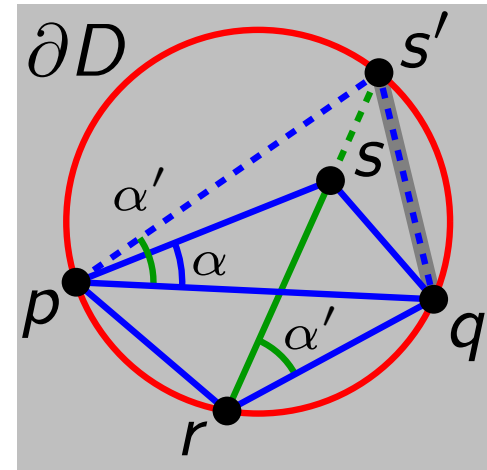
Legal Triangulations

Lemma: Let $\Delta prq, \Delta pqs \in \mathcal{T}$ and $p, q, r \in \partial D$.
Then edge pq is illegal iff $s \in \text{int}(D)$.

If p, q, r, s in convex position and $s \notin \partial D$,
then either pq or rs is illegal.

Proof: Show: $\forall \alpha' \text{ in } \mathcal{T}' \exists \alpha \text{ in } \mathcal{T} \text{ s.t. } \alpha < \alpha'$.
 (“ \Rightarrow ”) Use Thales++ w.r.t. qs' .

Note: Criterion symmetric in r and s
 \Rightarrow if $s \in \partial D$, both pq and rs legal.



Definition: A triangulation is *legal* if it has no illegal edge.

Existence? Algorithm: Let \mathcal{T} be any triangulation of P .
While \mathcal{T} has an illegal edge e , flip e . Return \mathcal{T} .

algorithm
terminates

$A(\mathcal{T})$ goes up! & $\#(\text{triangulations of } P) < \infty$

Legal vs. Angle-Optimal

Clearly... Every angle-optimal triangulation is legal.

But is every legal triangulation angle-optimal??

Let's see.

To clarify things, we'll introduce yet another type of triangulation...

Voronoi & Delaunay

Remember: Given a set P of n points in the plane...

$\text{Vor}(P)$ = subdivision of the plane into
Voronoi cells, edges, and vertices

$\mathcal{V}(p) = \{x \in \mathbb{R}^2 : |xp| < |xq| \text{ for all } q \in P \setminus \{p\}\}$
Voronoi cell of $p \in P$

Definition: The graph $\mathcal{G} = (P, E)$ with
 $\{p, q\} \in E \Leftrightarrow \mathcal{V}(p)$ and $\mathcal{V}(q)$ share an edge
is the *dual graph* of $\text{Vor}(P)$

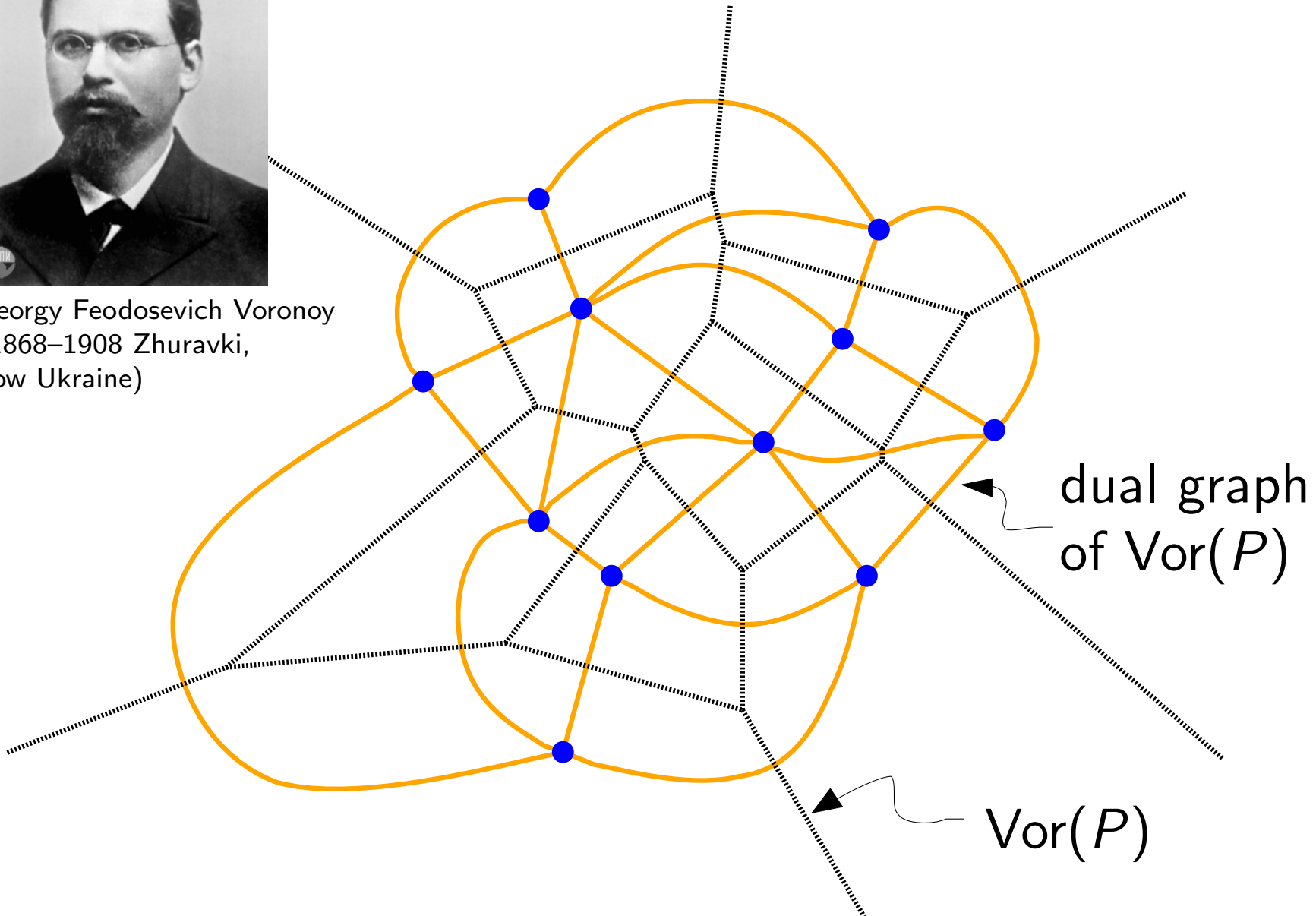
Definition: The *Delaunay graph* $\mathcal{DG}(P)$ is the straight-line drawing of \mathcal{G} .

From Voronoi to Delaunay

$$P \subset \mathbb{R}^2$$



Georgy Feodosevich Voronoy
(1868–1908 Zhuravki,
now Ukraine)



From Voronoi to Delaunay

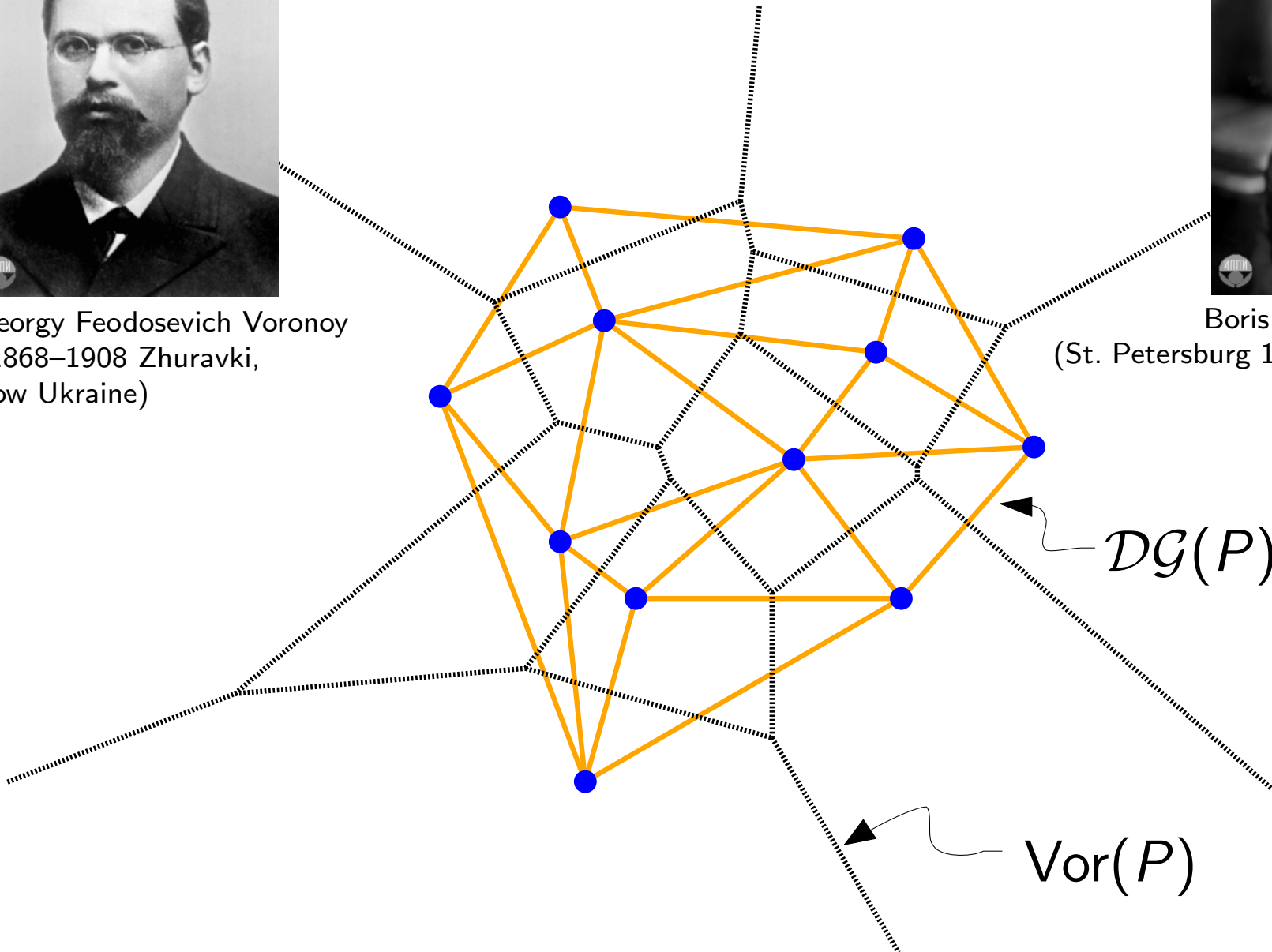
$$P \subset \mathbb{R}^2$$



Georgy Feodosevich Voronoy
(1868–1908 Zhuravki,
now Ukraine)



Boris Nikolaevich Delone
(St. Petersburg 1890–1980 Moscow)



Planarity

Theorem. $P \subset \mathbb{R}^2$ finite $\Rightarrow \mathcal{DG}(P)$ plane.

Proof.

Recall property of Voronoi edges:

Edge pq is in $\mathcal{DG}(P) \Leftrightarrow \exists D_{pq}$ closed disk s.t.

- $p, q \in \partial D_{pq}$ and
- $\{p, q\} = D_{pq} \cap P$.

$c = \text{center}(D_{pq})$ lies on edge betw. $\mathcal{V}(p)$ & $\mathcal{V}(q)$.

Suppose \exists edge $uv \neq pq$ in $\mathcal{DG}(P)$ that crosses pq .

$u, v \notin D_{pq} \Rightarrow u, v \notin t_{pq} \Rightarrow$

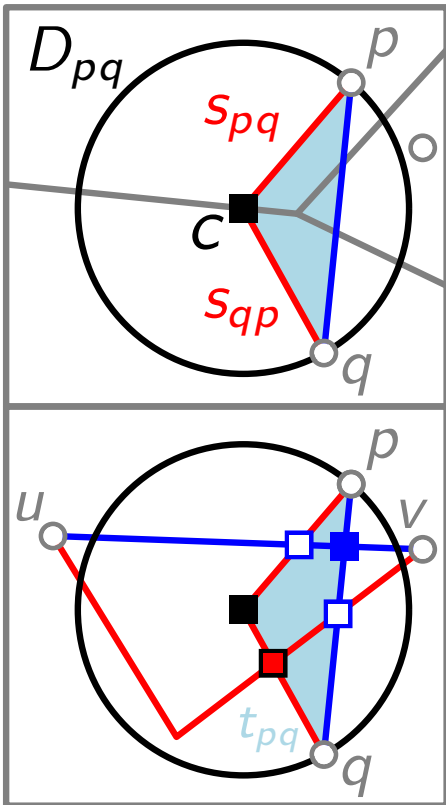
uv crosses another edge of t_{pq}

$p, q \notin D_{uv} \Rightarrow p, q \notin t_{uv} \Rightarrow$

pq crosses another edge of t_{uv}

\Rightarrow one of s_{pq} or s_{qp} crosses one of s_{uv} or s_{vu}

⚡ $s_{pq} \subset \mathcal{V}(p)$, $s_{qp} \subset \mathcal{V}(q)$, $s_{uv} \subset \mathcal{V}(u)$, $s_{vu} \subset \mathcal{V}(v)$.



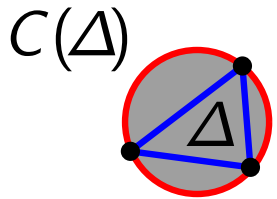
Characterization

Characterization of Voronoi vertices and Voronoi edges \Rightarrow

Theorem. $P \subset \mathbb{R}^2$ finite. Then

- (i) Three pts $p, q, r \in P$ are vertices of the same face of $\mathcal{DG}(P) \Leftrightarrow \text{int}(C(p, q, r)) \cap P = \emptyset$
- (ii) Two pts $p, q \in P$ form an edge of $\mathcal{DG}(P) \Leftrightarrow$ there is a disk D with
 - $\partial D \cap P = \{p, q\}$ and
 - $\text{int}(D) \cap P = \emptyset$.

Theorem. $P \subset \mathbb{R}^2$ finite, \mathcal{T} triangulation of P . Then
 \mathcal{T} Delaunay \Leftrightarrow for each triangle Δ of \mathcal{T} :
 $\text{int}(C(\Delta)) \cap P = \emptyset$.



(“empty-circumcircle property”)

Main Result

Theorem. $P \subset \mathbb{R}^2$ finite, \mathcal{T} triangulation of P .
Then \mathcal{T} legal $\Leftrightarrow \mathcal{T}$ Delaunay.

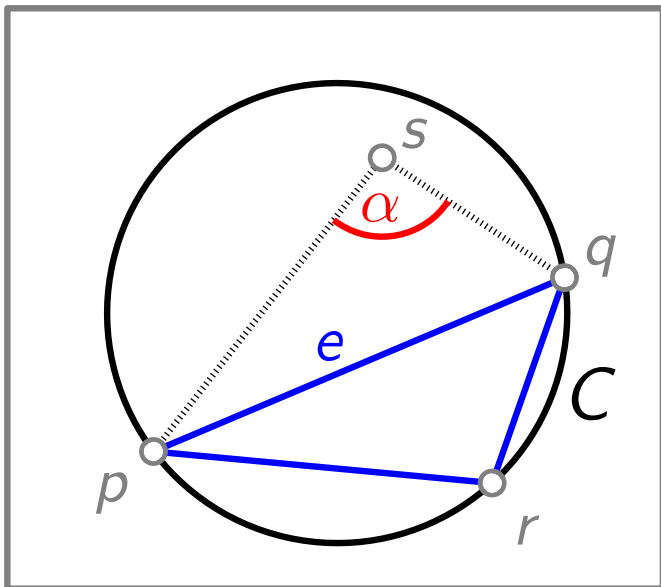
Proof. “ \Leftarrow ” implied by empty-circumcircle property & Thales++
“ \Rightarrow ” by contradiction:

Assume \mathcal{T} is legal triang. of P , but *not* Delaunay.

$\Rightarrow \exists \Delta pqr$ such that $\text{int}(C(\Delta pqr))$ contains $s \in P$.

Wlog. let $e = pq$ be the edge of Δpqr such that s “sees” pq before the other edges of Δpqr .

Among all such pairs $(\Delta pqr, s)$ in \mathcal{T} choose one that maximizes $\alpha = \angle psq$.



Proof of Main Result (cont'd)

Consider the triangle Δpqt adjacent to e in \mathcal{T} .

\mathcal{T} legal $\Rightarrow e$ legal $\Rightarrow t \notin \text{int}(C(\Delta pqr))$

$\Rightarrow C(\Delta pqt)$ contains $C(\Delta pqr) \cap e^+$. { halfplane supported by e that contains s

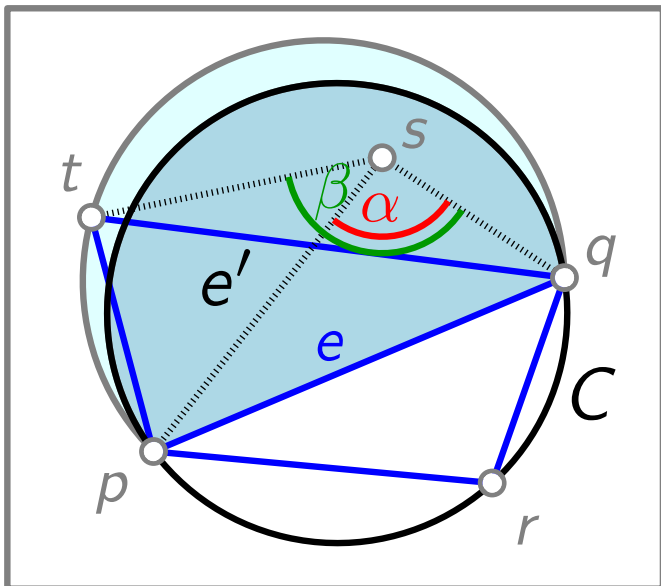
$\Rightarrow s \in C(\Delta pqt)$

Wlog. let $e' = qt$ be the edge of Δpqt that s sees.

$\Rightarrow \beta = \angle tsq > \alpha = \angle psq$



Contradiction to choice of the pair $(\Delta pqr, s)$.



Main Result

Theorem. $P \subset \mathbb{R}^2$ finite, \mathcal{T} triangulation of P .
Then \mathcal{T} legal $\Leftrightarrow \mathcal{T}$ Delaunay.

Observation. Suppose P is in general position... *no 4 pts on an empty circle!*
 \Rightarrow Delaunay triangulation unique [$\mathcal{DG}(P)$!]

\Rightarrow legal triangulation unique

\Downarrow angle-optimal \Rightarrow legal [by def.]

Delaunay triangulation is angle-optimal!

Suppose P is *not* in general position...

\Rightarrow Delaunay graph has convex "holes"
bounded by co-circular pts

\Downarrow Thales++ *homework exercise!*

All Delaunay triang. have same min. angle.

Computation

Fact. A Delaunay triangulation of an arbitrary set of n pts in the plane can be computed in $O(n \log n)$ time. [Compute dual of $\text{Vor}(P)$, fill holes.]

Corollary. An angle-optimal triangulation of a set of n pts in general position can be computed in $O(n \log n)$ time. [DG!]

Given an arbitrary set of n pts, a triangulation maximizing the minimum angle can be computed in $O(n \log n)$ time. [Use fact.]

An angle-optimal triangulation of an arbitrary set of n pts can be computed in $O(n^2)$ time. [How?]