Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Computational Geometry

Winter semester 2016/17

Height Interpolation

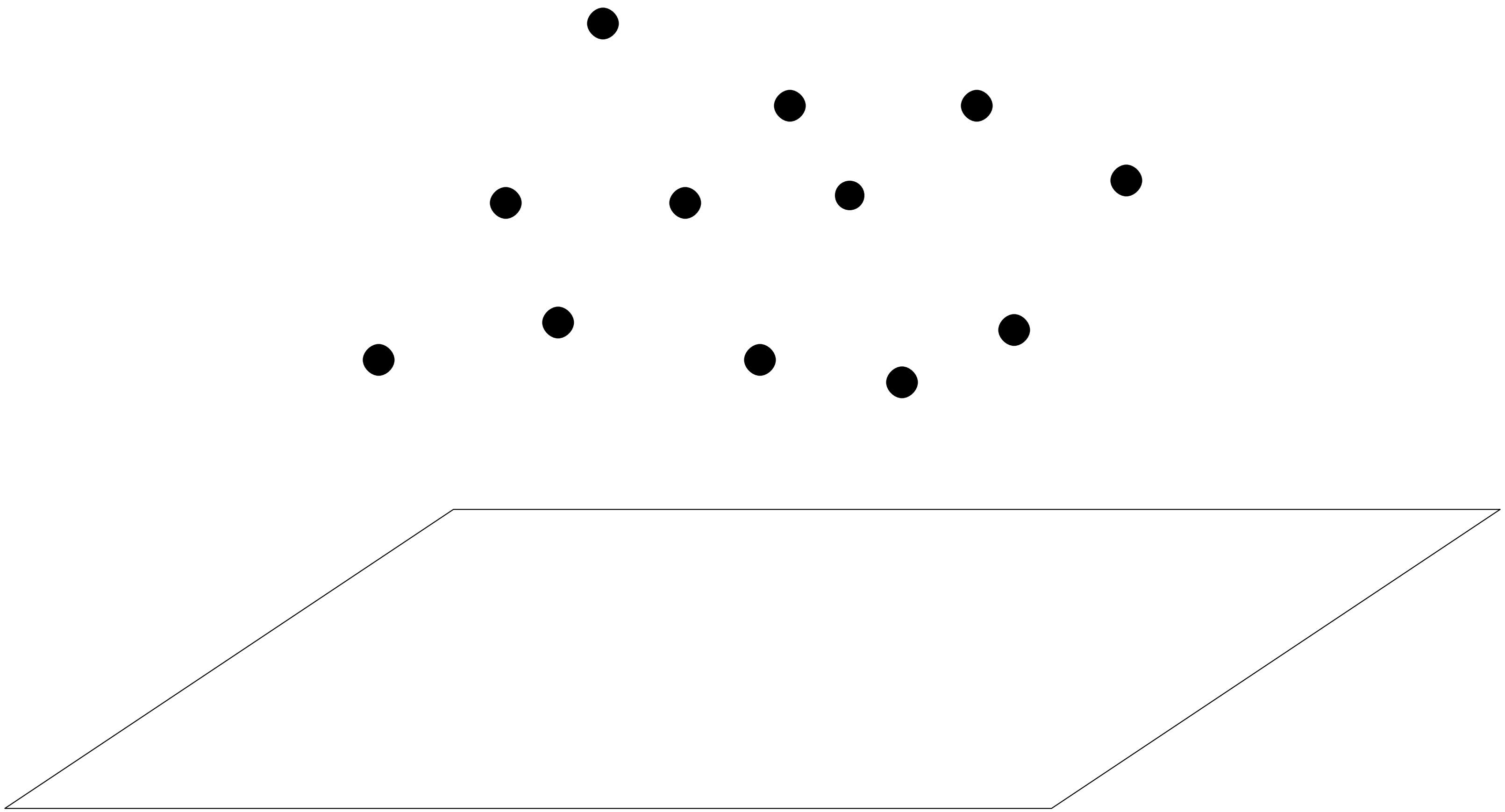
Lecture #8

(Chapter 9 in the textbook)

Joachim Spoerhase

Chair for Computer Science I

Height Interpolation



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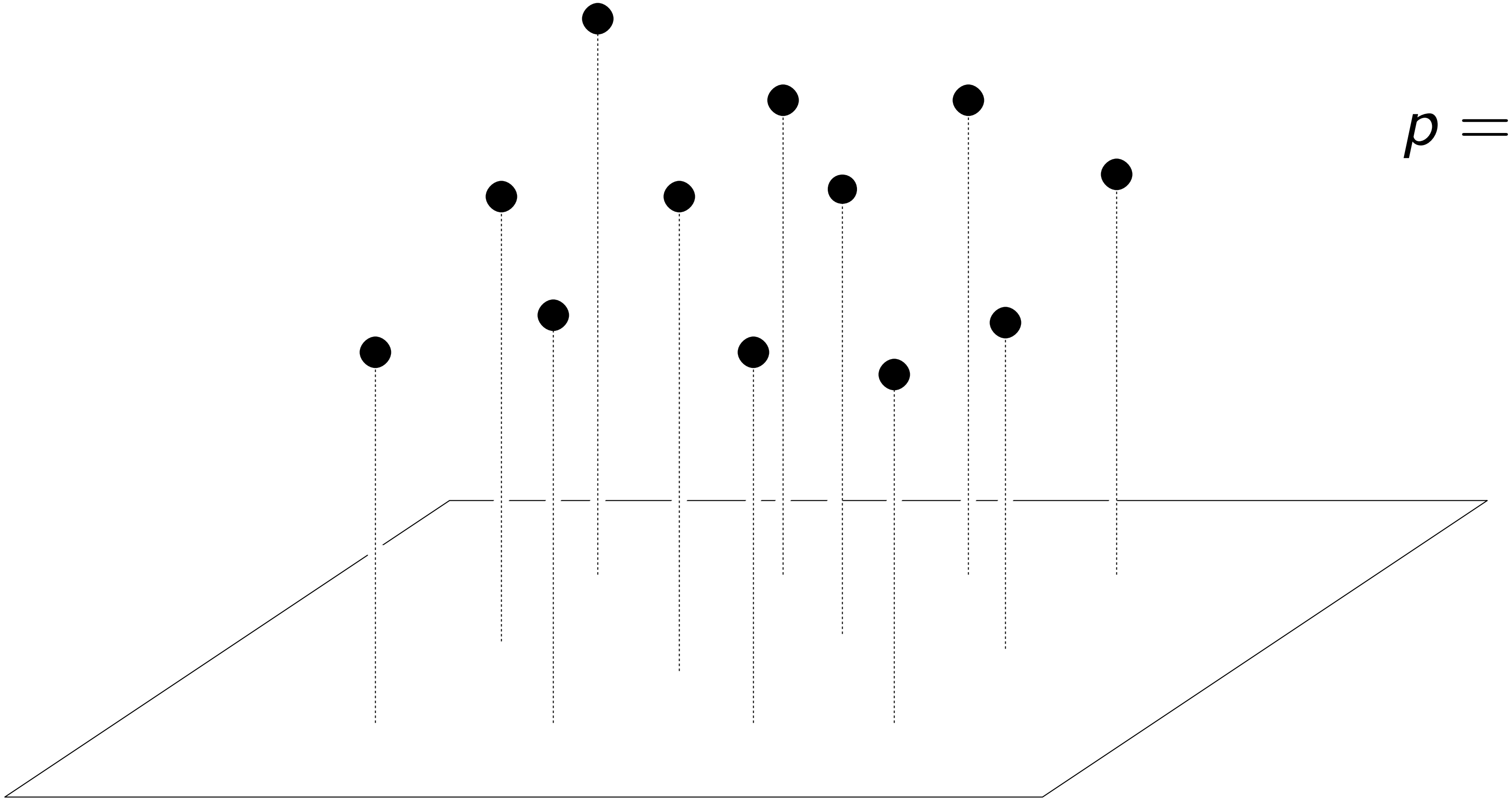

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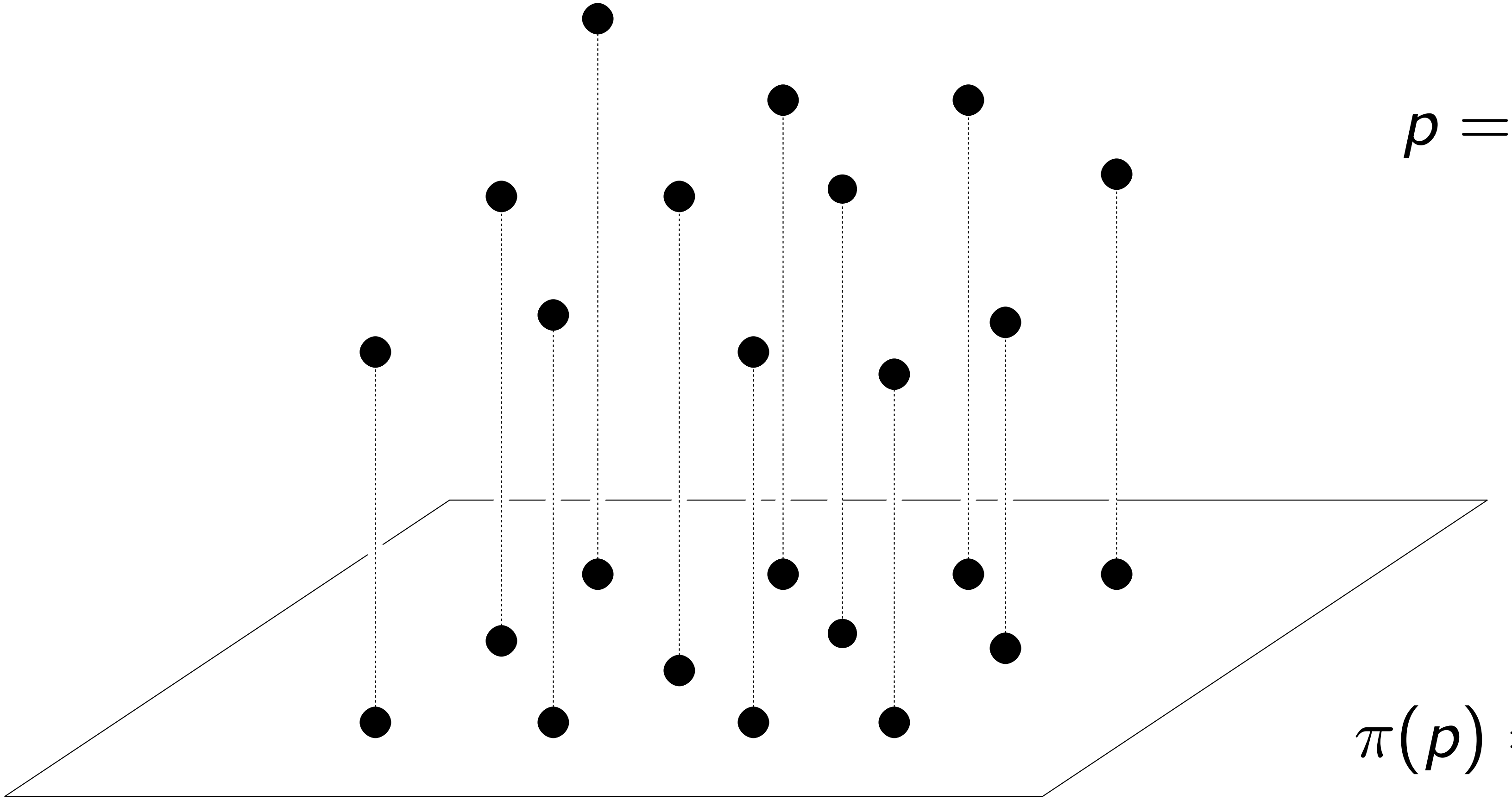

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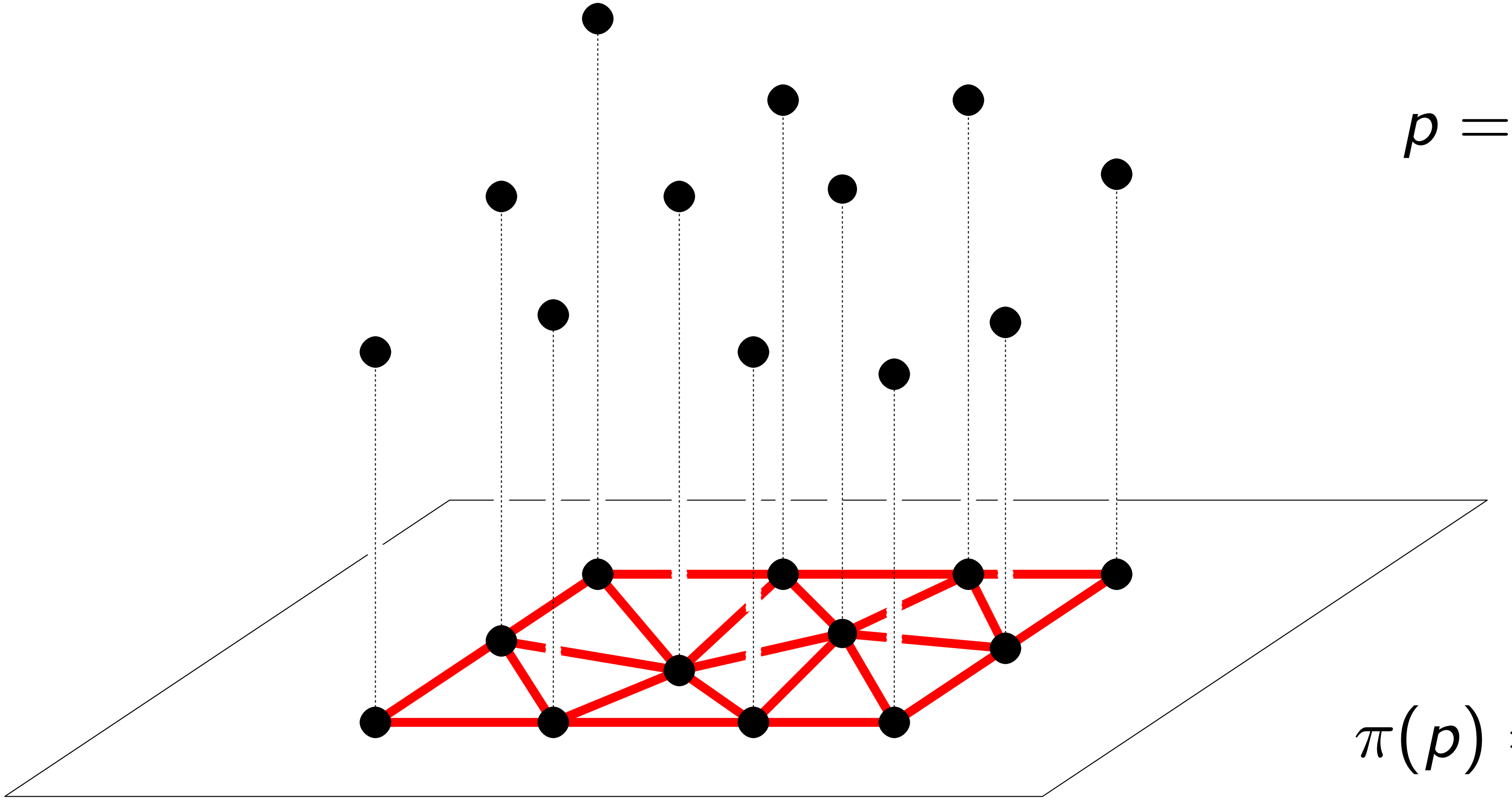

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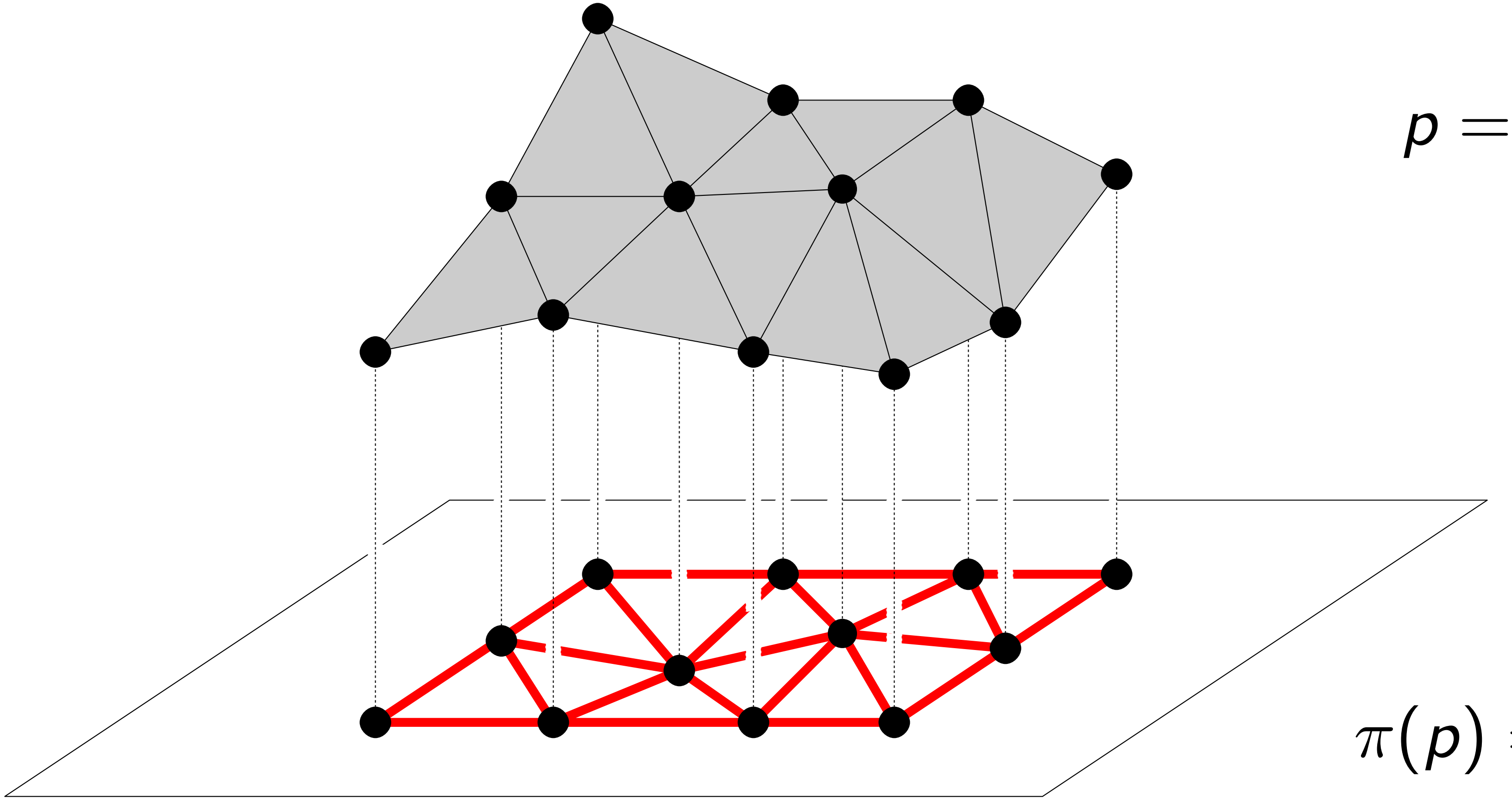

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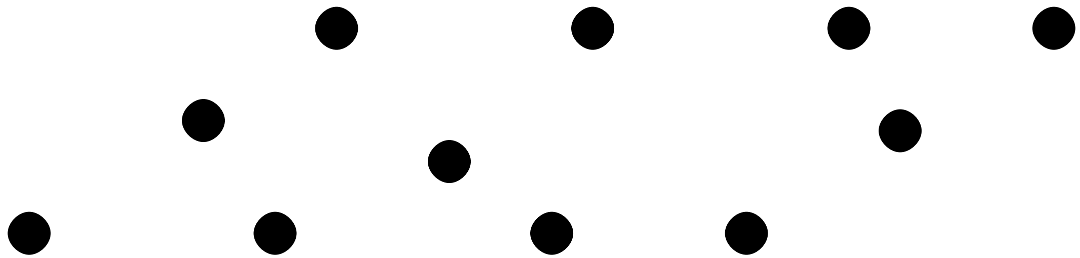
Triangulation of Planar Point Sets

Definition: Given $P \subset \mathbb{R}^2$, a *triangulation* of P is a maximal planar subdivision with vtx set P, that is, no edge can be added without crossing.



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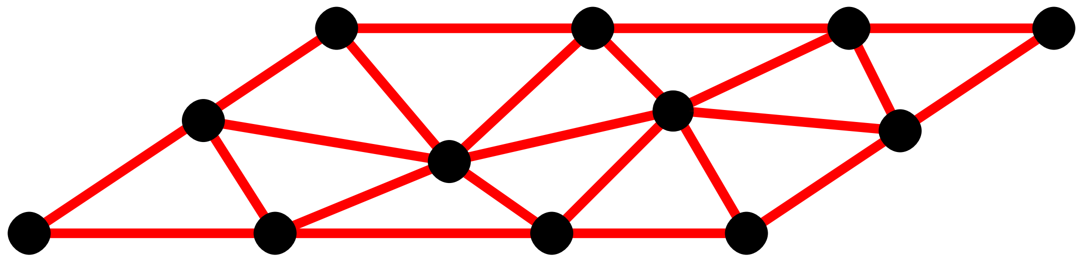

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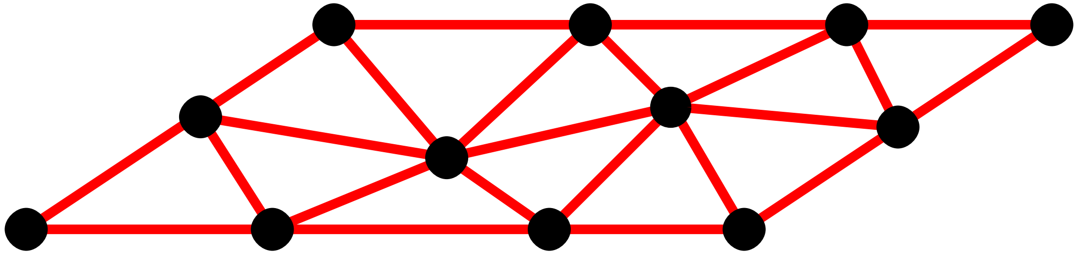

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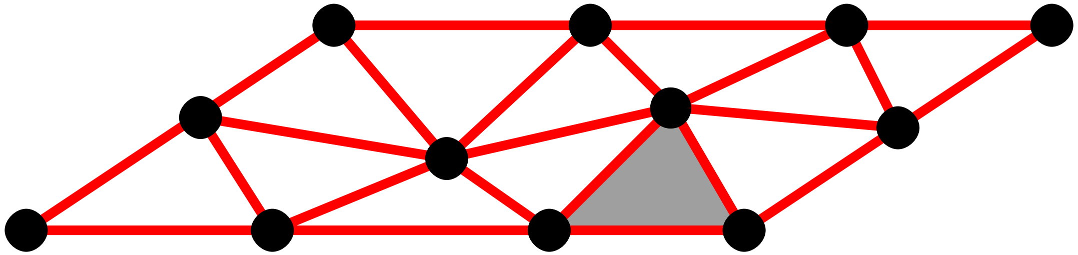

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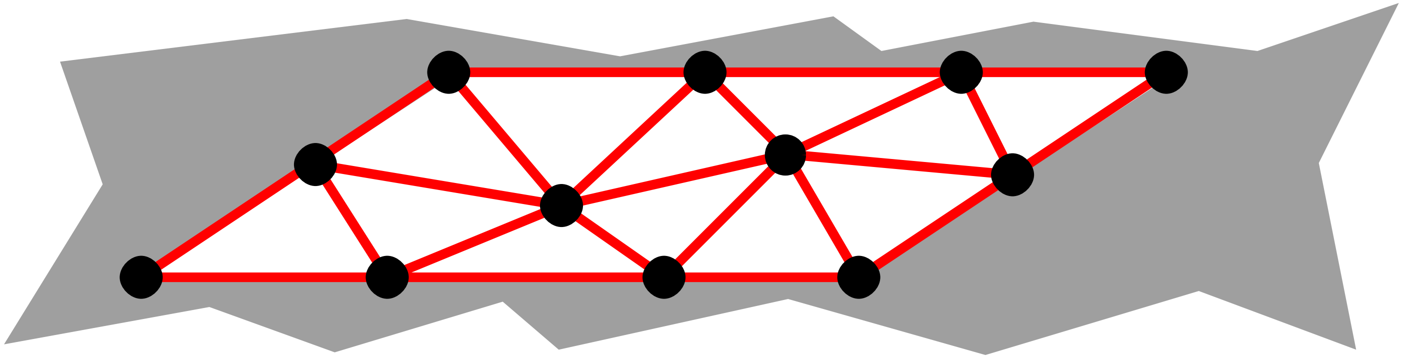

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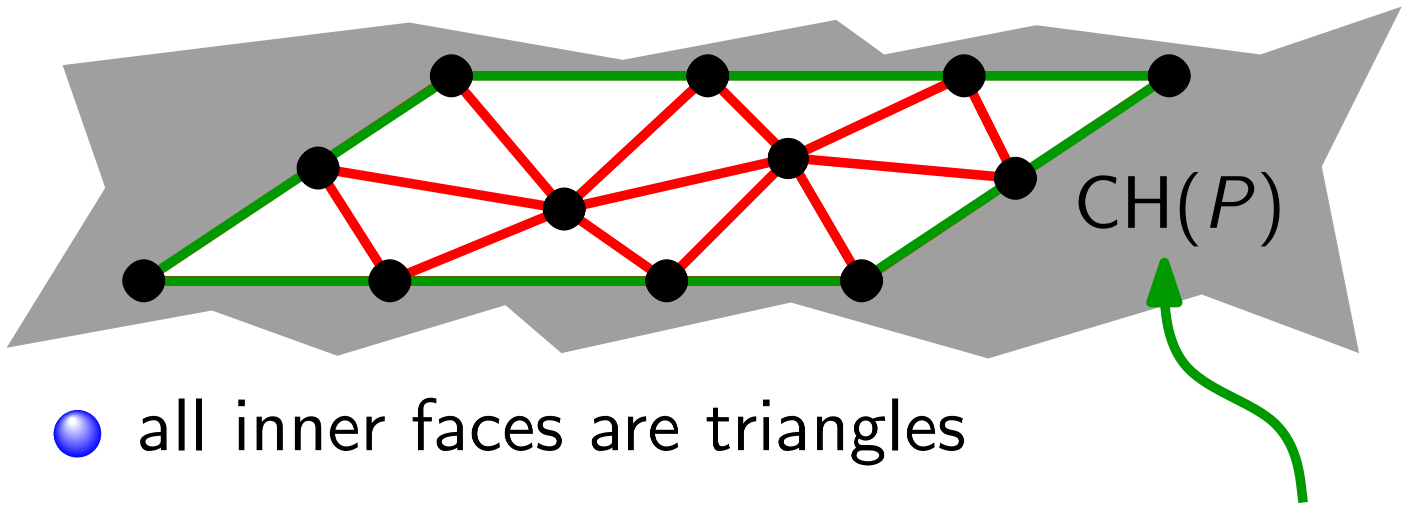

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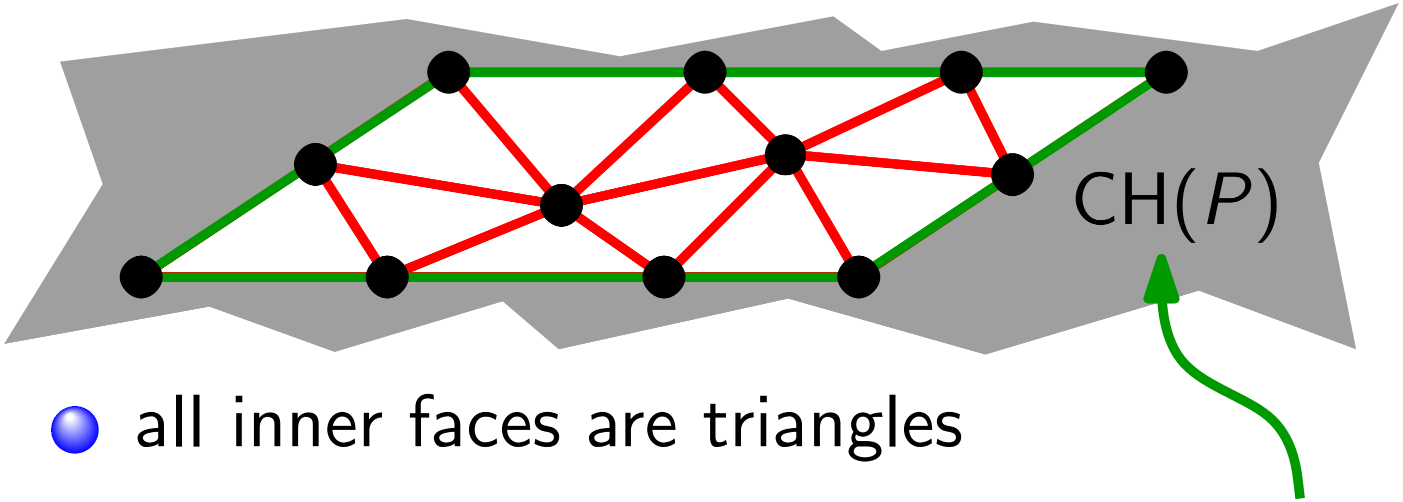

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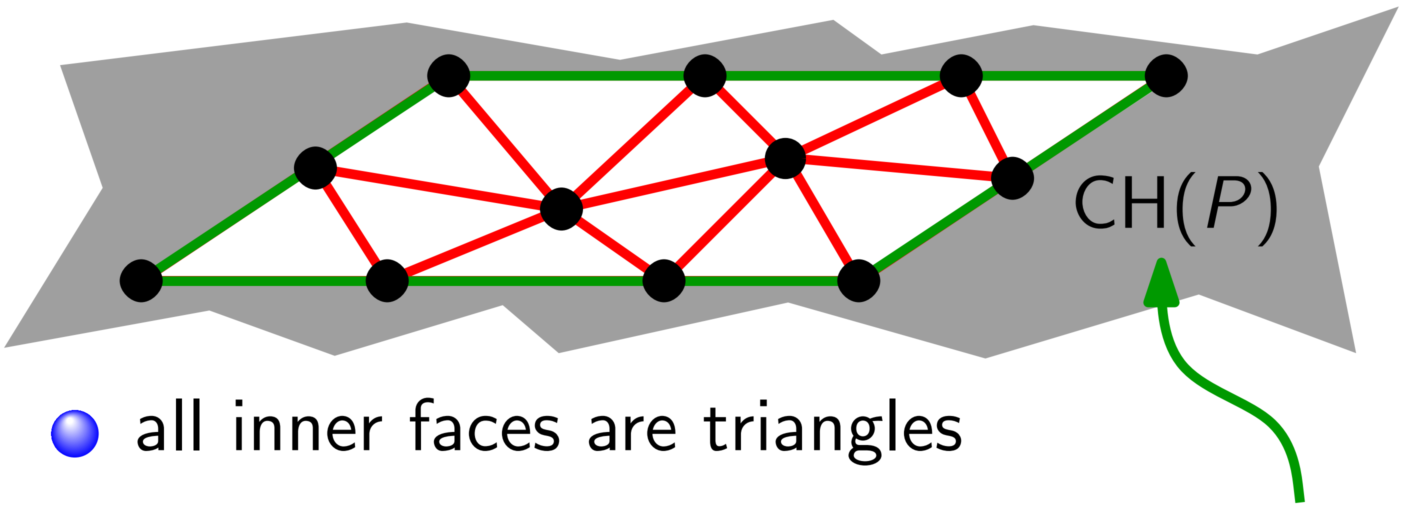

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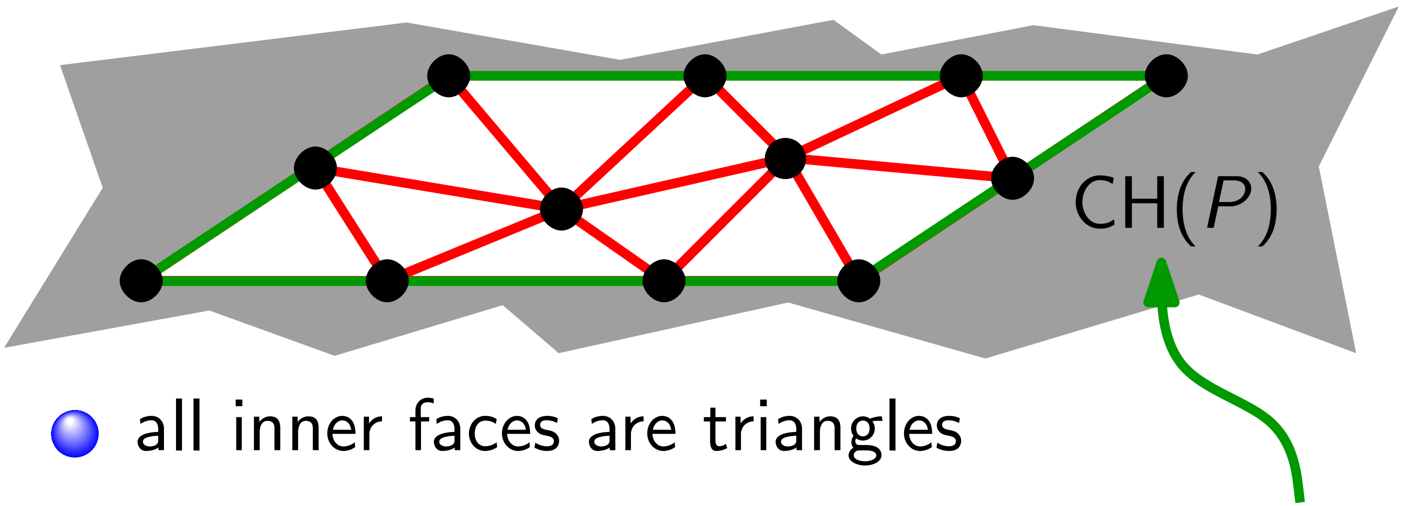

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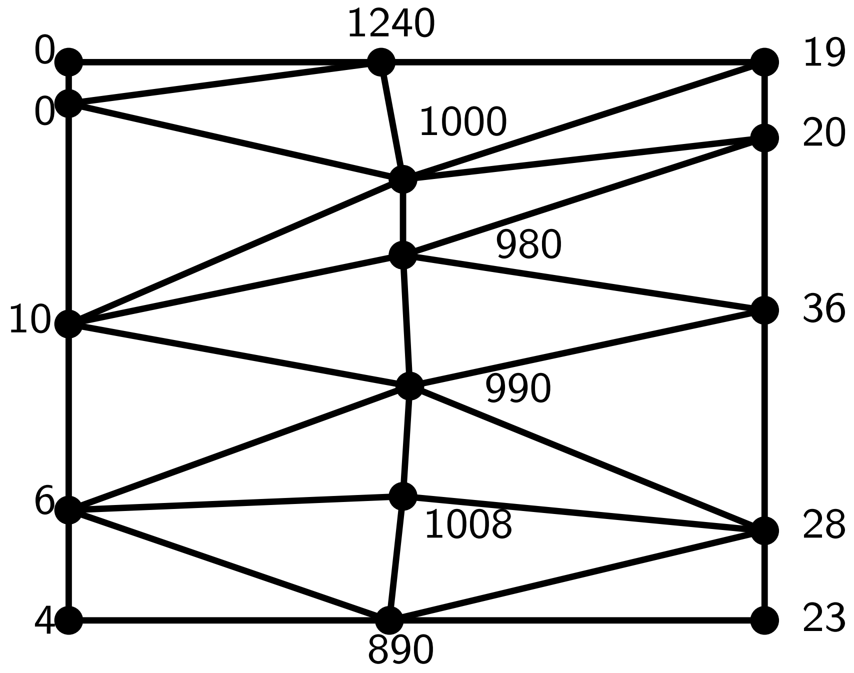
Then *any* triangulation of P has $t(n, h)$ triangles and $e(n, h)$ edges. **Task:** Compute t and e!

Back to Height Interpolation



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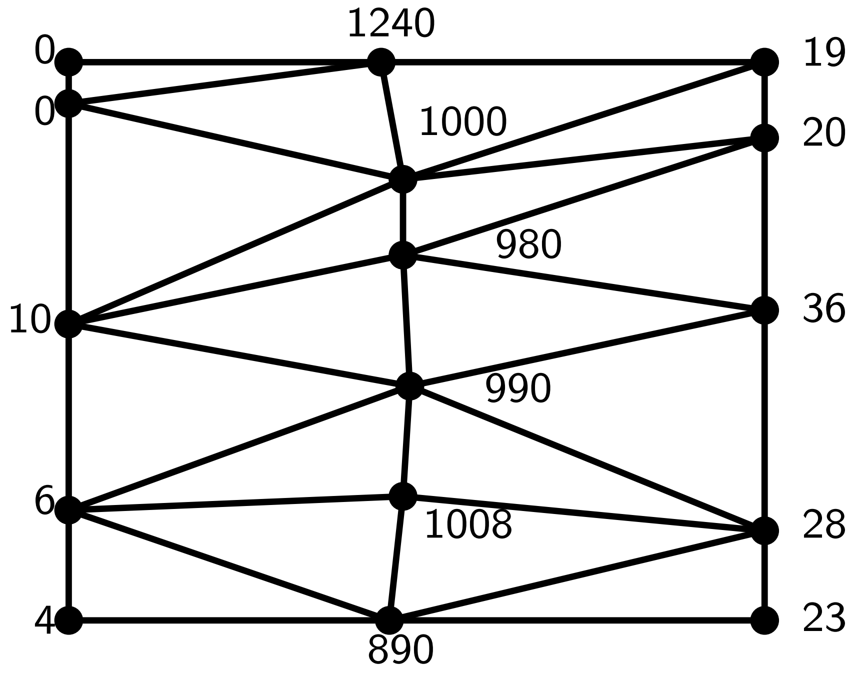

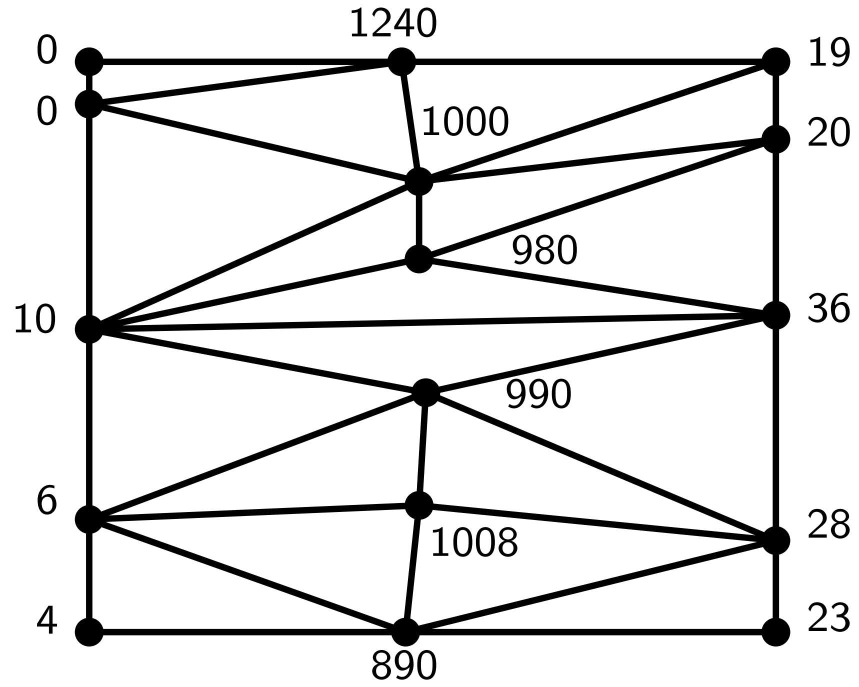

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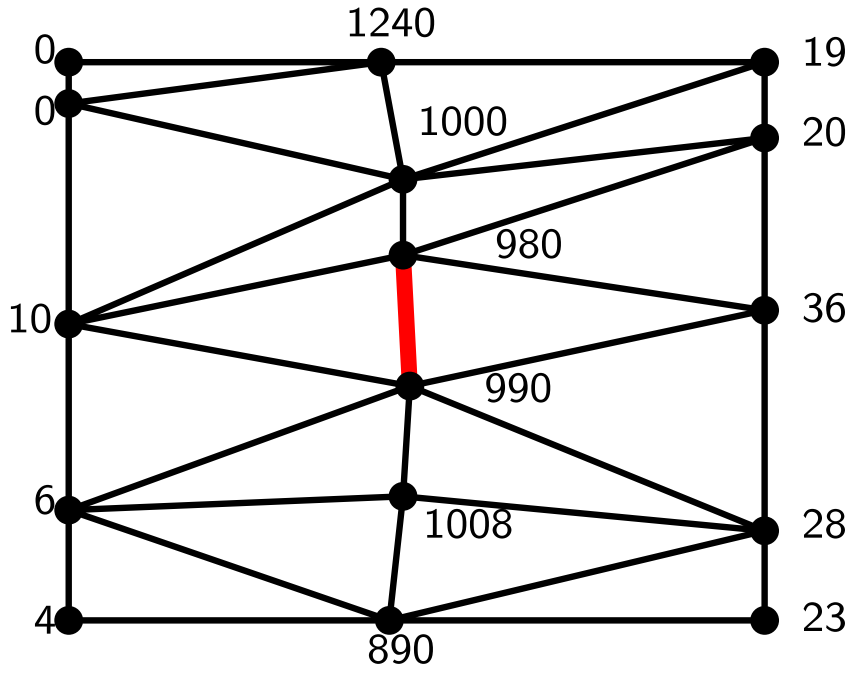

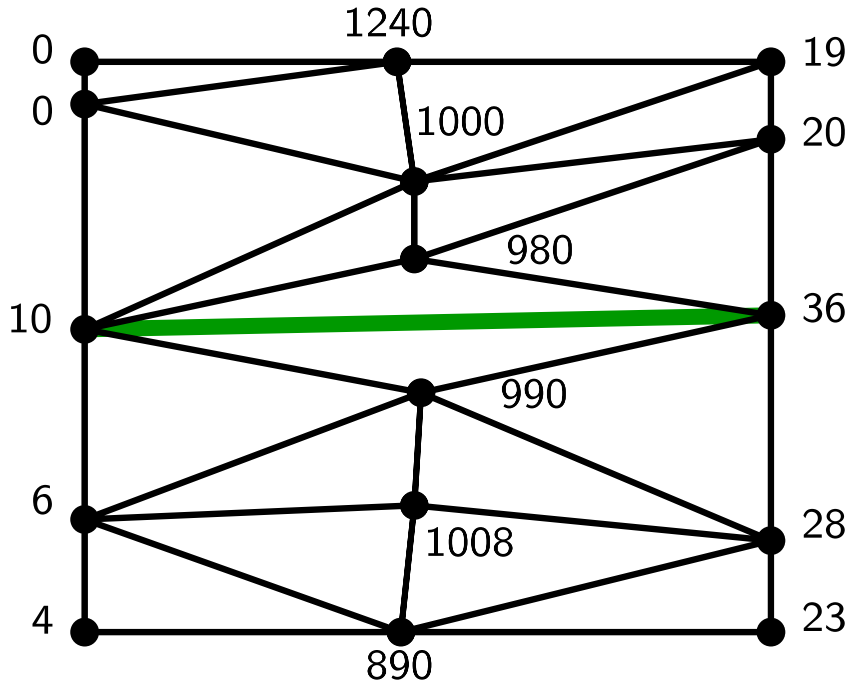

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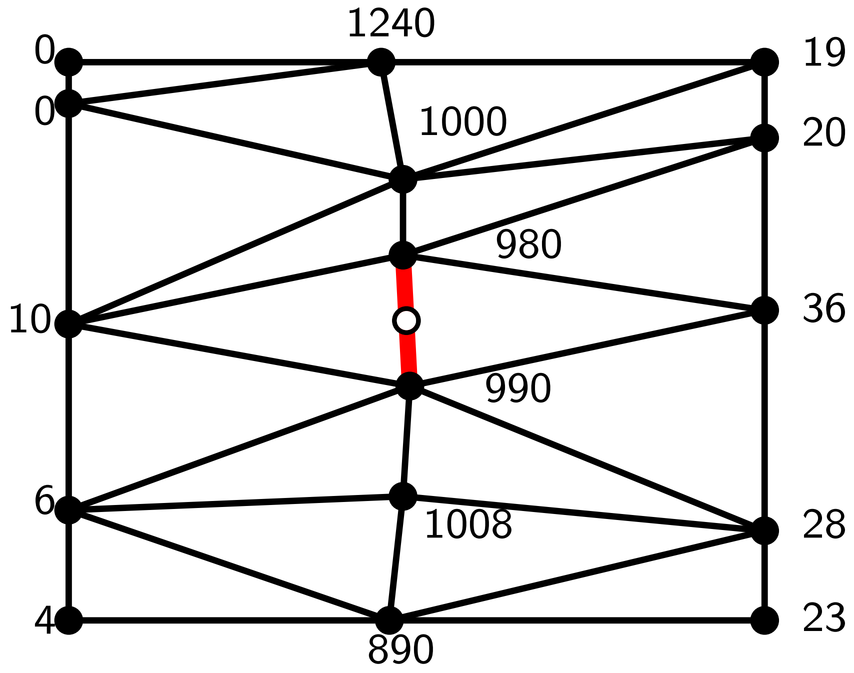

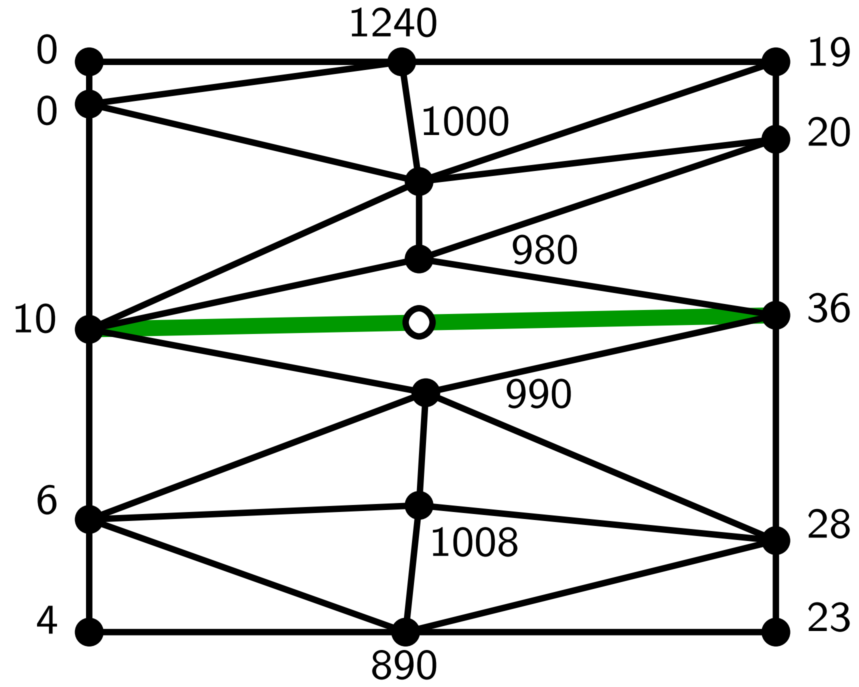

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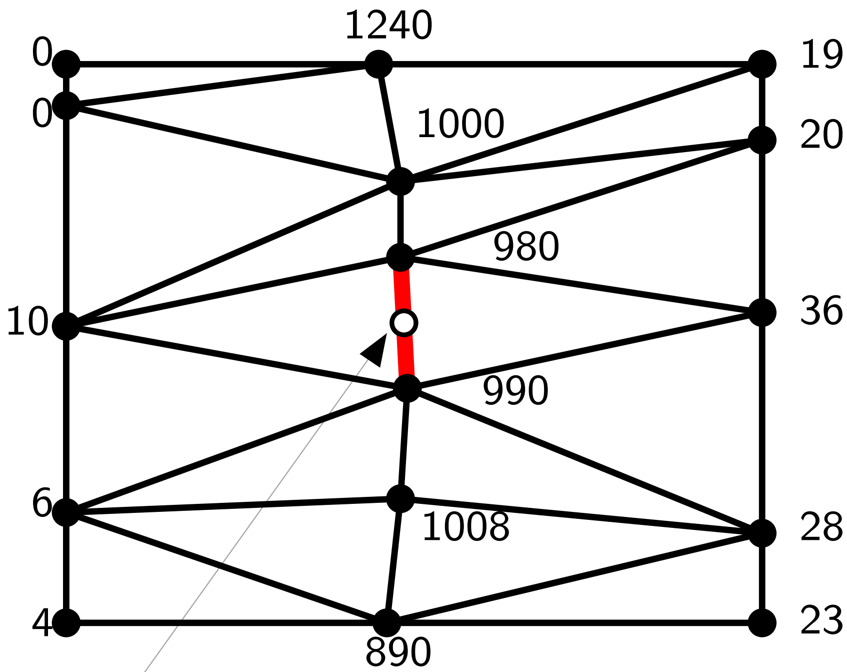
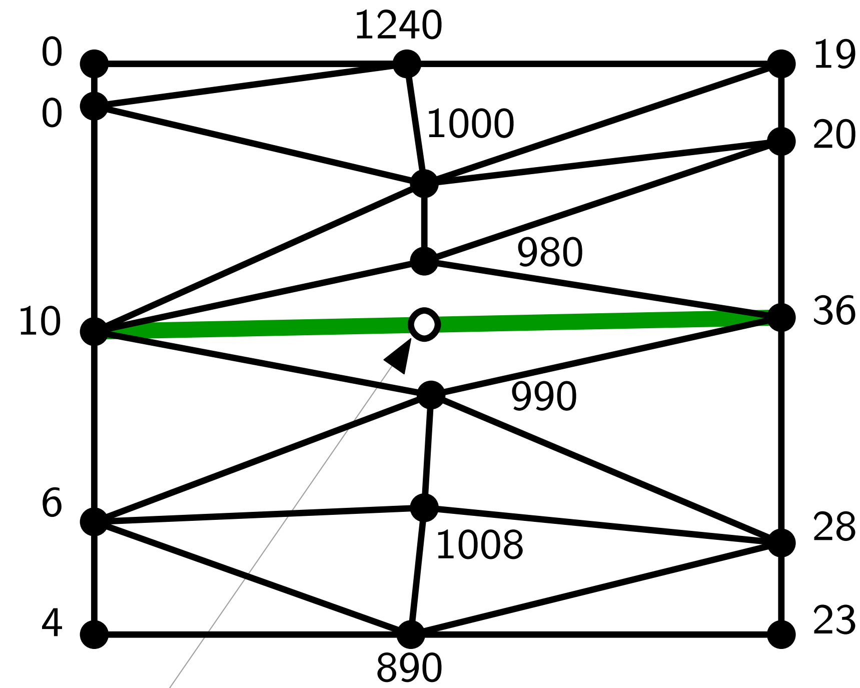
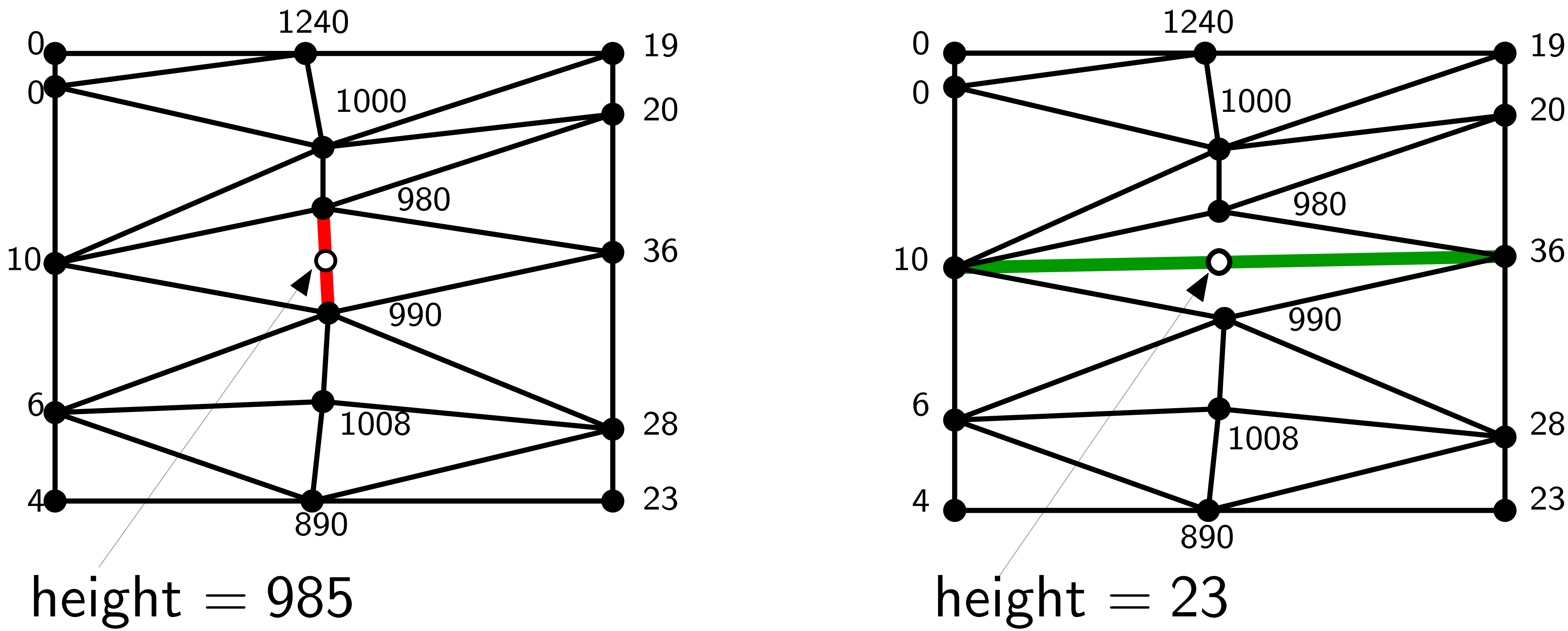


height $= 985$



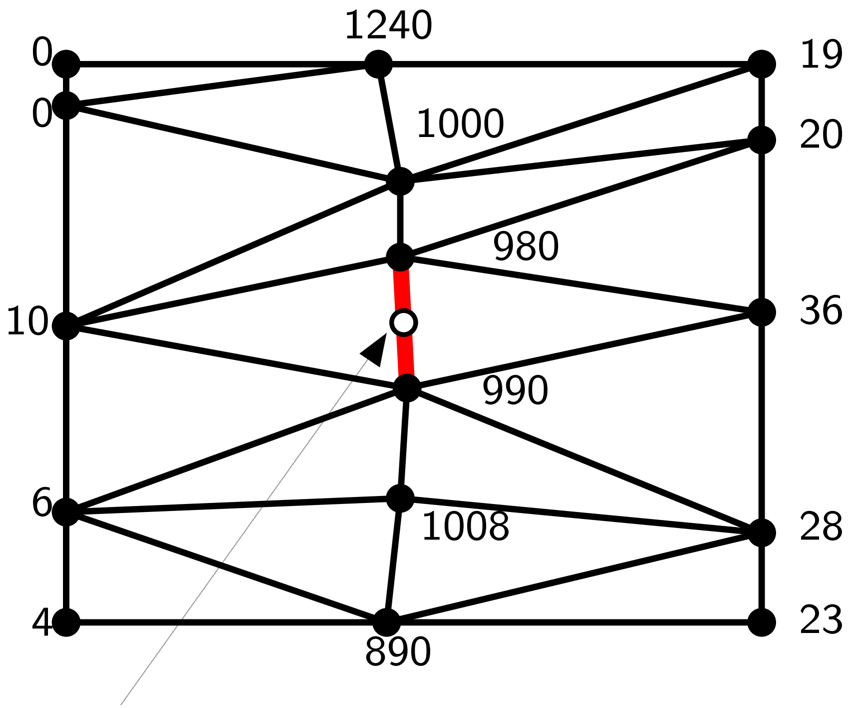
height $= 23$

Back to Height Interpolation



Intuition: Avoid "skinny" triangles!

Back to Height Interpolation



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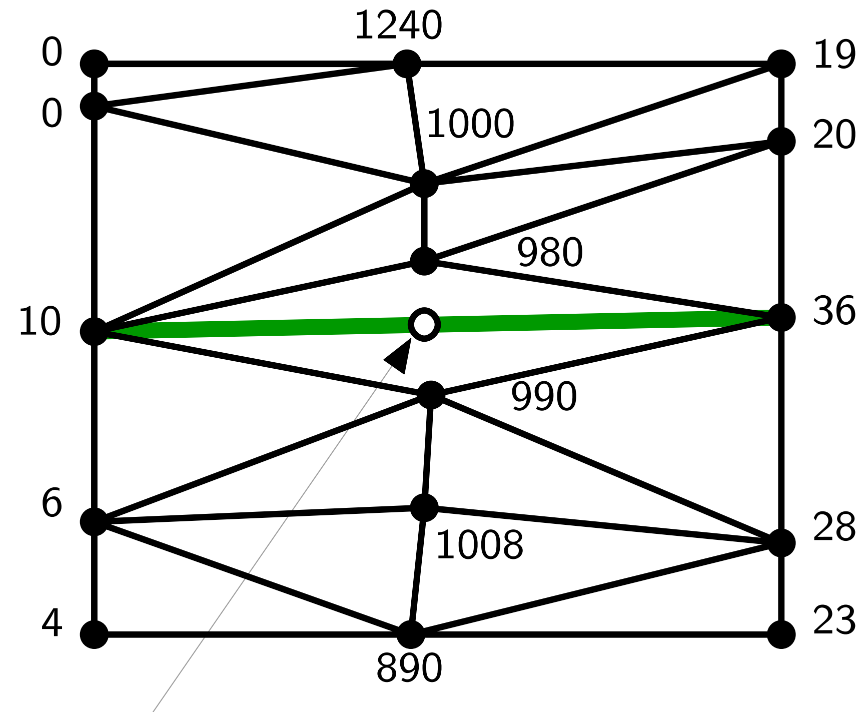

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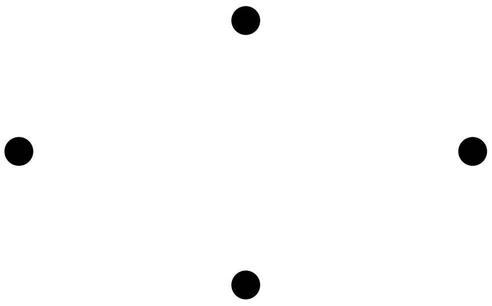
Intuition: Avoid "skinny" triangles!

In other words: avoid small angles!

Angle-Optimal Triangulations

Definition: Given a set $P \subset \mathbb{R}^2$



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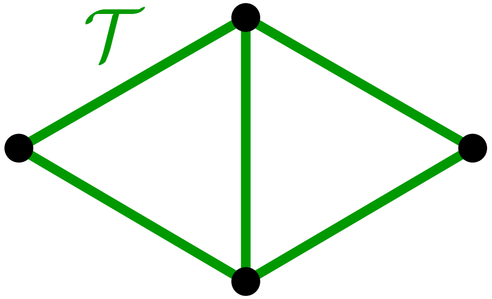

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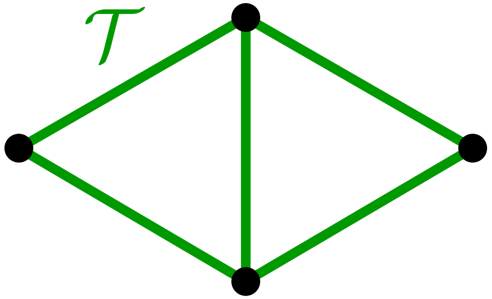

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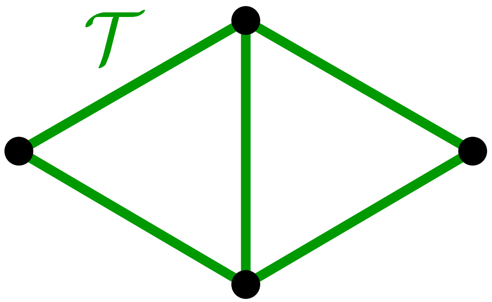

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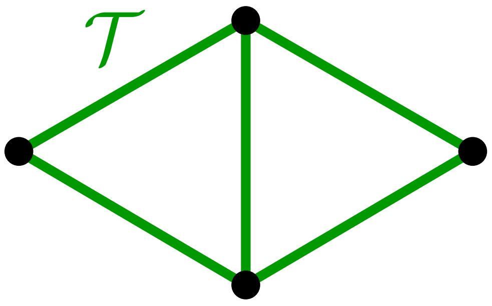

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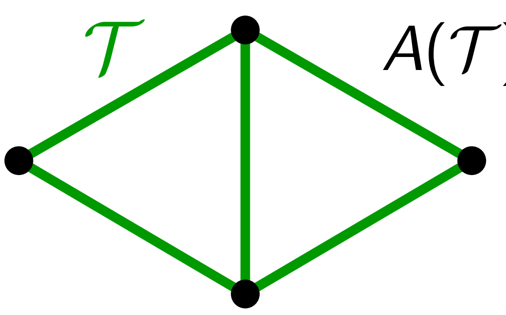

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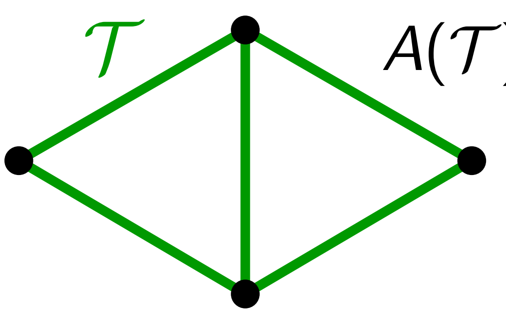

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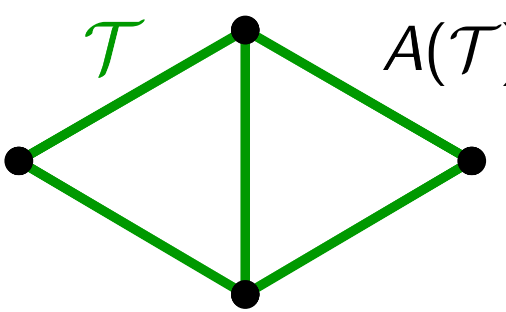

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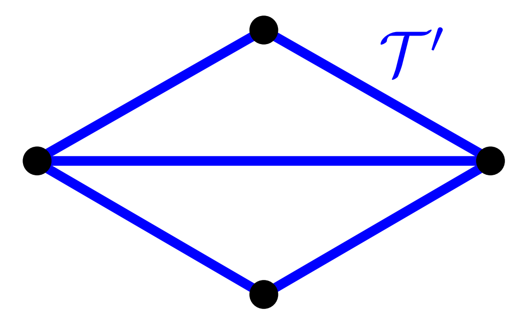

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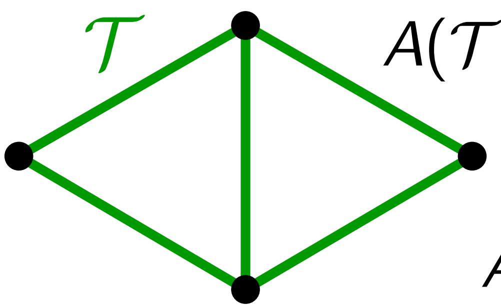

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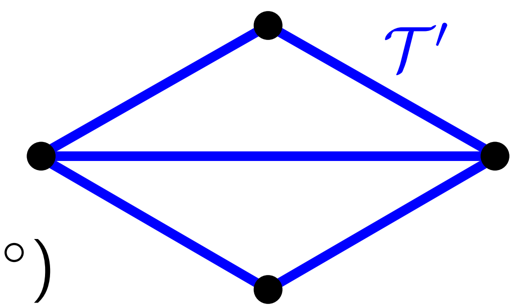

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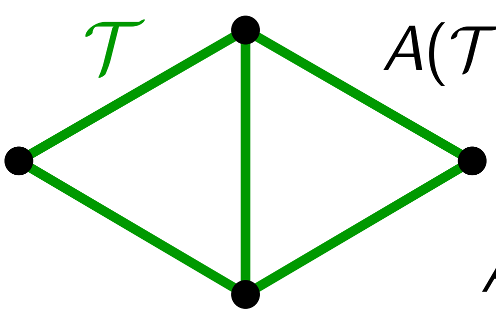

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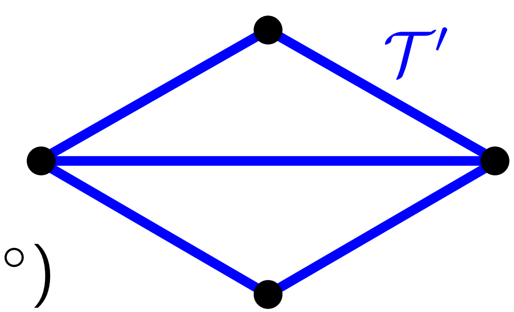

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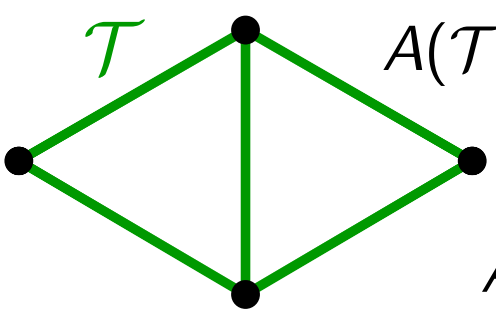

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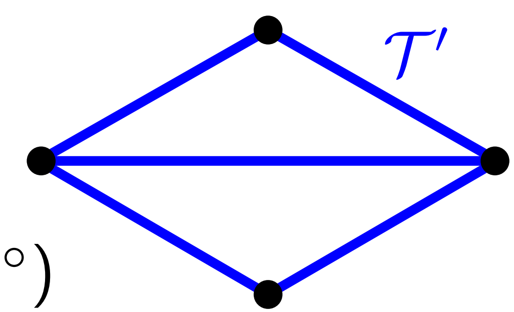

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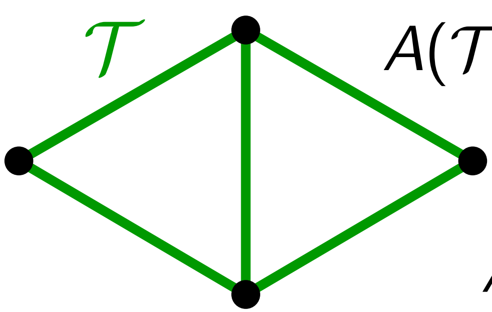

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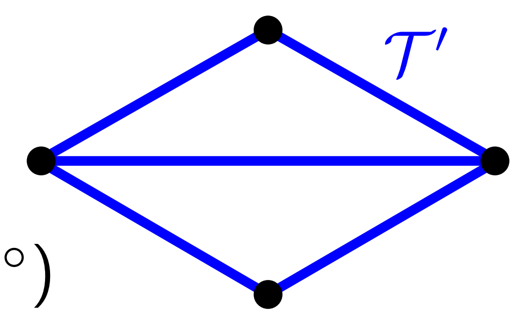

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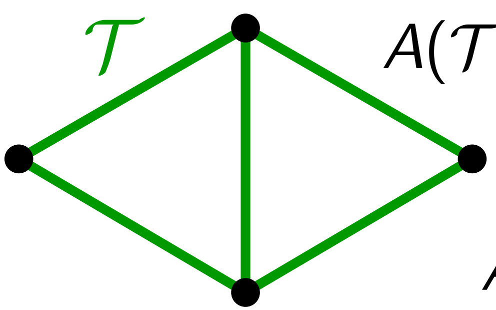

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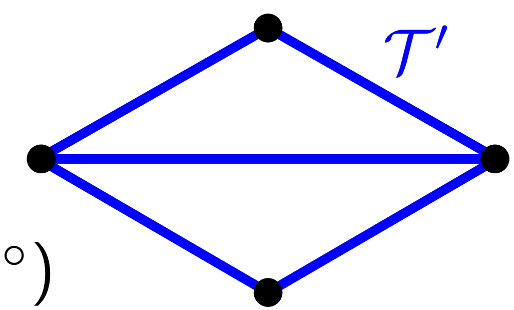

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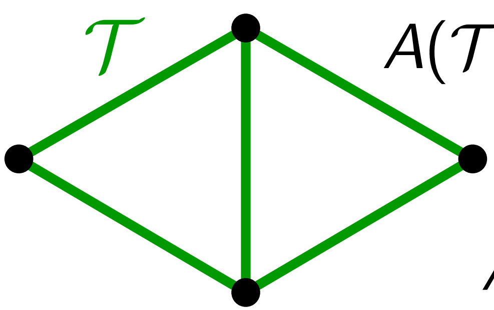
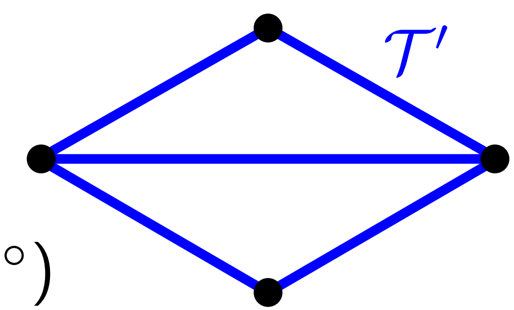
$\mathcal{T}$ is *angle-optimal* if



$A(\mathcal{T}) = (60°, 60°, 60°, 60°, 60°, 60°)$



$A(\mathcal{T}') = (30°, 30°, 30°, 30°, 120°, 120°)$

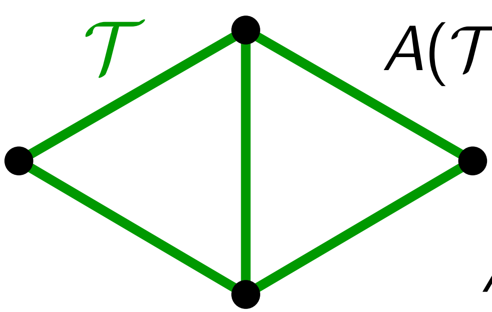
Angle-Optimal Triangulations

Definition: Given a set $P \subset \mathbb{R}^2$ and a triangulation $\mathcal{T}$ of P, let m be the number of triangles in $\mathcal{T}$ and let $A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m})$ be the *angle vector* of $\mathcal{T}$, where $\alpha_1 \leq \cdots \leq \alpha_{3m}$ are the angles in the triangles of $\mathcal{T}$.

We say $A(\mathcal{T}) > A(\mathcal{T}')$
if $\exists i \in \{1, ..., 3m\} : \alpha_i > \alpha'_i$ and $\forall j < i : \alpha_j = \alpha'_j$.

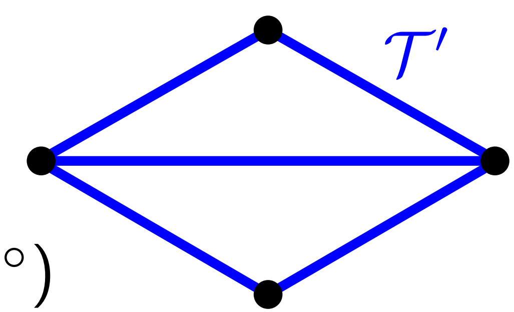
$\mathcal{T}$ is *angle-optimal* if
$A(\mathcal{T}) \geq A(\mathcal{T}')$ for all triangulations $\mathcal{T}'$ of P.

$\mathcal{T}$



$A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ)$

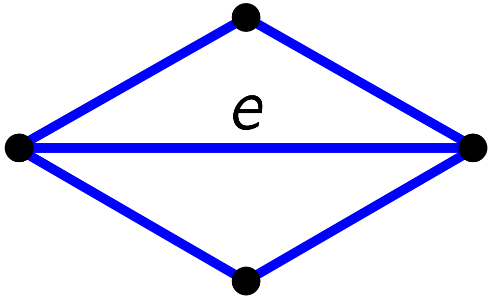
$\mathcal{T}'$



$A(\mathcal{T}') = (30^\circ, 30^\circ, 30^\circ, 30^\circ, 120^\circ, 120^\circ)$

Edge Flips

Definition: $\mathcal{T}$ a triangulation. An edge e of $\mathcal{T}$ is *illegal* if the minimum angle in the two triangles adjacent to e increases when flipping.



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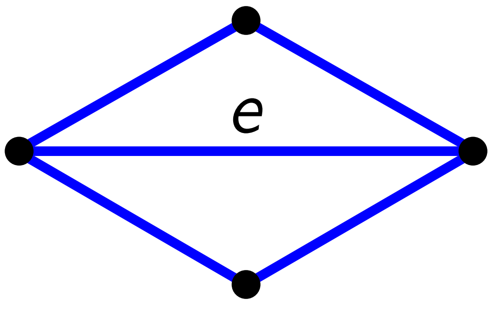
$\min_i \alpha_i = 30^\circ$



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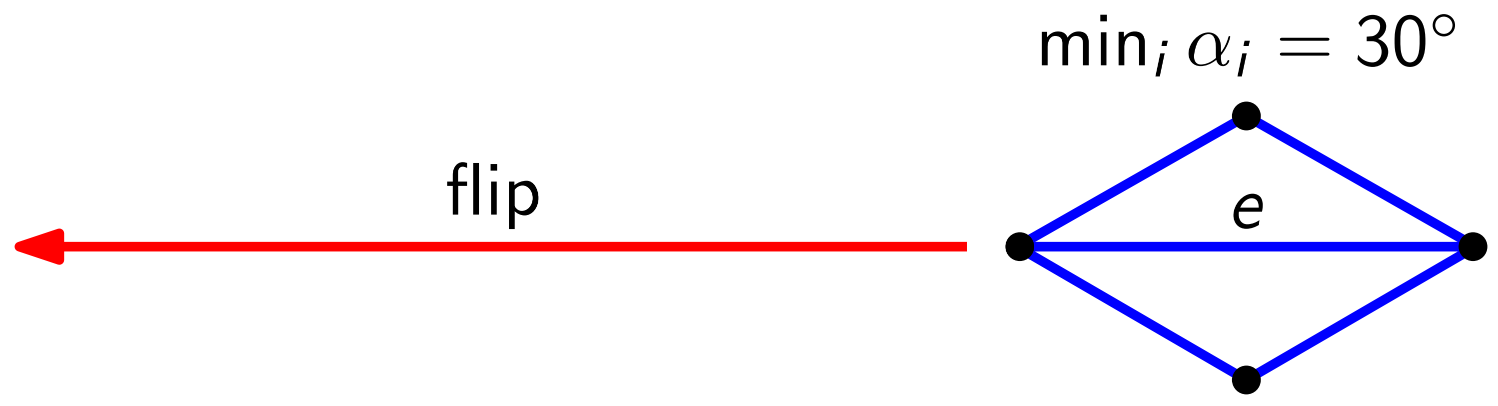

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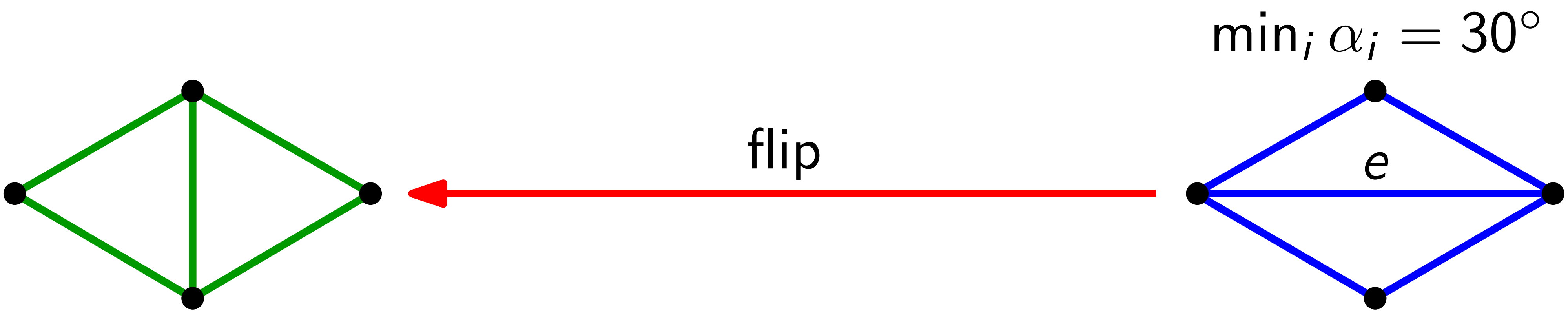

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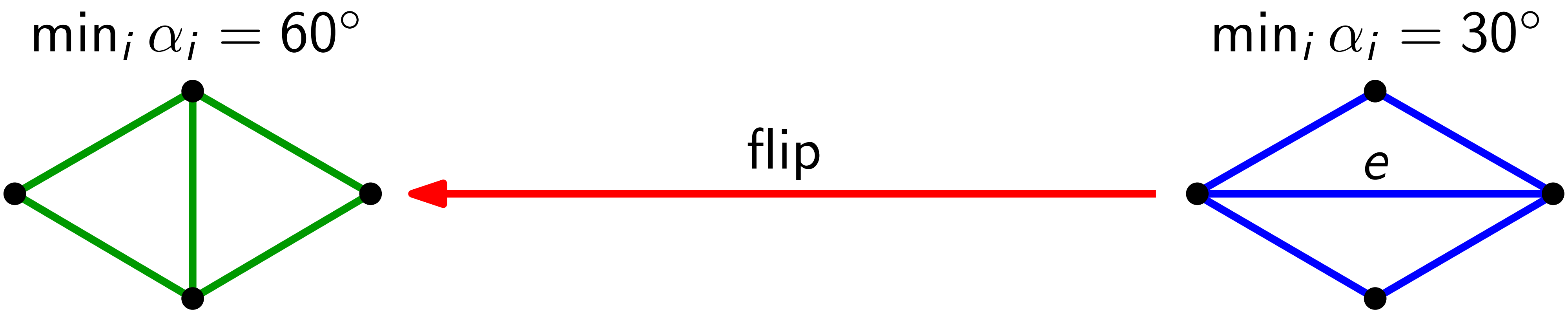

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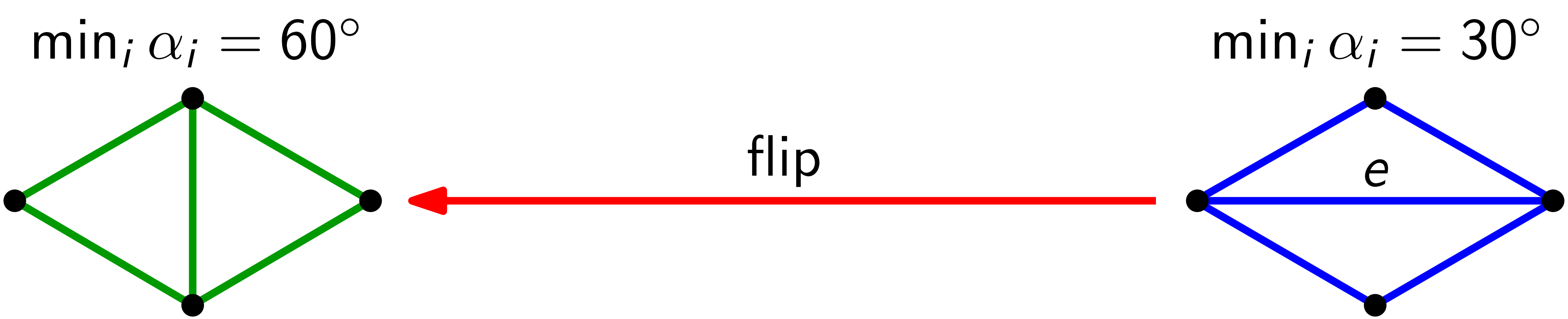
Observe: Let e be an illegal edge of $\mathcal{T}$, and $\mathcal{T}' = \text{flip}(\mathcal{T}, e)$.



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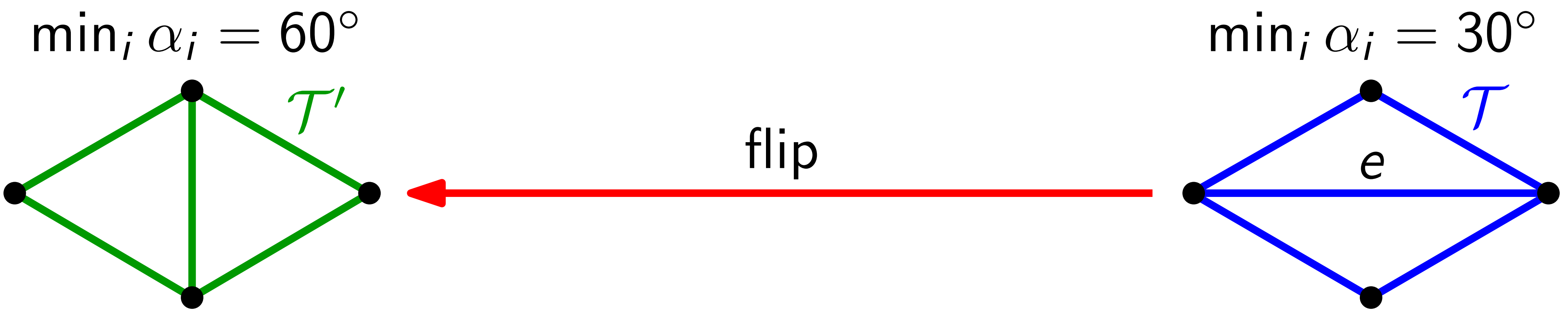

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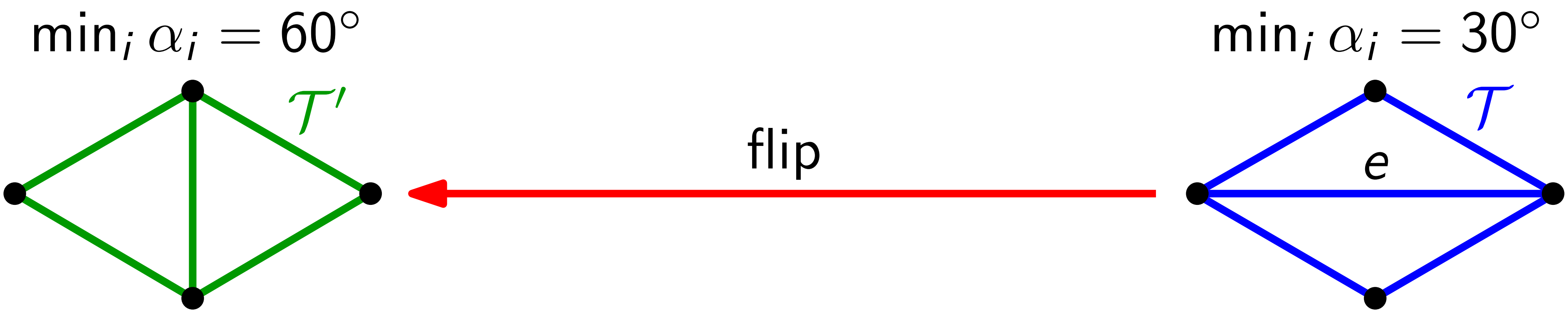
Observe: Let e be an illegal edge of $\mathcal{T}$, and $\mathcal{T}' = \text{flip}(\mathcal{T}, e)$. Then $A(\mathcal{T}') > A(\mathcal{T})$.



This is all Greek to me...

Theorem:

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Theorem: (Thales)
The diameter of a circle always subtends a right angle to any point on the circle.

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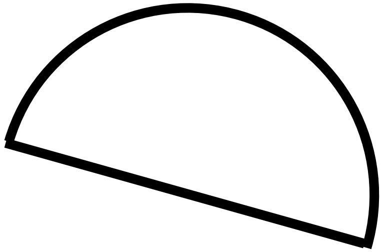

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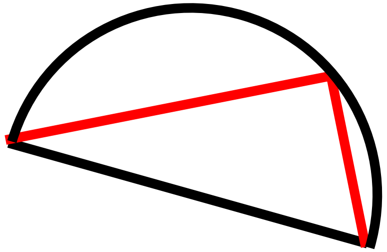

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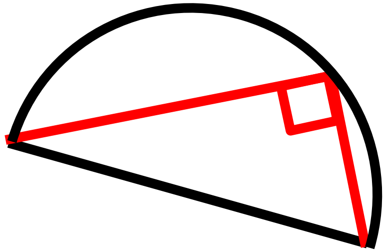

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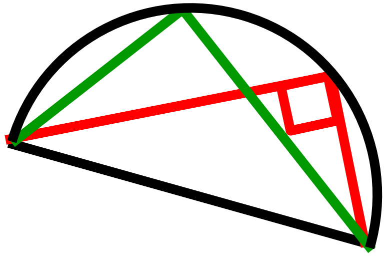

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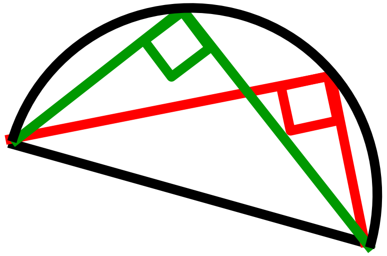

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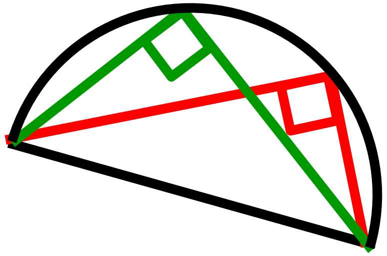

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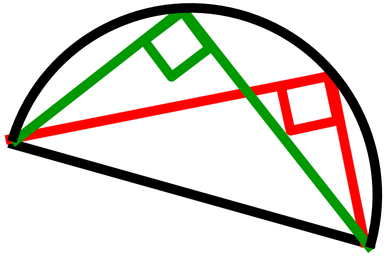

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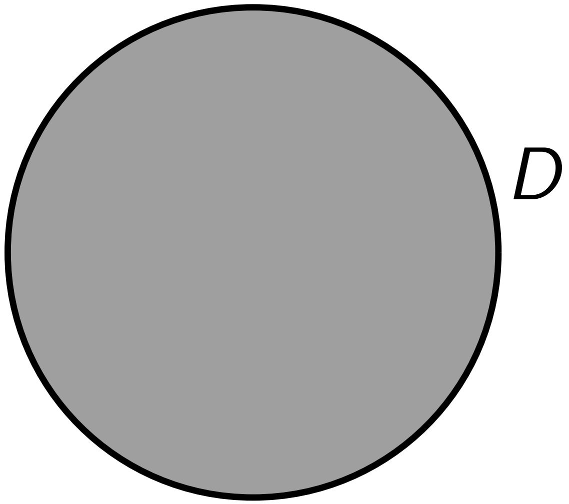

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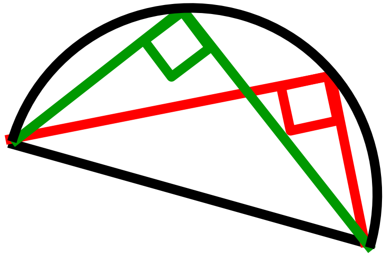

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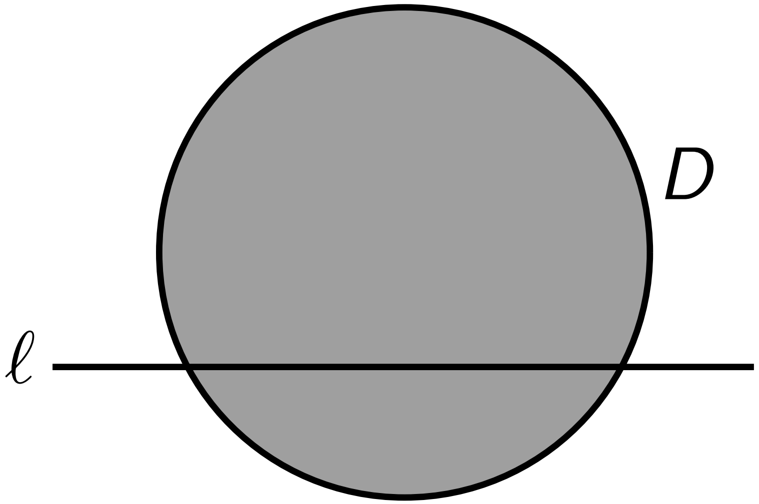

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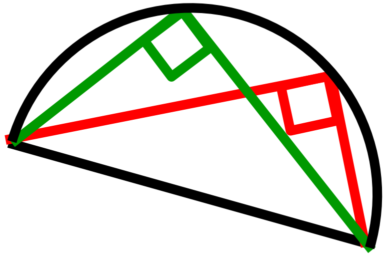

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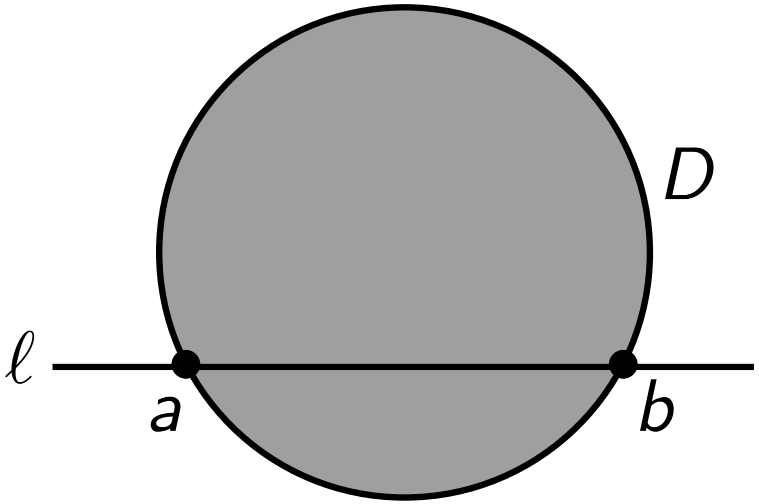

$$\{a, b\} := \ell \cap \partial D \ (a \neq b)$$

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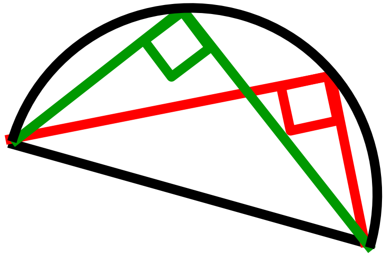

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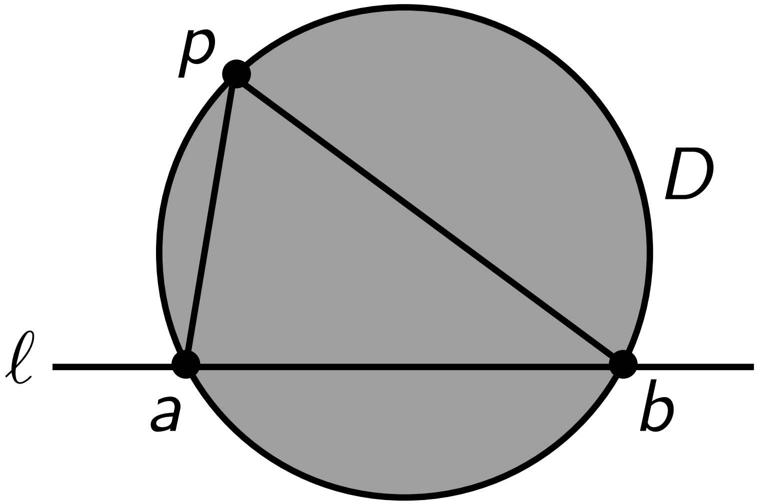

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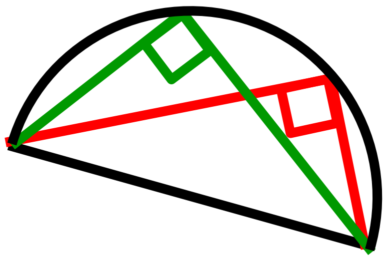

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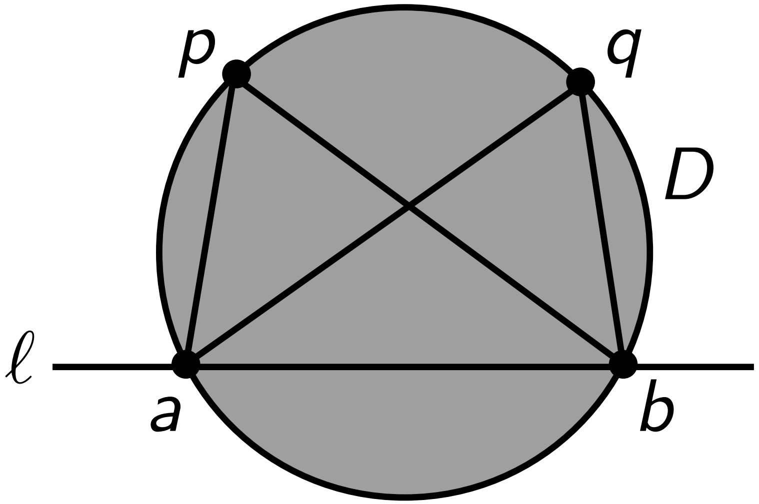

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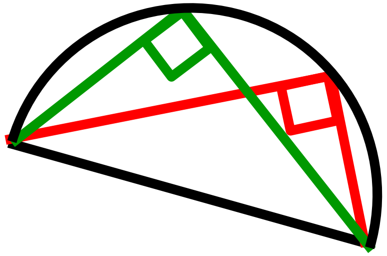

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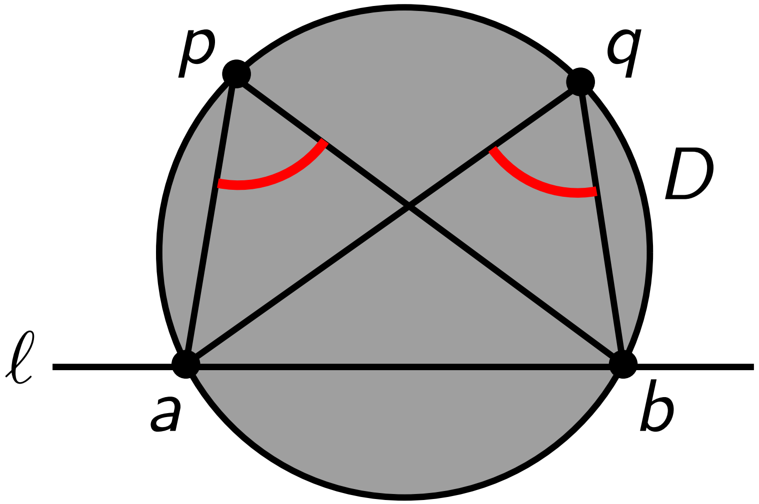

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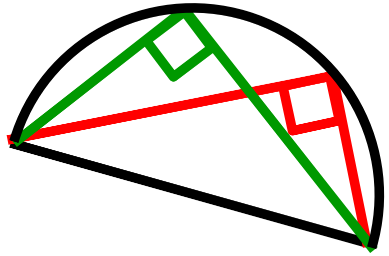

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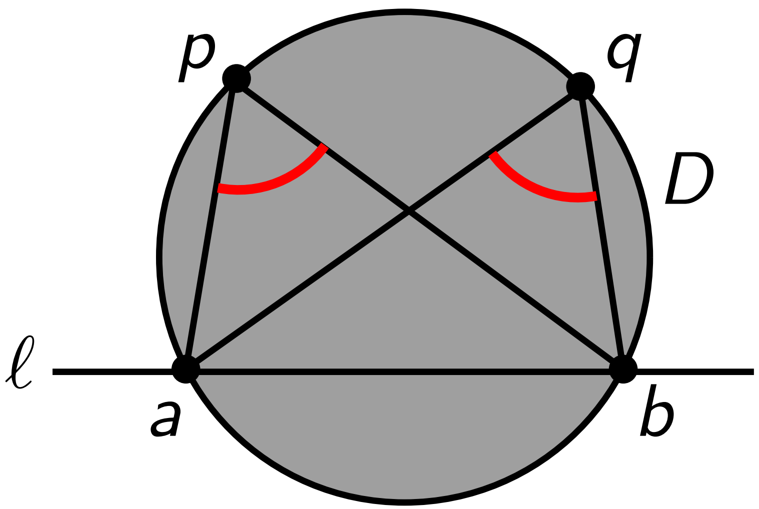

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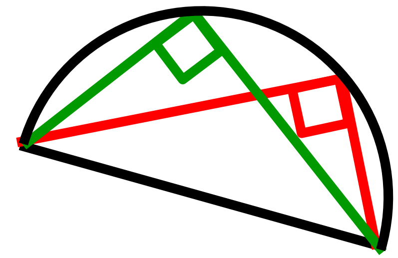

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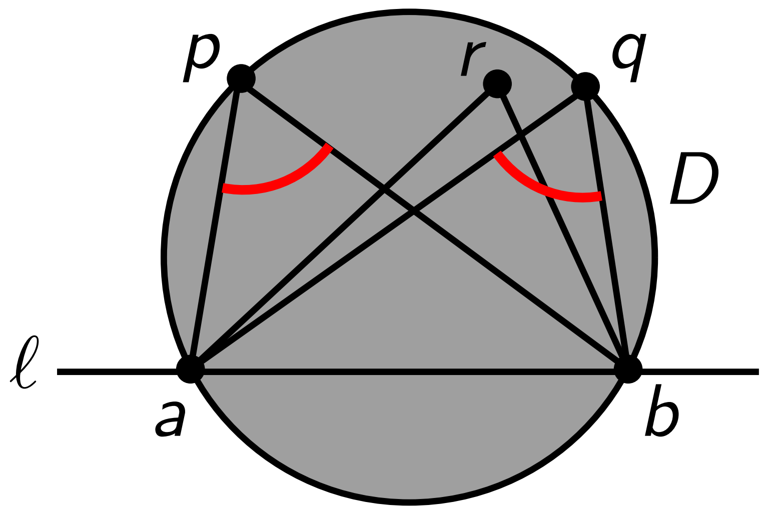

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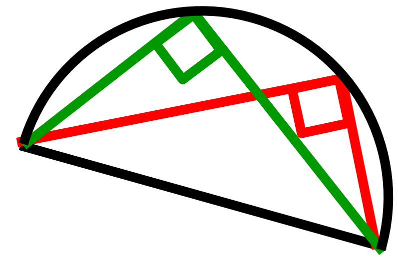

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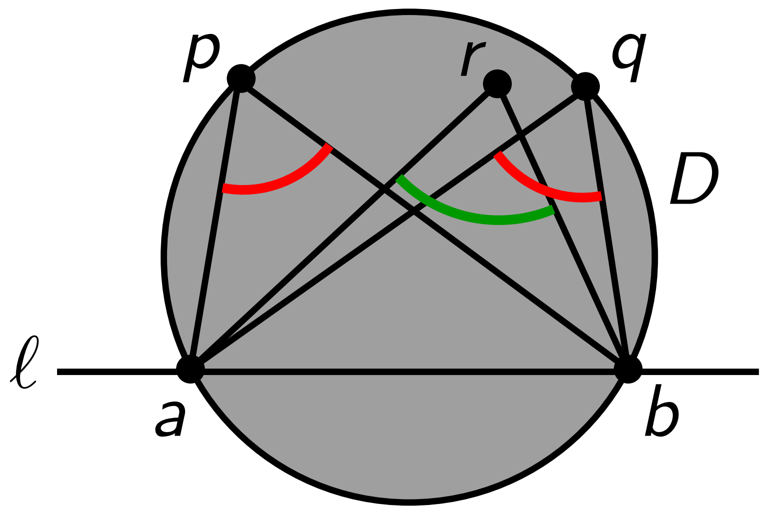

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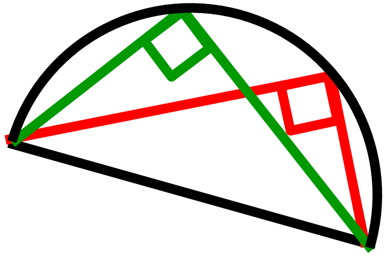

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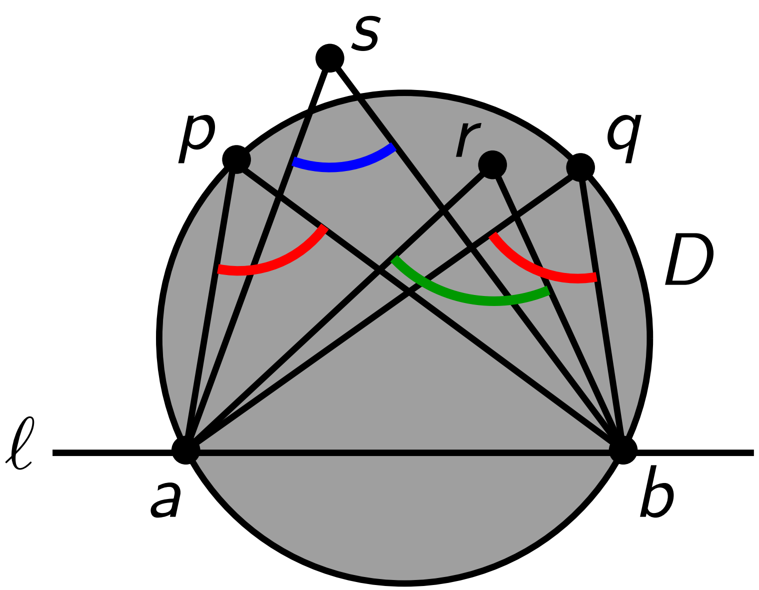

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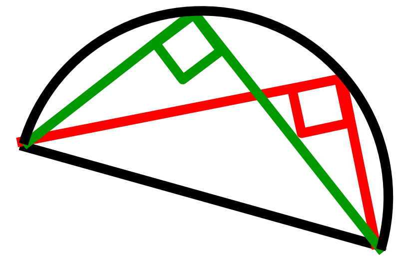

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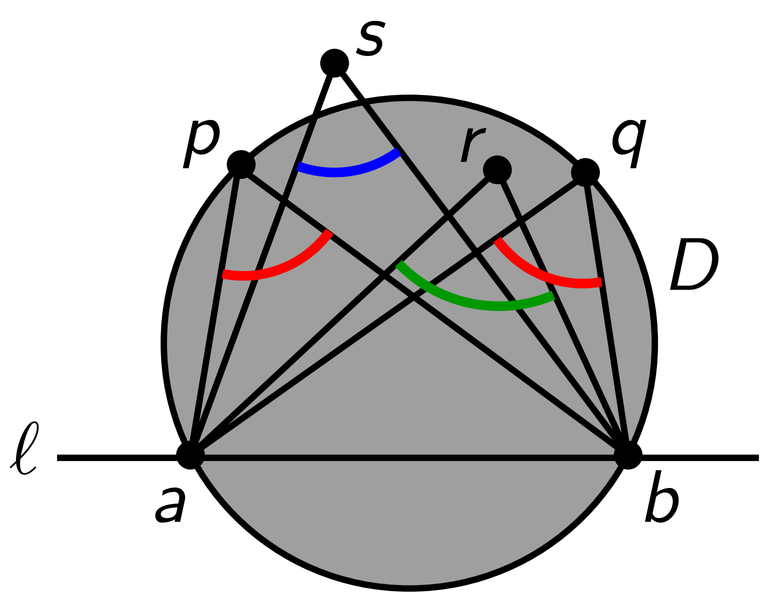

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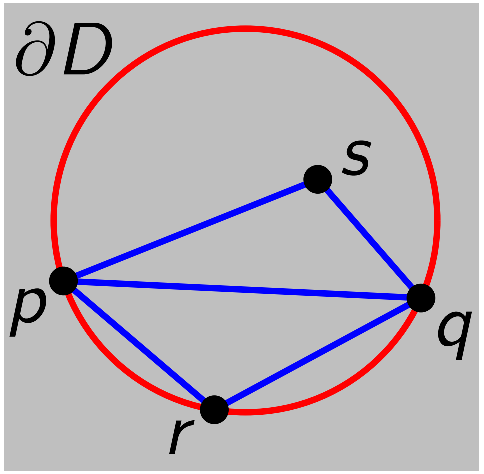
Legal Triangulations

Lemma: Let $\Delta prq, \Delta pqs \in \mathcal{T}$ and $p, q, r \in \partial D$.
Then edge pq is illegal iff $s \in \text{int}(D)$.



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If p, q, r, s in convex position and $s \notin \partial D$,
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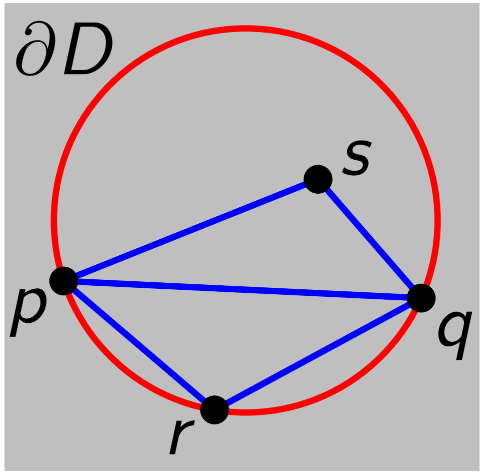

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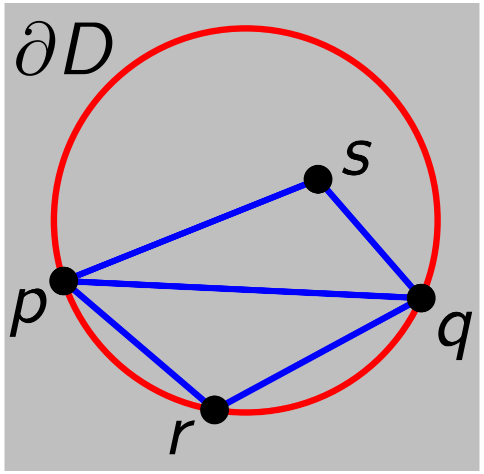

Proof:

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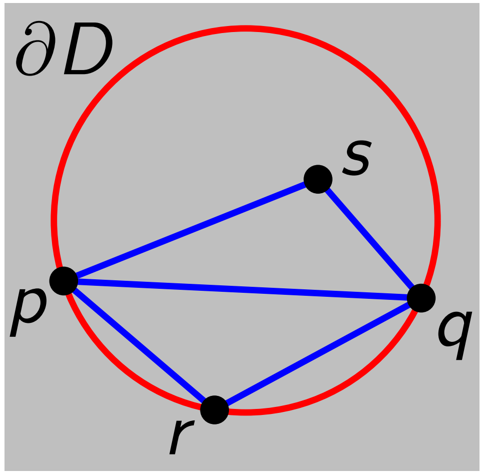

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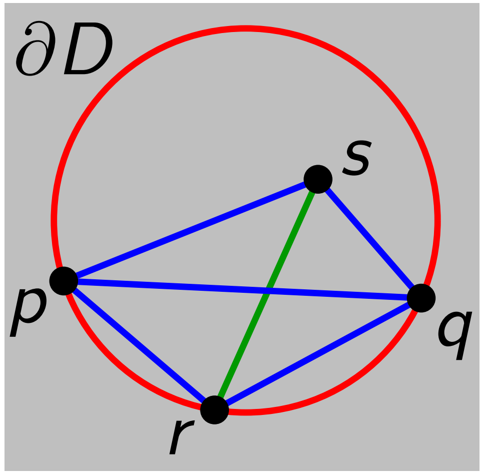

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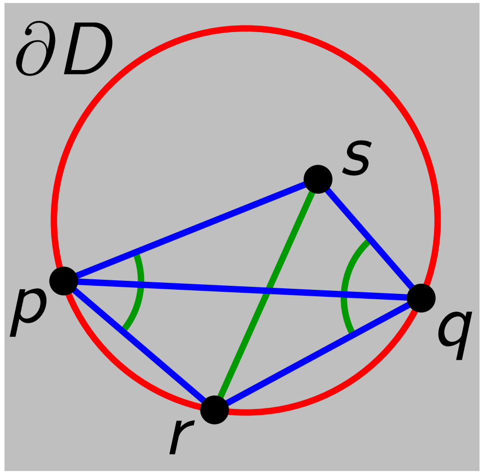

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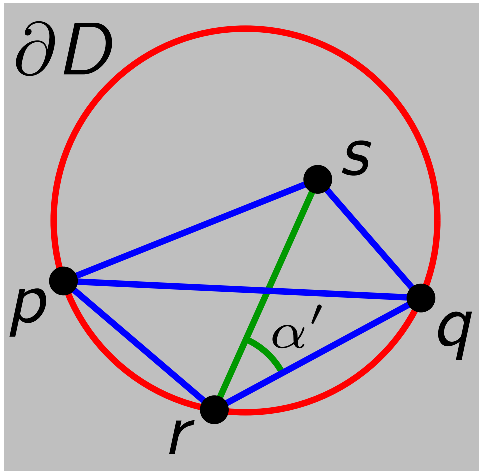

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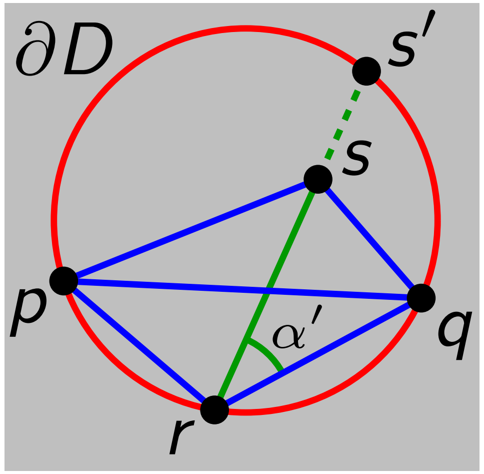

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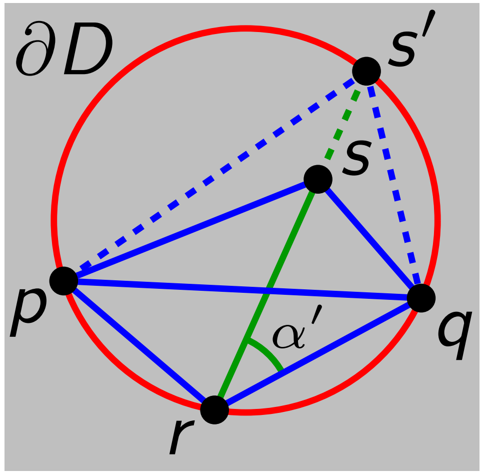

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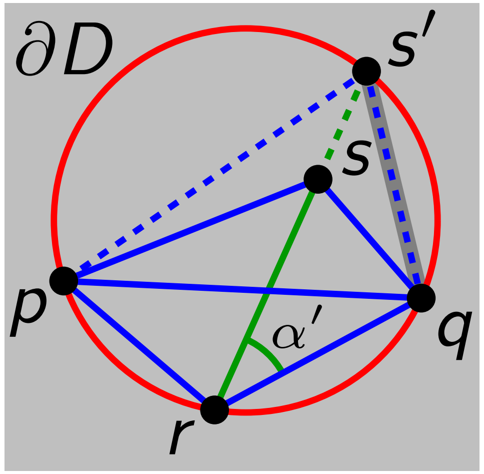

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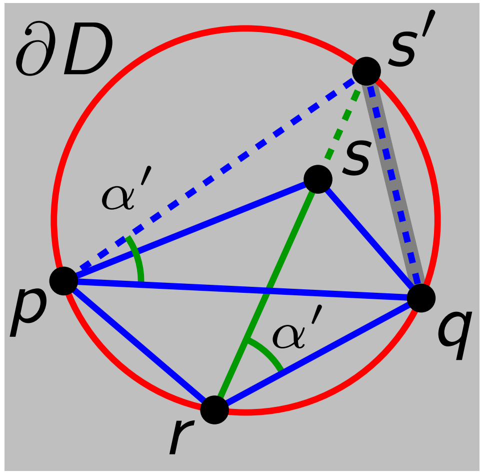

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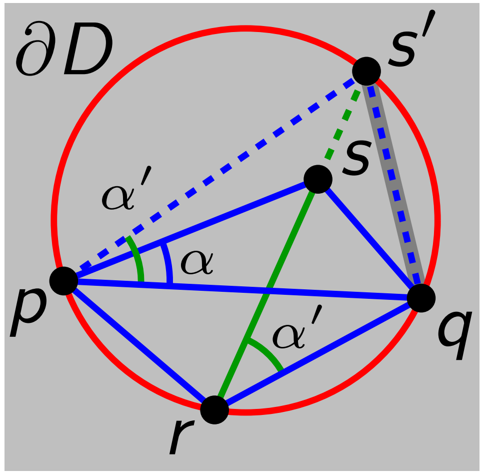

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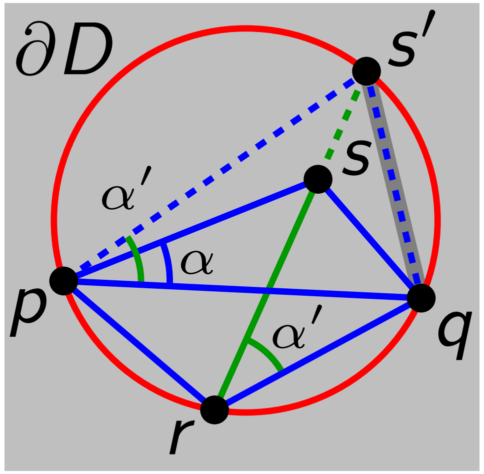

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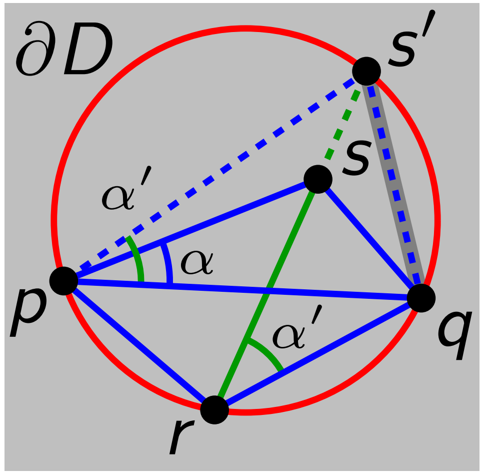

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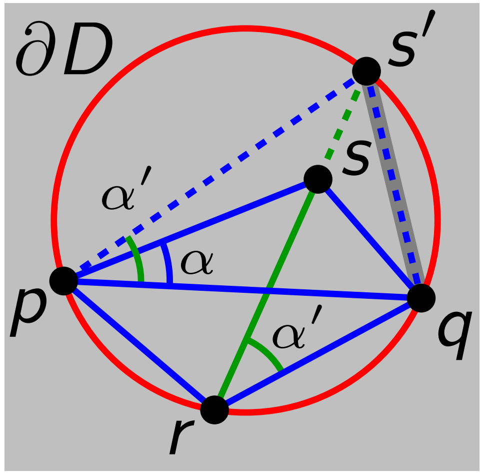

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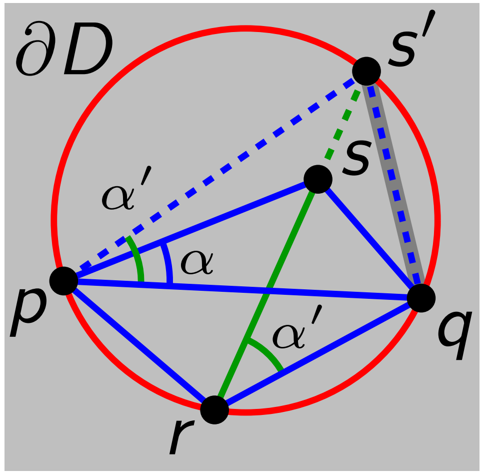

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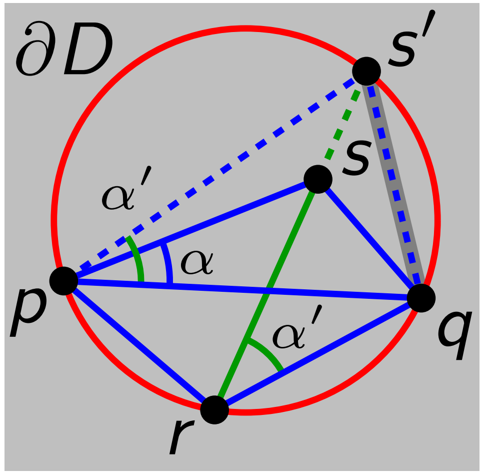

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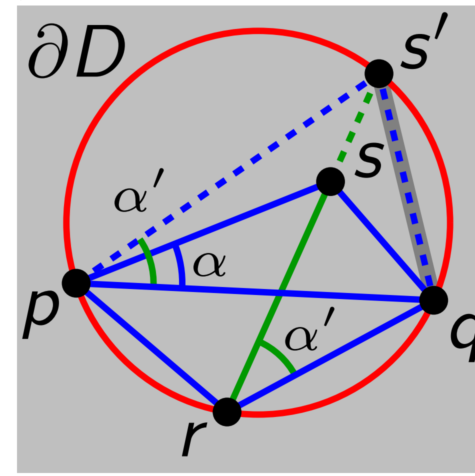

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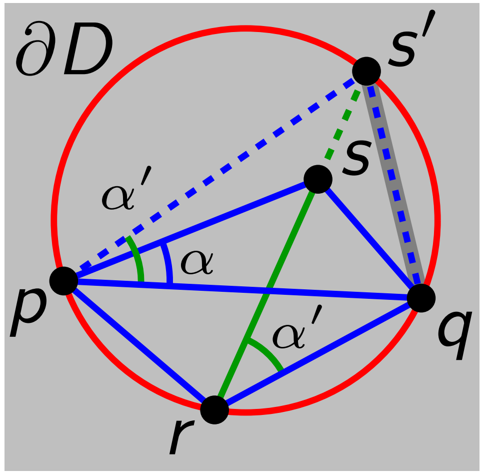

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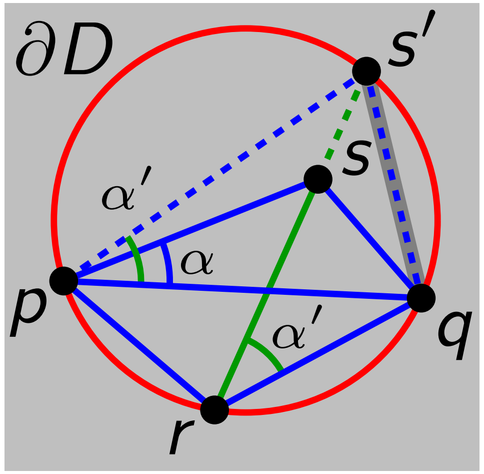

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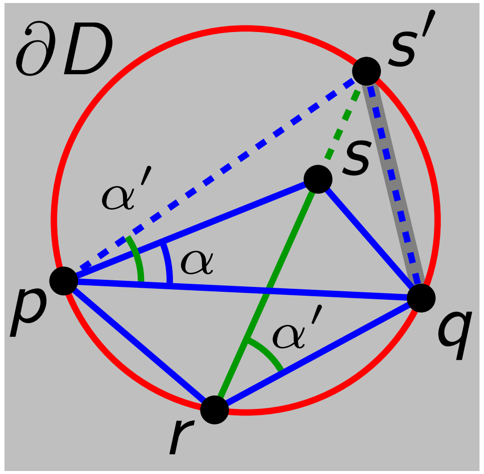

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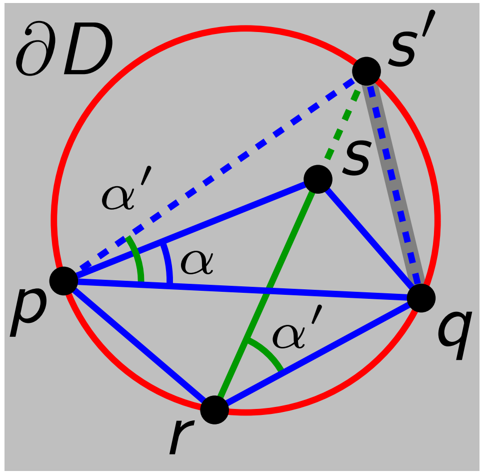

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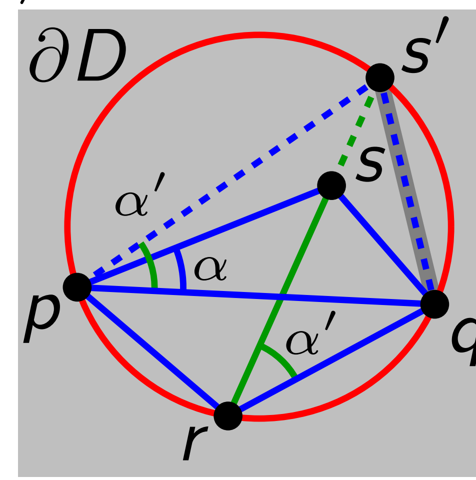

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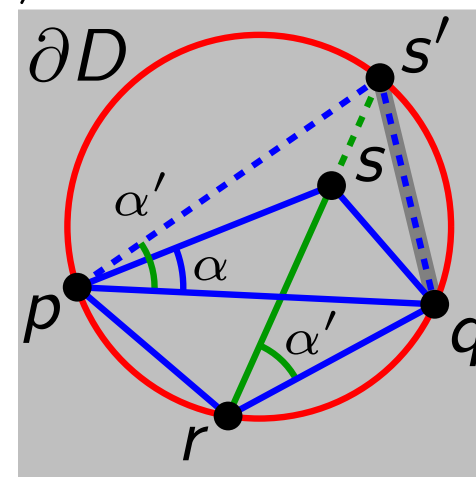

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To clarify things, we'll introduce yet another type of triangulation…

Voronoi & Delaunay

Remember: Given a set P of n points in the plane...

$\text{Vor}(P) =$ subdivision of the plane into Voronoi cells, edges, and vertices

$\mathcal{V}(p) = \{x \in \mathbb{R}^2 \colon |xp| < |xq| \text{ for all } q \in P \setminus \{p\}\}$

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Definition: The *Delaunay graph* $\mathcal{DG}(P)$ is the straight-line drawing of $\mathcal{G}$.

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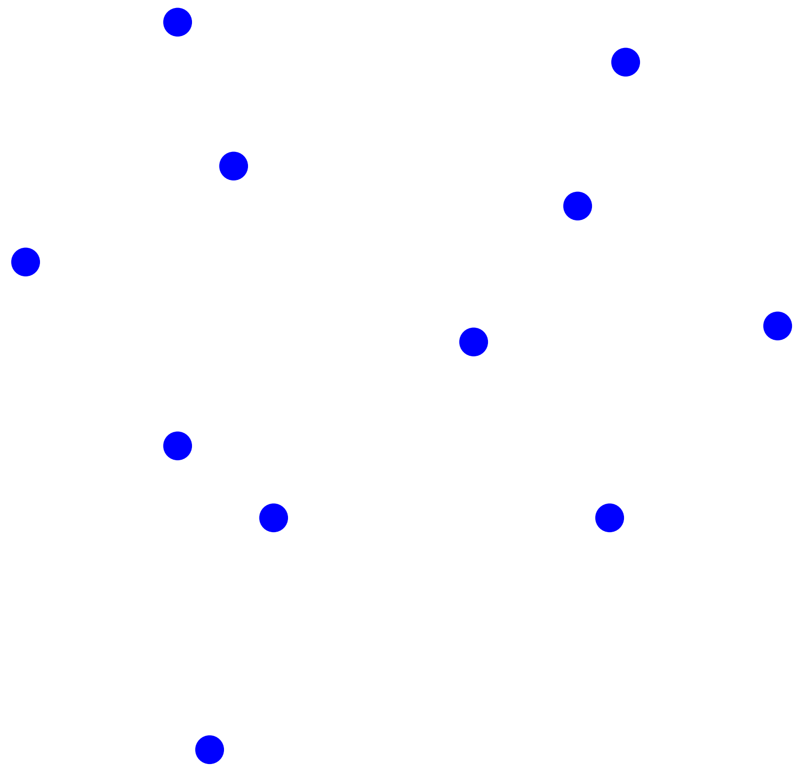

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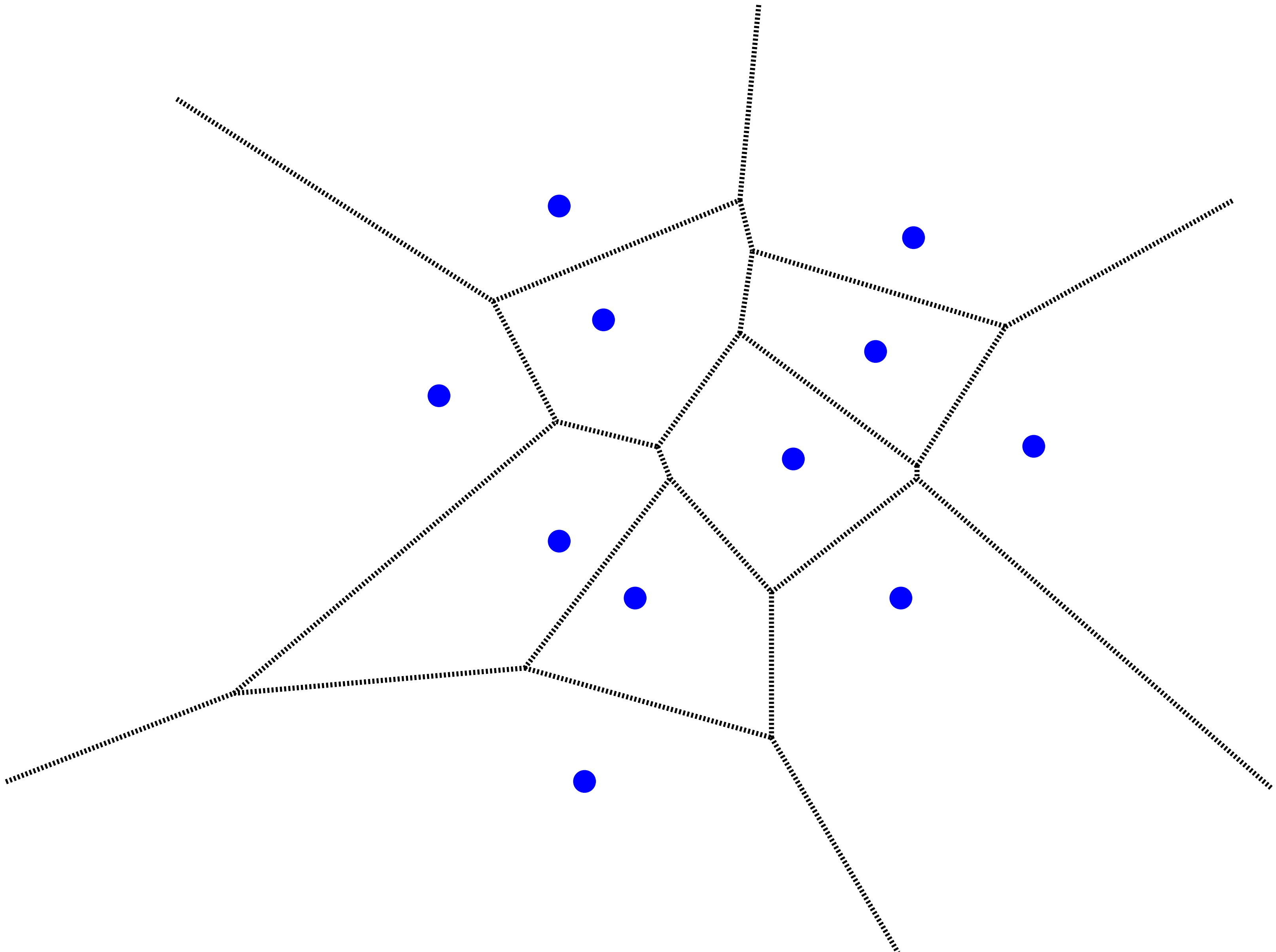

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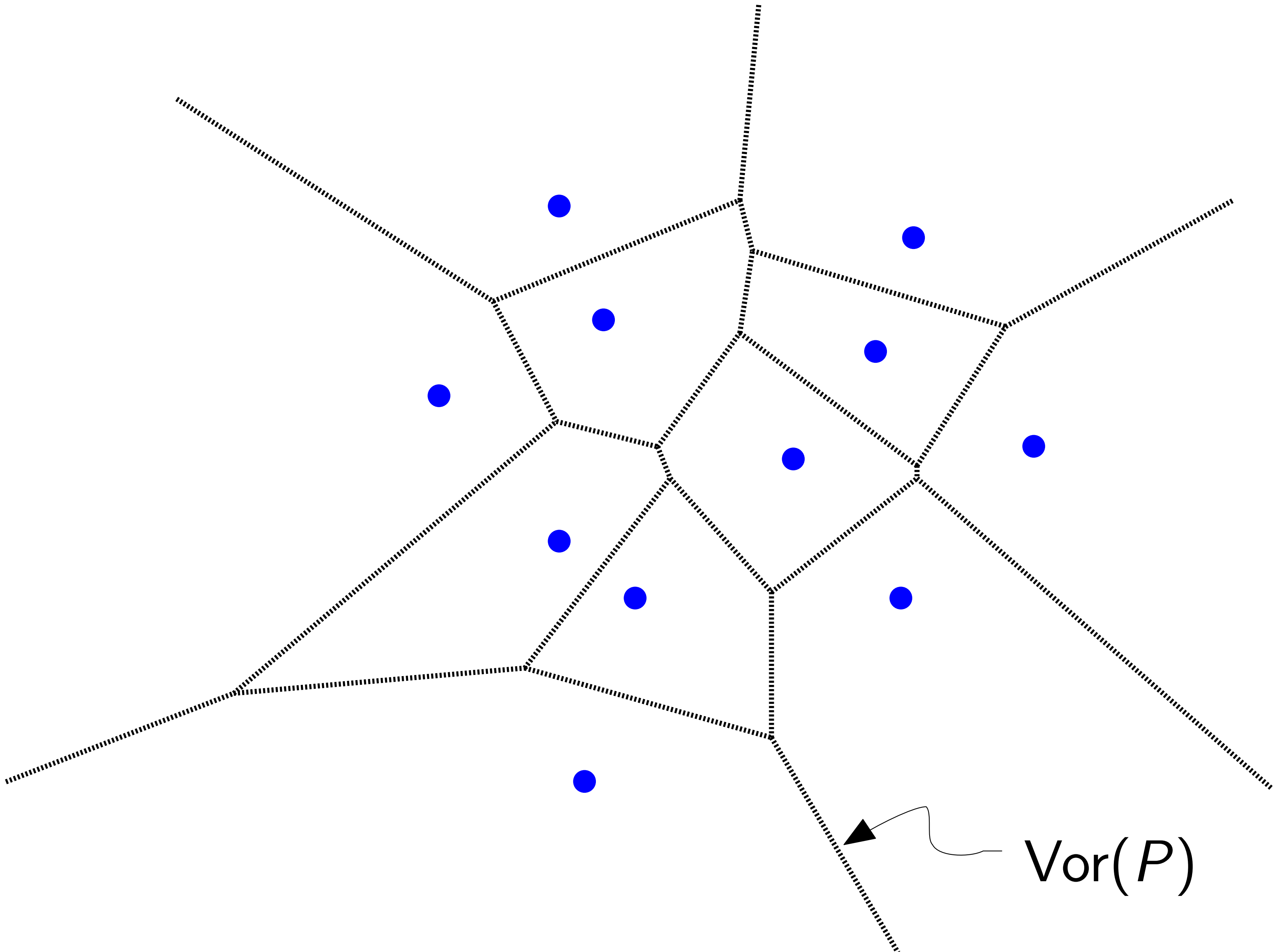

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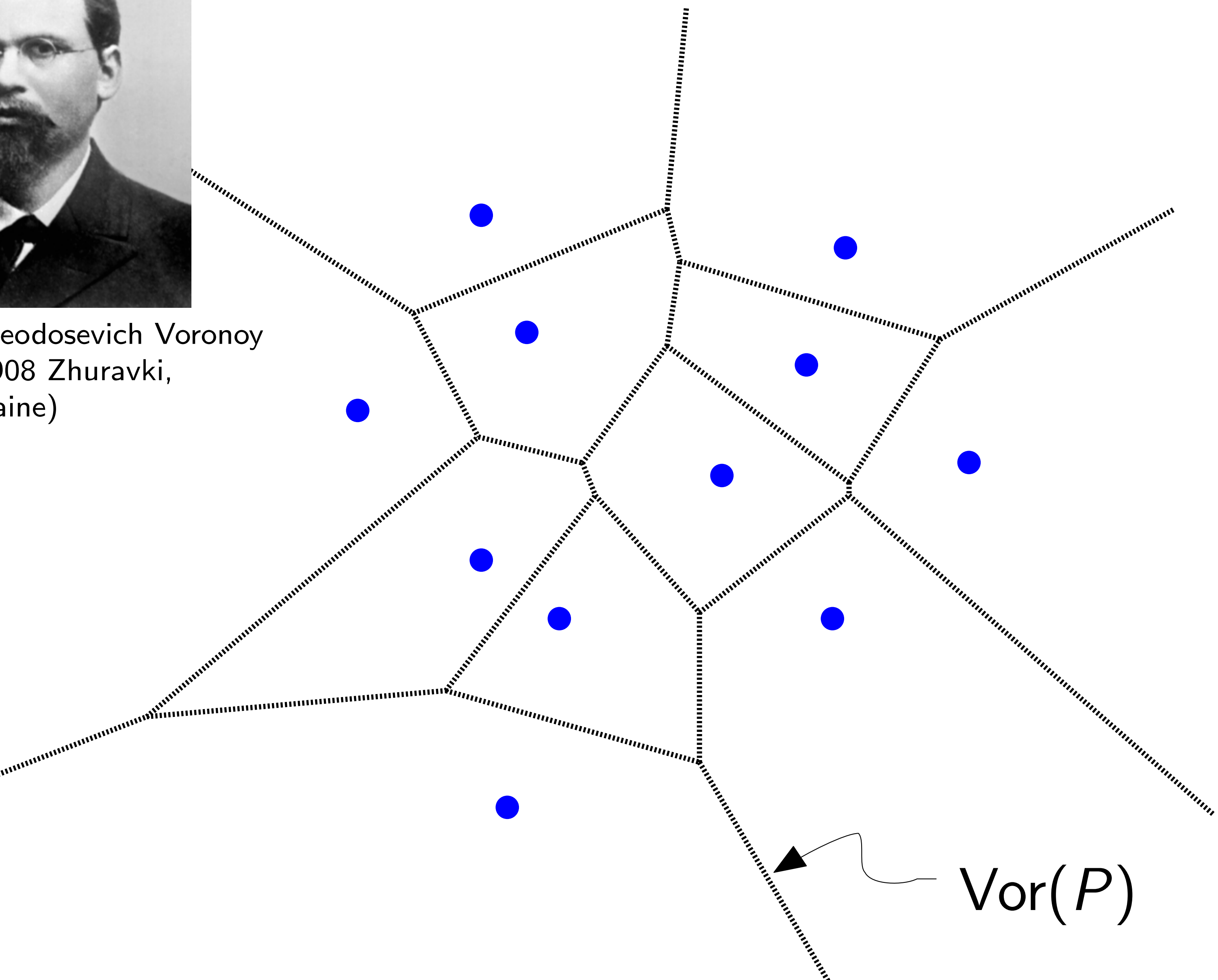

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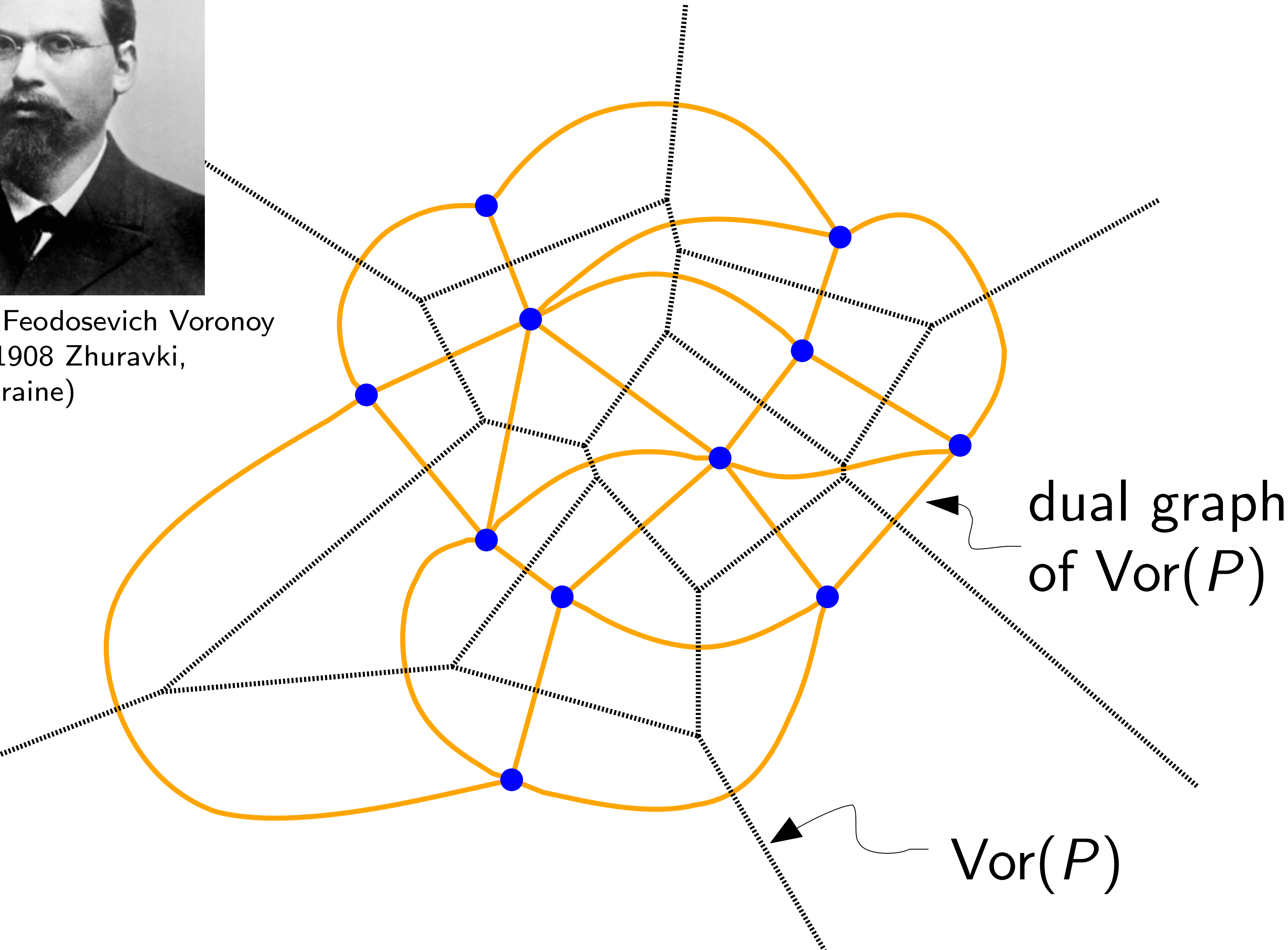

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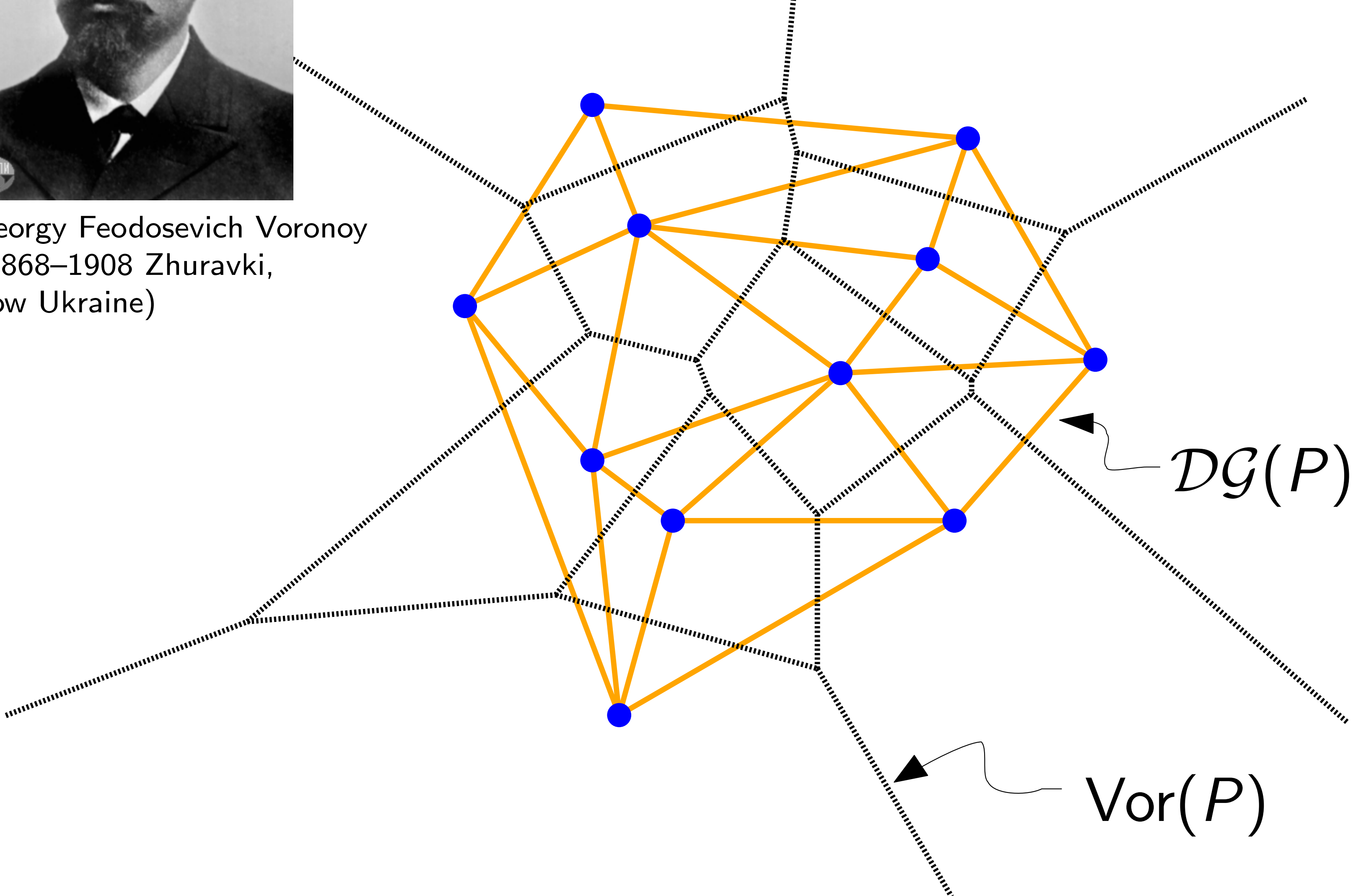

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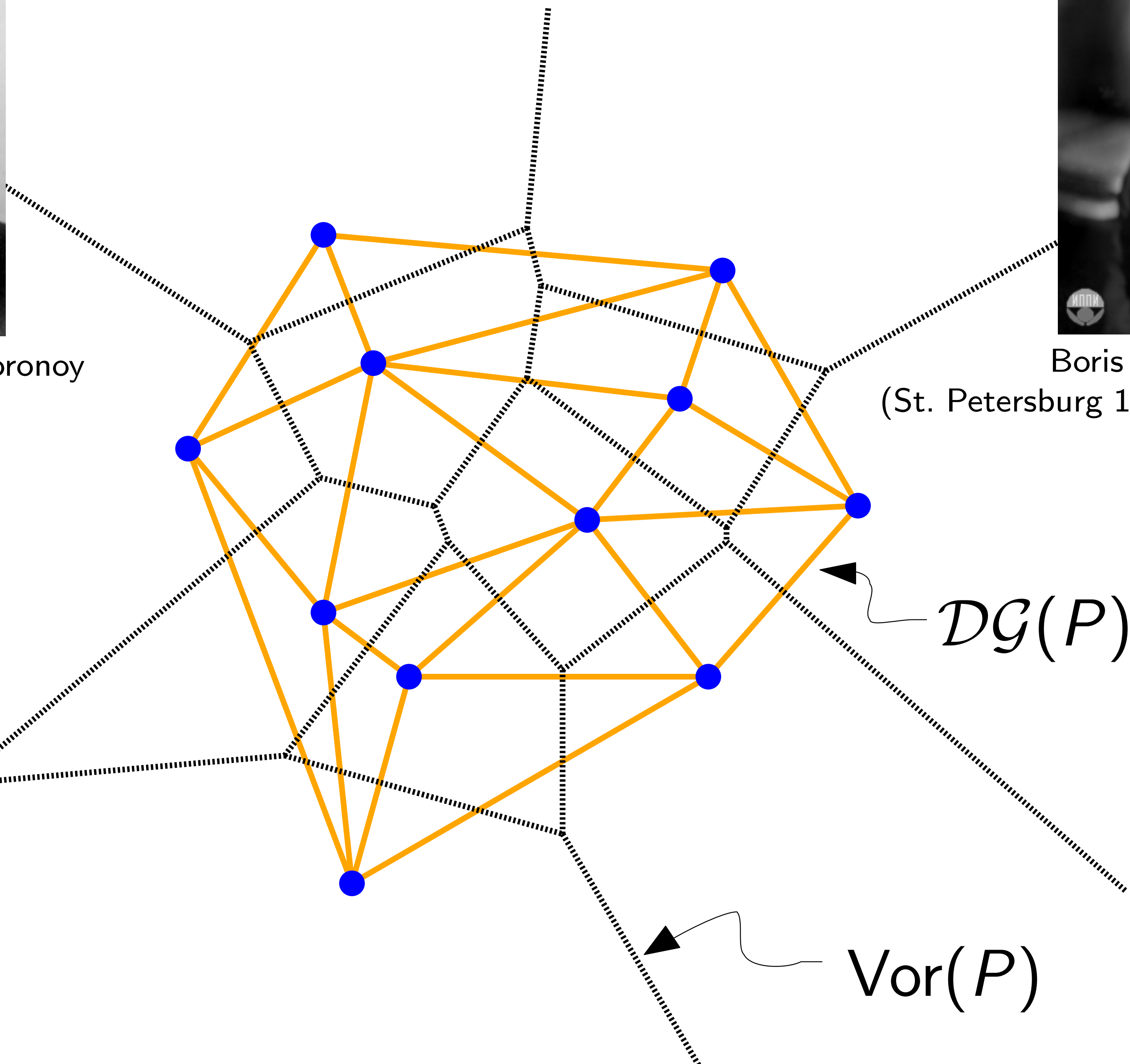

Boris Nikolaevich Delone
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Theorem. $P \subset \mathbb{R}^2$ finite $\Rightarrow$ $\mathcal{DG}(P)$ plane.

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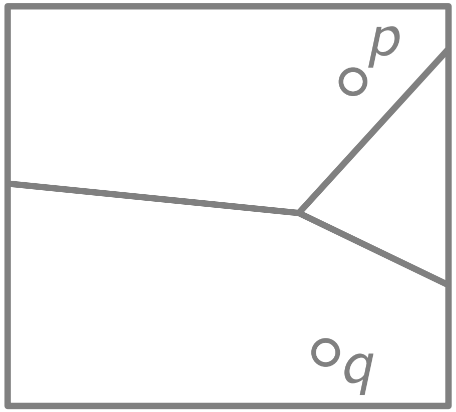

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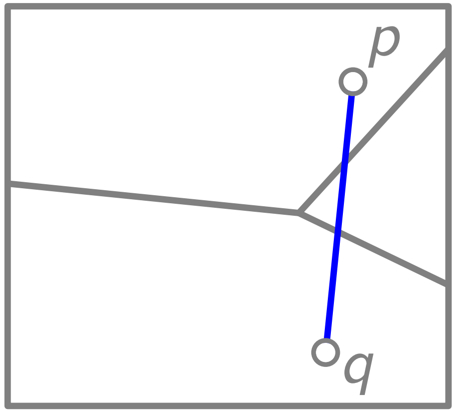

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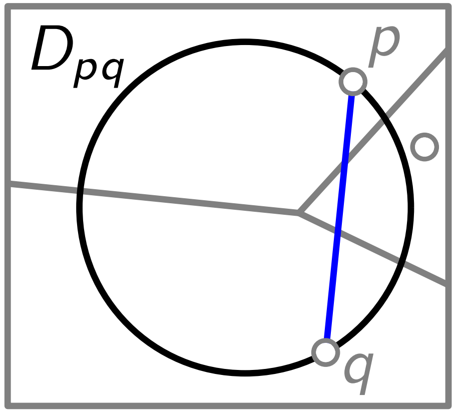

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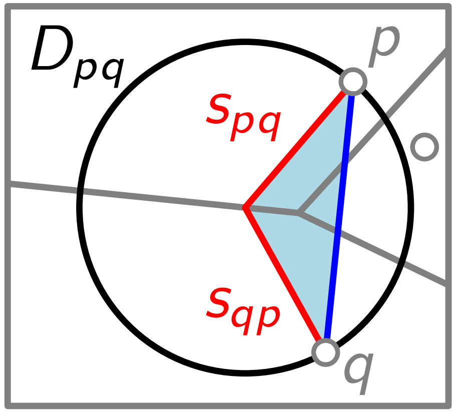

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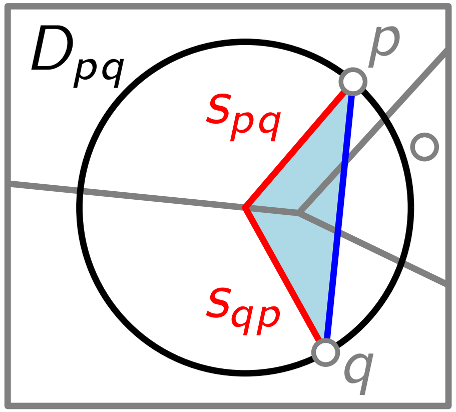

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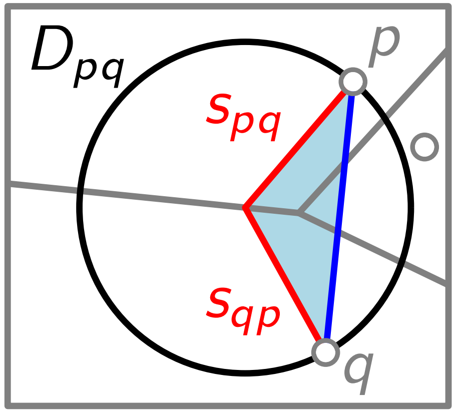

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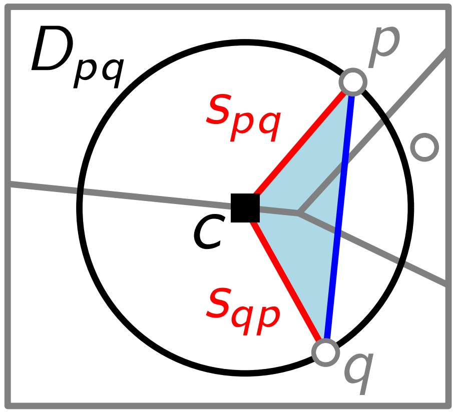

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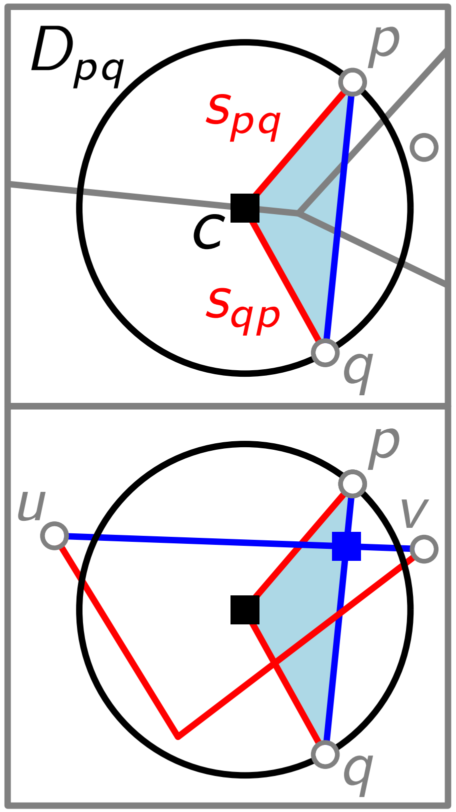

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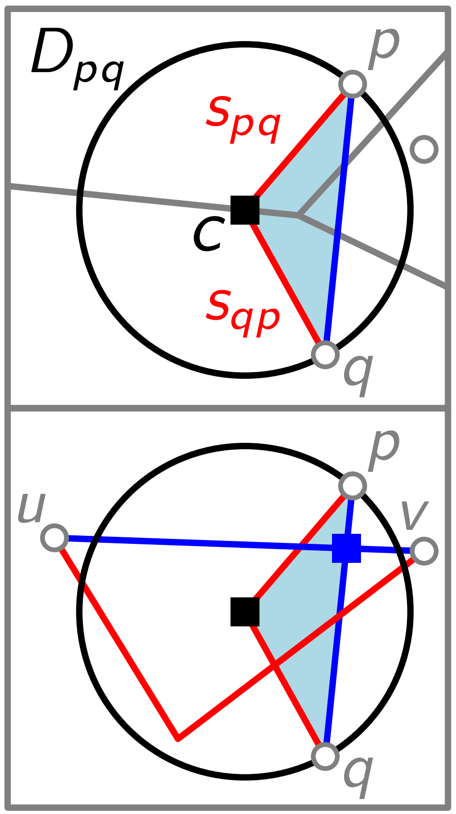

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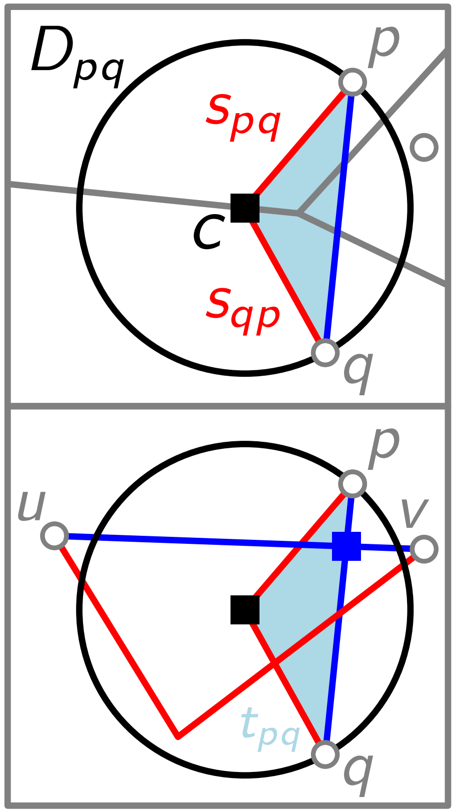

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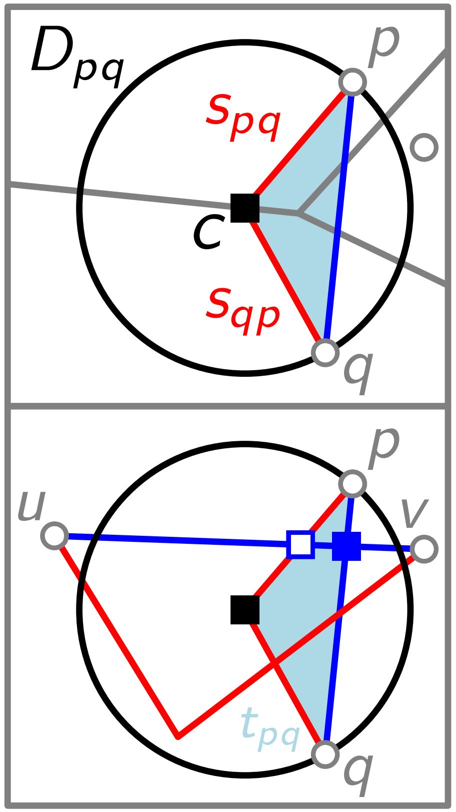

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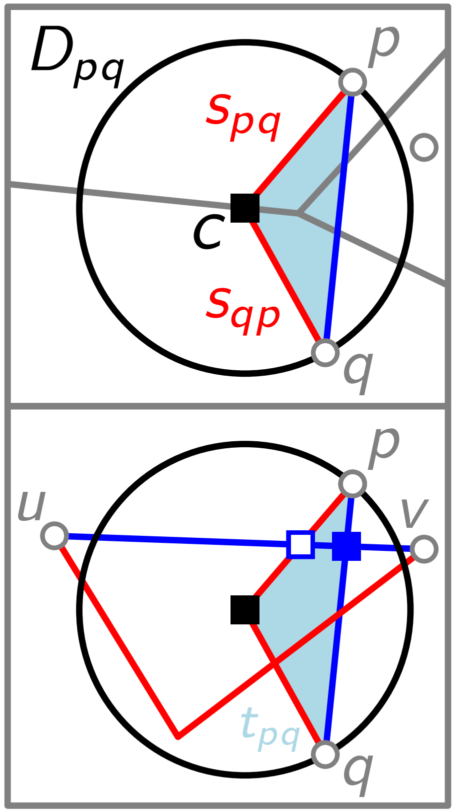

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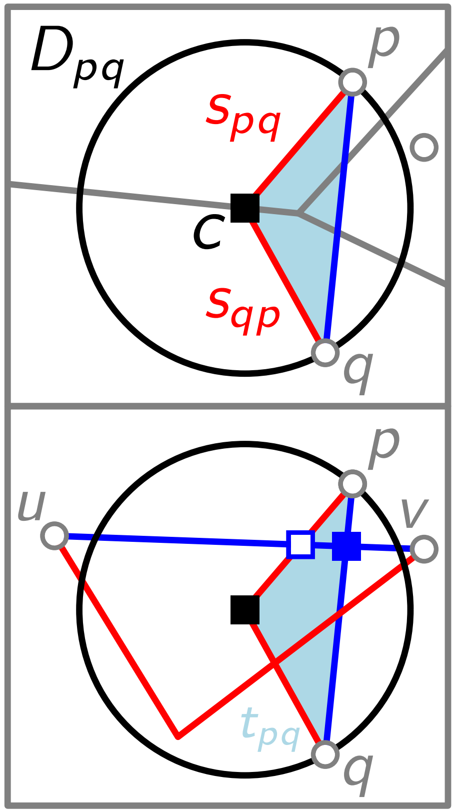

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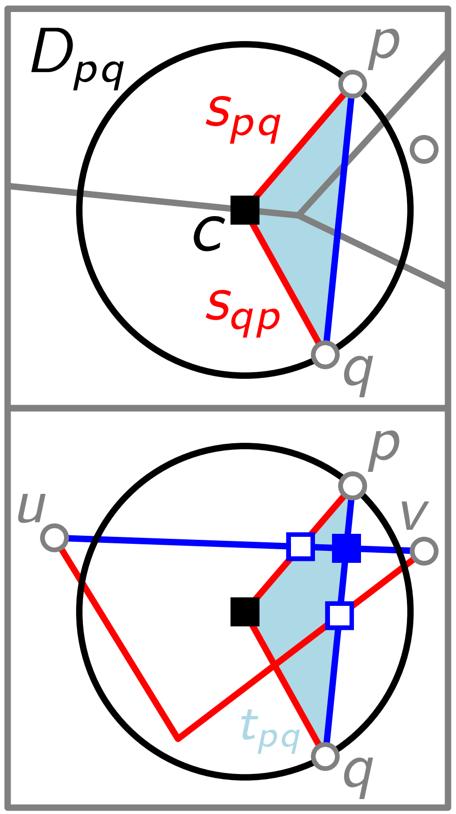

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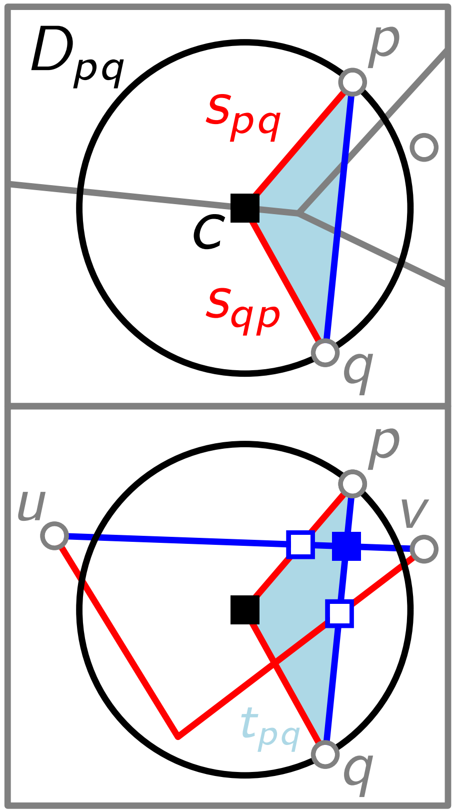

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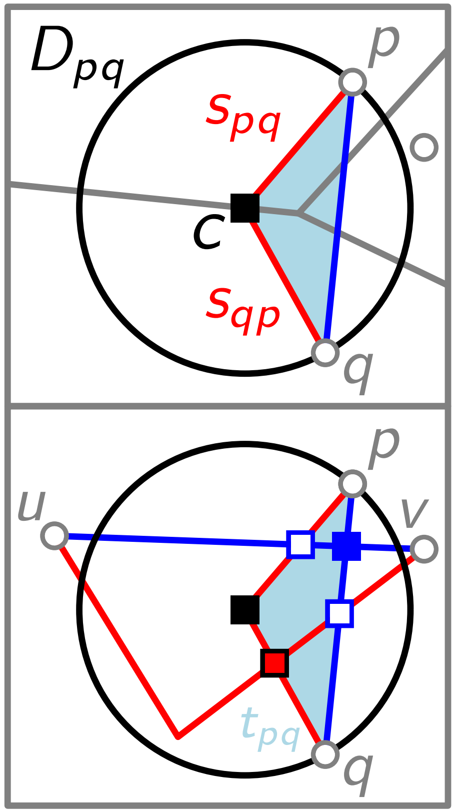

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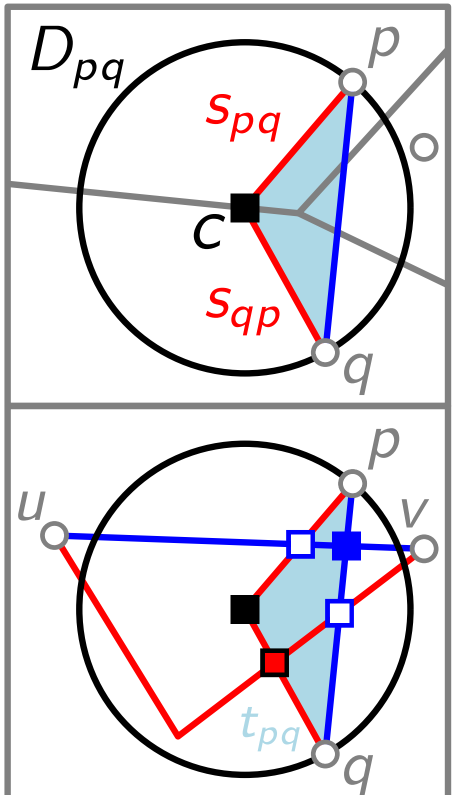

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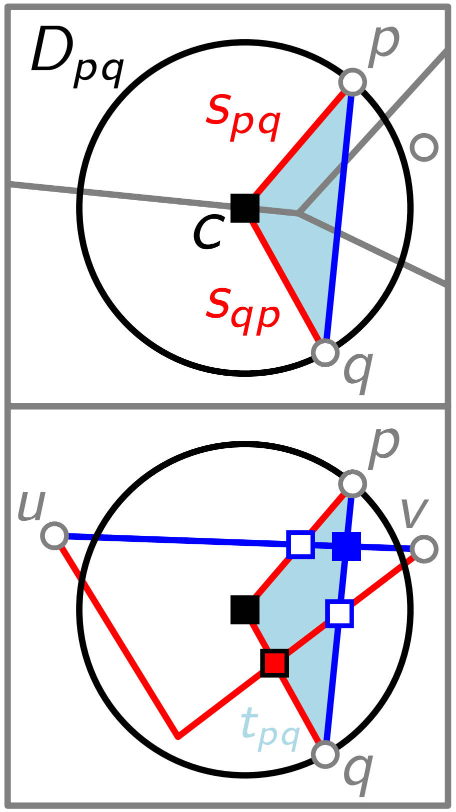

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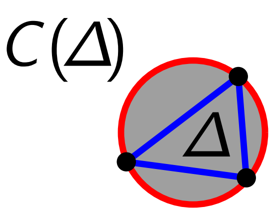

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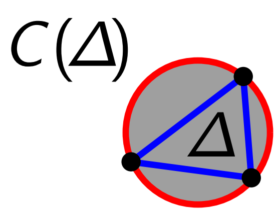

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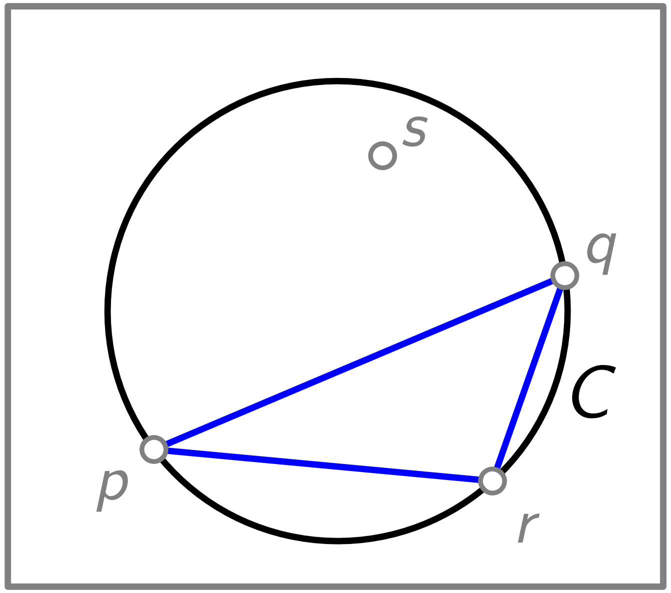

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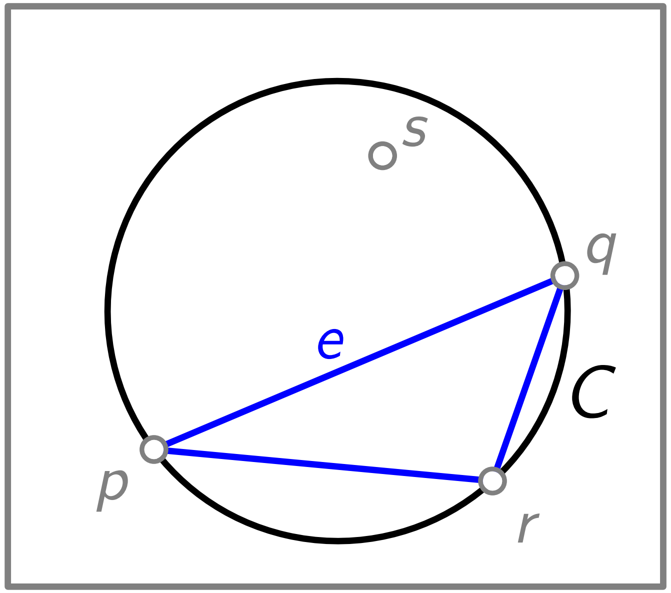

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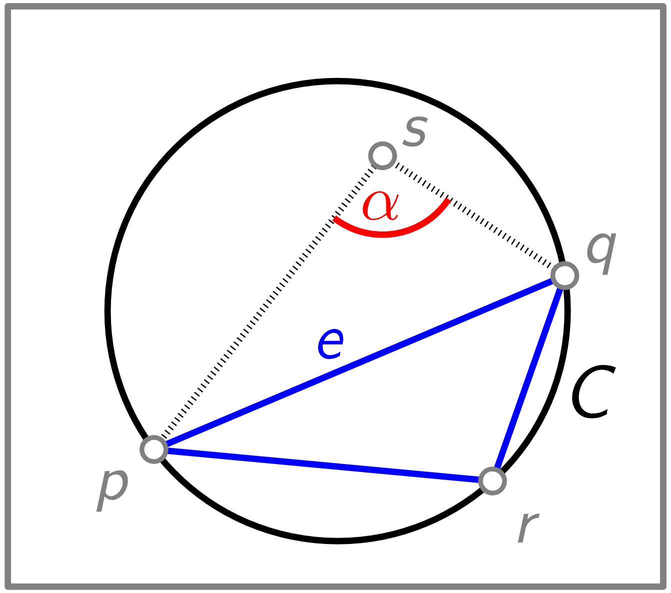

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Among all such pairs $(\Delta pqr, s)$ in $\mathcal{T}$ choose one that maximizes $\alpha = \angle psq$.

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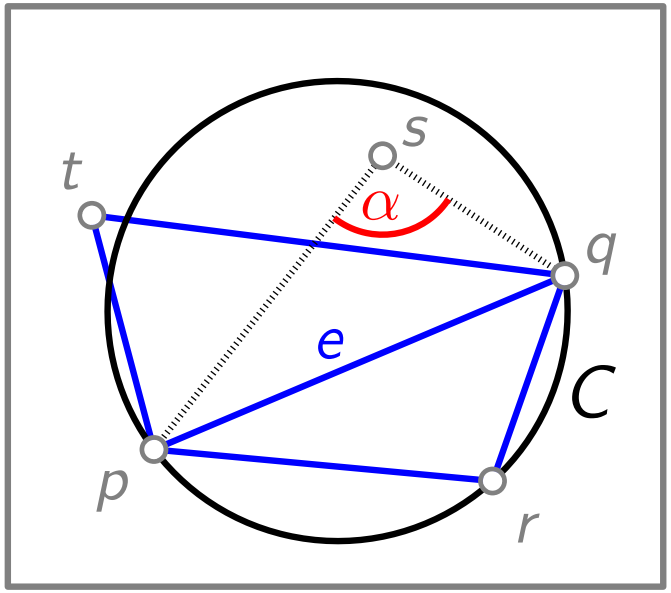

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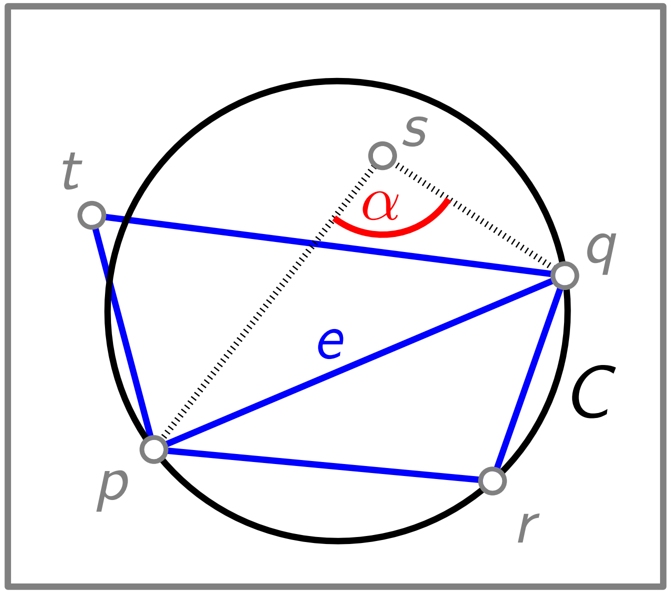

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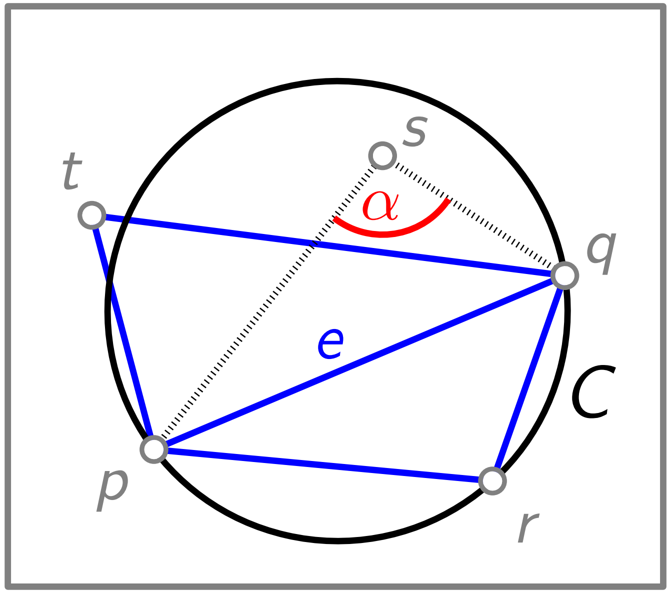

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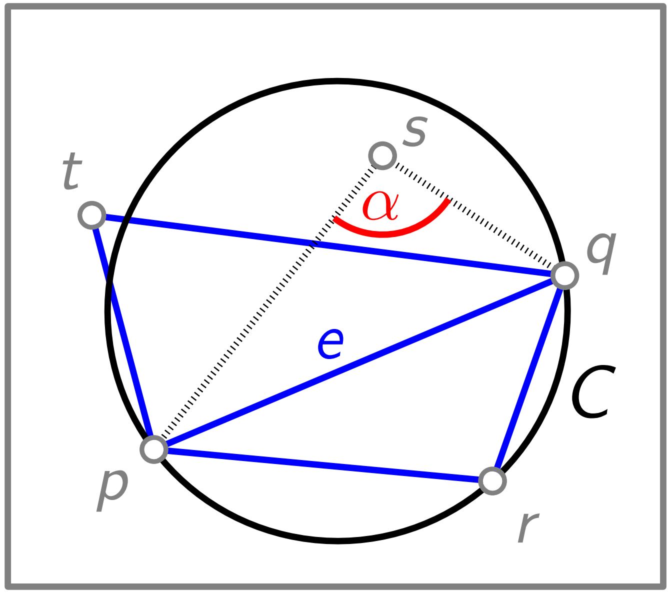

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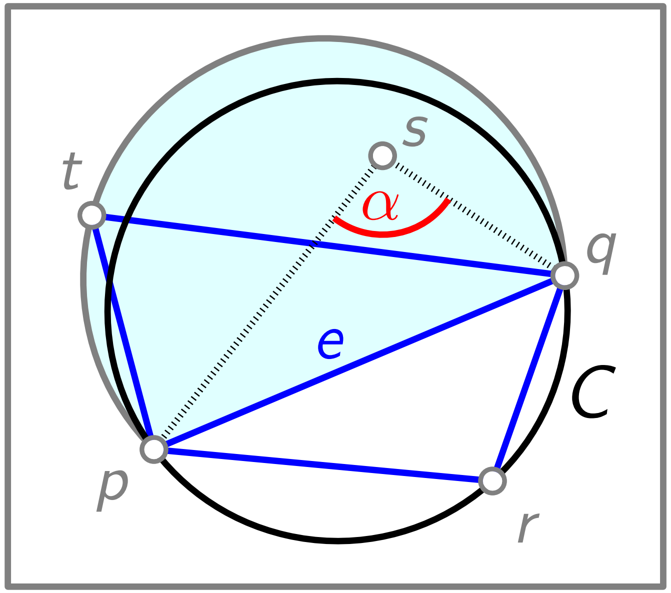

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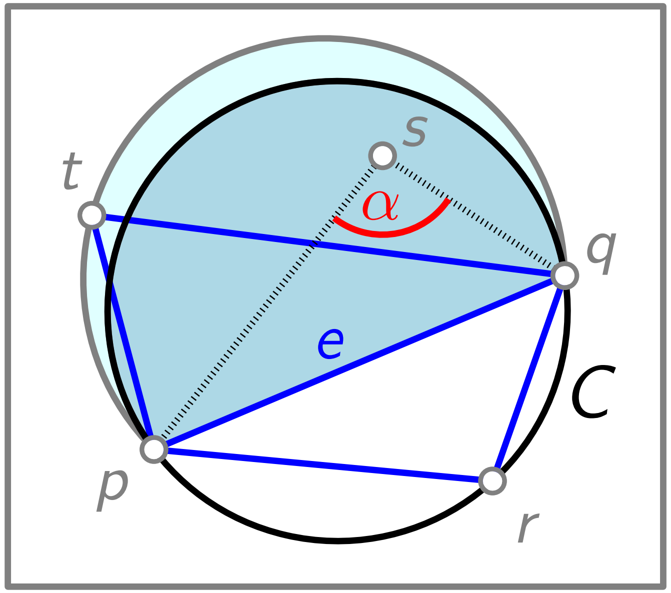

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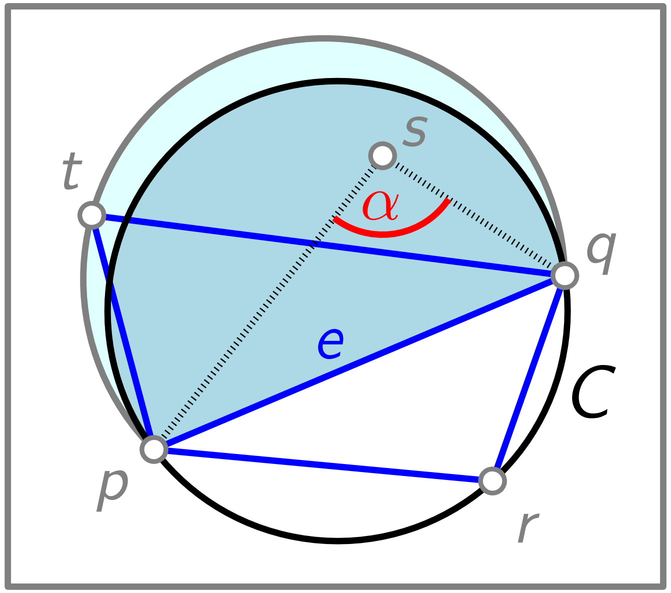

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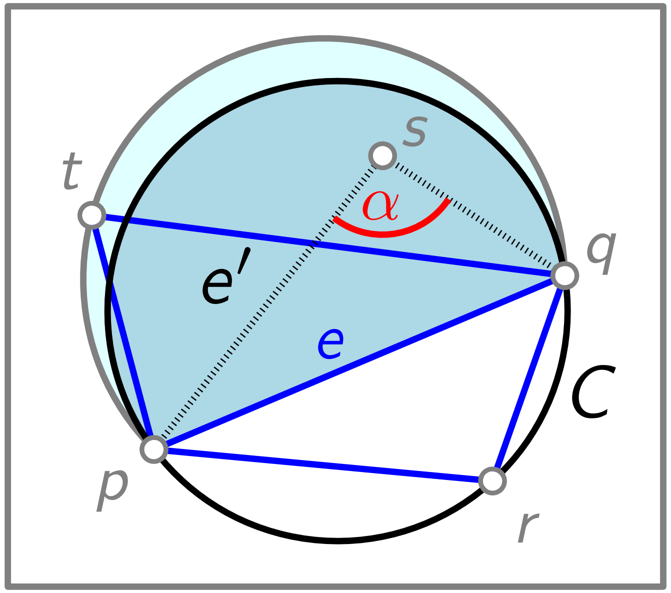

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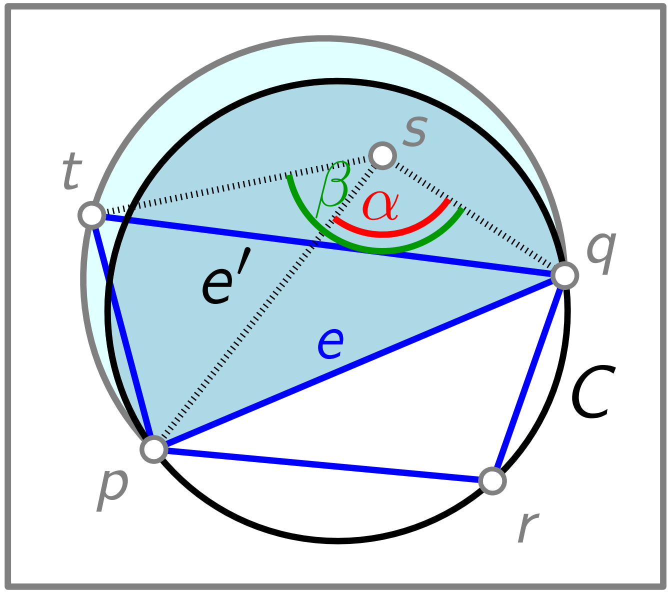

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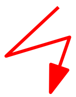

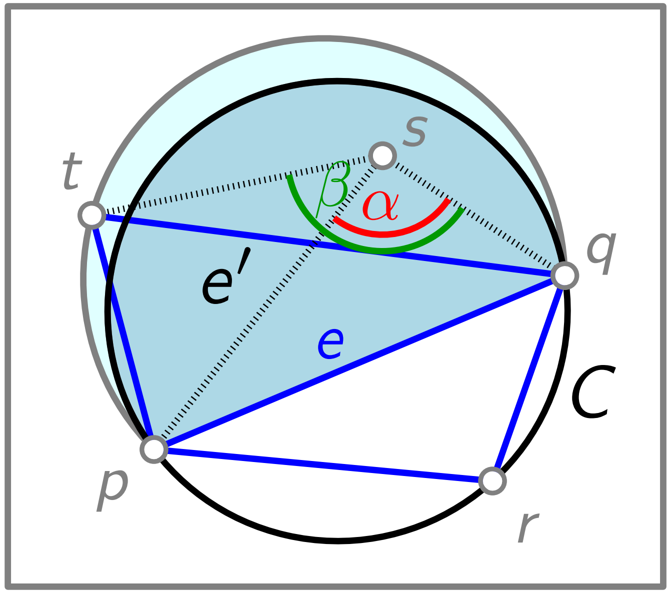

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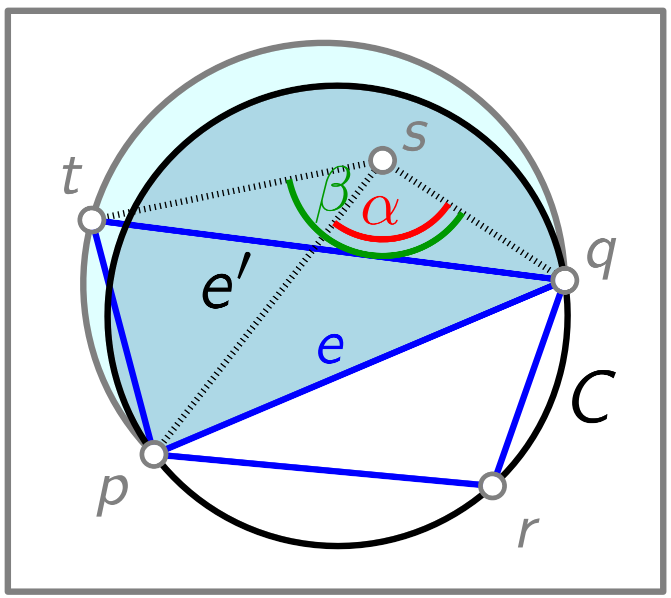
$\Rightarrow$ $C(\Delta pqt)$ contains $C(\Delta pqr) \cap e^+$. { halfplane supported by e that contains s

$\Rightarrow s \in C(\Delta pqt)$

Wlog. let $e' = qt$ be the edge of Δpqt that s sees.

$\Rightarrow \beta = \angle tsq \quad > \quad \alpha = \angle psq$



↯ Contradiction to choice of the pair $(\Delta pqr, s)$. □

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All Delaunay triang. have same min. angle.

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