Computational Geometry

Winter semester 2016/17

Height Interpolation

Lecture #8

(Chapter 9 in the textbook)
Height Interpolation
Height Interpolation

\[ p = (x_p, y_p, z_p) \]
Height Interpolation

\[ p = (x_p, y_p, z_p) \]

\[ \pi(p) = (x_p, y_p, 0) \]
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Triangulation of Planar Point Sets

**Definition:** Given \( P \subset \mathbb{R}^2 \), a *triangulation* of \( P \) is a maximal planar subdivision with vtx set \( P \), that is, no edge can be added without crossing.

\[
\begin{array}{c}
\text{\bullet} \\
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\end{array}
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Triangulation of Planar Point Sets

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**Observe:**
Triangulation of Planar Point Sets

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**Observe:** all inner faces are triangles
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Triangulation of Planar Point Sets

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Observe:
- All inner faces are triangles.
- Outer face is complement of a convex polygon.
Triangulation of Planar Point Sets

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**Theorem:** Let $P \subset \mathbb{R}^2$ be a set of $n$ sites, not all collinear, and let $h$ be the number of sites on $\partial \text{CH}(P)$.  

![Diagram of triangulation]
**Triangulation of Planar Point Sets**

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**Observe:**
- all inner faces are triangles
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**Theorem:** Let $P \subset \mathbb{R}^2$ be a set of $n$ sites, not all collinear, and let $h$ be the number of sites on $\partial \text{CH}(P)$. Then *any* triangulation of $P$ has $t(n, h)$ triangles and $e(n, h)$ edges.
Triangulation of Planar Point Sets

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**Theorem:** Let \( P \subset \mathbb{R}^2 \) be a set of \( n \) sites, not all collinear, and let \( h \) be the number of sites on \( \partial \text{CH}(P) \).
  
  Then *any* triangulation of \( P \) has \( t(n, h) \) triangles and \( e(n, h) \) edges.  

**Task:** Compute \( t \) and \( e \)!
Back to Height Interpolation
Back to Height Interpolation
Back to Height Interpolation
Back to Height Interpolation
Back to Height Interpolation

height = 985

height = 23
Back to Height Interpolation

Intuition: Avoid “skinny” triangles!
Back to Height Interpolation

**Intuition:** Avoid “skinny” triangles!

In other words: avoid small angles!
Angle-Optimal Triangulations

**Definition:** Given a set $P \subset \mathbb{R}^2$
Angle-Optimal Triangulations

**Definition:** Given a set $P \subset \mathbb{R}^2$ and a triangulation $\mathcal{T}$ of $P$, 
Angle-Optimal Triangulations

**Definition:** Given a set \( P \subset \mathbb{R}^2 \) and a triangulation \( \mathcal{T} \) of \( P \), let \( m \) be the number of triangles in \( \mathcal{T} \).
Angle-Optimal Triangulations

Definition: Given a set $P \subseteq \mathbb{R}^2$ and a triangulation $\mathcal{T}$ of $P$, let $m$ be the number of triangles in $\mathcal{T}$ and let $A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m})$ be the angle vector of $\mathcal{T}$.
**Angle-Optimal Triangulations**

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![Diagram](attachment://triangle.png)
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We say $A(\mathcal{T}) > A(\mathcal{T}')$
Angle-Optimal Triangulations

**Definition:** Given a set \( P \subset \mathbb{R}^2 \) and a triangulation \( \mathcal{T} \) of \( P \), let \( m \) be the number of triangles in \( \mathcal{T} \) and let \( A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m}) \) be the angle vector of \( \mathcal{T} \), where \( \alpha_1 \leq \cdots \leq \alpha_{3m} \) are the angles in the triangles of \( \mathcal{T} \).

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We say \( A(\mathcal{T}) > A(\mathcal{T}') \)

\[
\begin{align*}
\mathcal{T} & \\
A(\mathcal{T}) &= (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ) \\
\mathcal{T}' & \\
A(\mathcal{T}') &= (30^\circ, 30^\circ, 30^\circ, 30^\circ, 120^\circ, 120^\circ)
\end{align*}
\]
Angle-Optimal Triangulations

**Definition:** Given a set $P \subset \mathbb{R}^2$ and a triangulation $\mathcal{T}$ of $P$, let $m$ be the number of triangles in $\mathcal{T}$ and let $A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m})$ be the angle vector of $\mathcal{T}$, where $\alpha_1 \leq \cdots \leq \alpha_{3m}$ are the angles in the triangles of $\mathcal{T}$.

We say $A(\mathcal{T}) > A(\mathcal{T}')$ if $\exists i \in \{1, \ldots, 3m\}$:

$$A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ)$$

$$A(\mathcal{T}') = (30^\circ, 30^\circ, 30^\circ, 30^\circ, 120^\circ, 120^\circ)$$
Definition: Given a set $P \subset \mathbb{R}^2$ and a triangulation $\mathcal{T}$ of $P$, let $m$ be the number of triangles in $\mathcal{T}$ and let $A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m})$ be the angle vector of $\mathcal{T}$, where $\alpha_1 \leq \cdots \leq \alpha_{3m}$ are the angles in the triangles of $\mathcal{T}$.

We say $A(\mathcal{T}) > A(\mathcal{T}')$ if $\exists i \in \{1, \ldots, 3m\}: \alpha_i > \alpha'_i$.

\[
\mathcal{T} \quad A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ) \\
\mathcal{T}' \quad A(\mathcal{T}') = (30^\circ, 30^\circ, 30^\circ, 30^\circ, 120^\circ, 120^\circ)
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Angle-Optimal Triangulations

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$\mathcal{T}$

$A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ)$

$\mathcal{T}'$

$A(\mathcal{T}') = (30^\circ, 30^\circ, 30^\circ, 30^\circ, 120^\circ, 120^\circ)$
Angle-Optimal Triangulations

Definition: Given a set $P \subset \mathbb{R}^2$ and a triangulation $T$ of $P$, let $m$ be the number of triangles in $T$ and let $A(T) = (\alpha_1, \ldots, \alpha_{3m})$ be the angle vector of $T$, where $\alpha_1 \leq \cdots \leq \alpha_{3m}$ are the angles in the triangles of $T$.

We say $A(T) > A(T')$ if $\exists i \in \{1, \ldots, 3m\} : \alpha_i > \alpha'_i$ and $\forall j < i : \alpha_j = \alpha'_j$.

$T$

$A(T) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ)$

$T'$

$A(T') = (30^\circ, 30^\circ, 30^\circ, 30^\circ, 120^\circ, 120^\circ)$
Angle-Optimal Triangulations

**Definition:** Given a set \( P \subset \mathbb{R}^2 \) and a triangulation \( \mathcal{T} \) of \( P \), let \( m \) be the number of triangles in \( \mathcal{T} \) and let \( A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m}) \) be the *angle vector* of \( \mathcal{T} \), where \( \alpha_1 \leq \cdots \leq \alpha_{3m} \) are the angles in the triangles of \( \mathcal{T} \).

We say \( A(\mathcal{T}) > A(\mathcal{T}') \) if \( \exists i \in \{1, \ldots, 3m\} : \alpha_i > \alpha_i' \) and \( \forall j < i : \alpha_j = \alpha_j' \).

\( \mathcal{T} \) is *angle-optimal* if

\[
A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ)
\]

\[
A(\mathcal{T}') = (30^\circ, 30^\circ, 30^\circ, 30^\circ, 120^\circ, 120^\circ)
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Angle-Optimal Triangulations

**Definition:** Given a set $P \subset \mathbb{R}^2$ and a triangulation $\mathcal{T}$ of $P$, let $m$ be the number of triangles in $\mathcal{T}$ and let $A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m})$ be the angle vector of $\mathcal{T}$, where $\alpha_1 \leq \cdots \leq \alpha_{3m}$ are the angles in the triangles of $\mathcal{T}$.

We say $A(\mathcal{T}) > A(\mathcal{T}')$ if $\exists i \in \{1, \ldots, 3m\}: \alpha_i > \alpha_i'$ and $\forall j < i: \alpha_j = \alpha_j'$.

$\mathcal{T}$ is angle-optimal if $A(\mathcal{T}) \geq A(\mathcal{T}')$ for all triangulations $\mathcal{T}'$ of $P$.

$A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ)$

$A(\mathcal{T}') = (30^\circ, 30^\circ, 30^\circ, 30^\circ, 120^\circ, 120^\circ)$
Edge Flips

Definition: $\mathcal{T}$ a triangulation. An edge $e$ of $\mathcal{T}$ is \textit{illegal} if the minimum angle in the two triangles adjacent to $e$ increases when flipping.
Edge Flips

**Definition:** \( \mathcal{T} \) a triangulation. An edge \( e \) of \( \mathcal{T} \) is *illegal* if the minimum angle in the two triangles adjacent to \( e \) increases when flipping.

\[ \min_i \alpha_i = 30^\circ \]
**Definition:**  \( \mathcal{T} \) a triangulation. An edge \( e \) of \( \mathcal{T} \) is **illegal** if the minimum angle in the two triangles adjacent to \( e \) increases when flipping.

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\[ \min_i \alpha_i = 30^\circ \]
Edge Flips

**Definition:** $T$ a triangulation. An edge $e$ of $T$ is **illegal** if the minimum angle in the two triangles adjacent to $e$ increases when flipping.

$\min_i \alpha_i = 60^\circ$  

$\min_i \alpha_i = 30^\circ$
**Edge Flips**

**Definition:** \( \mathcal{T} \) a triangulation. An edge \( e \) of \( \mathcal{T} \) is *illegal* if the minimum angle in the two triangles adjacent to \( e \) increases when flipping.

**Observe:** Let \( e \) be an illegal edge of \( \mathcal{T} \), and \( \mathcal{T}' = \text{flip}(\mathcal{T}, e) \).

\[
\min_i \alpha_i = 60^\circ \quad \text{min}_i \alpha_i = 30^\circ
\]
Edge Flips

Definition: \( T \) a triangulation. An edge \( e \) of \( T \) is **illegal** if the minimum angle in the two triangles adjacent to \( e \) increases when flipping.

Observe: Let \( e \) be an illegal edge of \( T \), and \( T' = \text{flip}(T, e) \).

\[
\min_i \alpha_i = 60^\circ \quad \text{and} \quad \min_i \alpha_i = 30^\circ
\]
**Edge Flips**

**Definition:** \( \mathcal{T} \) a triangulation. An edge \( e \) of \( \mathcal{T} \) is **illegal** if the minimum angle in the two triangles adjacent to \( e \) increases when flipping.

**Observe:** Let \( e \) be an illegal edge of \( \mathcal{T} \), and \( \mathcal{T}' = \text{flip}(\mathcal{T}, e) \). Then \( A(\mathcal{T}') > A(\mathcal{T}) \).

\[
\min_i \alpha_i = 60^\circ
\]

\[
\min_i \alpha_i = 30^\circ
\]
This is all Greek to me...

Theorem:
This is all Greek to me...

**Theorem:** (Thales)

The diameter of a circle always subtends a right angle to any point on the circle.
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The diameter of a circle always subtends a right angle to any point on the circle.

**Theorem: (Thales++)**

\[
\{a, b\} := \ell \cap \partial \mathbb{D} \quad (a \neq b)
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p, q \in \partial D
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\[ \angle apb = \angle aqb \]
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\[
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\]

\[
r \in \text{int}(D)
\]

\[\angle apb = \angle aqb\]
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\[p, q \in \partial D\]
\[r \in \text{int}(D)\]

\[\angle apb = \angle aqb < \angle arb\]
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**Theorem:** (Thales)
The diameter of a circle always subtends a right angle to any point on the circle.

**Theorem:** (Thales++)

\[ \{a, b\} := \ell \cap \partial D \quad (a \neq b) \]

\[ p, q \in \partial D \]
\[ r \in \text{int}(D) \]
\[ s \notin D \]

\[ \angle apb = \angle aqb < \angle arb \]
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r \in \text{int}(D)
\]

\[
s \notin D
\]

\[\angle asb < \angle apb = \angle aqb < \angle arb\]
Legal Triangulations

**Lemma:** Let $\Delta prq, \Delta pqs \in \mathcal{T}$ and $p, q, r \in \partial D$. Then edge $pq$ is illegal iff $s \in \text{int}(D)$. 

\[
\begin{align*}
\partial D \\
p & \quad r \\
\quad s \\
q
\end{align*}
\]
Legal Triangulations

Lemma: Let $\Delta prq, \Delta pqs \in \mathcal{T}$ and $p, q, r \in \partial D$. Then edge $pq$ is illegal iff $s \in \text{int}(D)$.

If $p, q, r, s$ in convex position and $s \not\in \partial D$, then either $pq$ or $rs$ is illegal.
Lemma: Let $\Delta prq, \Delta pqs \in \mathcal{T}$ and $p, q, r \in \partial D$. Then edge $pq$ is illegal iff $s \in \text{int}(D)$.

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Proof:
Lemma: Let $\Delta prq, \Delta pqs \in \mathcal{T}$ and $p, q, r \in \partial D$. Then edge $pq$ is illegal iff $s \in \text{int}(D)$.

If $p, q, r, s$ in convex position and $s \notin \partial D$, then either $pq$ or $rs$ is illegal.

Proof: Show: $\forall \alpha'$ in $\mathcal{T}' \exists \alpha$ in $\mathcal{T}$ s.t. $\alpha < \alpha'$.
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\[ p \quad r \quad q \quad s \]
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If $p, q, r, s$ in convex position and $s \not\in \partial D$, then either $pq$ or $rs$ is illegal.

**Proof:** Show: $\forall \alpha' \text{ in } \mathcal{T}' \exists \alpha \text{ in } \mathcal{T} \text{ s.t. } \alpha < \alpha'$.\hfill ("$\Rightarrow$")
Legal Triangulations

**Lemma:** Let \( \Delta prq, \Delta pqs \in \mathcal{T} \) and \( p, q, r \in \partial D \). Then edge \( pq \) is illegal iff \( s \in \text{int}(D) \).

If \( p, q, r, s \) in convex position and \( s \notin \partial D \), then either \( pq \) or \( rs \) is illegal.

**Proof:**

Show: \( \forall \alpha' \in \mathcal{T}' \exists \alpha \in \mathcal{T} \text{ s.t. } \alpha < \alpha' \).

\( \Rightarrow \)
Legal Triangulations

Lemma: Let $\Delta prq, \Delta pqs \in \mathcal{T}$ and $p, q, r \in \partial D$. Then edge $pq$ is illegal iff $s \in \text{int}(D)$.

If $p, q, r, s$ in convex position and $s \notin \partial D$, then either $pq$ or $rs$ is illegal.

Proof: Show: $\forall \alpha' \in T' \exists \alpha \in T$ s.t. $\alpha < \alpha'$.
Legal Triangulations

**Lemma:** Let $\Delta prq, \Delta pqs \in \mathcal{T}$ and $p, q, r \in \partial D$. Then edge $pq$ is illegal iff $s \in \text{int}(D)$.

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**Proof:** Show: $\forall \alpha' \in \mathcal{T}' \exists \alpha \in \mathcal{T}$ s.t. $\alpha < \alpha'$.
**Legal Triangulations**

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If \( p, q, r, s \) in convex position and \( s \notin \partial D \), then either \( pq \) or \( rs \) is illegal.

**Proof:**

Show: \( \forall \alpha' \in \mathcal{T}' \exists \alpha \in \mathcal{T} \text{ s.t. } \alpha < \alpha' \). ("\( \Rightarrow \)"")

Use Thales++ w.r.t. \( qs' \).
Legal Triangulations

**Lemma:** Let $\triangle prq, \triangle pqs \in \mathcal{T}$ and $p, q, r \in \partial D$. Then edge $pq$ is illegal iff $s \in \text{int}(D)$.

If $p, q, r, s$ in convex position and $s \notin \partial D$, then either $pq$ or $rs$ is illegal.

**Proof:** Show: $\forall \alpha' \in \mathcal{T}' \exists \alpha \in \mathcal{T}$ s.t. $\alpha < \alpha'$.

"$\Rightarrow$"

Use Thales++ w.r.t. $qs'$. 

\[\partial D \quad \bullet \quad s' \quad \bullet \quad \alpha' \quad \bullet \quad s \quad \alpha' \quad \bullet \quad p \quad q \quad r\]
Legal Triangulations

Lemma: Let $\triangle prq, \triangle pqs \in \mathcal{T}$ and $p, q, r \in \partial D$. Then edge $pq$ is illegal iff $s \in \text{int}(D)$.

If $p, q, r, s$ in convex position and $s \notin \partial D$, then either $pq$ or $rs$ is illegal.

Proof: Show: $\forall \alpha' \in \mathcal{T}' \exists \alpha \in \mathcal{T}$ s.t. $\alpha < \alpha'$.

("$\Rightarrow$") Use Thales++ w.r.t. $qs'$.
Legal Triangulations

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If $p, q, r, s$ in convex position and $s \notin \partial D$, then either $pq$ or $rs$ is illegal.

**Proof:**

Show: $\forall \alpha' \text{ in } \mathcal{T}' \exists \alpha \text{ in } \mathcal{T}$ s.t. $\alpha < \alpha'$.

("$\Rightarrow$"") Use Thales++ w.r.t. $qs'$.

**Note:** Criterion symmetric in $r$ and $s$
Legal Triangulations

**Lemma:** Let Δprq, Δpqrs ∈ ℳ and p, q, r ∈ ∂D. Then edge pq is illegal iff s ∈ int(D).

If p, q, r, s in convex position and s ∉ ∂D, then either pq or rs is illegal.

**Proof:**

Show: ∀α' in ℳ' ∃α in ℳ s.t. α < α'.

("⇒") Use Thales++ w.r.t. qs'.

**Note:** Criterion symmetric in r and s
⇒ if s ∈ ∂D, both pq and rs legal.
Legal Triangulations

Lemma: Let $\Delta prq, \Delta pqs \in \mathcal{T}$ and $p, q, r \in \partial D$. Then edge $pq$ is illegal iff $s \in \text{int}(D)$.

If $p, q, r, s$ in convex position and $s \not\in \partial D$, then either $pq$ or $rs$ is illegal.

Proof: Show: $\forall \alpha' \in \mathcal{T}' \exists \alpha \in \mathcal{T}$ s.t. $\alpha < \alpha'$.

(“$\Rightarrow$”) Use Thales++ w.r.t. $qs'$.

Note: Criterion symmetric in $r$ and $s$.

$\Rightarrow$ if $s \in \partial D$, both $pq$ and $rs$ legal.

Definition: A triangulation is legal if it has no illegal edge.
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**Existence?**
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algorithm terminates

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Legal vs. Angle-Optimal

Clearly... Every angle-optimal triangulation is legal.
Legal vs. Angle-Optimal

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Let’s see.

To clarify things, we’ll introduce yet another type of triangulation...
**Voronoi & Delaunay**

**Remember:** Given a set $P$ of $n$ points in the plane...

Vor($P$) = subdivision of the plane into

Voronoi cells, edges, and vertices

$\mathcal{V}(p) = \{x \in \mathbb{R}^2 : |xp| < |xq| \text{ for all } q \in P \setminus \{p\}\}$

Voronoi cell of $p \in P$
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**Definition:** The graph $\mathcal{G} = (P, E)$ with

$\{p, q\} \in E \Leftrightarrow \mathcal{V}(p) \text{ and } \mathcal{V}(q) \text{ share an edge}$

is the *dual graph* of $\text{Vor}(P)$
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**Remember:** Given a set $P$ of $n$ points in the plane . . .

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**Definition:** The Delaunay graph $\mathcal{DG}(P)$ is the straight-line drawing of $\mathcal{G}$. 
From Voronoi to Delaunay

\[ P \subset \mathbb{R}^2 \]
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Georgy Feodosevich Voronoy (1868–1908 Zhuravki, now Ukraine)
From Voronoi to Delaunay

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Boris Nikolaevich Delone (St. Petersburg 1890–1980 Moscow)
Planarity

**Theorem.** \( P \subset \mathbb{R}^2 \) finite \( \Rightarrow \mathcal{DG}(P) \) plane.
Planarity

**Theorem.** \( P \subset \mathbb{R}^2 \) finite \( \Rightarrow \mathcal{DG}(P) \) plane.

**Proof.** Recall property of Voronoi edges:
Planarity

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Edge \( pq \) is in \( \mathcal{DG}(P) \) \( \iff \)
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**Theorem.** $P \subset \mathbb{R}^2$ finite $\Rightarrow \mathcal{DG}(P)$ plane.

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Edge $pq$ is in $\mathcal{DG}(P) \iff \exists D_{pq}$ closed disk s.t.
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Suppose \( \exists \) edge \( uv \neq pq \) in \( DG(P) \) that crosses \( pq. \)
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Suppose $\exists$ edge $uv \neq pq$ in $\mathcal{DG}(P)$ that crosses $pq$.

$u, v \not\in D_{pq}$ $\Rightarrow$
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**Proof.**

Recall property of Voronoi edges:

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- \( c = \text{center}(D_{pq}) \) lies on edge betw. \( \mathcal{V}(p) \) & \( \mathcal{V}(q) \).

Suppose \( \exists \) edge \( uv \neq pq \) in \( \mathcal{DG}(P) \) that crosses \( pq \).

\[ u, v \notin D_{pq} \Rightarrow u, v \notin t_{pq} \Rightarrow \]
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\[
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& p, q \in \partial D_{pq} \text{ and } \\
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\end{align*}
\]
c = center(\( D_{pq} \)) lies on edge betw. \( \mathcal{V}(p) \) & \( \mathcal{V}(q) \).
Suppose \( \exists \) edge \( uv \neq pq \) in \( \mathcal{DG}(P) \) that crosses \( pq \).
\[
\begin{align*}
u, v \notin D_{pq} \Rightarrow u, v \notin t_{pq} \Rightarrow uv \text{ crosses another edge of } t_{pq}
\end{align*}
\]
Planarity

Theorem. \( P \subset \mathbb{R}^2 \) finite \( \Rightarrow \mathcal{D}\mathcal{G}(P) \) plane.

Proof. Recall property of Voronoi edges:

Edge \( pq \) is in \( \mathcal{D}\mathcal{G}(P) \) \( \iff \) \( \exists D_{pq} \) closed disk s.t.

\[ p, q \in \partial D_{pq} \text{ and } \{p, q\} = D_{pq} \cap P. \]

\( c = \text{center}(D_{pq}) \) lies on edge betw. \( \mathcal{V}(p) \) & \( \mathcal{V}(q) \).

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\( u, v \notin D_{pq} \Rightarrow u, v \notin t_{pq} \Rightarrow \)

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\( p, q \notin D_{uv} \Rightarrow \)
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**Theorem.** \( P \subset \mathbb{R}^2 \) finite \( \Rightarrow \mathcal{DG}(P) \) plane.

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\( p, q \notin D_{uv} \Rightarrow p, q \notin t_{uv} \Rightarrow \)
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**Theorem.** \[ P \subset \mathbb{R}^2 \text{ finite } \Rightarrow DG(P) \text{ plane.} \]

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Planarity

Theorem. $P \subset \mathbb{R}^2$ finite $\Rightarrow \mathcal{DG}(P)$ plane.

Proof. Recall property of Voronoi edges:

Edge $pq$ is in $\mathcal{DG}(P)$ $\iff$ $\exists D_{pq}$ closed disk s.t.

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$u, v \not\in D_{pq} \Rightarrow u, v \not\in t_{pq} \Rightarrow$

$uv$ crosses another edge of $t_{pq}$

$p, q \not\in D_{uv} \Rightarrow p, q \not\in t_{uv} \Rightarrow$

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\( p, q \notin D_{uv} \Rightarrow p, q \notin t_{uv} \Rightarrow \)

- \( pq \) crosses another edge of \( t_{uv} \)

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Planarity

**Theorem.** $P \subset \mathbb{R}^2$ finite $\Rightarrow$ $\mathcal{DG}(P)$ plane.

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Edge $pq$ is in $\mathcal{DG}(P) \iff \exists D_{pq}$ closed disk s.t.

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\[
egin{align*}
    u, v \notin D_{pq} & \Rightarrow u, v \notin t_{pq} \Rightarrow \\
    & uv \text{ crosses another edge of } t_{pq} \\
    p, q \notin D_{uv} & \Rightarrow p, q \notin t_{uv} \Rightarrow \\
    & pq \text{ crosses another edge of } t_{uv} \\
    & \Rightarrow \text{one of } s_{pq} \text{ or } s_{qp} \text{ crosses one of } s_{uv} \text{ or } s_{vu} \\
    & s_{pq} \subset \mathcal{V}(p), \ s_{qp} \subset \mathcal{V}(q), \ s_{uv} \subset \mathcal{V}(u), \ s_{vu} \subset \mathcal{V}(v).
\end{align*}
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Planarity

**Theorem.** \( P \subset \mathbb{R}^2 \) finite \( \Rightarrow \mathcal{DG}(P) \) plane.

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Suppose \( \exists \) edge \( uv \neq pq \) in \( \mathcal{DG}(P) \) that crosses \( pq \).

\[
\begin{align*}
& u, v \notin D_{pq} \Rightarrow u, v \notin t_{pq} \Rightarrow \\
& \quad uv \text{ crosses another edge of } t_{pq} \\
& p, q \notin D_{uv} \Rightarrow p, q \notin t_{uv} \Rightarrow \\
& \quad pq \text{ crosses another edge of } t_{uv} \\
& \Rightarrow \text{ one of } s_{pq} \text{ or } s_{qp} \text{ crosses one of } s_{uv} \text{ or } s_{vu} \\
& \quad \triangleleft s_{pq} \subset \mathcal{V}(p), s_{qp} \subset \mathcal{V}(q), s_{uv} \subset \mathcal{V}(u), s_{vu} \subset \mathcal{V}(v).
\end{align*}
\]
Characterization

Characterization of Voronoi vertices and Voronoi edges ⇒

**Theorem.** \( P \subset \mathbb{R}^2 \) finite. Then

(i) Three pts \( p, q, r \in P \) are vertices of the same face of \( \mathcal{DG}(P) \) \( \iff \) \( \text{int}(C(p, q, r)) \cap P = \emptyset \)
Characterization

Characterization of Voronoi vertices and Voronoi edges

**Theorem.** $P \subset \mathbb{R}^2$ finite. Then

(i) Three pts $p, q, r \in P$ are vertices of the same face of $DG(P) \iff \text{int}(C(p, q, r)) \cap P = \emptyset$

(ii) Two pts $p, q \in P$ form an edge of $DG(P) \iff$

there is a disk $D$ with

- $\partial D \cap P = \{p, q\}$ and
- $\text{int}(D) \cap P = \emptyset$. 
Characterization

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- \( \partial D \cap P = \{p, q\} \)
- \( \text{int}(D) \cap P = \emptyset \).

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) Delaunay \( \iff \) for each triangle \( \Delta \) of \( \mathcal{T} \):
\[
\text{int}(C(\Delta)) \cap P = \emptyset.
\]
Characterization

Characterization of Voronoi vertices and Voronoi edges ⇒

**Theorem.** $P \subset \mathbb{R}^2$ finite. Then

(i) Three pts $p, q, r \in P$ are vertices of the same face of $DG(P)$ ⇔ $\text{int}(C(p, q, r)) \cap P = \emptyset$

(ii) Two pts $p, q \in P$ form an edge of $DG(P)$ ⇔ there is a disk $D$ with

- $\partial D \cap P = \{p, q\}$ and
- $\text{int}(D) \cap P = \emptyset$.

**Theorem.** $P \subset \mathbb{R}^2$ finite, $\mathcal{T}$ triangulation of $P$. Then $\mathcal{T}$ Delaunay ⇔ for each triangle $\Delta$ of $\mathcal{T}$:

$\text{int}(C(\Delta)) \cap P = \emptyset$.

(“empty-circumcircle property”)
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \).
Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \)
Main Result

Theorem. \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \) \( \mathcal{T} \) Delaunay.
Main Result

**Theorem.** \( P \subseteq \mathbb{R}^2 \) finite, \( T \) triangulation of \( P \).

Then \( T \) legal \( \iff \) \( T \) Delaunay.

**Proof.** “\( \Leftarrow \)”
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \).

Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Proof.** “\( \leftarrow \)” implied by empty-circumcircle property & Thales+++.
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \).

Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Proof.** “\( \Leftarrow \)” implied by empty-circumcircle property & Thales++

“\( \Rightarrow \)”
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \).
Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Proof.** “\( \Leftarrow \)” implied by empty-circumcircle property & Thales++
“\( \Rightarrow \)” by contradiction:
Main Result

**Theorem.** $P \subset \mathbb{R}^2$ finite, $\mathcal{T}$ triangulation of $P$. Then $\mathcal{T}$ legal $\iff \mathcal{T}$ Delaunay.

**Proof.** “$\Leftarrow$” implied by empty-circumcircle property & Thales++

“$\Rightarrow$” by contradiction:

Assume $\mathcal{T}$ is legal triang. of $P$, but *not* Delaunay.
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Proof.** “\( \Leftarrow \)” implied by empty-circumcircle property & Thales\( ++ \)

“\( \Rightarrow \)” by contradiction:

Assume \( \mathcal{T} \) is legal triang. of \( P \), but *not* Delaunay.

\[ \Rightarrow \exists \Delta pqr \text{ such that } \text{int}(C(\Delta pqr)) \text{ contains } s \in P. \]
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \).
Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Proof.** “\( \Leftarrow \)” implied by empty-circumcircle property & Thales++
“\( \Rightarrow \)” by contradiction:
Assume \( \mathcal{T} \) is legal triang. of \( P \), but *not* Delaunay.
\( \Rightarrow \exists \Delta pqr \) such that \( \text{int}(C(\Delta pqr)) \) contains \( s \in P \).
Wlog. let \( e = pq \) be the edge of \( \Delta pqr \) such that \( s \) “sees” \( pq \) before the other edges of \( \Delta pqr \).
Main Result

**Theorem.** $P \subset \mathbb{R}^2$ finite, $\mathcal{T}$ triangulation of $P$. Then $\mathcal{T}$ legal $\iff$ $\mathcal{T}$ Delaunay.

**Proof.** "$\Leftarrow$" implied by empty-circumcircle property & Thales+++ 

"$\Rightarrow$" by contradiction: 
Assume $\mathcal{T}$ is legal triang. of $P$, but *not* Delaunay. 
$\Rightarrow \exists \Delta pqr$ such that int($C(\Delta pqr)$) contains $s \in P$. 
Wlog. let $e = pq$ be the edge of $\Delta pqr$ such that $s$ "sees" $pq$ before the other edges of $\Delta pqr$.

Among all such pairs $(\Delta pqr, s)$ in $\mathcal{T}$ choose one that maximizes $\alpha = \angle psq$. 
Proof of Main Result (cont’d)

Consider the triangle $\Delta pqt$ adjacent to $e$ in $\mathcal{T}$. 
Proof of Main Result (cont’d)

Consider the triangle $\Delta pqt$ adjacent to $e$ in $\mathcal{T}$.

$\mathcal{T}$ legal $\Rightarrow$
Proof of Main Result (cont’d)

Consider the triangle $\Delta pqt$ adjacent to $e$ in $\mathcal{T}$.

$\mathcal{T}$ legal $\Rightarrow$ $e$ legal $\Rightarrow$
Proof of Main Result (cont’d)

Consider the triangle $\Delta pqt$ adjacent to $e$ in $\mathcal{T}$. $\mathcal{T}$ legal $\Rightarrow$ $e$ legal $\Rightarrow$ $t \notin \text{int}(C(\Delta pqr))$
Proof of Main Result (cont’d)

Consider the triangle $\Delta pqt$ adjacent to $e$ in $\mathcal{T}$. 
$\mathcal{T}$ legal $\Rightarrow e$ legal $\Rightarrow t \notin \text{int}(C(\Delta pqr))$
$\Rightarrow C(\Delta pqt)$ contains $C(\Delta pqr) \cap e^+$. 
Proof of Main Result (cont’d)

Consider the triangle $\Delta pqt$ adjacent to $e$ in $\mathcal{T}$.

$\mathcal{T}$ legal $\Rightarrow$ $e$ legal $\Rightarrow$ $t \notin \text{int}(C(\Delta pqr))$

$\Rightarrow$ $C(\Delta pqt)$ contains $C(\Delta pqr) \cap e^+$. 

halfplane supported by $e$ that contains $s$
Consider the triangle $\Delta pqt$ adjacent to $e$ in $T$.

$T$ legal $\Rightarrow$ $e$ legal $\Rightarrow$ $t \notin \text{int}(C(\Delta pqr))$

$\Rightarrow$ $C(\Delta pqt)$ contains $C(\Delta pqr) \cap e^+$. 

$\Rightarrow$ $s \in C(\Delta pqt)$
Proof of Main Result (cont’d)

Consider the triangle $\Delta pqt$ adjacent to $e$ in $\mathcal{T}$.  

$\mathcal{T}$ legal $\Rightarrow$ $e$ legal $\Rightarrow$ $t \notin \text{int}(C(\Delta pqr))$

$\Rightarrow$ $C(\Delta pqt)$ contains $C(\Delta pqr) \cap e^+$. 

$\Rightarrow$ $s \in C(\Delta pqt)$

Wlog. let $e' = qt$ be the edge of $\Delta pqt$ that $s$ sees.
Proof of Main Result (cont’d)

Consider the triangle $\Delta pqt$ adjacent to $e$ in $\mathcal{T}$.

$\mathcal{T}$ legal $\Rightarrow$ $e$ legal $\Rightarrow$ $t \notin \text{int}(C(\Delta pqr))$

$\Rightarrow$ $C(\Delta pqt)$ contains $C(\Delta pqr) \cap e^+$.

$\Rightarrow$ $s \in C(\Delta pqt)$

Wlog. let $e' = qt$ be the edge of $\Delta pqt$ that $s$ sees.

$\Rightarrow$ $\beta = \angle tsq > \alpha = \angle psq$
Proof of Main Result (cont’d)

Consider the triangle $\Delta pqt$ adjacent to $e$ in $\mathcal{T}$. $\mathcal{T}$ legal $\Rightarrow$ $e$ legal $\Rightarrow$ $t \notin \text{int}(C(\Delta pqr))$ $\Rightarrow$ $C(\Delta pqt)$ contains $C(\Delta pqr) \cap e^+$. $\Rightarrow$ $s \in C(\Delta pqt)$.

Wlog. let $e' = qt$ be the edge of $\Delta pqt$ that $s$ sees. $\Rightarrow \beta = \angle tsq \ > \ \alpha = \angle psq$.
Consider the triangle $\Delta pqt$ adjacent to $e$ in $\mathcal{T}$.

$\mathcal{T}$ legal $\Rightarrow$ $e$ legal $\Rightarrow$ $t \not\in \text{int}(C(\Delta pqr))$

$\Rightarrow C(\Delta pqt)$ contains $C(\Delta pqr) \cap e^+$.

$\Rightarrow s \in C(\Delta pqt)$

Wlog. let $e' = qt$ be the edge of $\Delta pqt$ that $s$ sees.

$\Rightarrow \beta = \angle tsq > \alpha = \angle psq$

$\Rightarrow$ Contradiction to choice of the pair $(\Delta pqr, s)$.
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \) \( \mathcal{T} \) Delaunay.
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Observation.** Suppose \( P \) is in general position. . .
Main Result

Theorem. $P \subset \mathbb{R}^2$ finite, $\mathcal{T}$ triangulation of $P$. Then $\mathcal{T}$ legal $\iff$ $\mathcal{T}$ Delaunay.

Observation. Suppose $P$ is in general position... no 4 pts on an empty circle!
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Observation.** Suppose \( P \) is in general position. …
\[ \implies \] Delaunay triangulation unique

no 4 pts on an empty circle!
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \) \( \mathcal{T} \) Delaunay.

**Observation.** Suppose \( P \) is in general position. . .
\[ \Rightarrow \text{Delaunay triangulation unique} \quad [\mathcal{DG}(P)!] \]
Main Result

**Theorem.** $P \subset \mathbb{R}^2$ finite, $\mathcal{T}$ triangulation of $P$.
Then $\mathcal{T}$ legal $\iff$ $\mathcal{T}$ Delaunay.

**Observation.** Suppose $P$ is in general position. . .
$\Rightarrow$ Delaunay triangulation unique $[\mathcal{DG}(P)!]$
$\Rightarrow$ legal triangulation unique

*no 4 pts on an empty circle!*
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Observation.** Suppose \( P \) is in general position. . .
\[\Rightarrow \text{Delaunay triangulation unique} \]
\[\Rightarrow \text{legal triangulation unique} \]
\[\Downarrow \]
Main Result

Theorem. \( P \subseteq \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

Observation. Suppose \( P \) is in general position. . .
\[ \Rightarrow \text{Delaunay triangulation unique} \]
\[ \Rightarrow \text{legal triangulation unique} \]
\[ \Downarrow \text{angle-optimal} \Rightarrow \text{legal} \]
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Observation.** Suppose \( P \) is in general position. . .
\[ \Rightarrow \text{Delaunay triangulation unique} \quad [\mathcal{DG}(P)!] \]
\[ \Rightarrow \text{legal triangulation unique} \]
\[ \Downarrow \quad \text{angle-optimal} \Rightarrow \text{legal} \quad [\text{by def.}] \]
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Observation.** Suppose \( P \) is in general position.

\[ \Rightarrow \text{Delaunay triangulation unique} \quad [\mathcal{D}(P)!] \]

\[ \Rightarrow \text{legal triangulation unique} \]

\[ \Downarrow \text{angle-optimal} \Rightarrow \text{legal} \quad [\text{by def.}] \]

Delaunay triangulation is angle-optimal!
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( T \) triangulation of \( P \). Then \( T \) legal \( \iff T \) Delaunay.

**Observation.** Suppose \( P \) is in general position.

\[ DG(P)! \]

\( \Rightarrow \) Delaunay triangulation unique

\( \Rightarrow \) legal triangulation unique

\( \Downarrow \) angle-optimal \( \Rightarrow \) legal [by def.]

Delaunay triangulation is angle-optimal!

Suppose \( P \) is *not* in general position...
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Observation.** Suppose \( P \) is in general position. . .

\[ \Rightarrow \text{Delaunay triangulation unique \[ DG(P) \!] } \]

\[ \Rightarrow \text{legal triangulation unique} \]

\[ \Downarrow \text{angle-optimal} \Rightarrow \text{legal [by def.]} \]

Delaunay triangulation is angle-optimal!

Suppose \( P \) is *not* in general position. . .

\[ \Rightarrow \text{Delaunay graph has convex “holes” bounded by co-circular pts} \]

no 4 pts on an empty circle!
Main Result

Theorem. \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

Observation. Suppose \( P \) is in general position. . .
\( \Rightarrow \) Delaunay triangulation unique \( [\mathcal{DG}(P)!] \)
\( \Rightarrow \) legal triangulation unique
\( \Downarrow \) angle-optimal \( \Rightarrow \) legal \( [\text{by def.}] \)
Delaunay triangulation is angle-optimal!

Suppose \( P \) is not in general position. . .
\( \Rightarrow \) Delaunay graph has convex “holes” bounded by co-circular pts
\( \Downarrow \) Thales++ \( \text{homework exercise!} \)
Main Result

**Theorem.** \( P \subseteq \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \).
Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Observation.** Suppose \( P \) is in general position.

\[ \Rightarrow \text{Delaunay triangulation unique} \quad [\mathcal{DG}(P)!] \]
\[ \Rightarrow \text{legal triangulation unique} \]
\[ \Downarrow \text{angle-optimal} \Rightarrow \text{legal} \quad [\text{by def.}] \]
Delaunay triangulation is angle-optimal!

Suppose \( P \) is *not* in general position.

\[ \Rightarrow \text{Delaunay graph has convex “holes” bounded by co-circular pts} \]
\[ \Downarrow \text{Thales++} \quad \text{homework exercise!} \]
All Delaunay triang. have same min. angle.
Computation

Fact. A Delaunay triangulation of an arbitrary set of $n$ pts in the plane can be computed in $O(n \log n)$ time.
Computation

**Fact.** A Delaunay triangulation of an arbitrary set of $n$ pts in the plane can be computed in $O(n \log n)$ time.  

[Compute dual of Vor($P$), fill holes.]
Computation

**Fact.** A Delaunay triangulation of an arbitrary set of $n$ pts in the plane can be computed in $O(n \log n)$ time. [Compute dual of Vor($P$), fill holes.]

**Corollary.** An angle-optimal triangulation of a set of $n$ pts in general position can be computed in $O(n \log n)$ time.
Computation

**Fact.** A Delaunay triangulation of an arbitrary set of $n$ pts in the plane can be computed in $O(n \log n)$ time. [Compute dual of Vor($P$), fill holes.]

**Corollary.** An angle-optimal triangulation of a set of $n$ pts in general position can be computed in $O(n \log n)$ time. [$DG!$]
Computation

**Fact.** A Delaunay triangulation of an arbitrary set of $n$ pts in the plane can be computed in $O(n \log n)$ time. [Compute dual of Vor($P$), fill holes.]

**Corollary.** An angle-optimal triangulation of a set of $n$ pts in general position can be computed in $O(n \log n)$ time.

Given an arbitrary set of $n$ pts, a triangulation maximizing the minimum angle can be computed in $O(n \log n)$ time.
Computation

**Fact.** A Delaunay triangulation of an arbitrary set of $n$ pts in the plane can be computed in $O(n \log n)$ time. [Compute dual of Vor($P$), fill holes.]

**Corollary.** An angle-optimal triangulation of a set of $n$ pts in general position can be computed in $O(n \log n)$ time.

Given an arbitrary set of $n$ pts, a triangulation maximizing the minimum angle can be computed in $O(n \log n)$ time. [Use fact.]
Computation

**Fact.** A Delaunay triangulation of an arbitrary set of $n$ pts in the plane can be computed in $O(n \log n)$ time.\[Compute \ dual \ of \ Vor(\mathcal{P}), \ fill \ holes.\]

**Corollary.** An angle-optimal triangulation of a set of $n$ pts in general position can be computed in $O(n \log n)$ time.

Given an arbitrary set of $n$ pts, a triangulation maximizing the minimum angle can be computed in $O(n \log n)$ time.\[\text{DG!}\]

An angle-optimal triangulation of an arbitrary set of $n$ pts can be computed in $O(n^2)$ time.
Computation

Fact. A Delaunay triangulation of an arbitrary set of $n$ pts in the plane can be computed in $O(n \log n)$ time. [Compute dual of Vor$(P)$, fill holes.]

Corollary. An angle-optimal triangulation of a set of $n$ pts in general position can be computed in $O(n \log n)$ time.

Given an arbitrary set of $n$ pts, a triangulation maximizing the minimum angle can be computed in $O(n \log n)$ time. [Use fact.]

An angle-optimal triangulation of an arbitrary set of $n$ pts can be computed in $O(n^2)$ time. [How?]