Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Computational Geometry

Winter term 2016/17

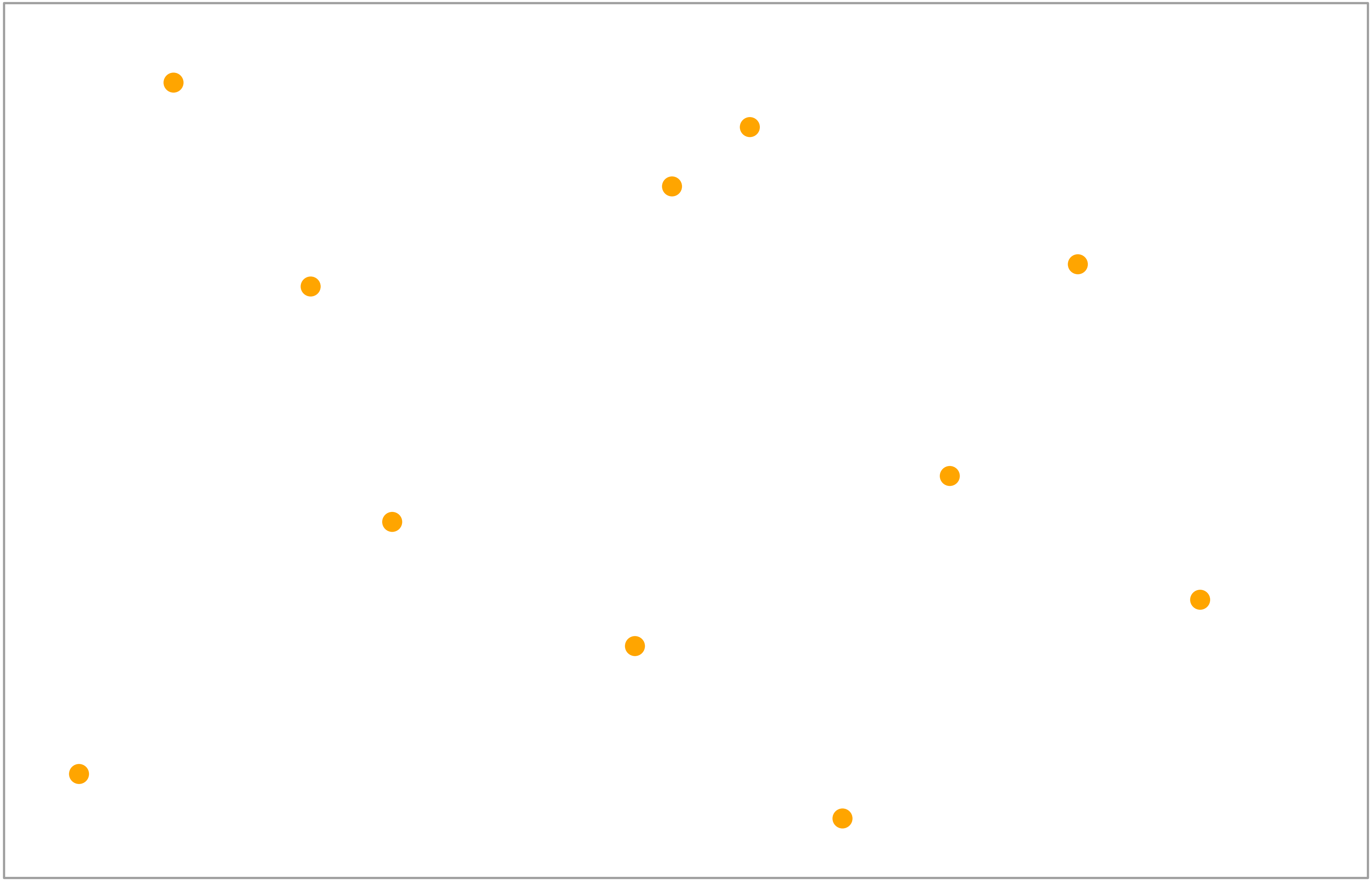
The Post-Office Problem

Lecture #7

Joachim Spoerhase

Chair for Computer Science I

The Post-Office Problem



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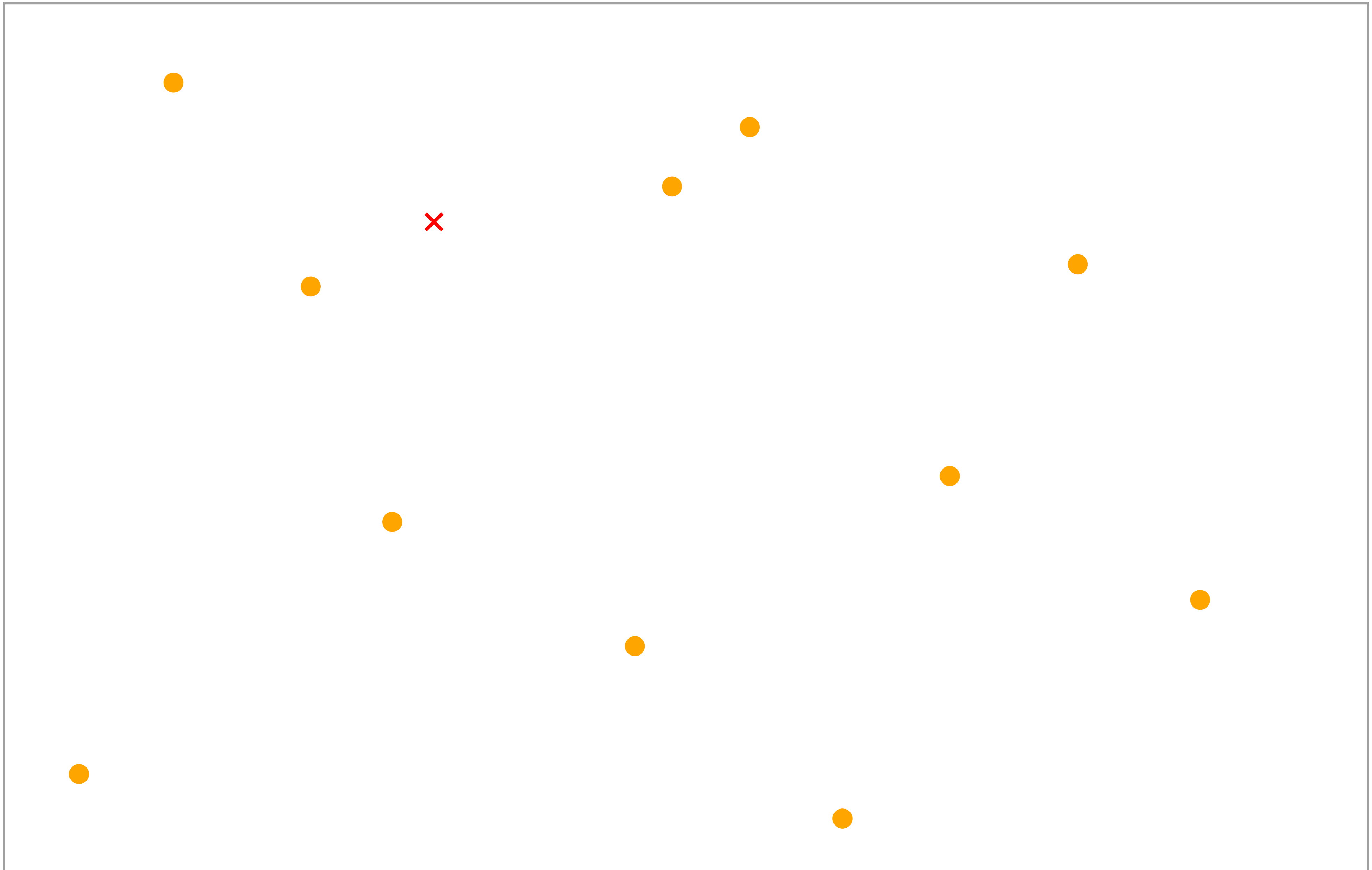

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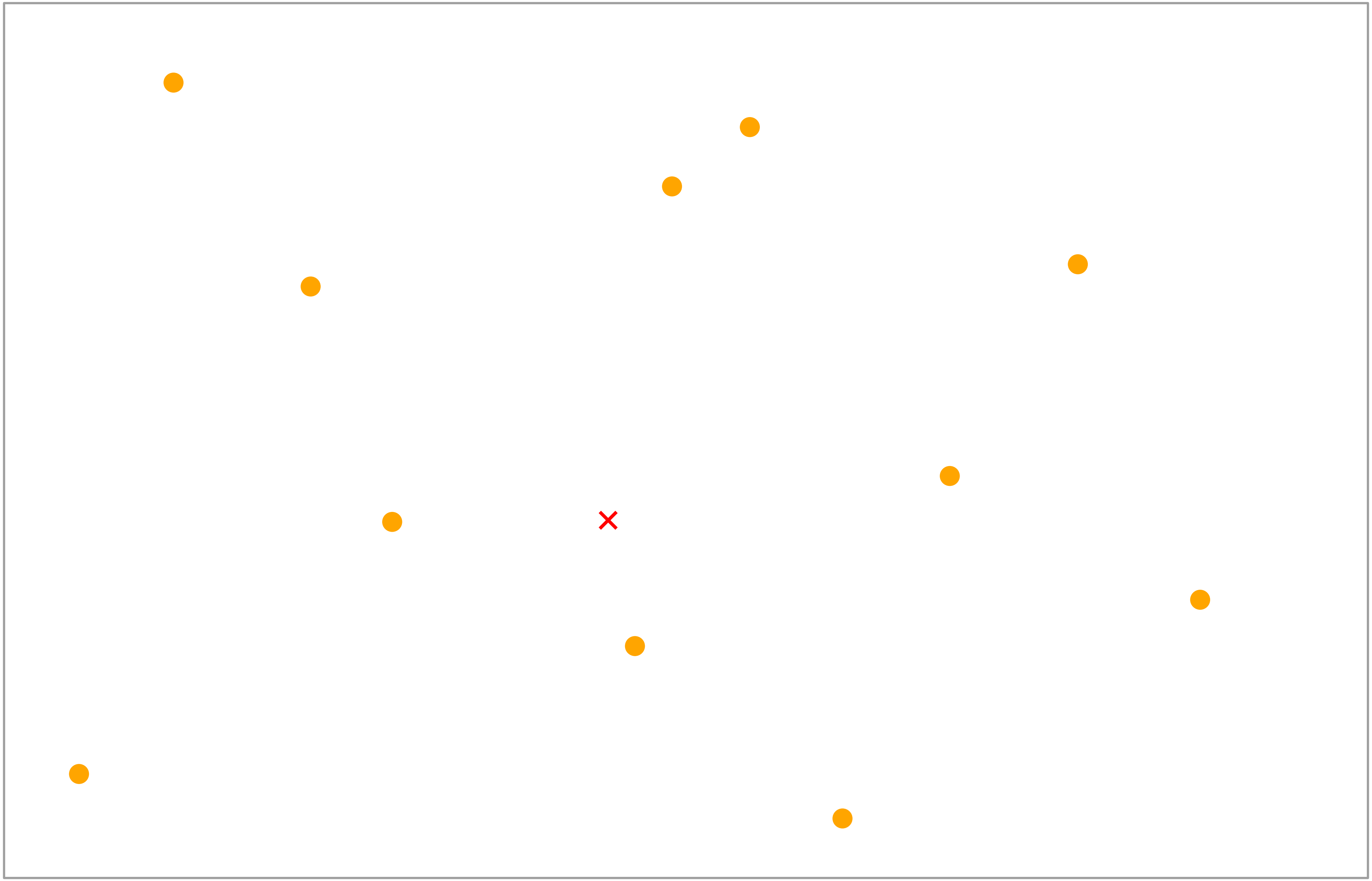

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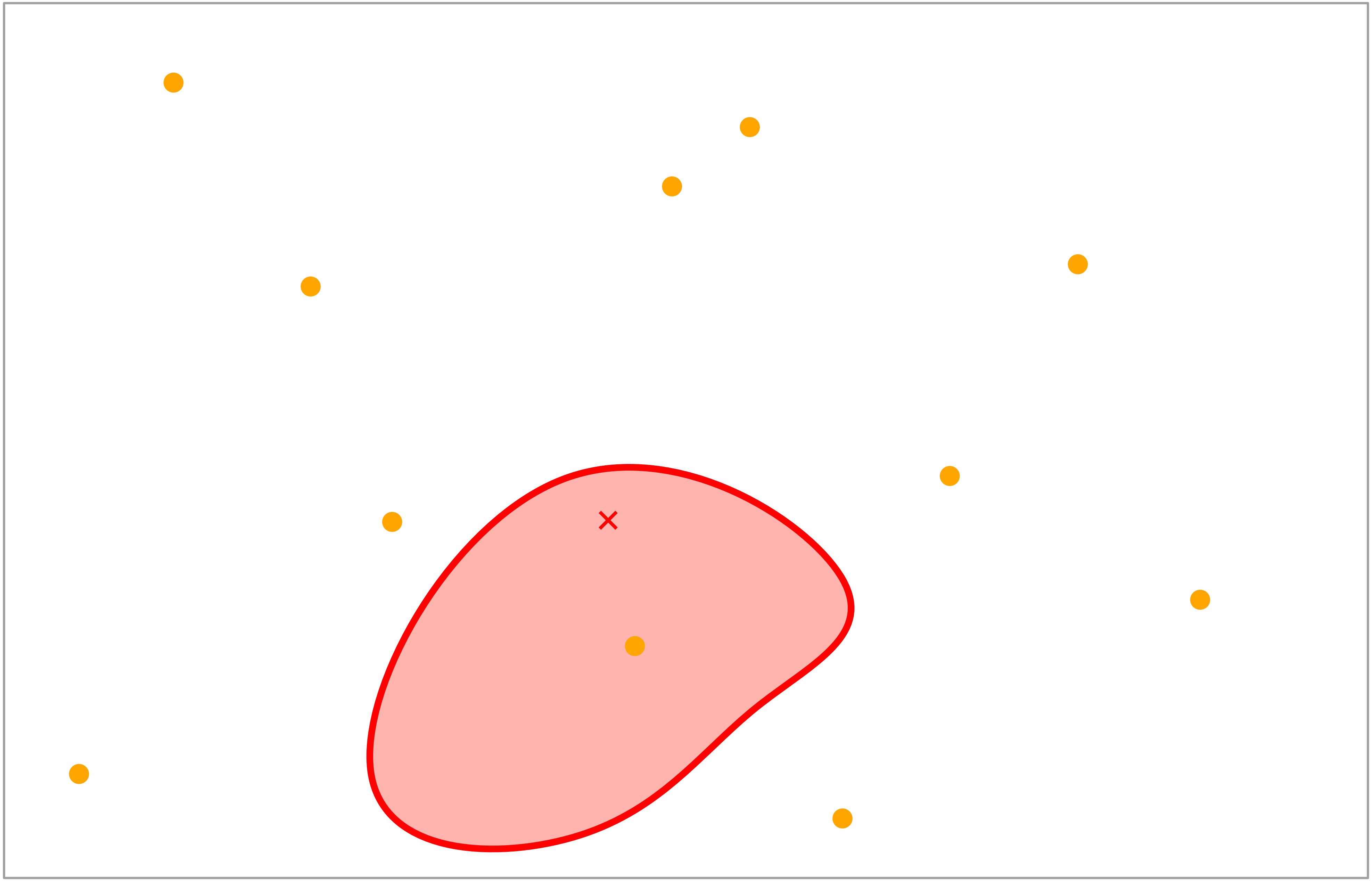

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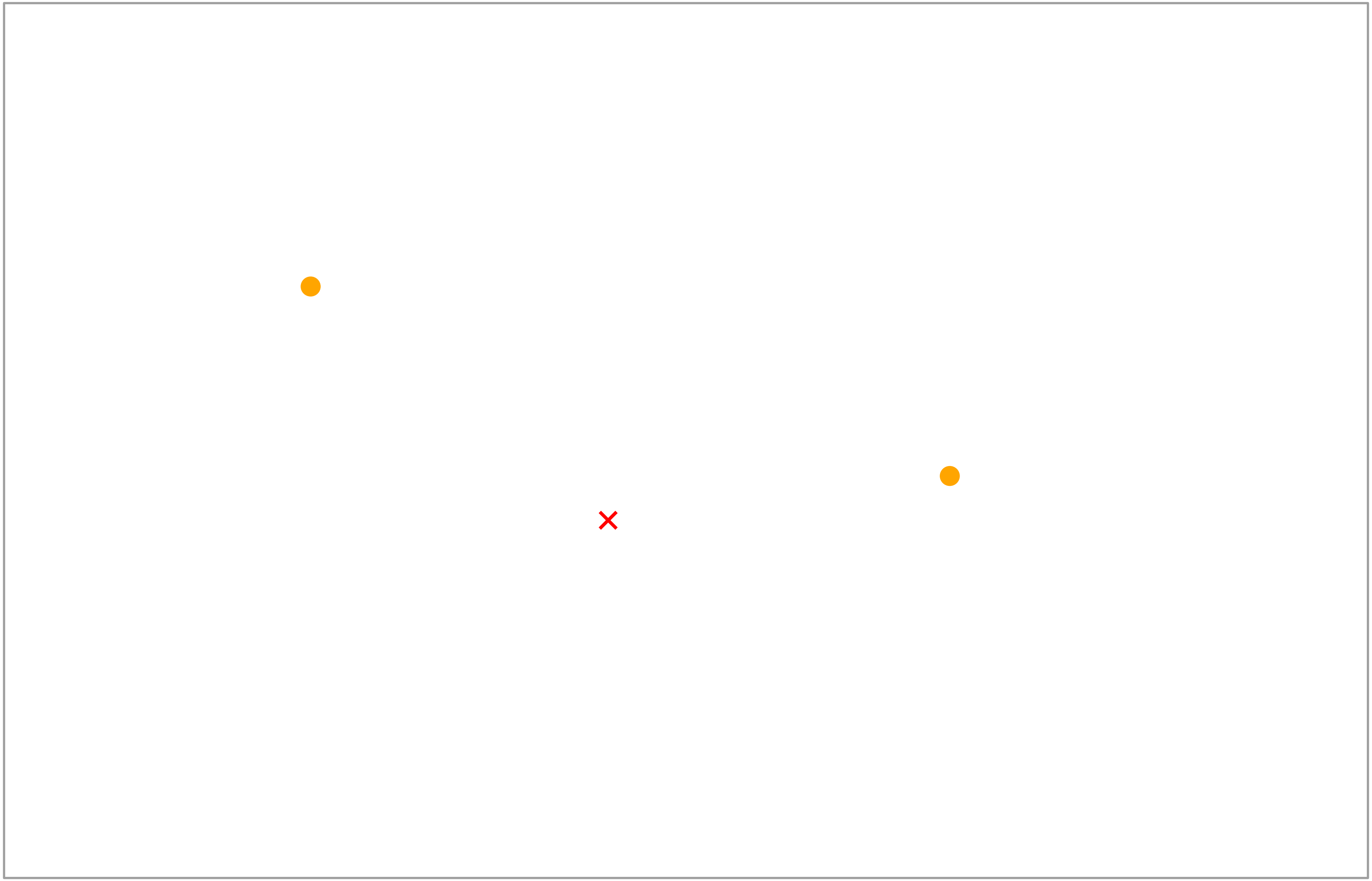

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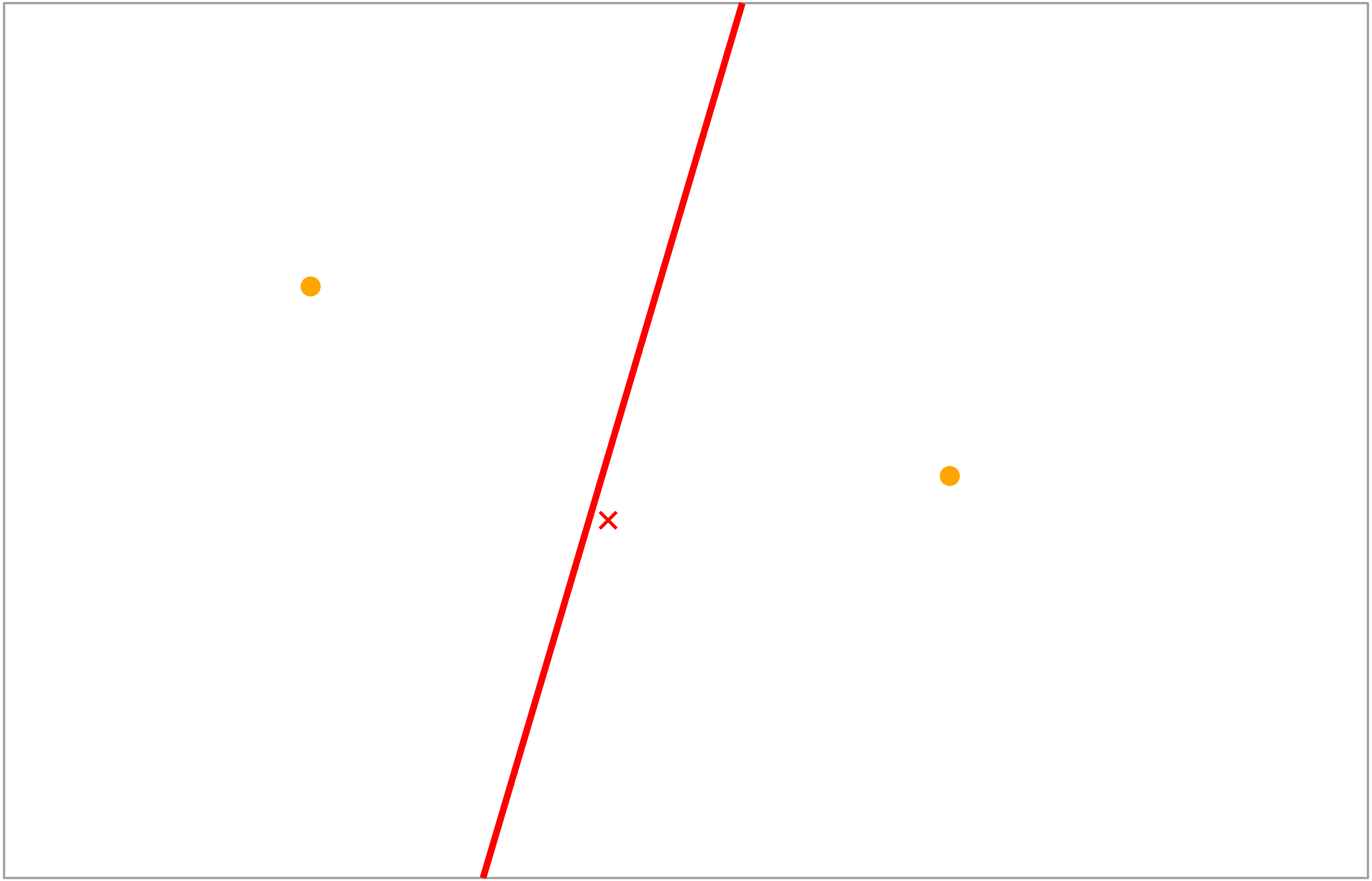

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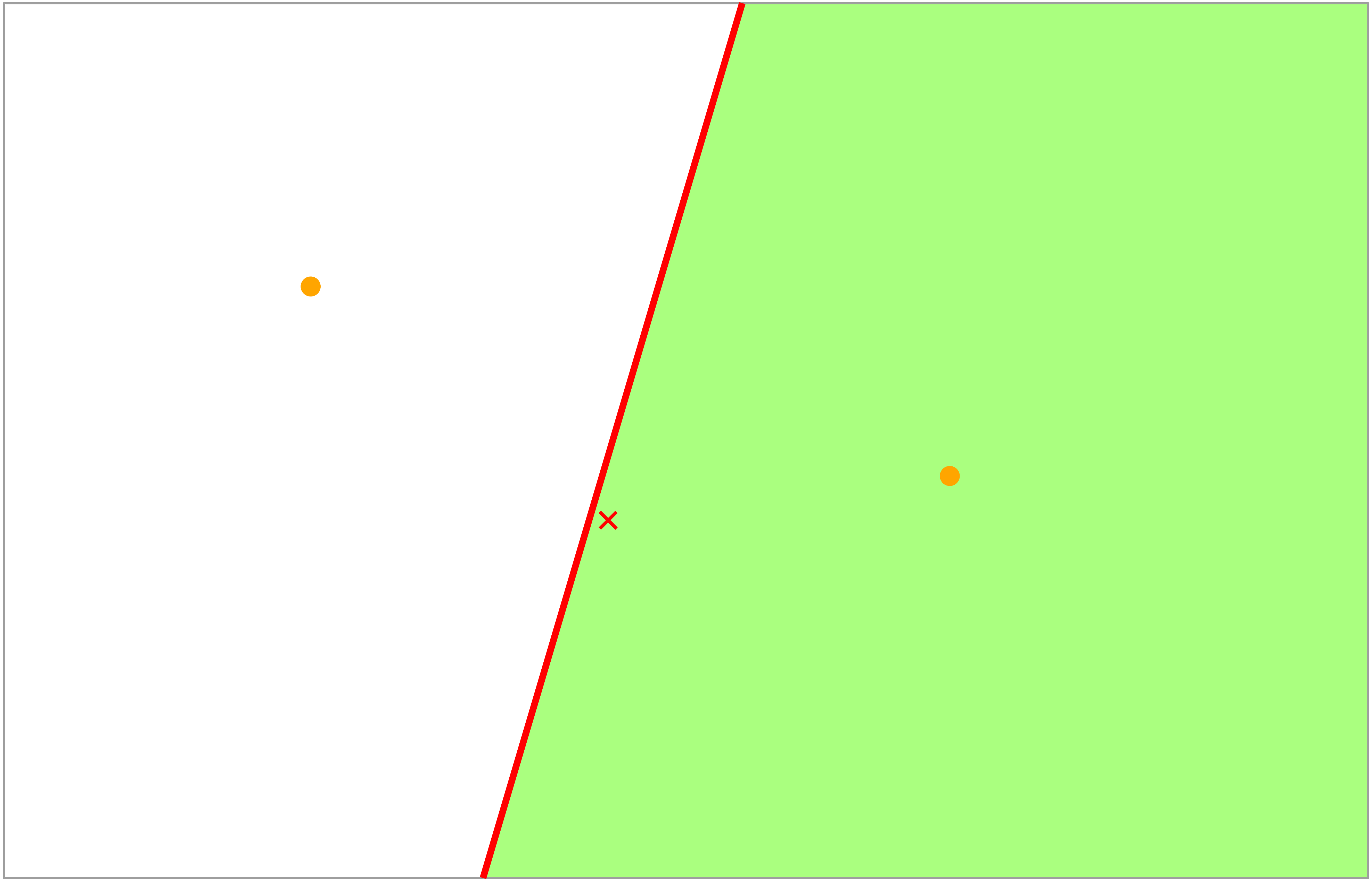

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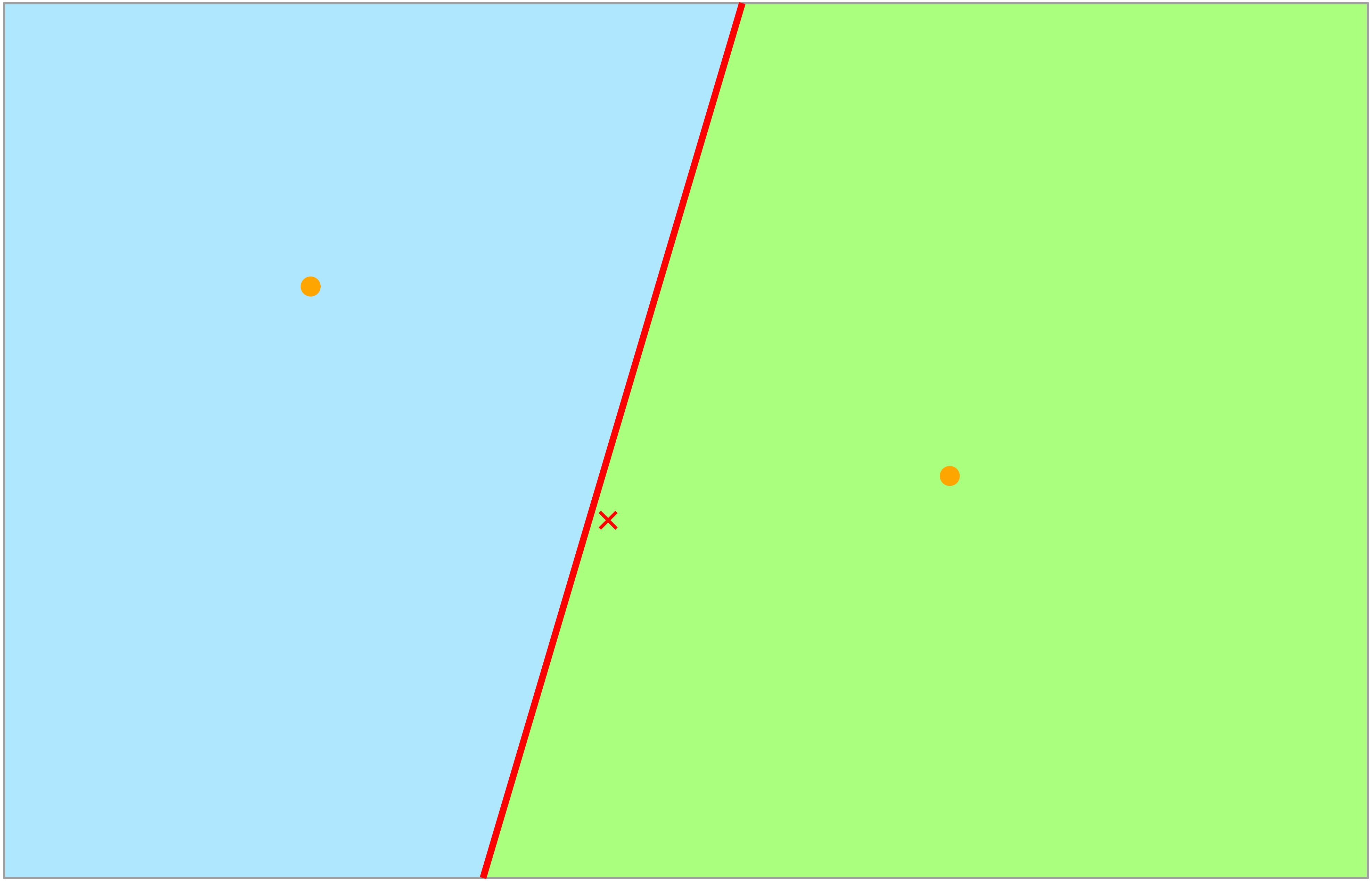

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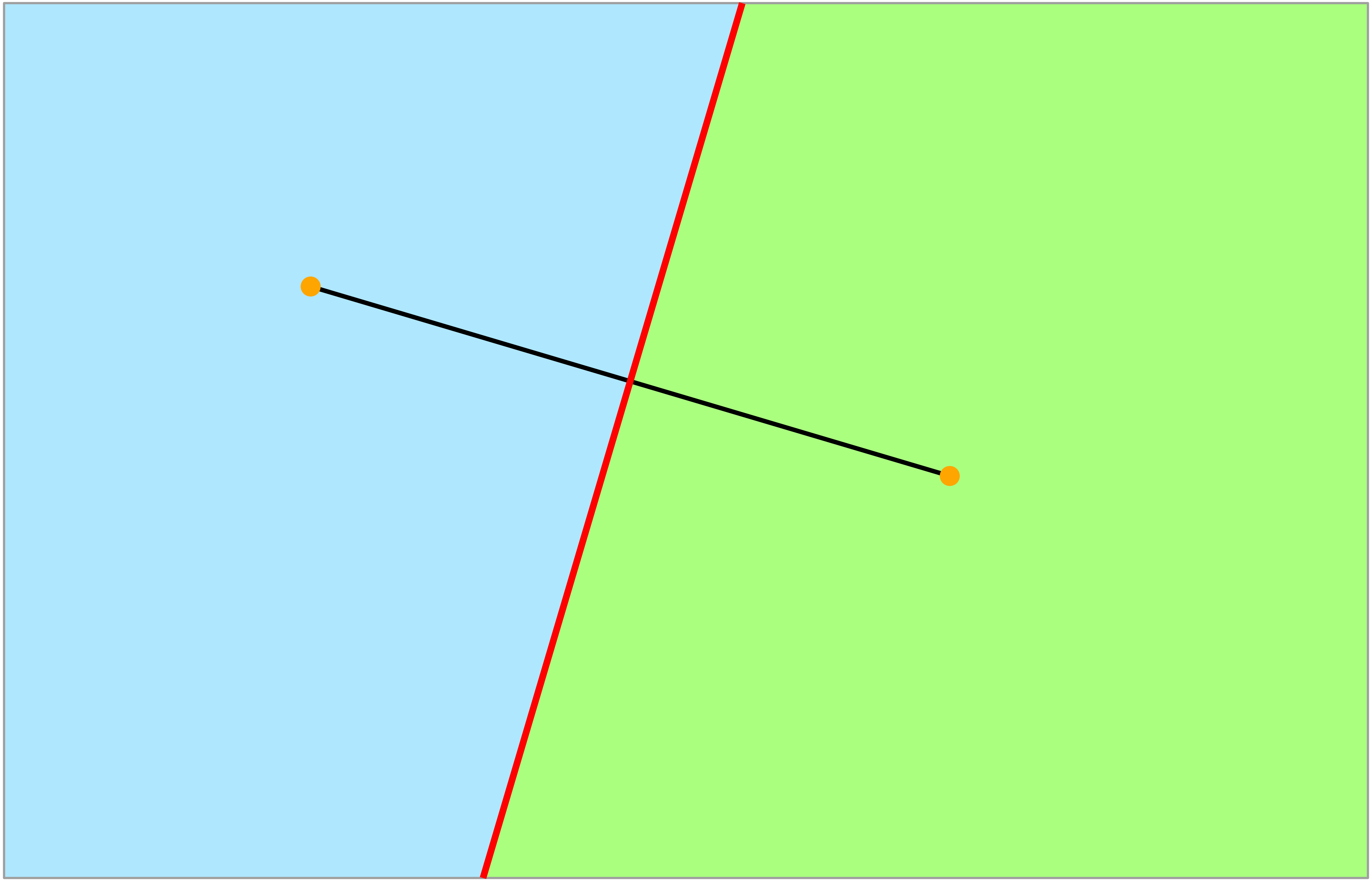

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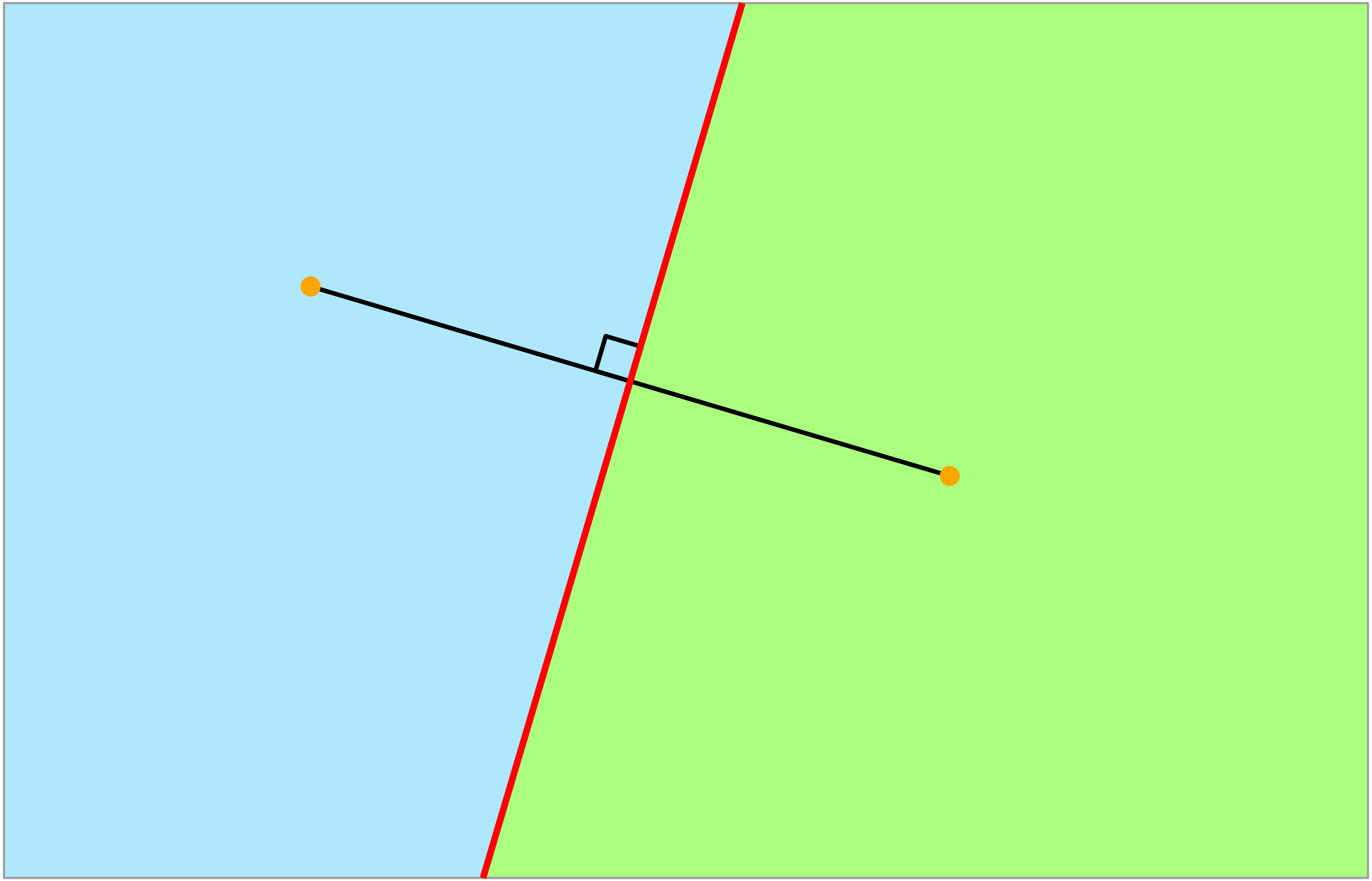

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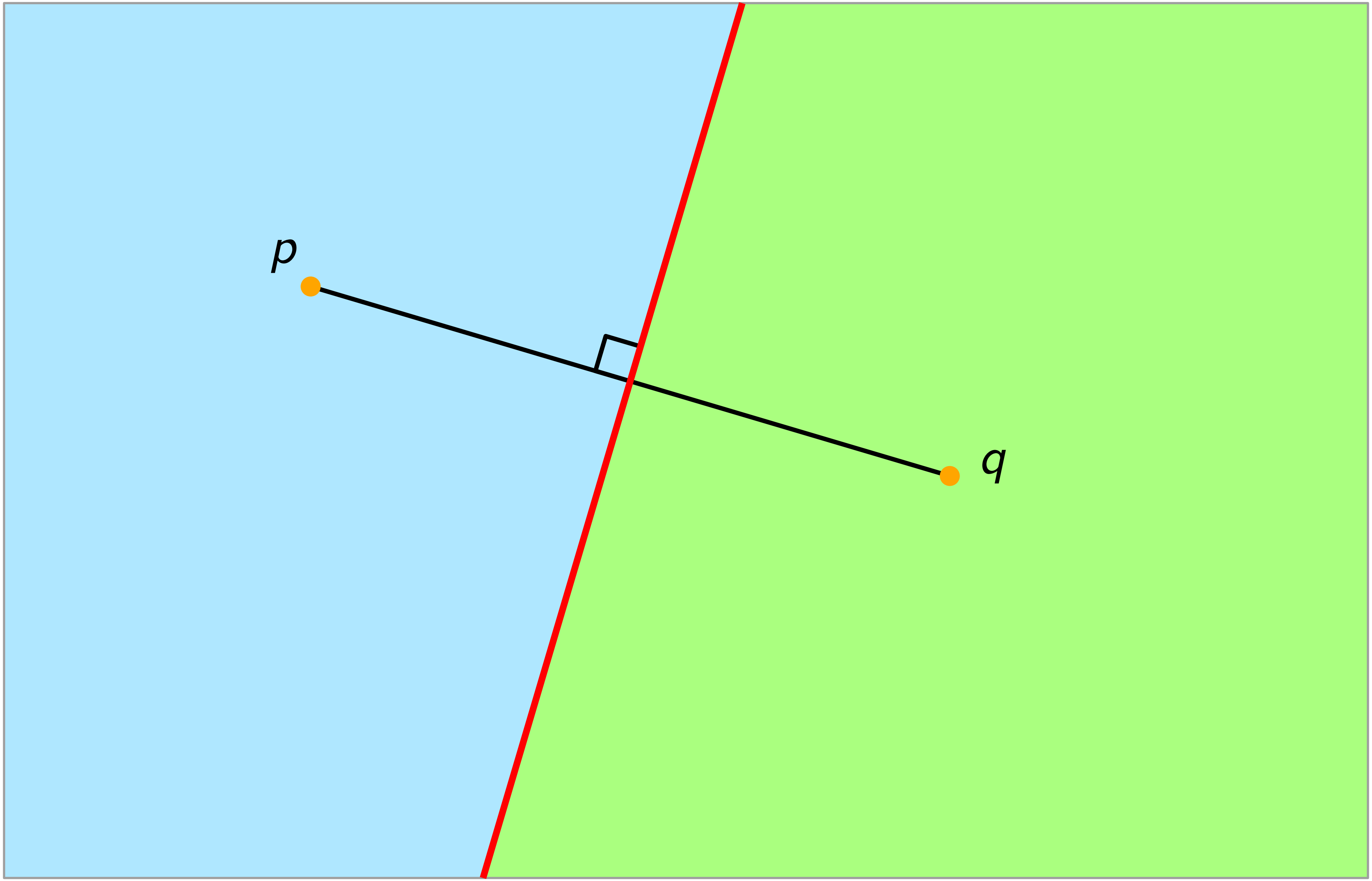

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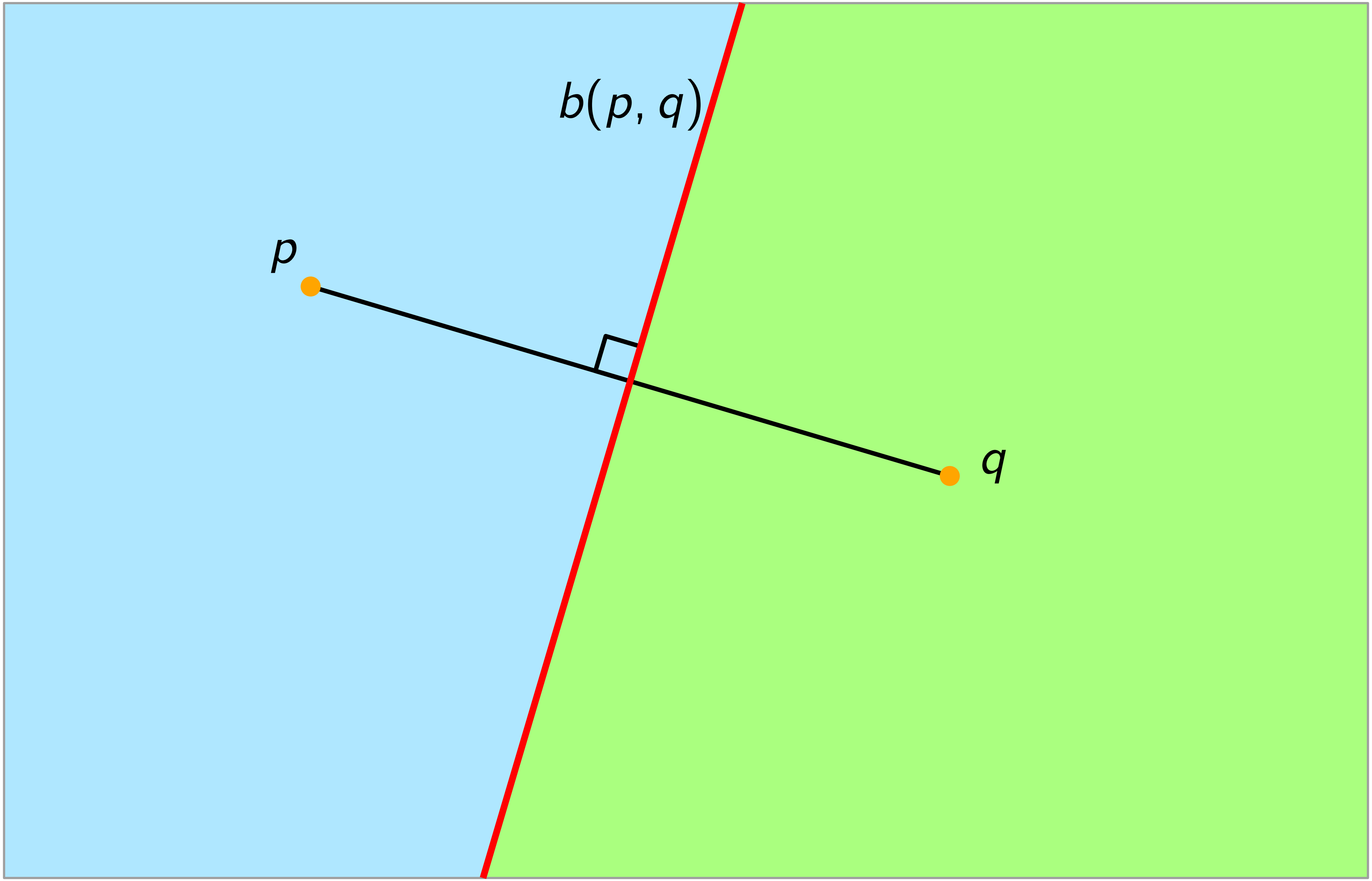

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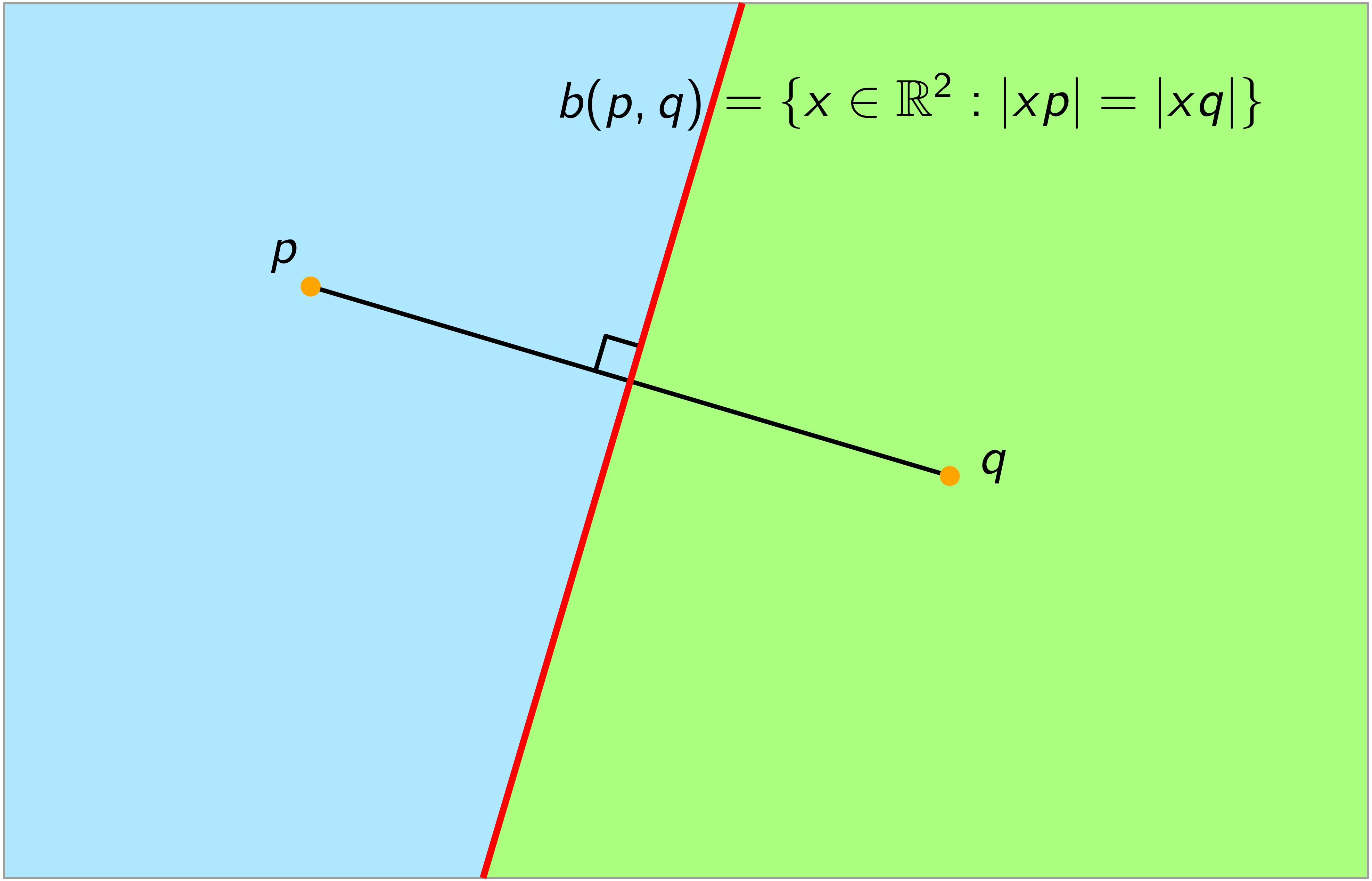

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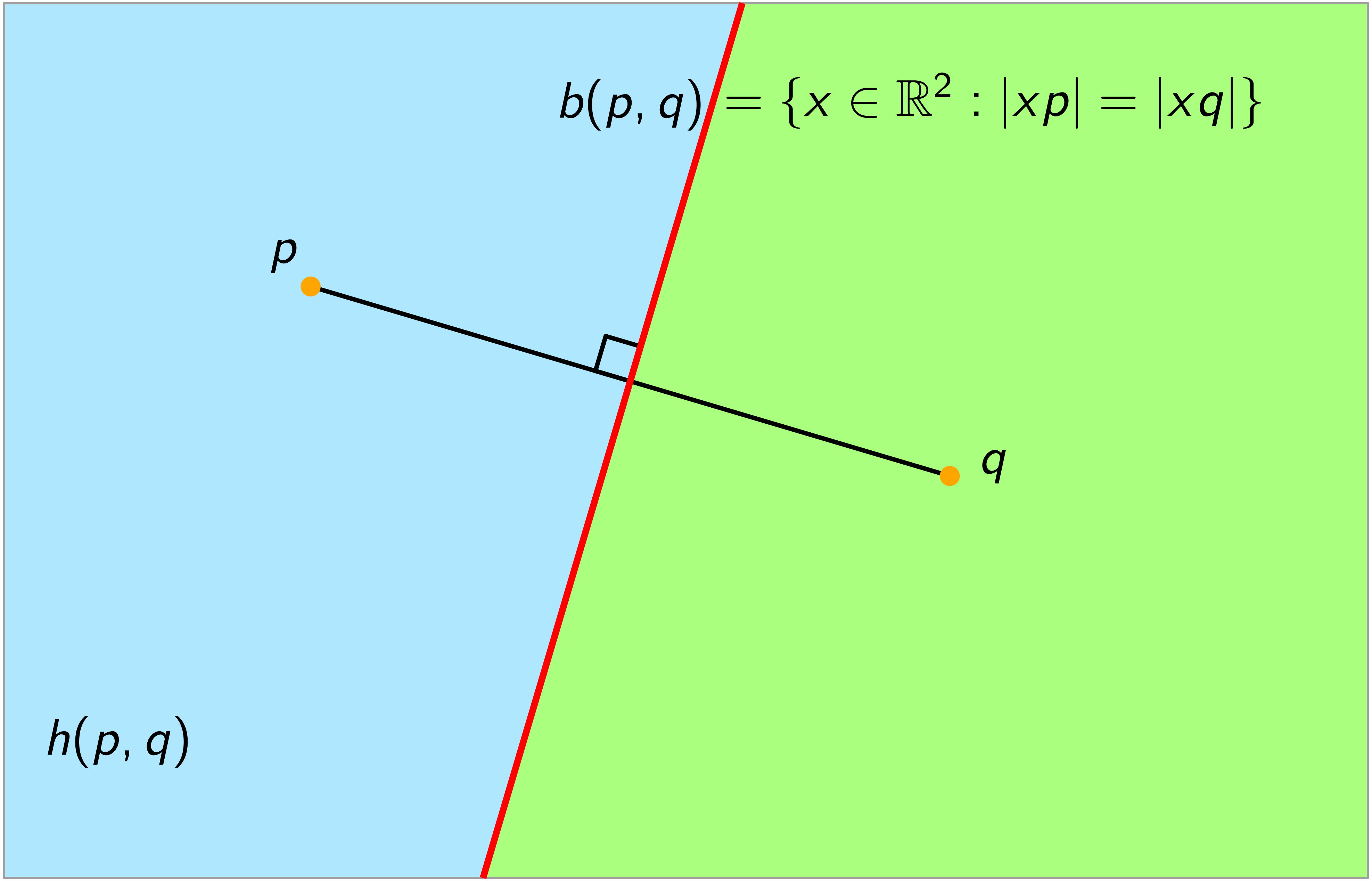

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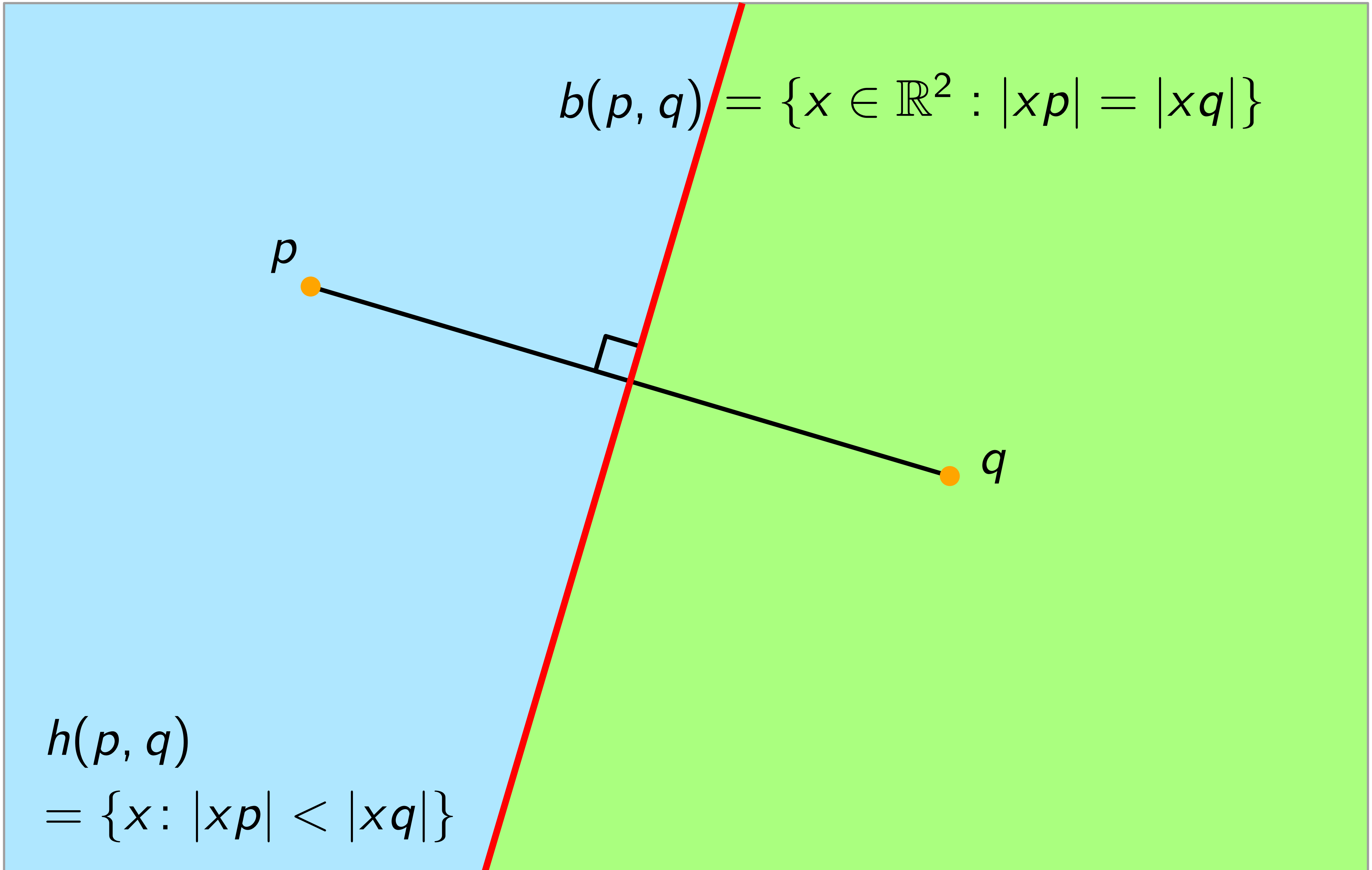

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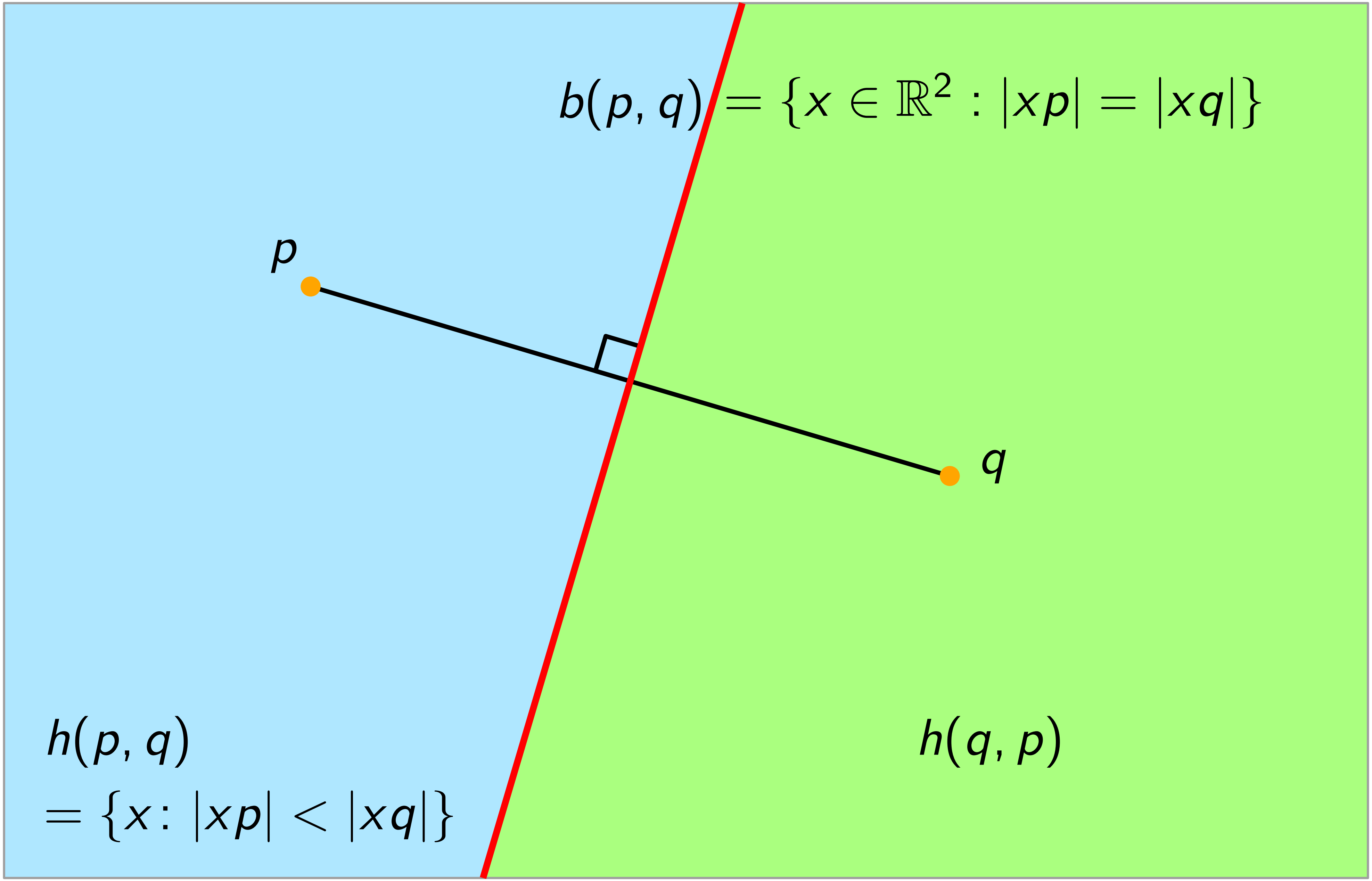

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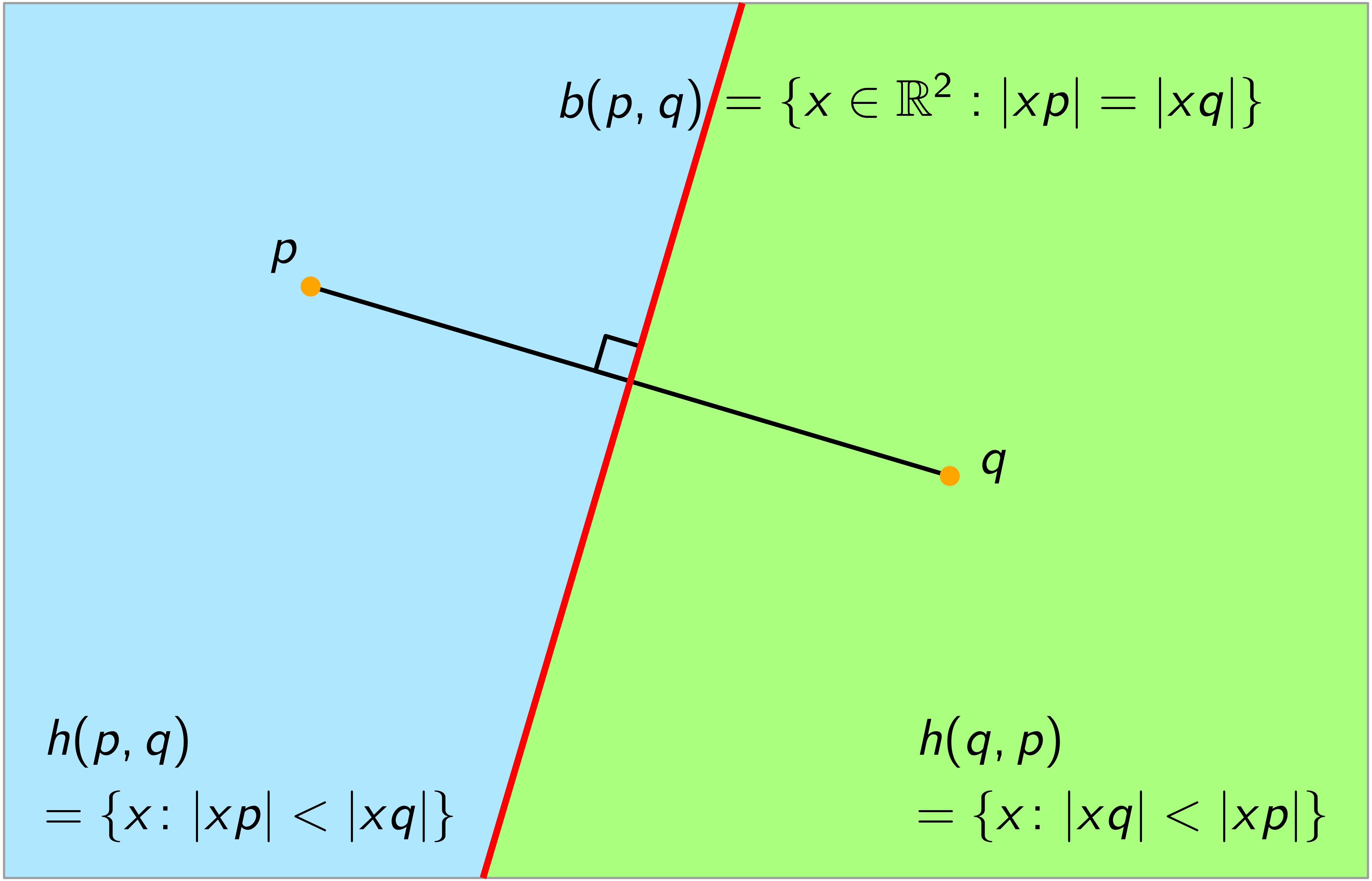

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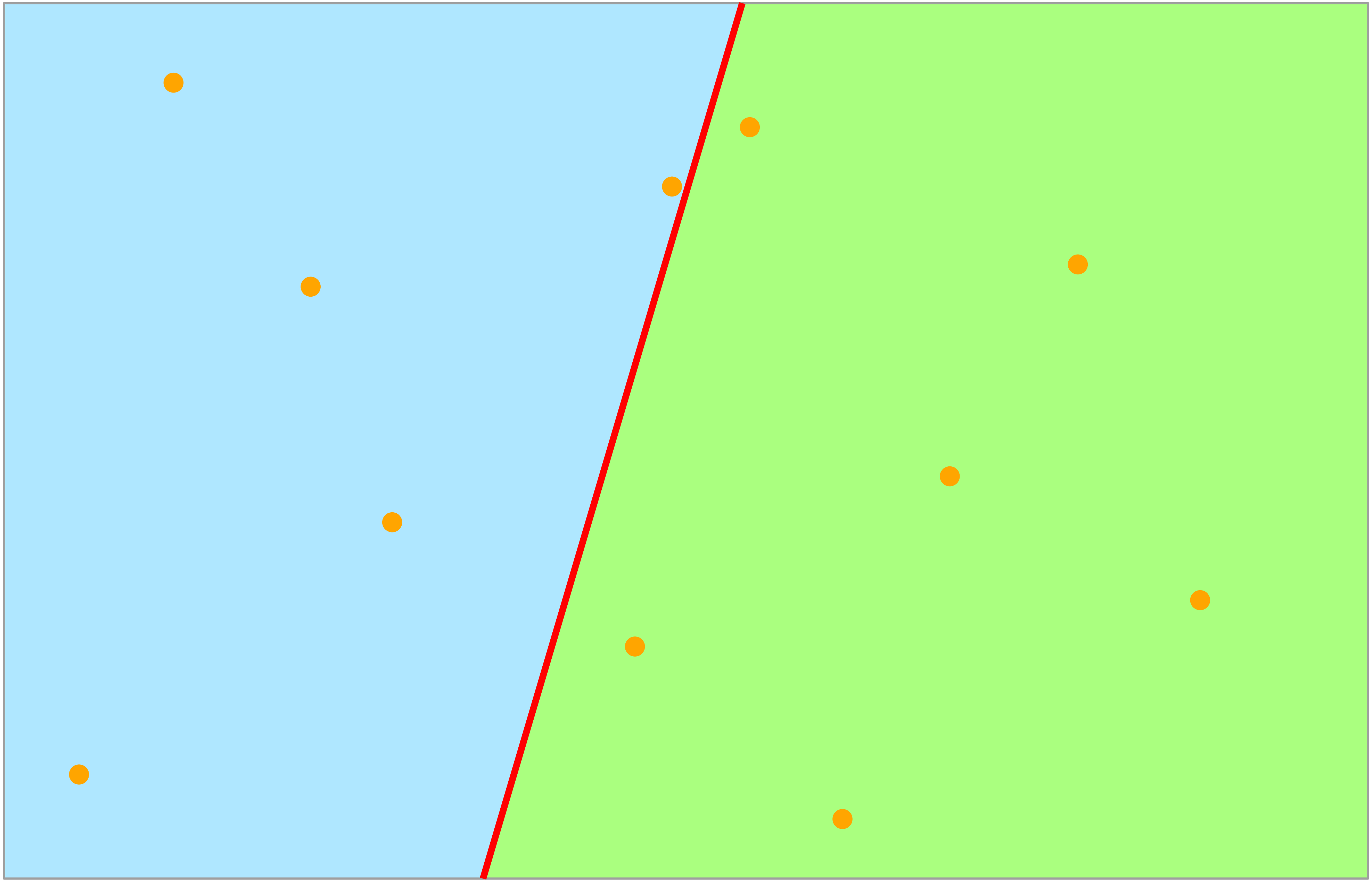

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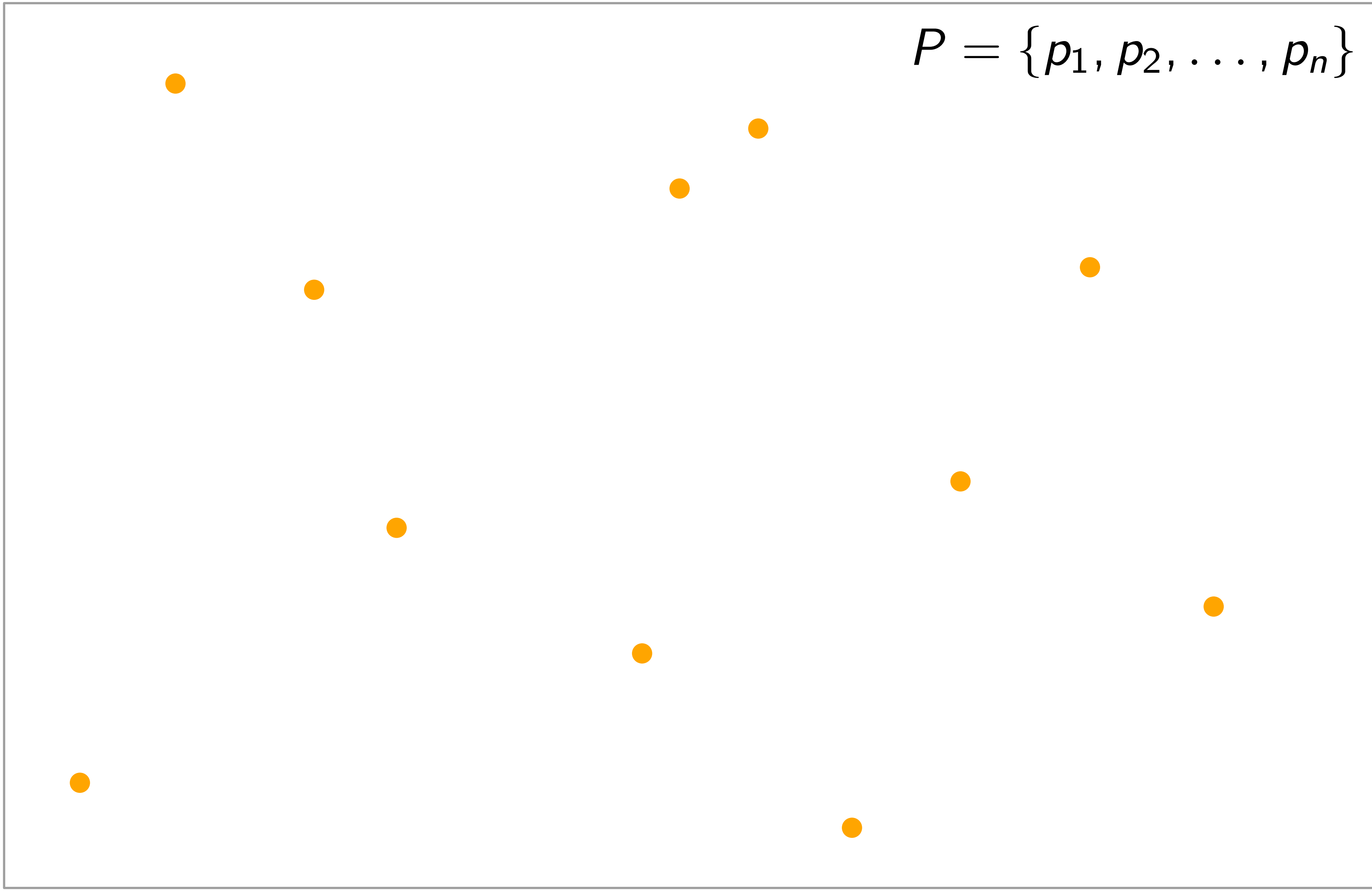
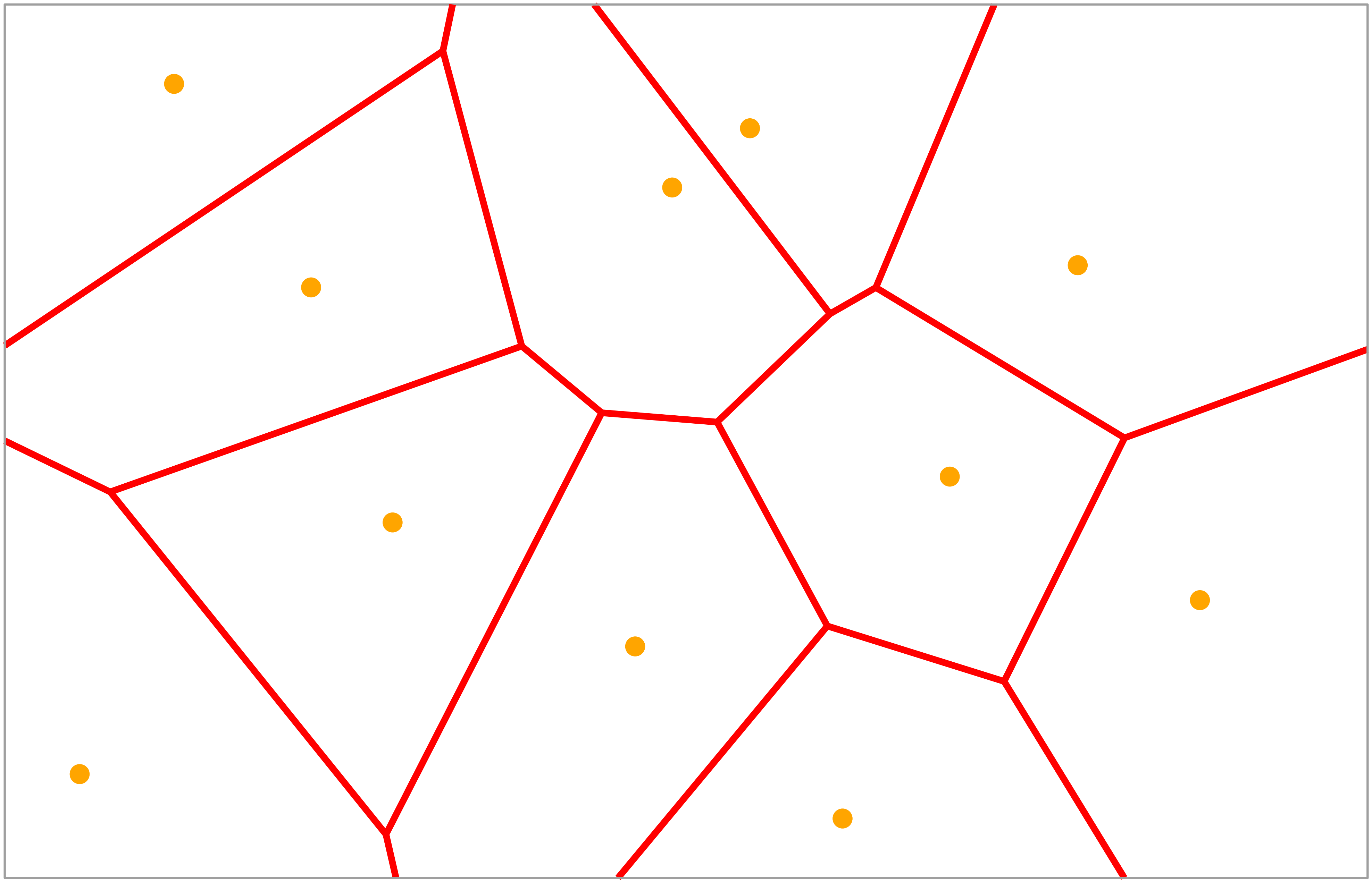


$$P = \{p_1, p_2, \ldots, p_n\}$$

The Post-Office Problem



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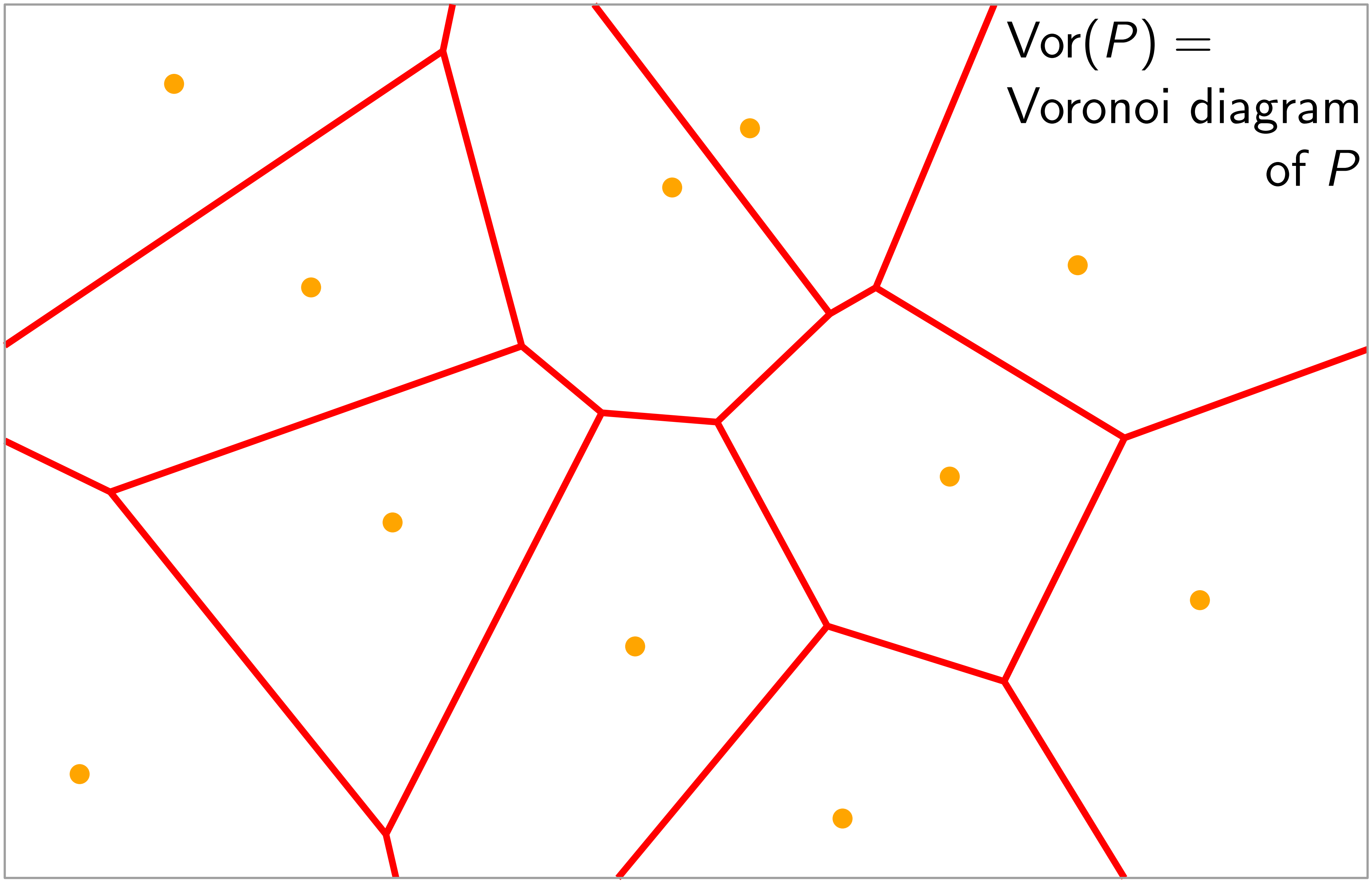

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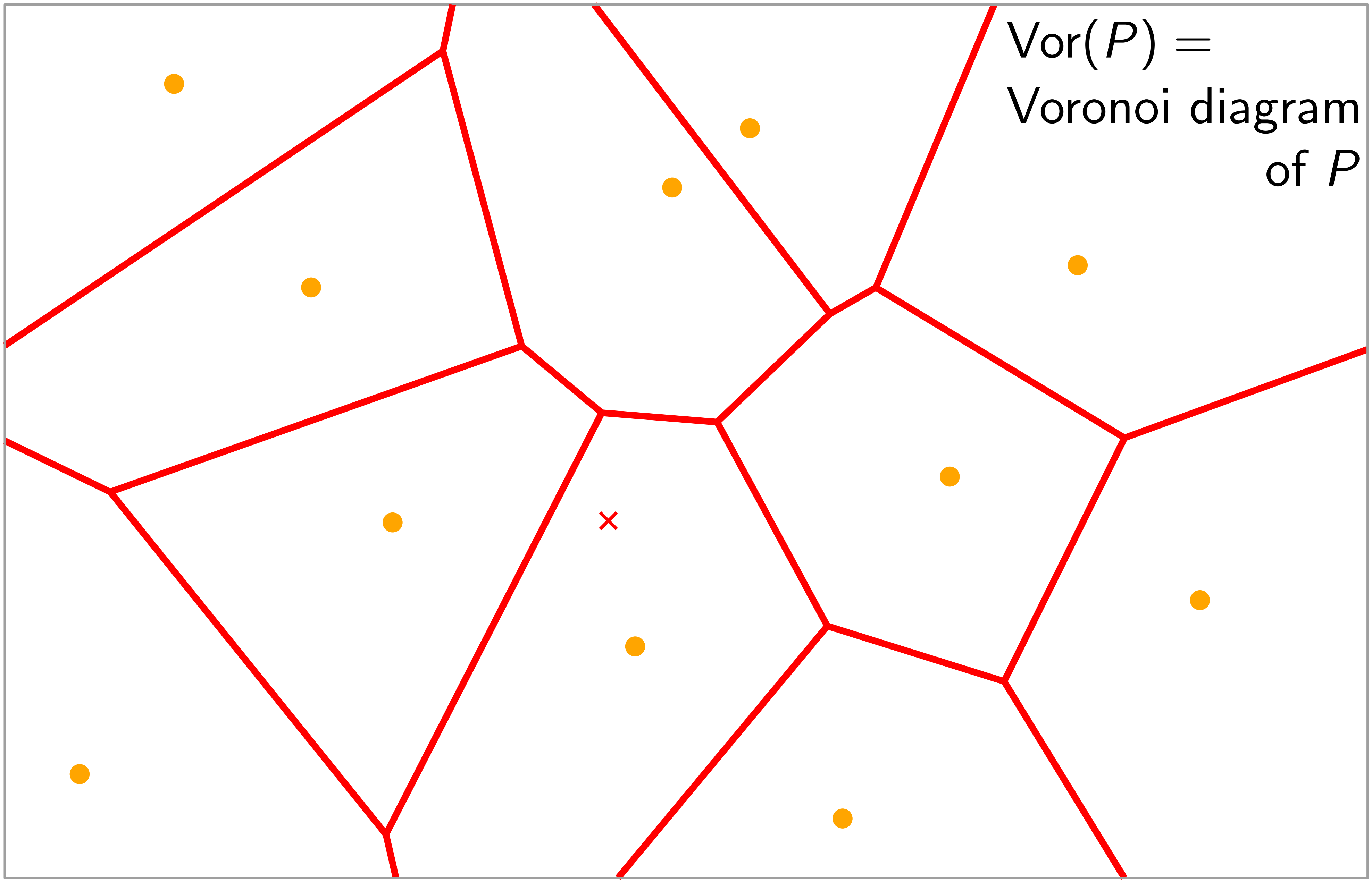

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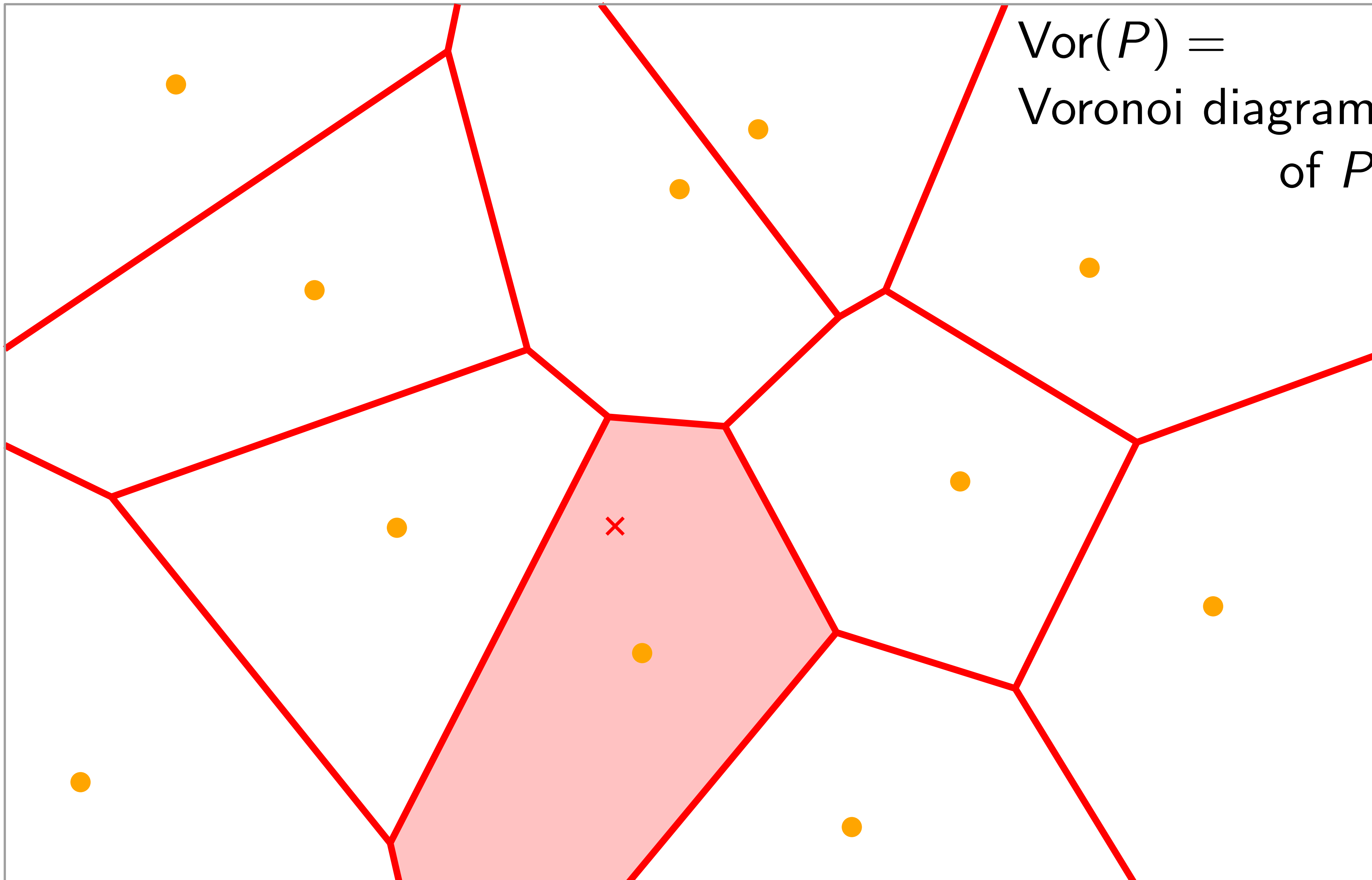
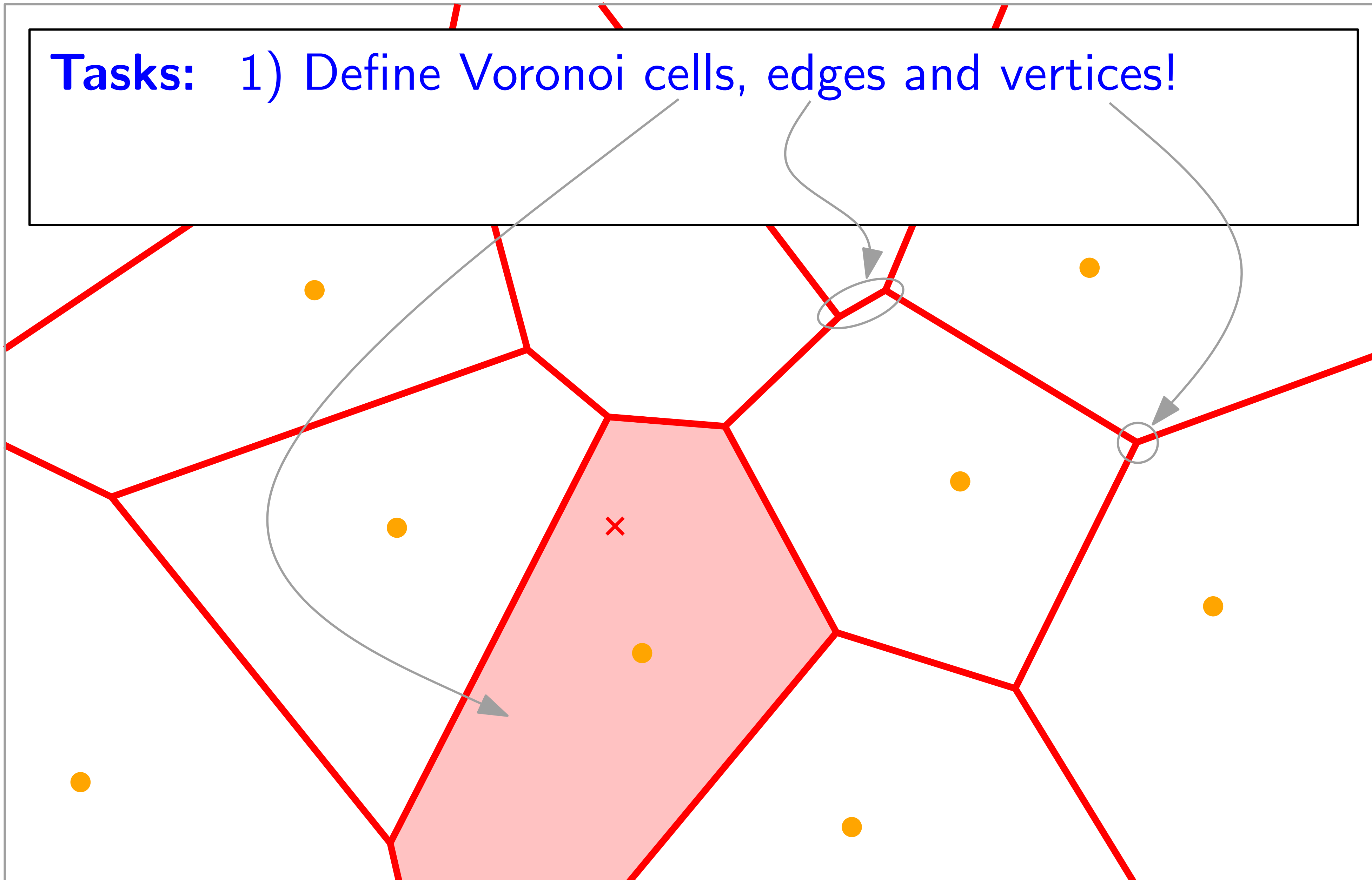


The Post-Office Problem

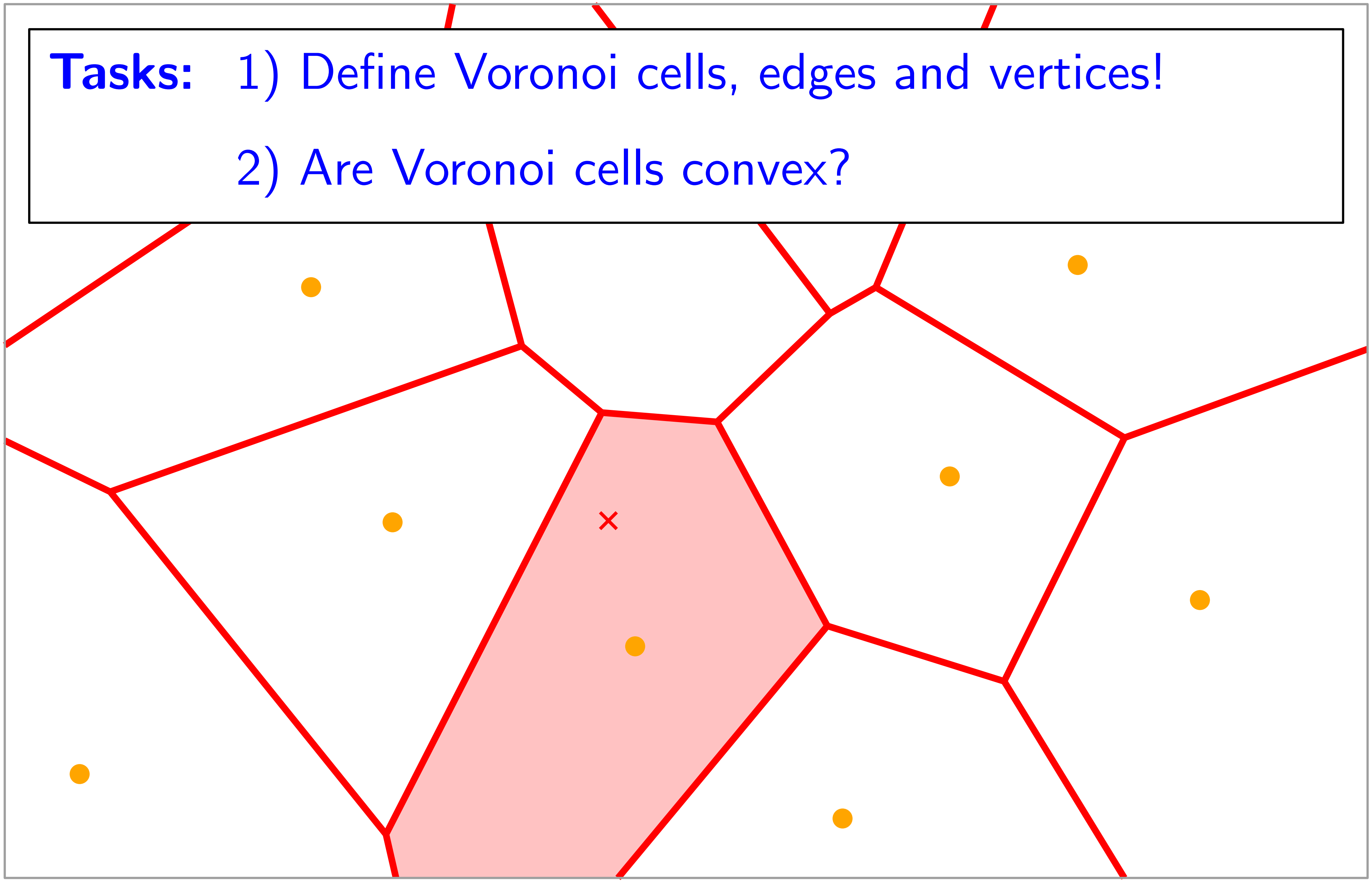
Tasks: 1) Define Voronoi cells, edges and vertices!



The Post-Office Problem

Tasks: 1) Define Voronoi cells, edges and vertices!

2) Are Voronoi cells convex?



The Voronoi diagram

Let P be a set of points in the plane and let $p, p', p'' \in P$.

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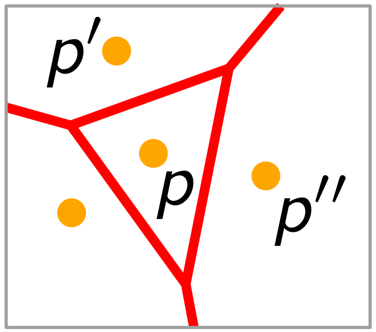

Vor(P)

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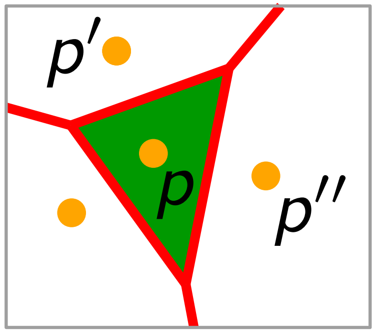

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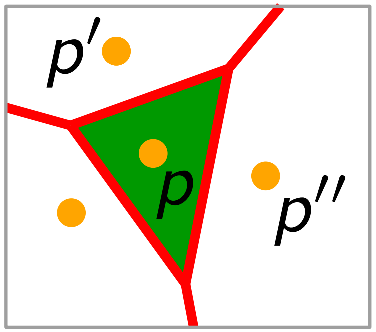

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$\mathcal{V}(\{p\}) =$

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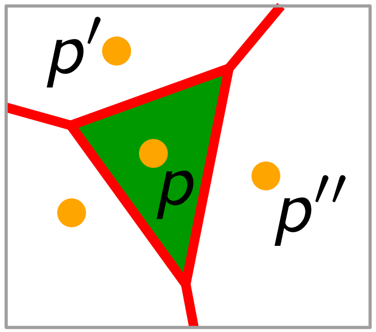

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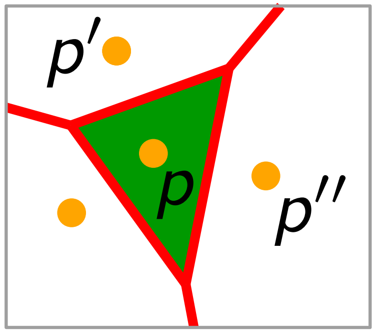

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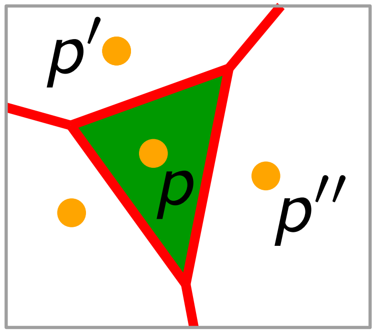

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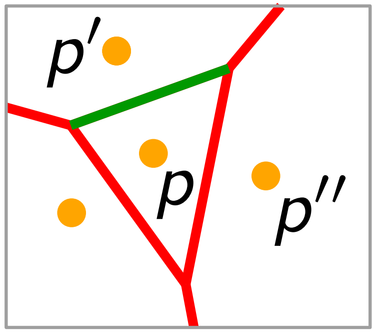

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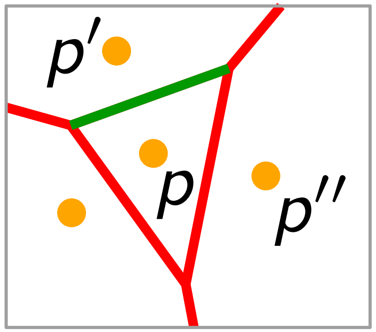

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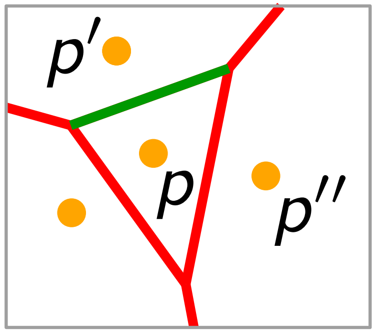

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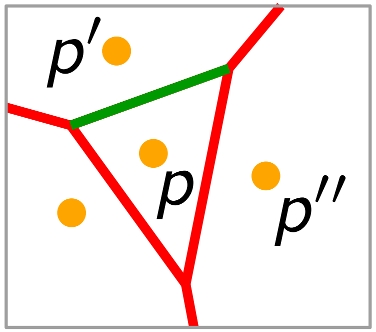

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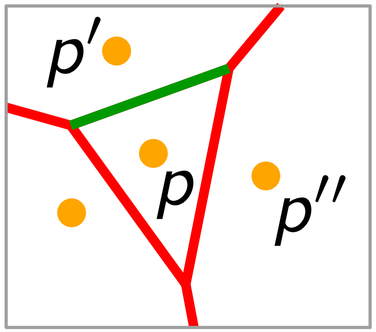

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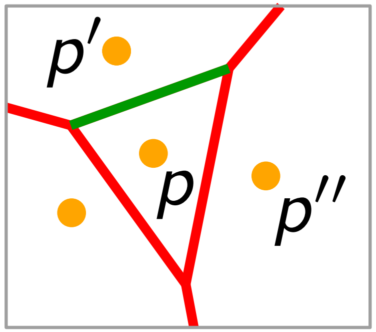

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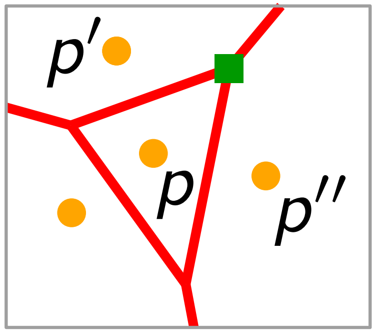

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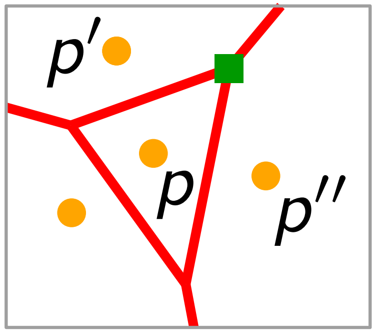

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[Voronoi vertex]

$\mathcal{V}(\{p, p', p''\})$

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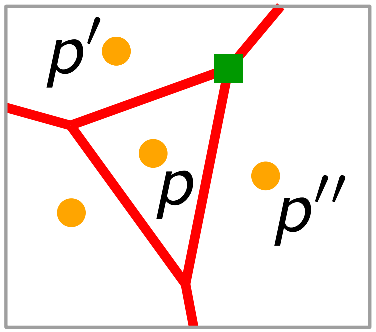

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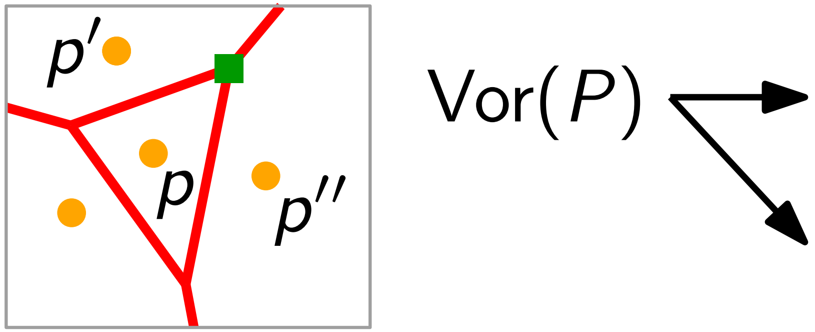
[*Voronoi diagram*] 

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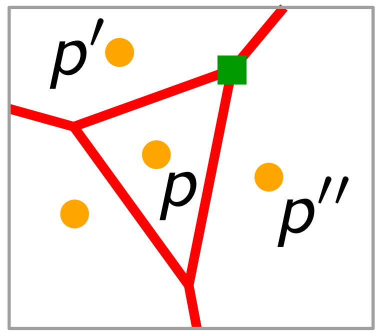

Vor(P) → subdivision of $\mathbb{R}^2$

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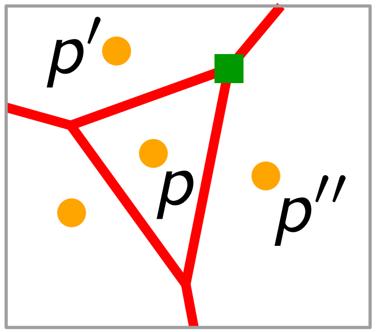

Vor(P) → subdivision of $\mathbb{R}^2$
→ geometric graph

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$$= \text{rel-int}\big(\partial\mathcal{V}(p) \cap \partial\mathcal{V}(p')\big), \text{ i.e., w/o the endpts}$$

[*Voronoi vertex*]

$$\mathcal{V}(\{p, p', p''\}) = \partial\mathcal{V}(p) \cap \partial\mathcal{V}(p') \cap \partial\mathcal{V}(p'')$$

Overall Shape of Vor(P)

Theorem. Let $P \subset \mathbb{R}^2$ be a set of n pts (called *sites*). If all sites are collinear, Vor(P) consists of $n-1$ parallel lines. Otherwise, Vor(P) is connected and its edges are line segments or half-lines.

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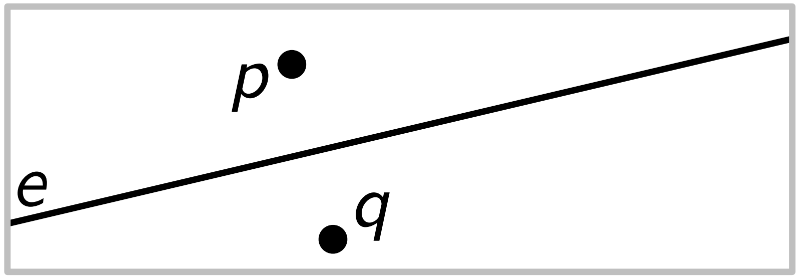

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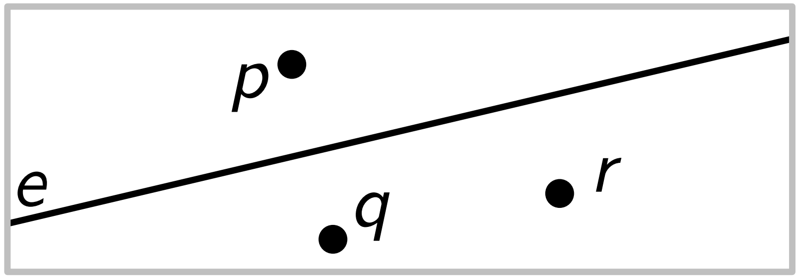

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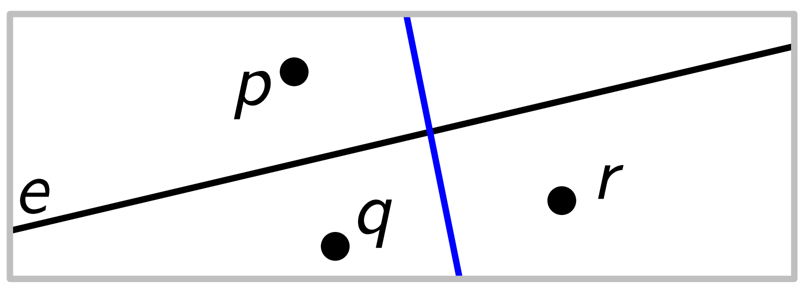

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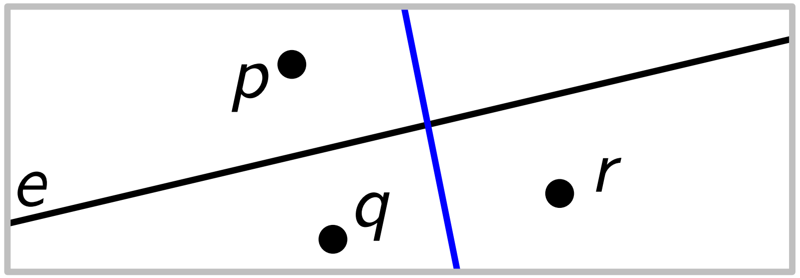

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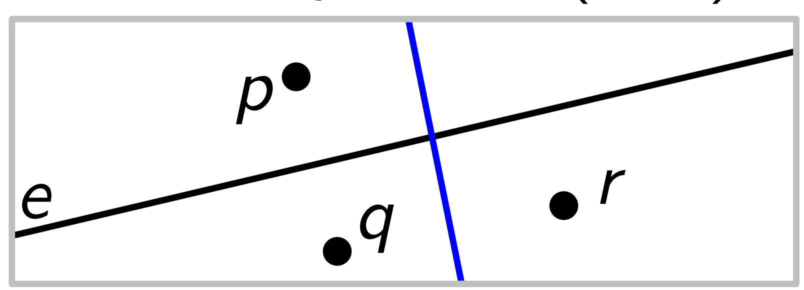

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$\Rightarrow$ e is bounded on at least one side. $\square$

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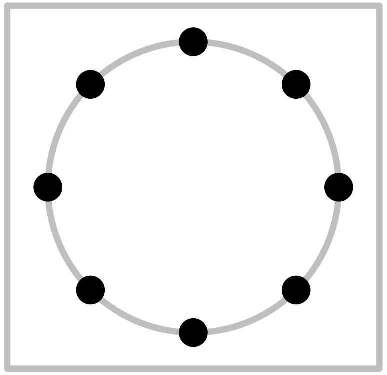

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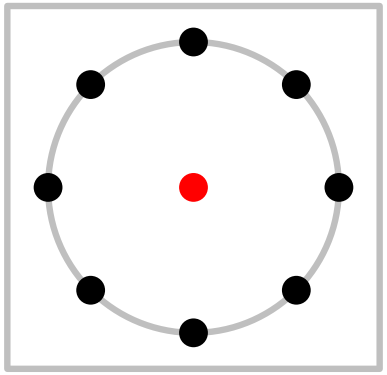

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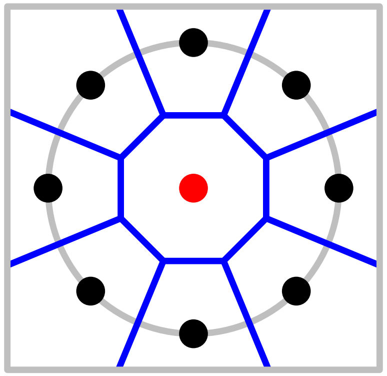

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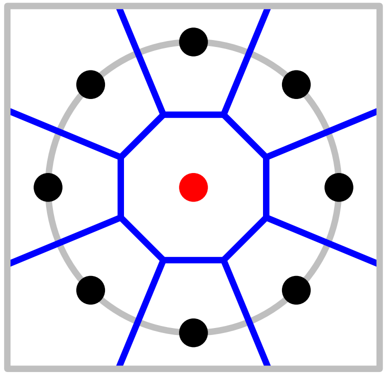

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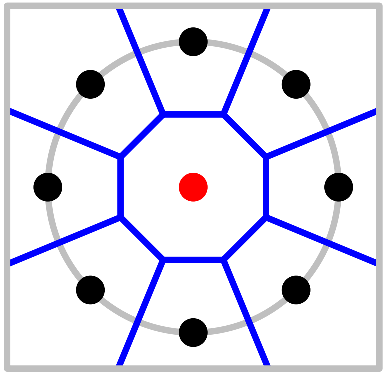

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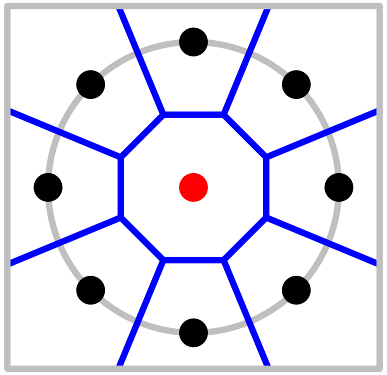

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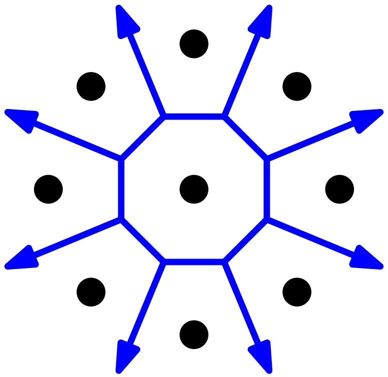

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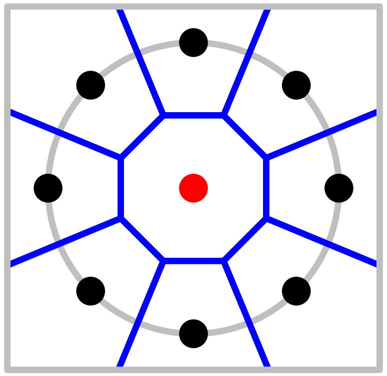

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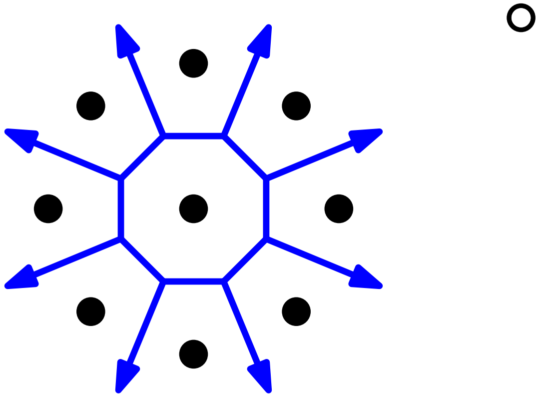

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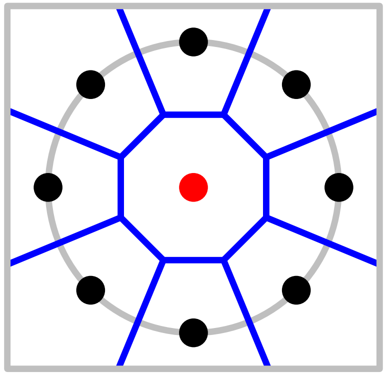

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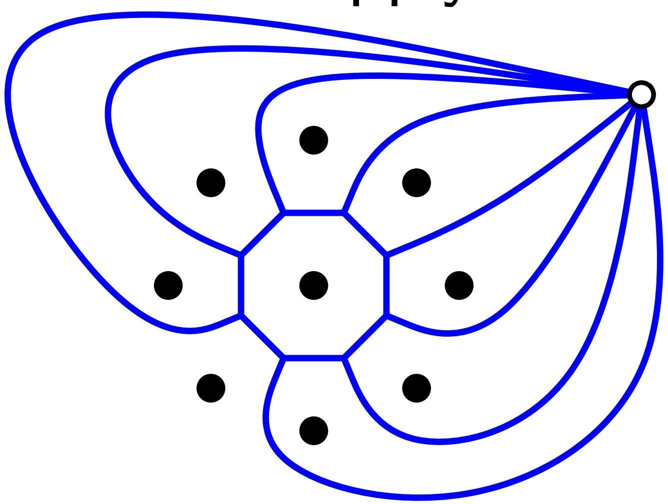
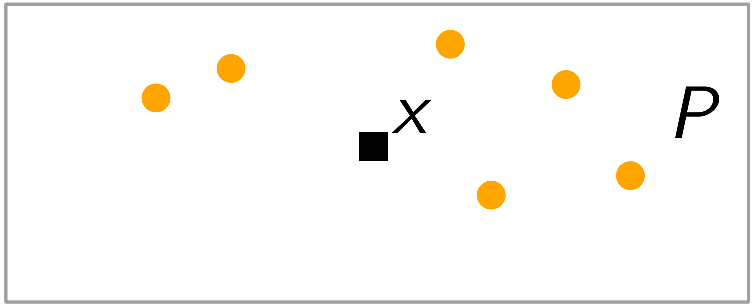


Characterization of Voronoi vtc and edges

$C_P(x) :=$ largest circle centered at x w/o sites in its interior



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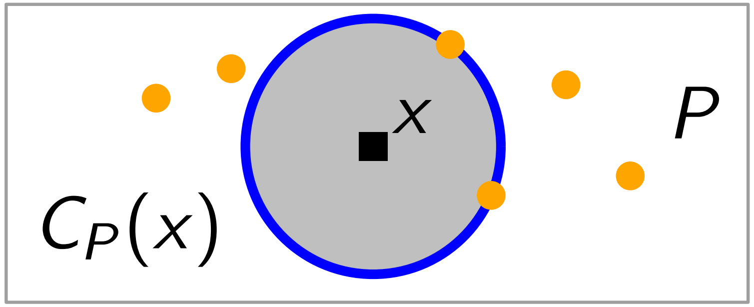

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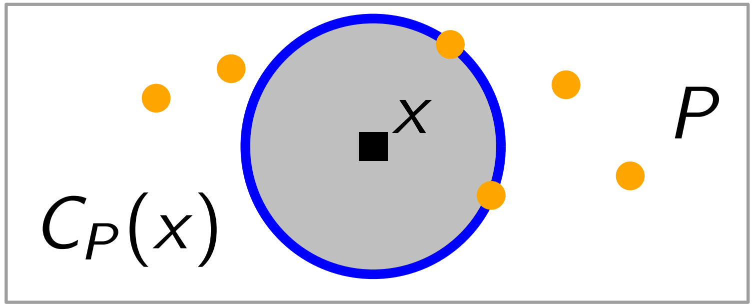

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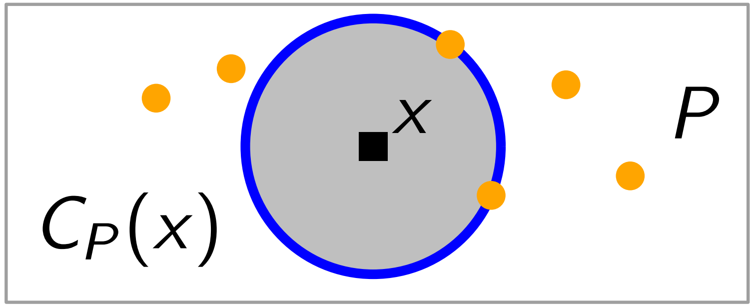

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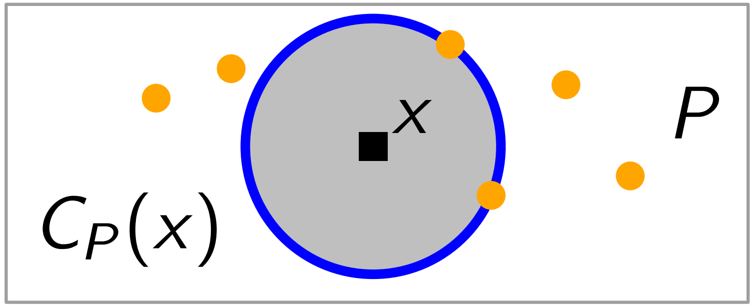

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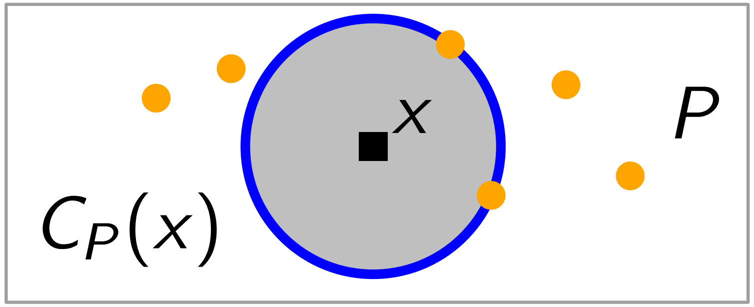

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[Chapter 4] $O(n \log n)$ time

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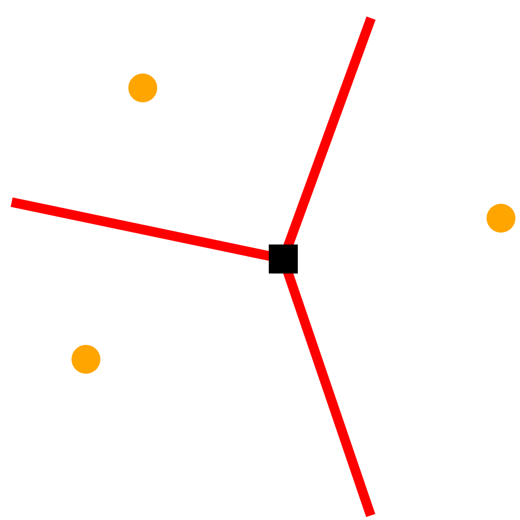

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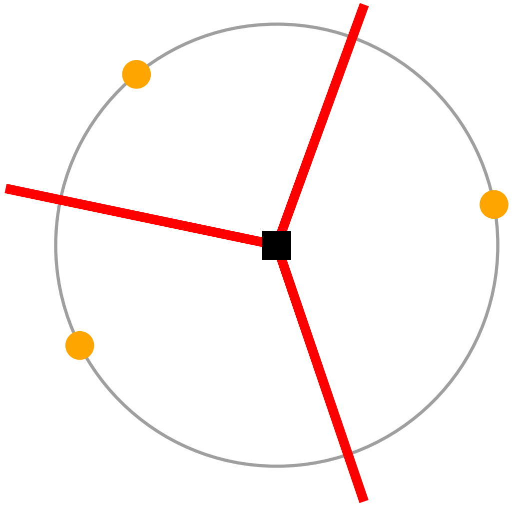

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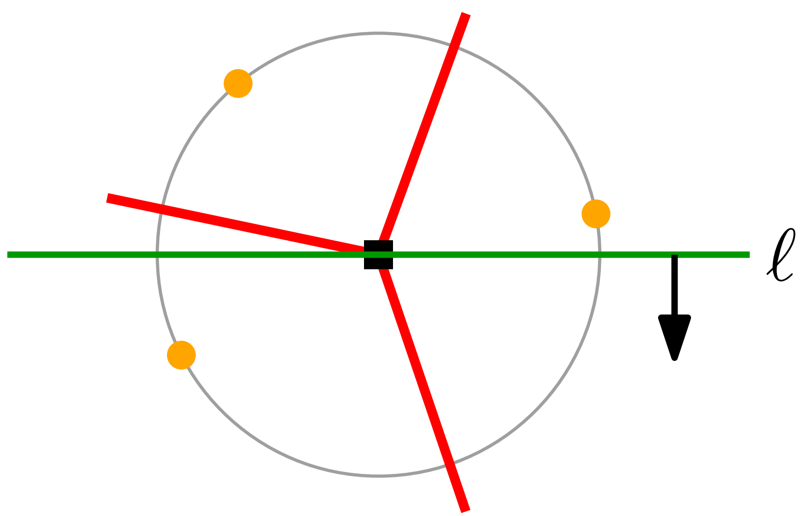

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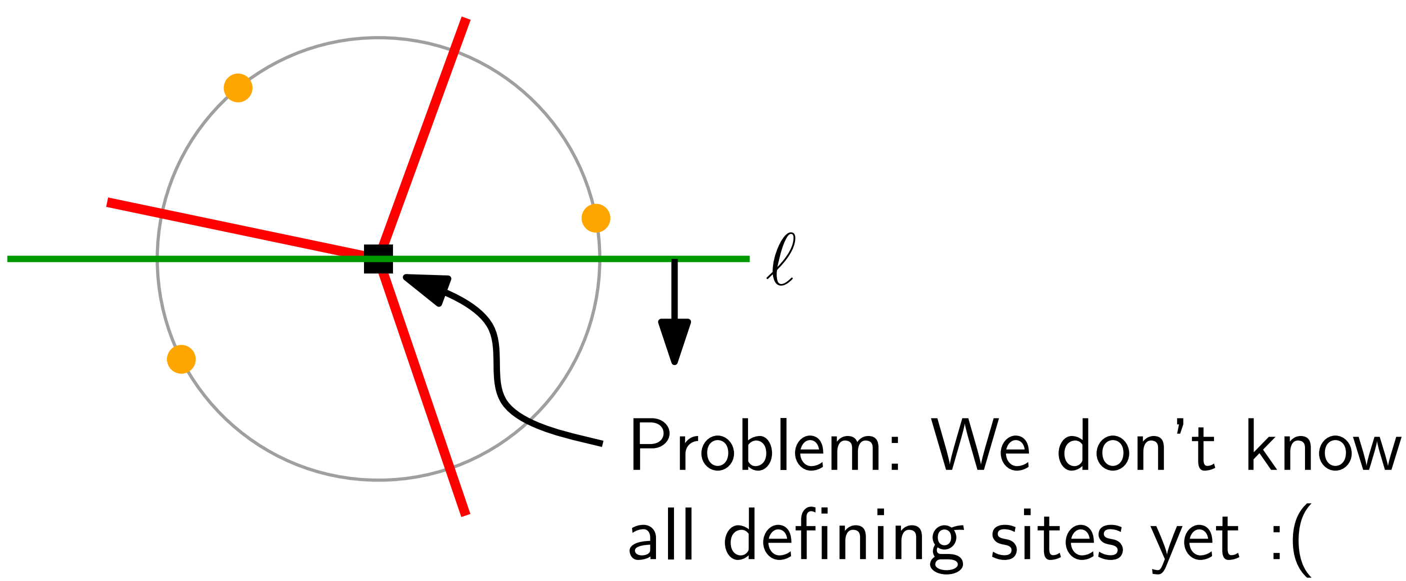

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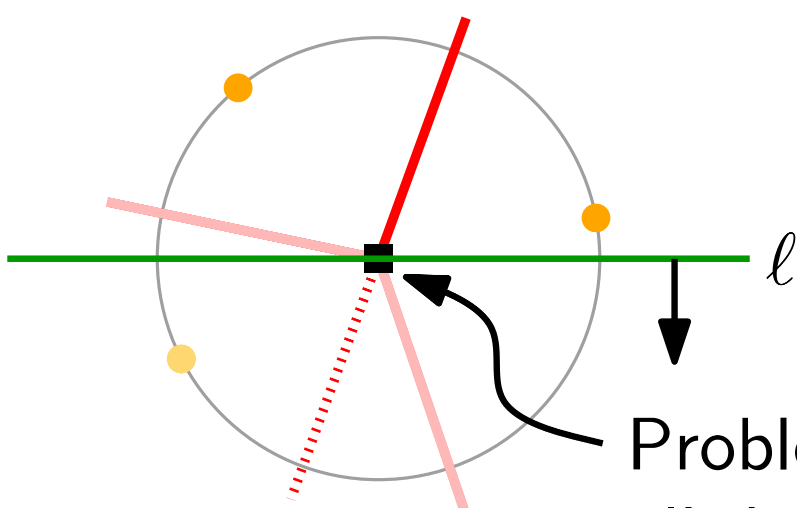

Problem: We don't know all defining sites yet :(

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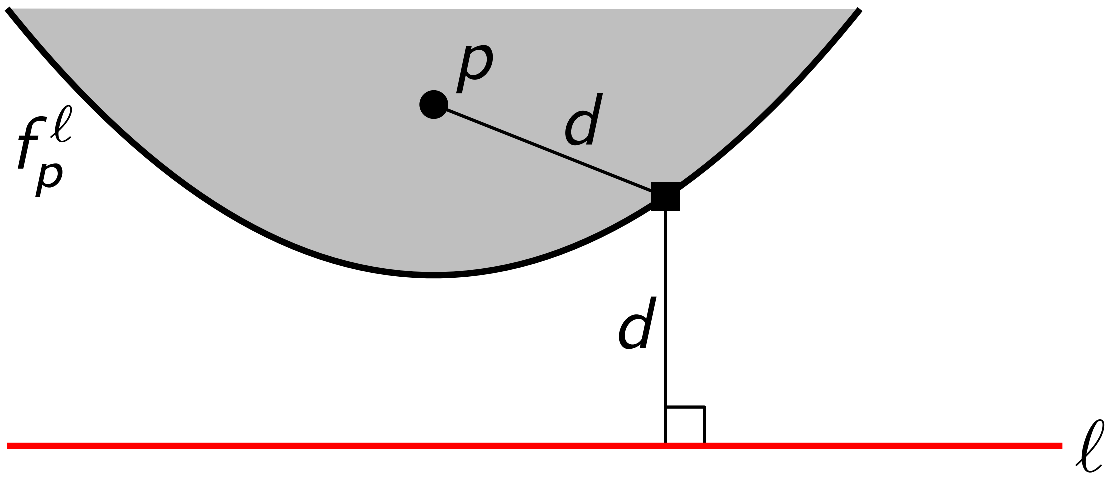

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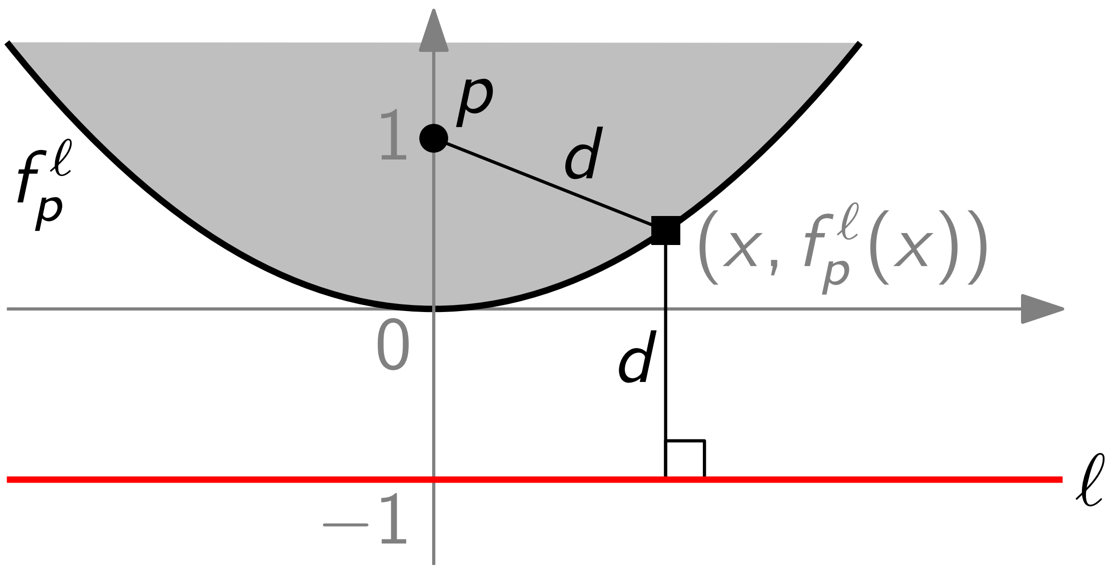

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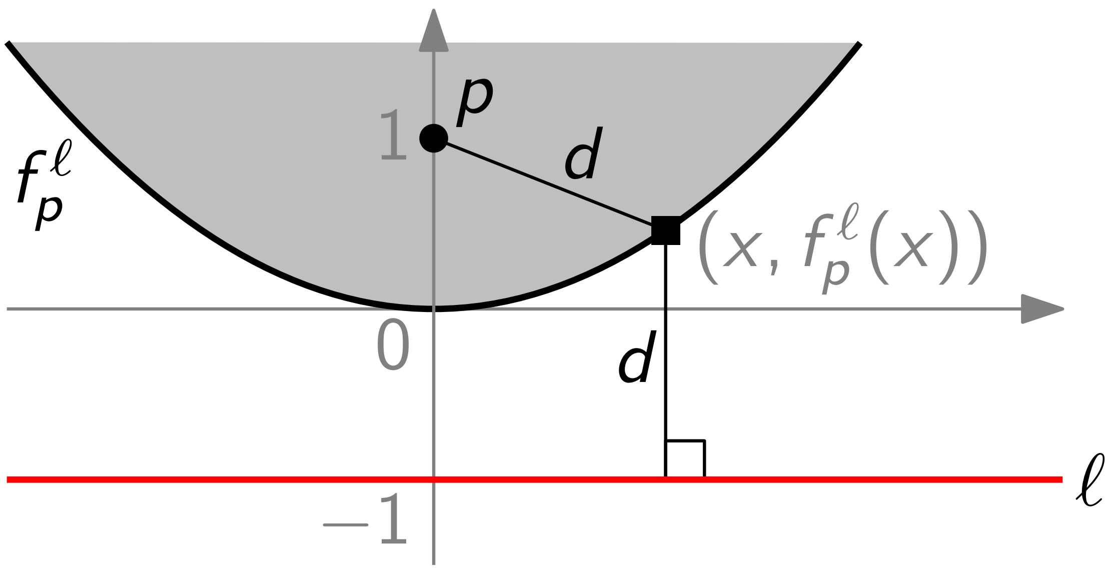

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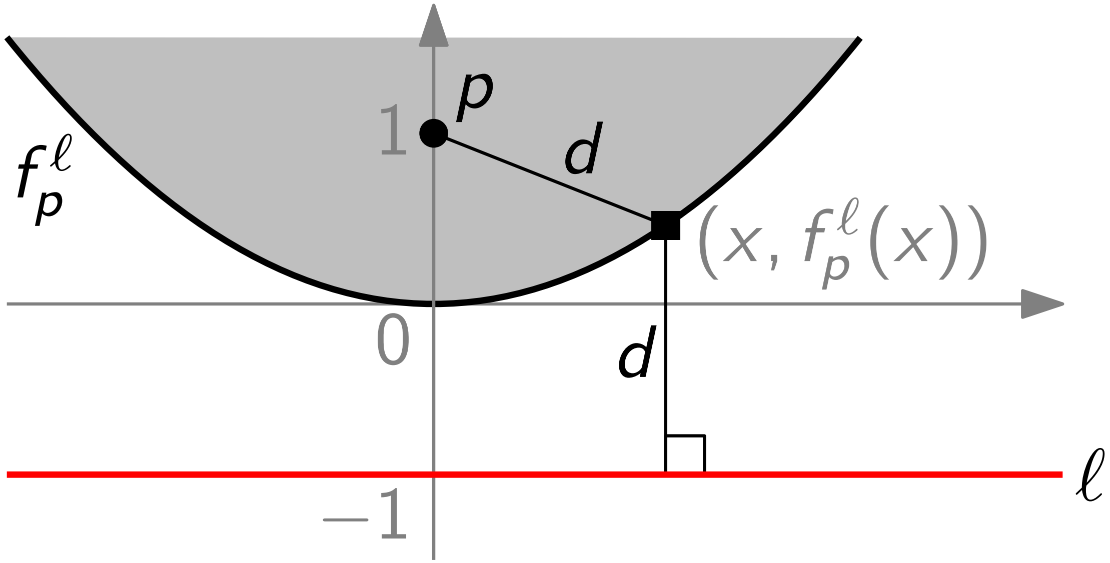

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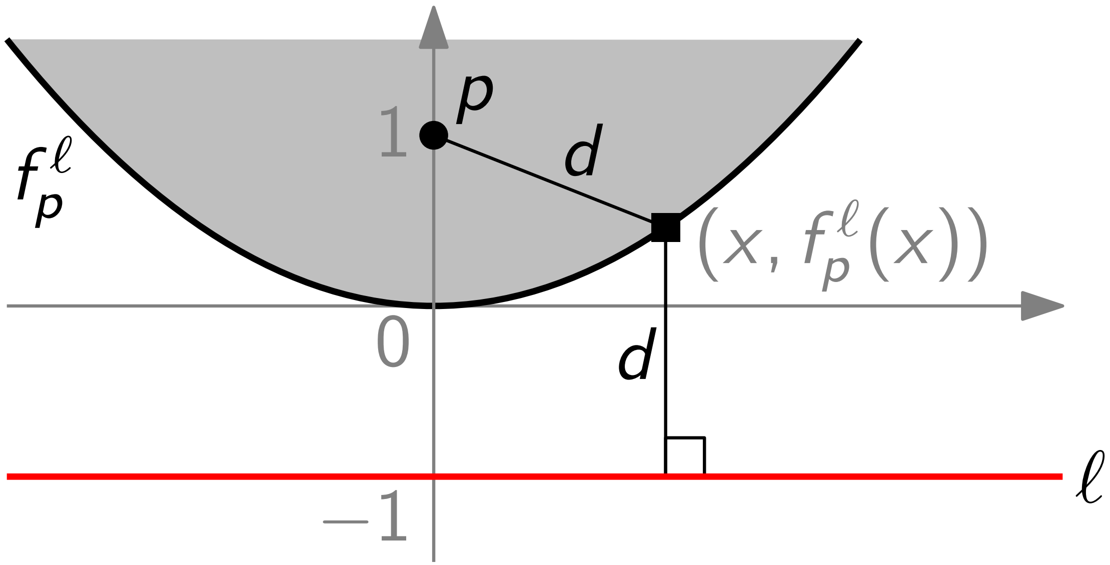

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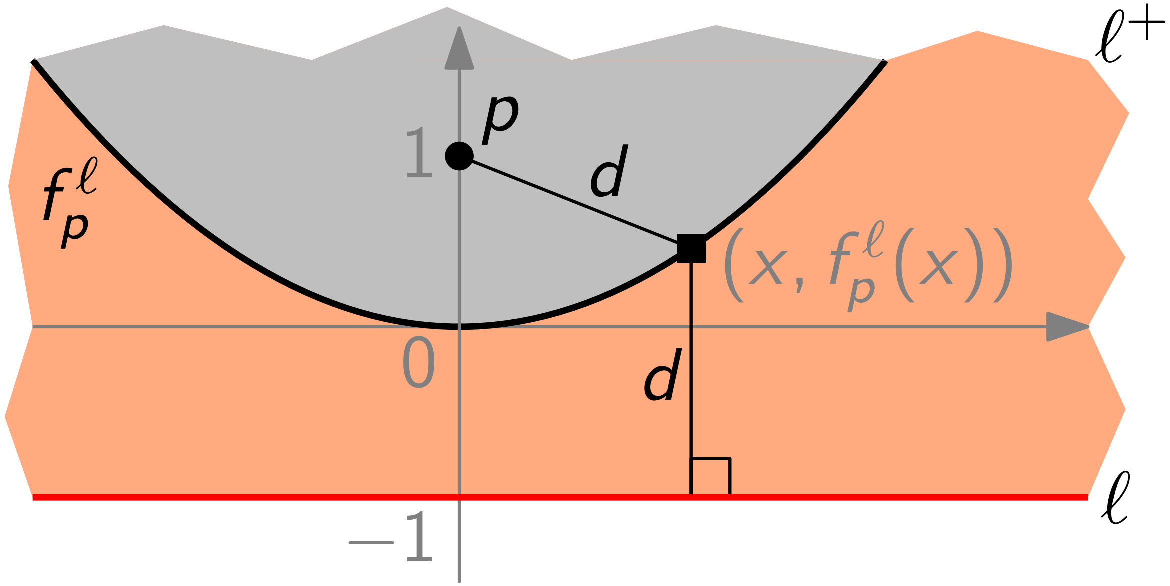

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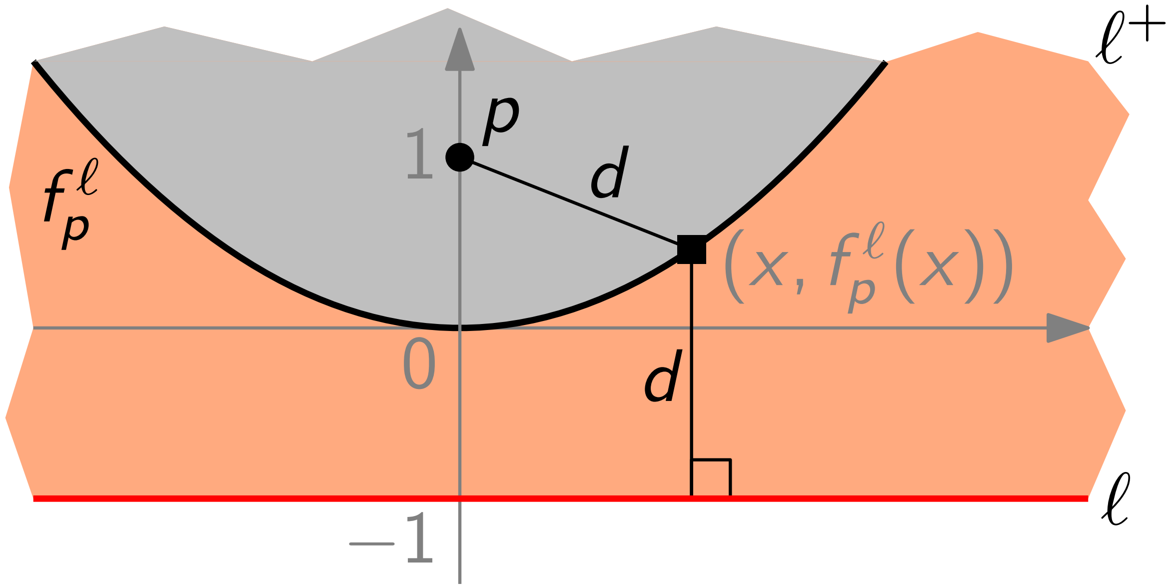

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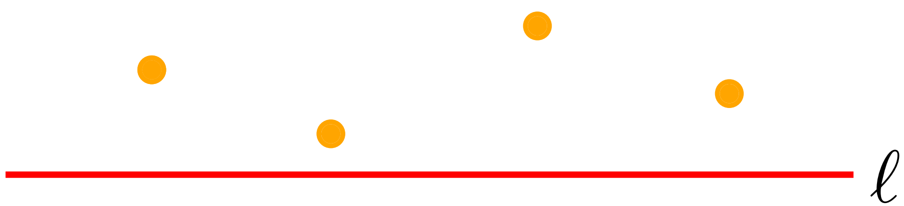

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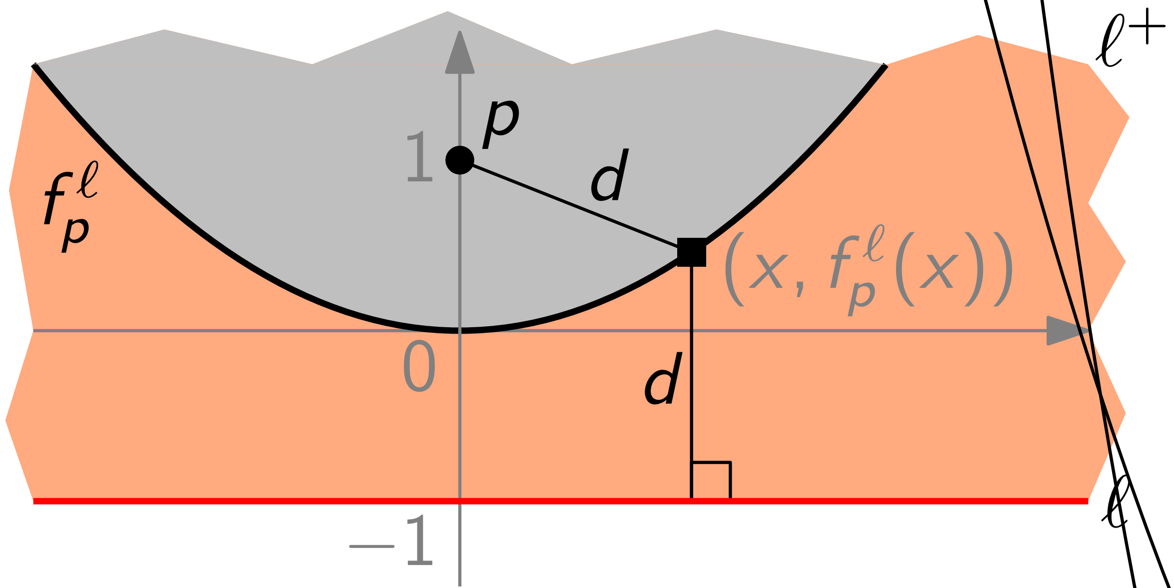

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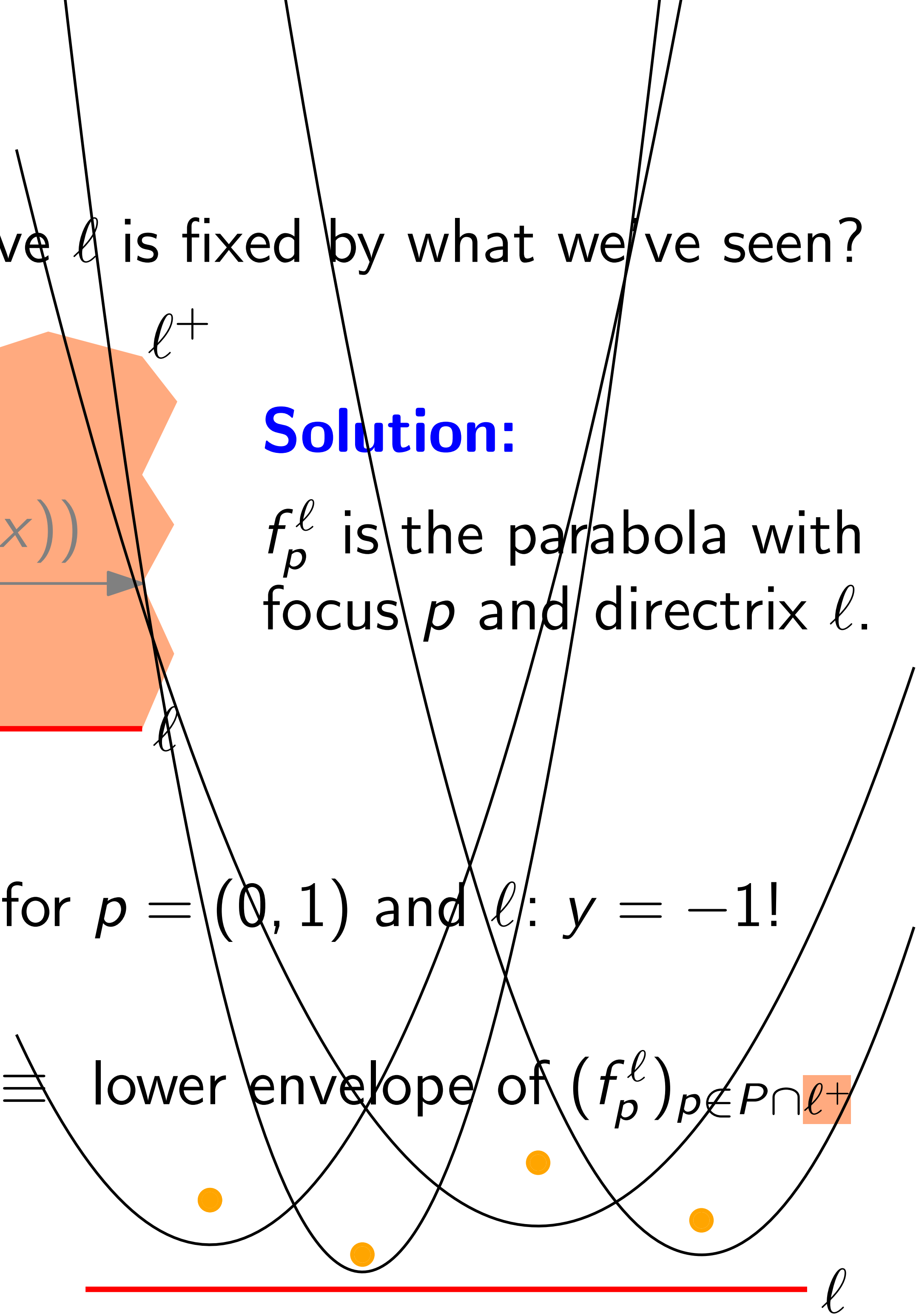

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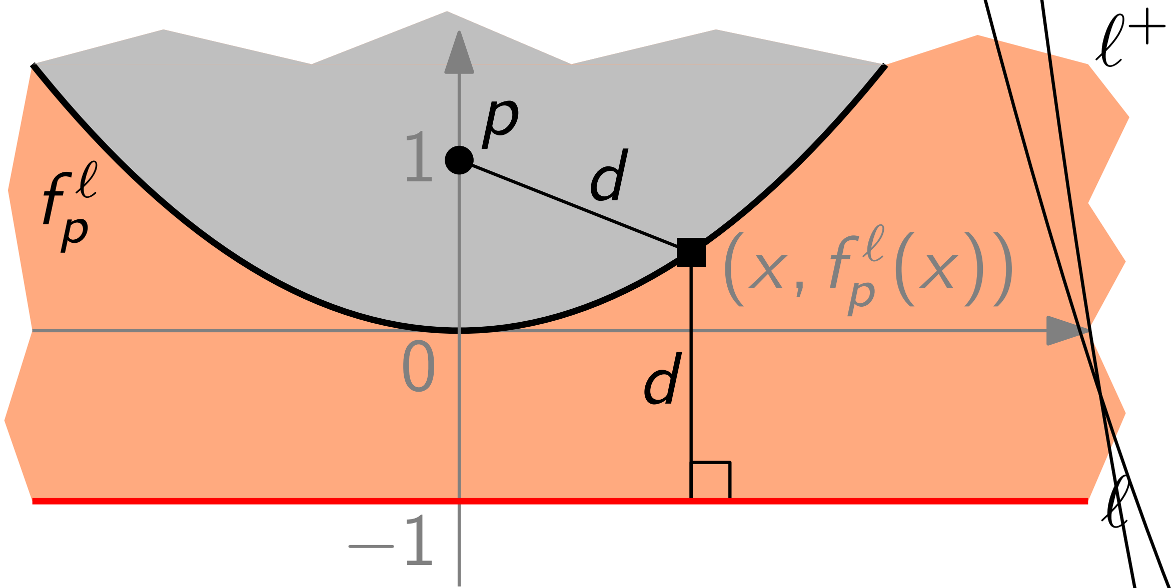

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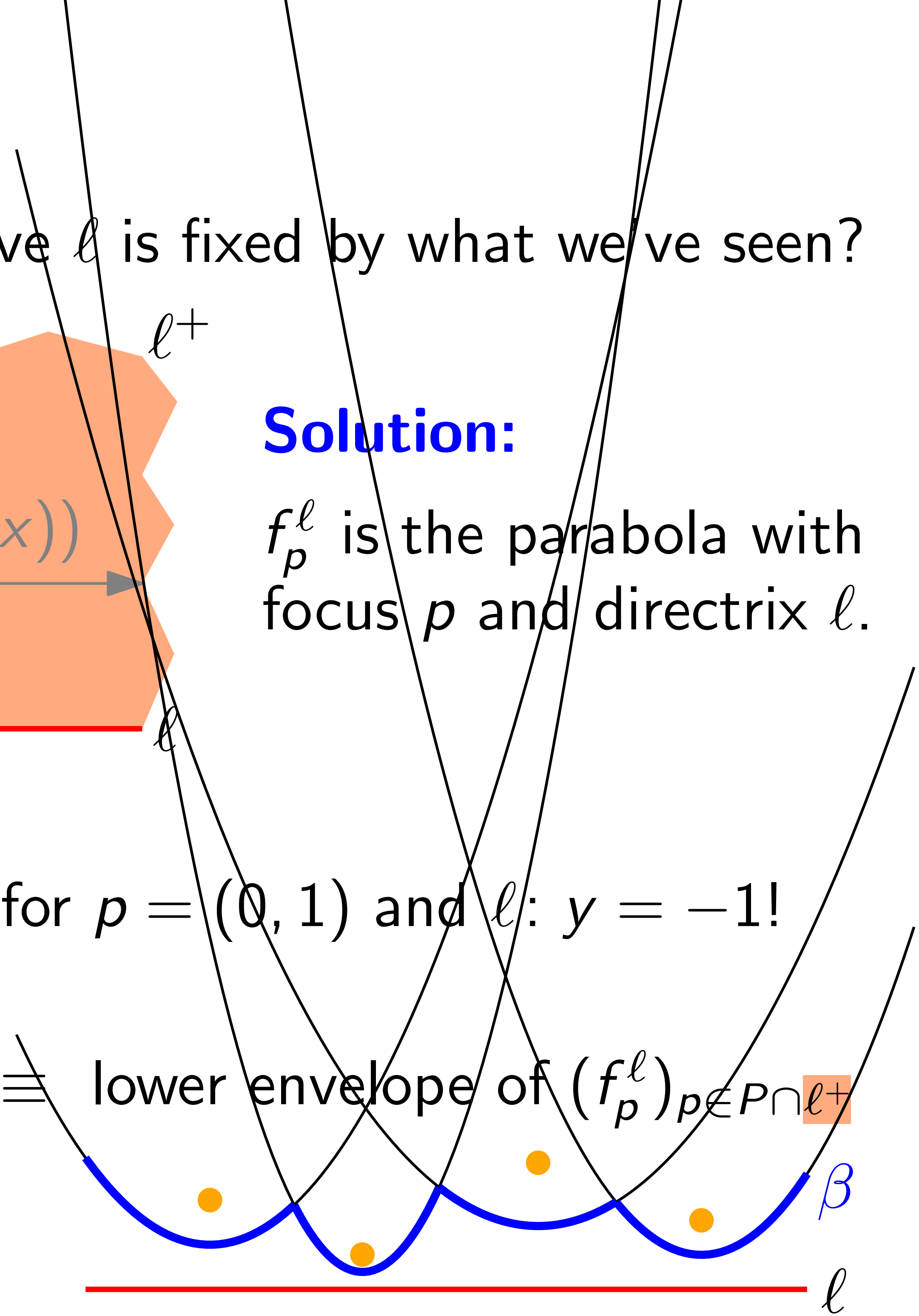
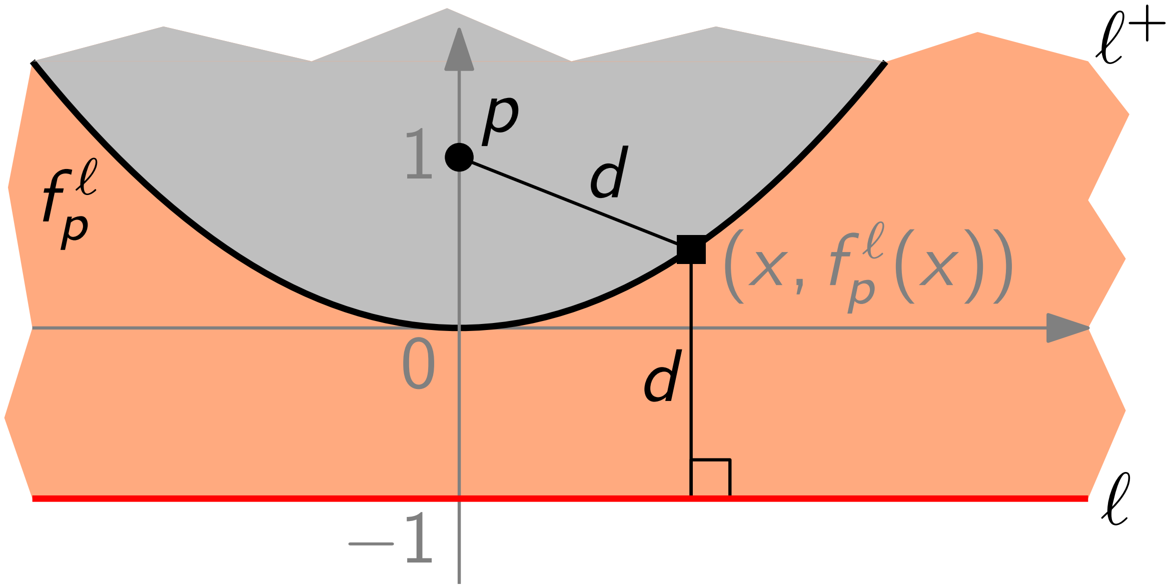


Sweep?

Which part of the plane above ℓ is fixed by what we've seen?



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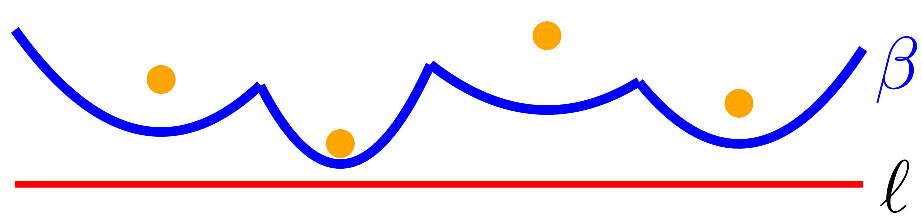

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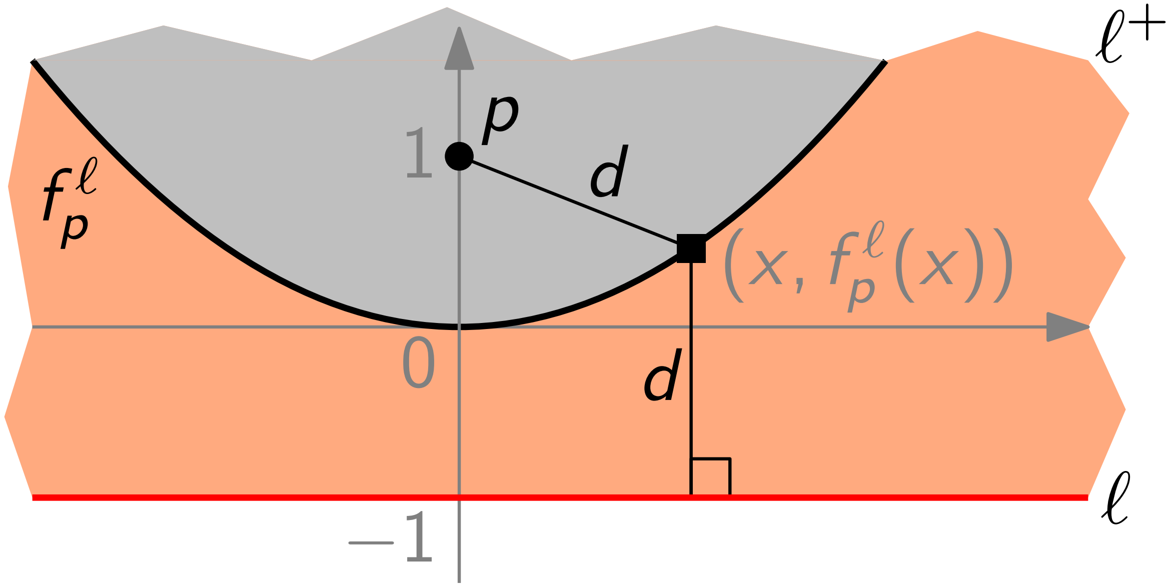

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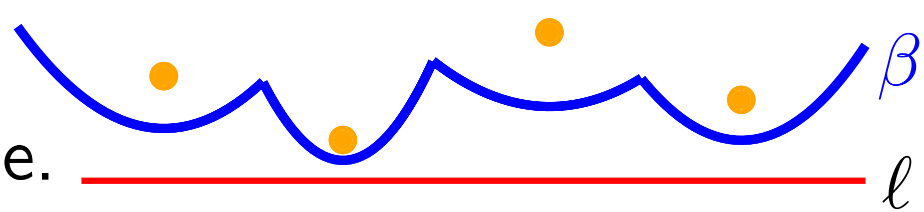
Observation. β is x-monotone.



The beachline β

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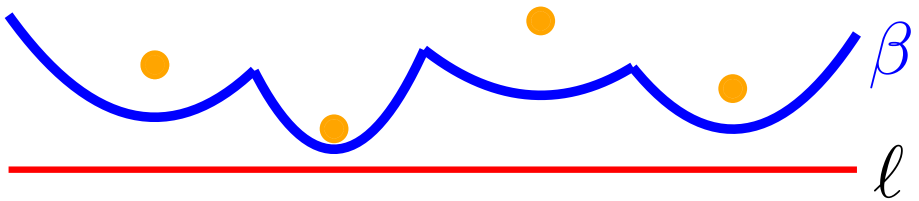

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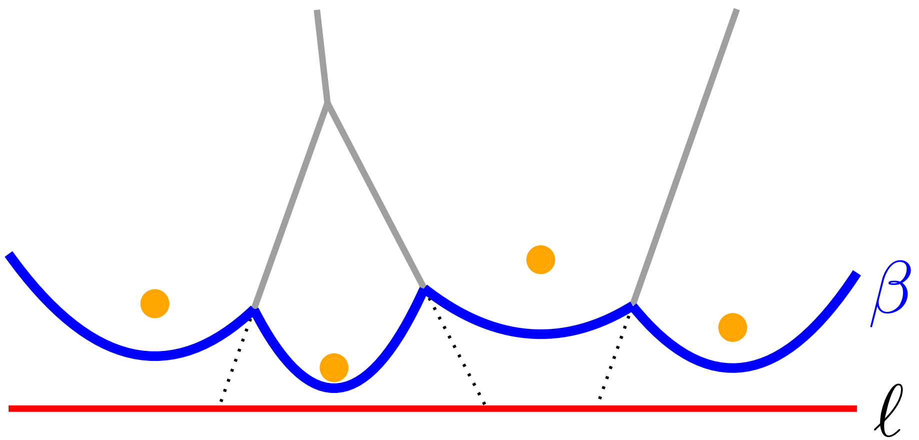

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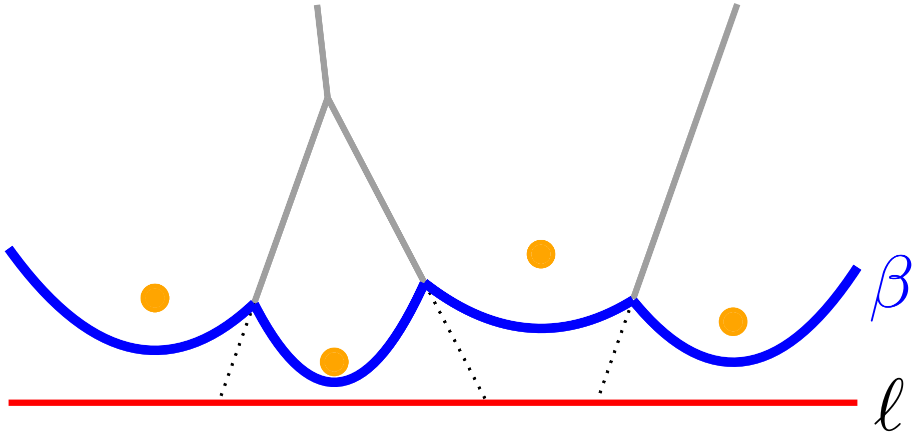

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Lemma. The Voronoi vtc correspond 1:1 to circle events.

Fortune's Sweep

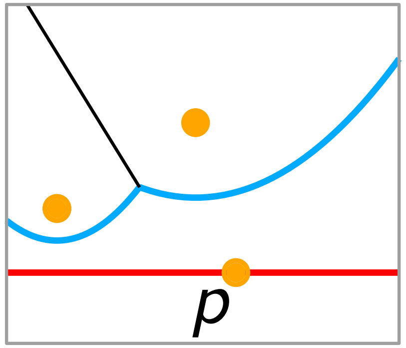
VoronoiDiagram($P \subset \mathbb{R}^2$)
$\mathcal{Q} \leftarrow$ new PriorityQueue(P) // site events sorted acc. y-coord.
$\mathcal{T} \leftarrow$ new BalancedBinarySearchTree() // sweep status (β)
$\mathcal{D} \leftarrow$ new DCEL() // to-be Vor(P)
while not $\mathcal{Q}$.empty() **do**

treat remaining internal nodes of $\mathcal{T}$ ($\equiv$ unbnd. edges of Vor(P))
return $\mathcal{D}$

Fortune's Sweep

```
VoronoiDiagram(P ⊂ ℝ²)
  𝒬 ← new PriorityQueue(P)   // site events sorted acc. y-coord.
  𝒯 ← new BalancedBinarySearchTree()   // sweep status (β)
  𝒟 ← new DCEL()   // to-be Vor(P)
  while not 𝒬.empty() do
      p ← 𝒬.ExtractMax()
      if p site event then
          HandleSiteEvent(p)
      else
          α ← arc on β that will disappear
          HandleCircleEvent(α)
  treat remaining internal nodes of 𝒯 (≡ unbnd. edges of Vor(P))
  return 𝒟
```

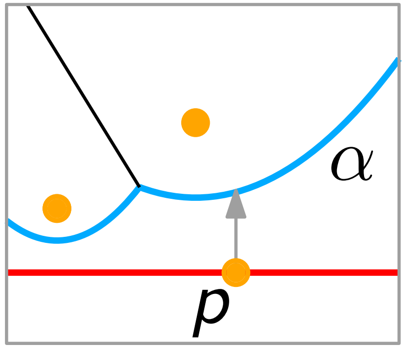
Handling Events



HandleSiteEvent(point p)

HandleCircleEvent(arc α)

Handling Events



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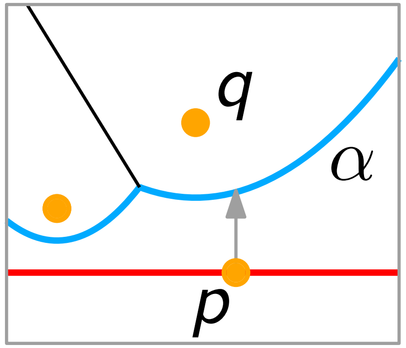

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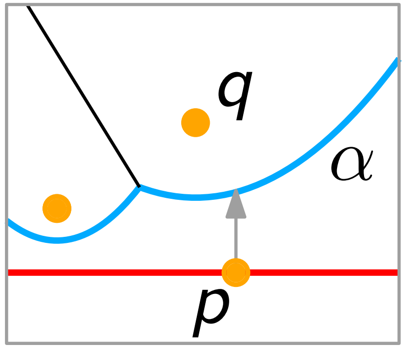

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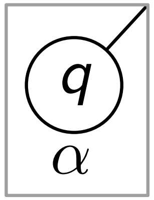

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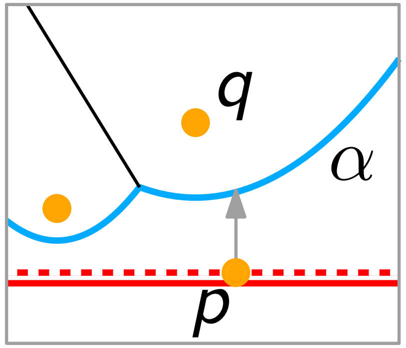

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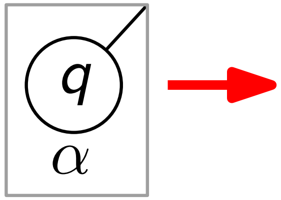

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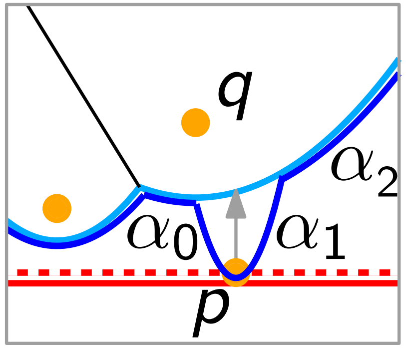

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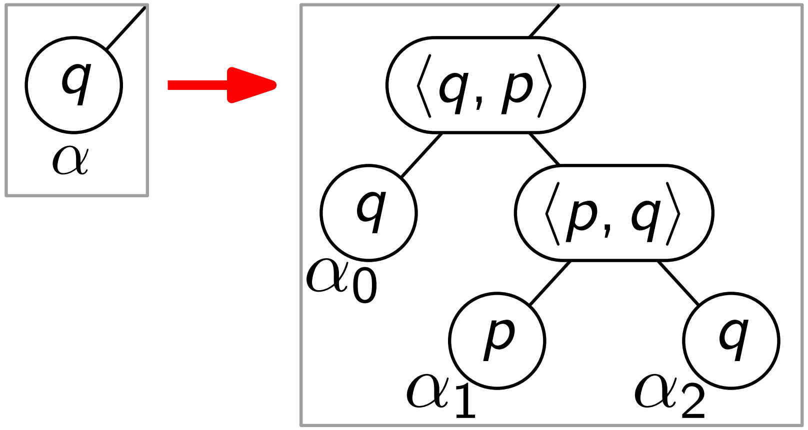

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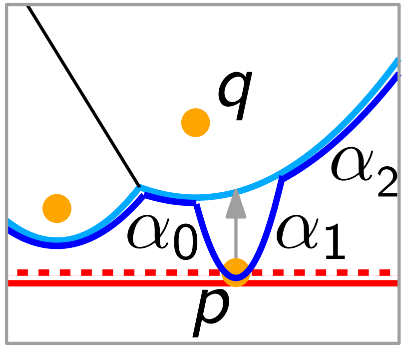

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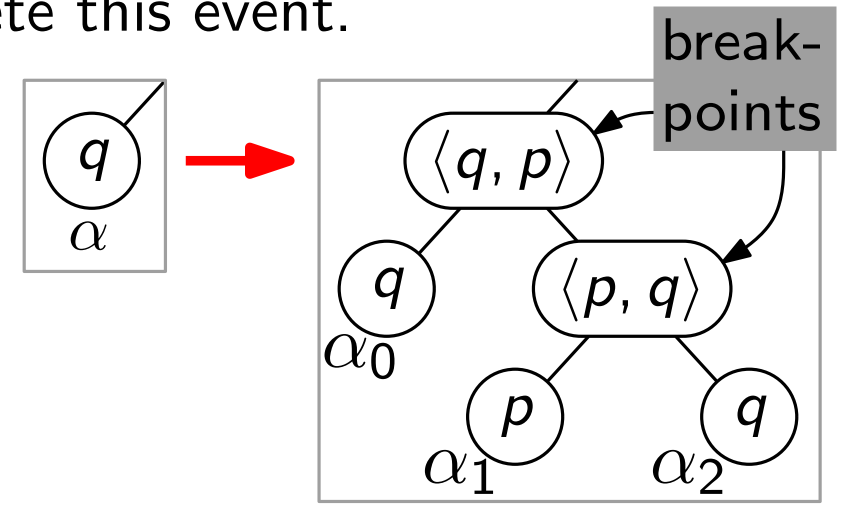

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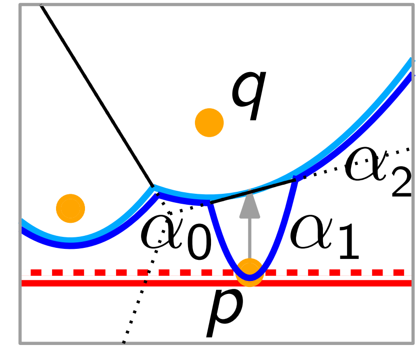

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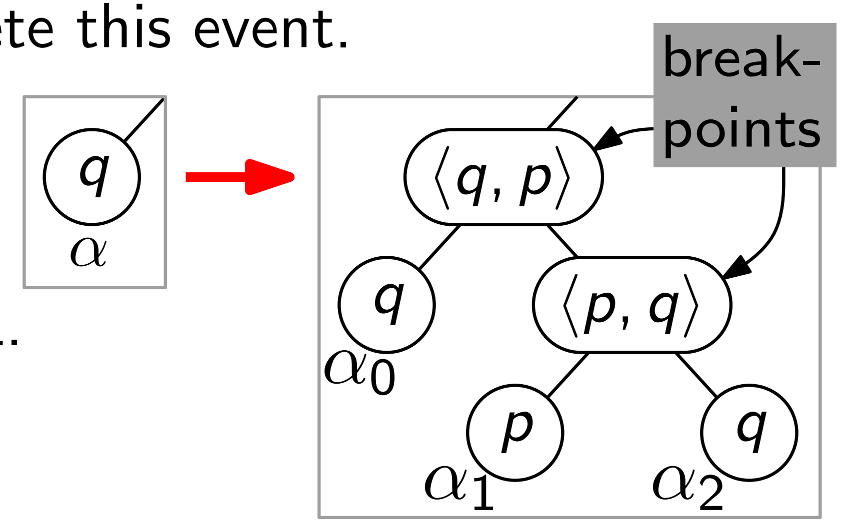

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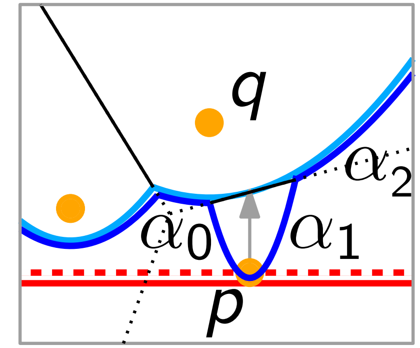

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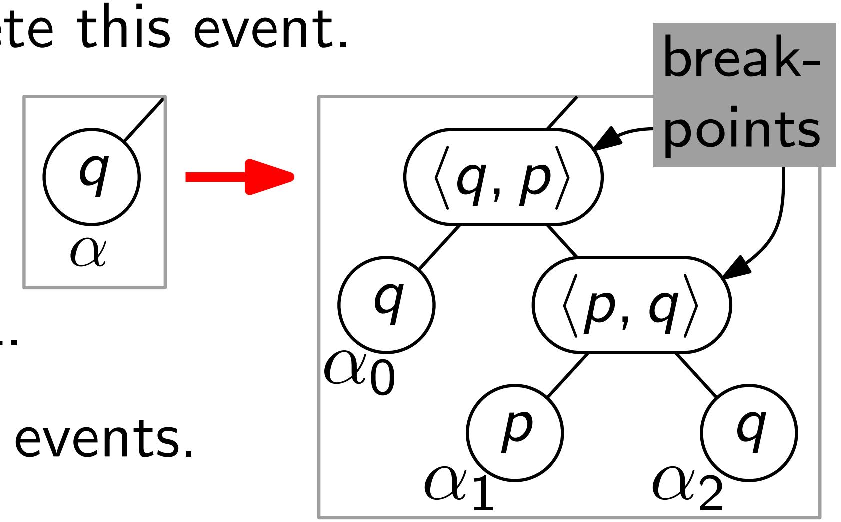

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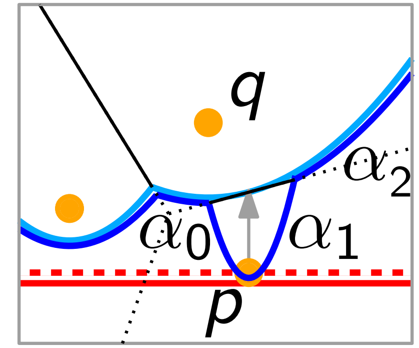

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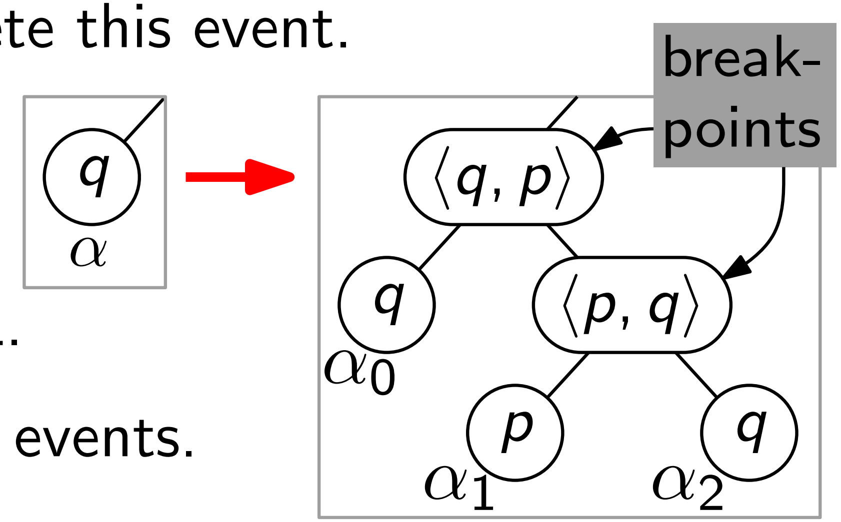

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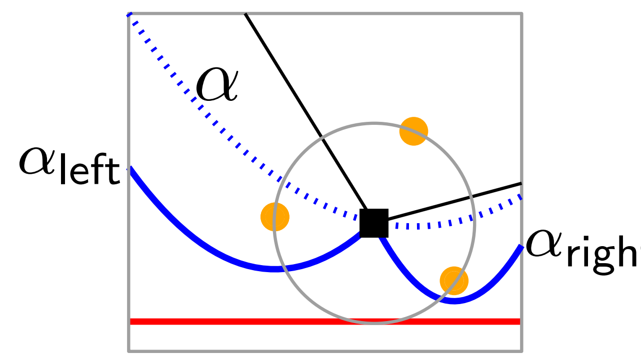

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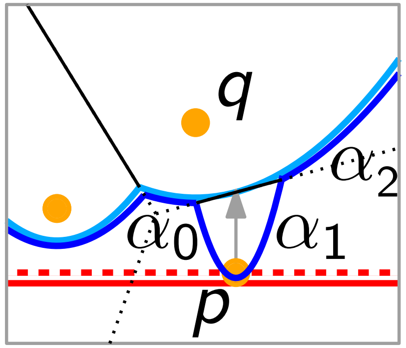

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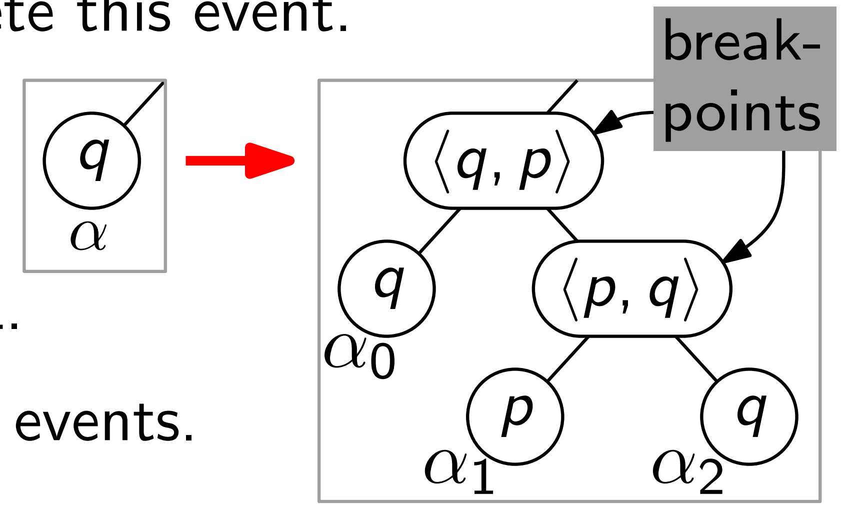

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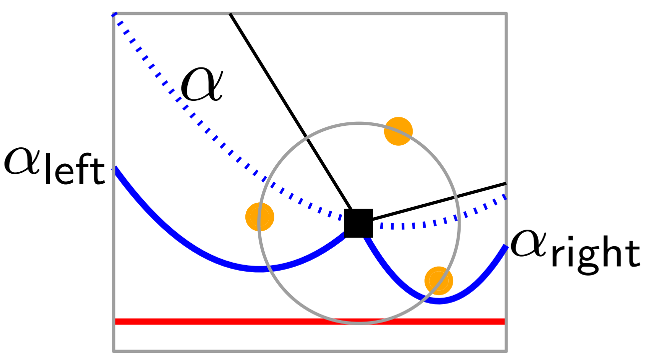

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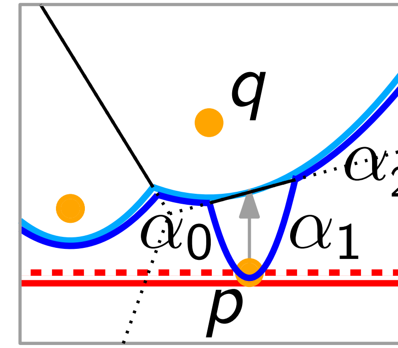

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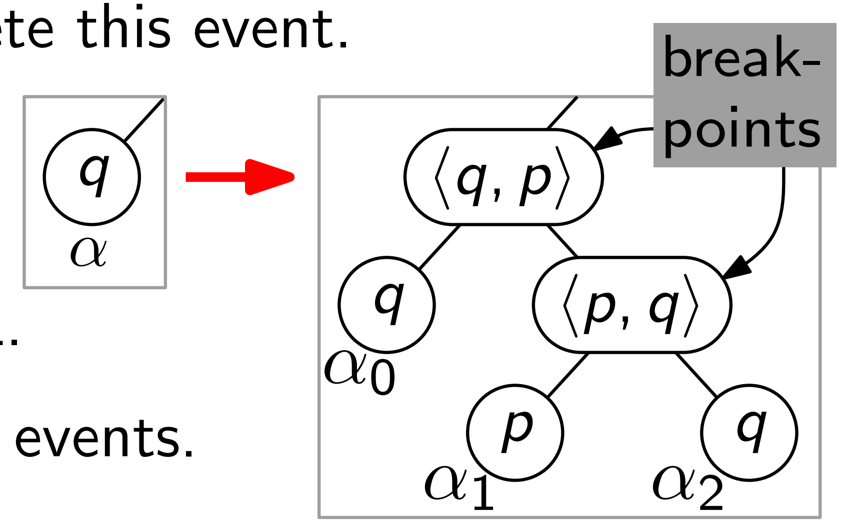

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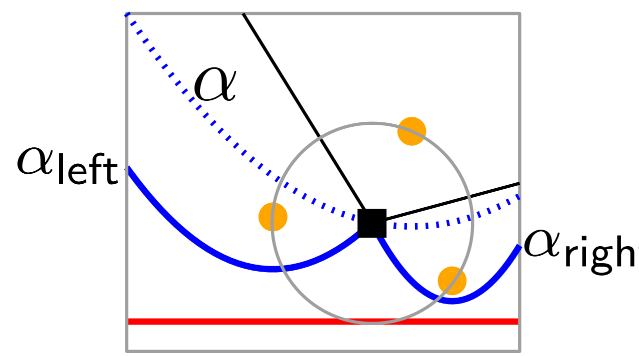

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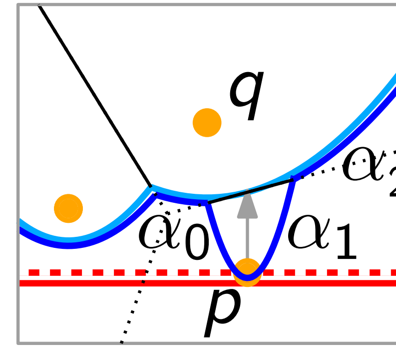

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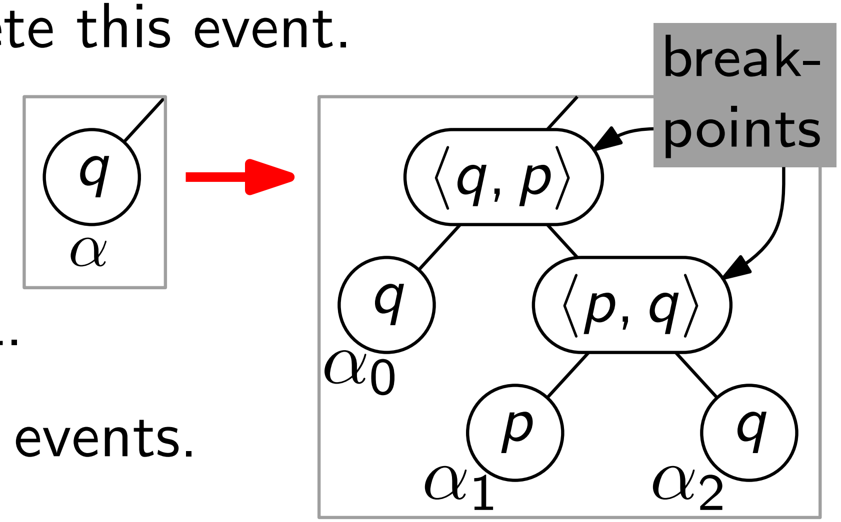

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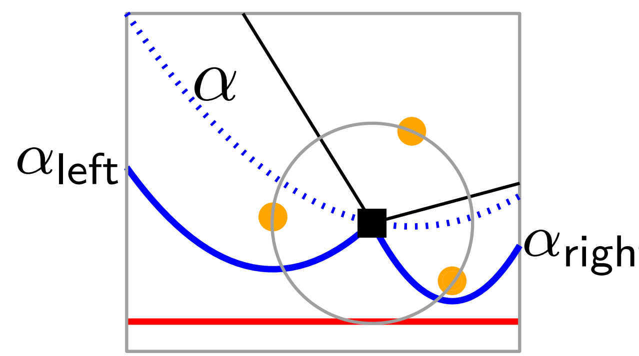

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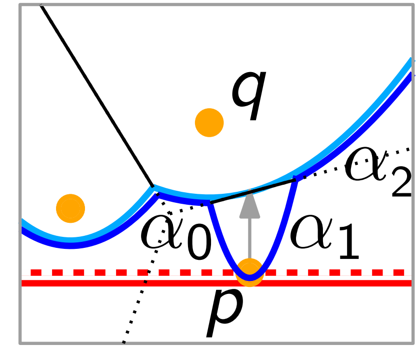

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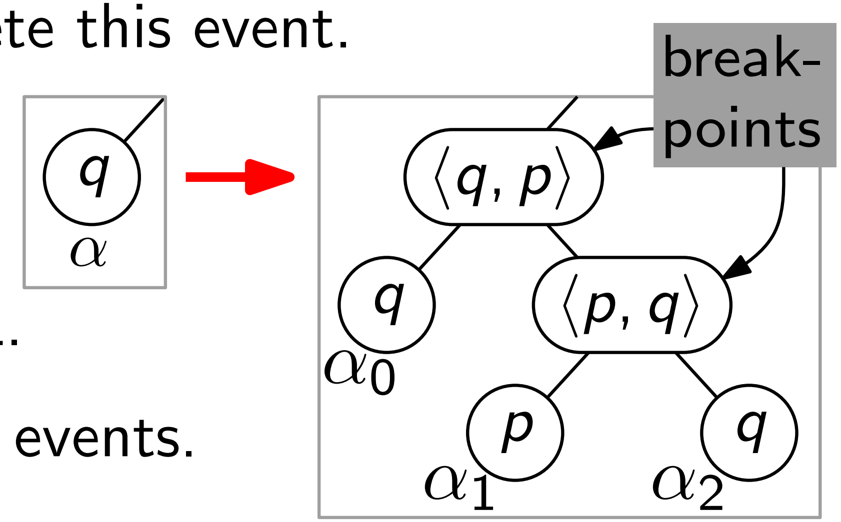

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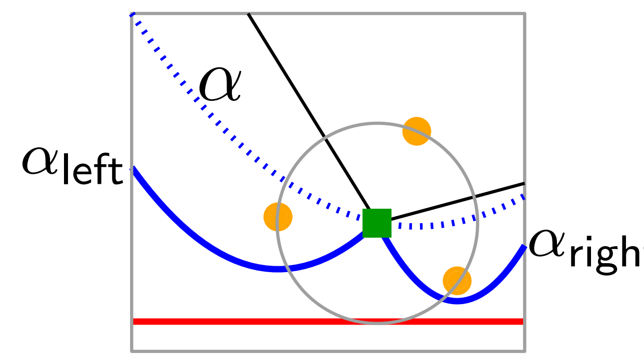

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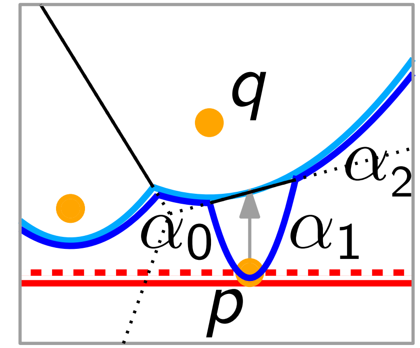

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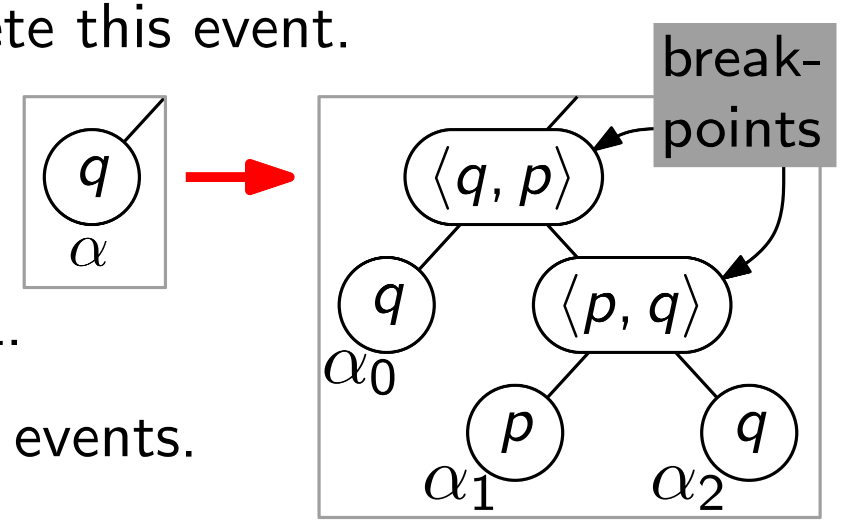

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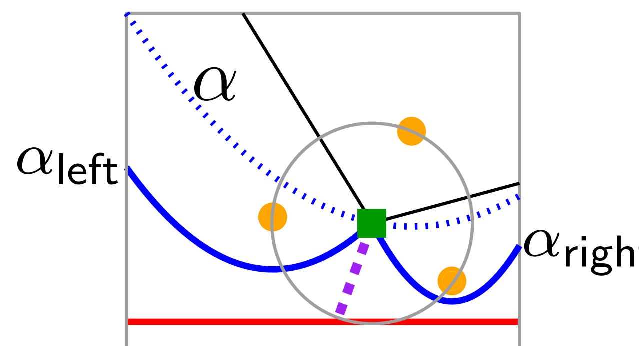

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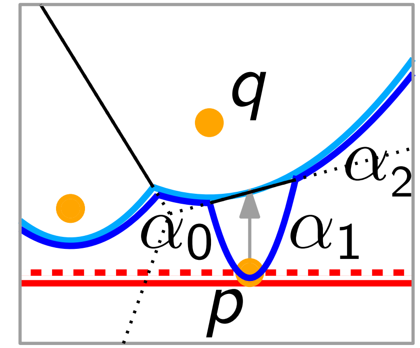

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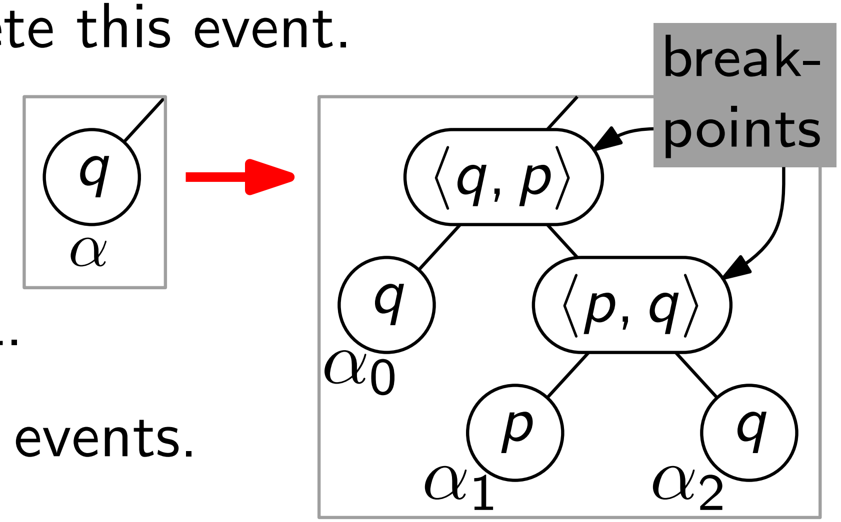

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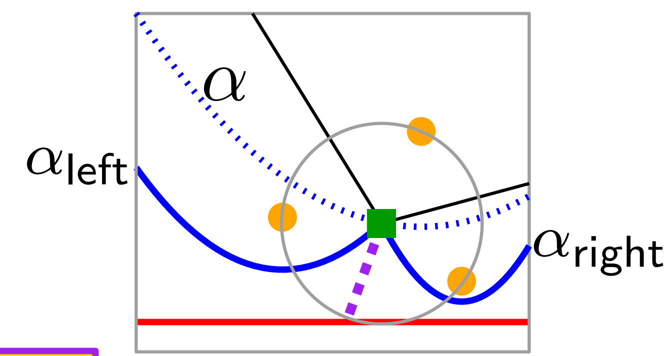
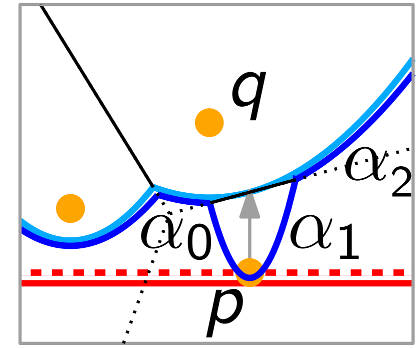


- $\mathcal{T}$.delete(α); update breakpts
- Delete all circle events involving α from $\mathcal{Q}$.
- Add Vor-vtx $\alpha_{\text{left}} \cap \alpha_{\text{right}}$ and Vor-edge $\langle \alpha_{\text{left}}, \alpha_{\text{right}} \rangle$ to DCEL.
- Check $\langle \cdot, \alpha_{\text{left}}, \alpha_{\text{right}} \rangle$ and $\langle \alpha_{\text{left}}, \alpha_{\text{right}}, \cdot \rangle$ for circle events.

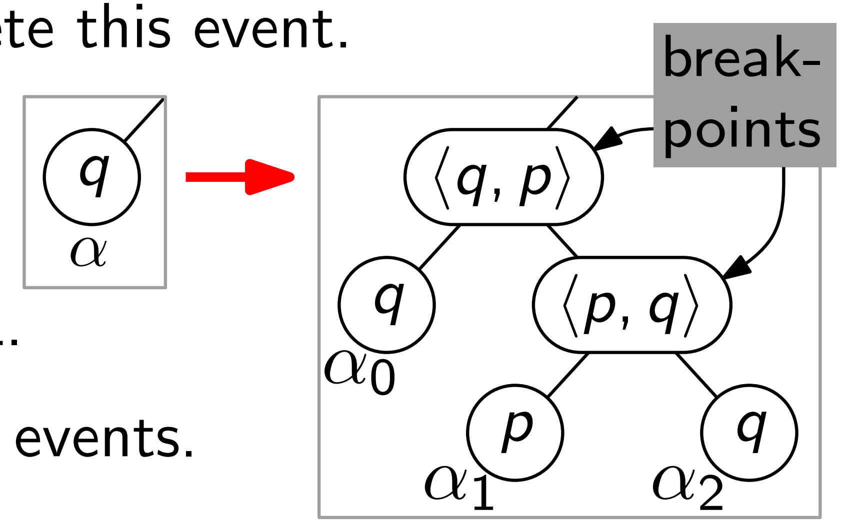
Handling Events



HandleSiteEvent(point p)

- Search in $\mathcal{T}$ for the arc α vertically above p.
  If α has pointer to circle event in $\mathcal{Q}$, delete this event.
- Split α into α_0 and α_2.
  Let α_1 be the new arc of p.
- Add Vor-edges $\langle q, p \rangle$ and $\langle p, q \rangle$ to DCEL.
- Check $\langle \cdot, \alpha_0, \alpha_1 \rangle$ and $\langle \alpha_1, \alpha_2, \cdot \rangle$ for circle events.

In $\mathcal{T}$:



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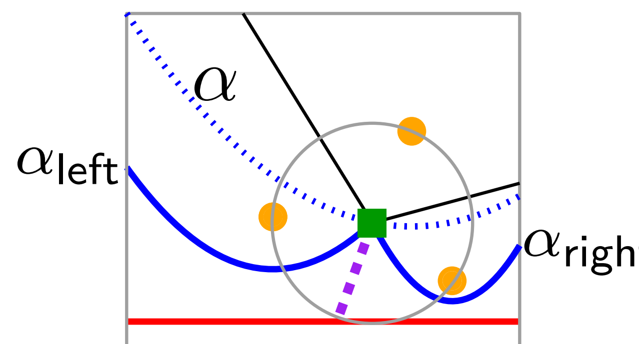

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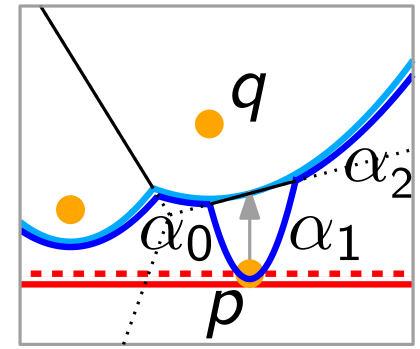
- Check $\langle \cdot, \alpha_{\text{left}}, \alpha_{\text{right}} \rangle$ and $\langle \alpha_{\text{left}}, \alpha_{\text{right}}, \cdot \rangle$ for circle events.

Running time?

Handling Events



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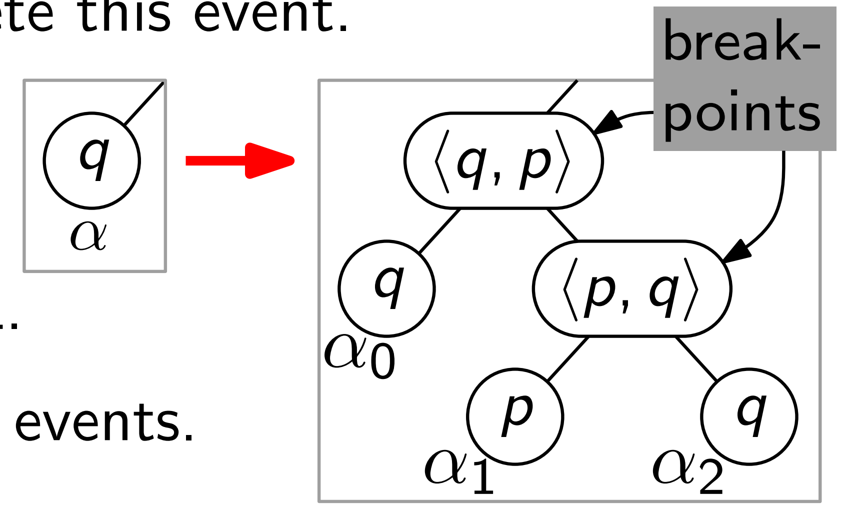
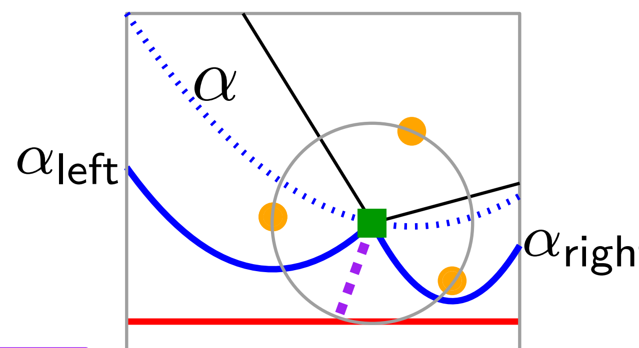
In $\mathcal{T}$:



HandleCircleEvent(arc α)



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- Check $\langle \cdot, \alpha_{\text{left}}, \alpha_{\text{right}} \rangle$ and $\langle \alpha_{\text{left}}, \alpha_{\text{right}}, \cdot \rangle$ for circle events.

Running time? $O(\log n)$ per event...

Running Time?

```
VoronoiDiagram(P ⊂ ℝ²)
 𝒬 ← new PriorityQueue(P)   // site events sorted acc. y-coord.
 𝒯 ← new BalancedBinarySearchTree()   // sweep status (β)
 𝒟 ← new DCEL()   // to-be Vor(P)
 while not 𝒬.empty() do
     p ← 𝒬.ExtractMax()
     if p site event then
         HandleSiteEvent(p)
     else
         α ← arc on β that will disappear
         HandleCircleEvent(α)
 treat remaining internal nodes of 𝒯 (≡ unbnd. edges of Vor(P))
 return 𝒟
```

Running Time?

```
VoronoiDiagram(P ⊂ ℝ²)
 Q ← new PriorityQueue(P)   // site events sorted acc. y-coord.
 T ← new BalancedBinarySearchTree()   // sweep status (β)
 D ← new DCEL()   // to-be Vor(P)
 while not Q.empty() do
     p ← Q.ExtractMax()
     if p site event then
         HandleSiteEvent(p)      exactly n such events
     else
         α ← arc on β that will disappear
         HandleCircleEvent(α)
 treat remaining internal nodes of T (≡ unbnd. edges of Vor(P))
 return D
```

Running Time?

VoronoiDiagram($P \subset \mathbb{R}^2$)

$\mathcal{Q} \leftarrow$ new PriorityQueue(P) // site events sorted acc. y-coord.
$\mathcal{T} \leftarrow$ new BalancedBinarySearchTree() // sweep status (β)
$\mathcal{D} \leftarrow$ new DCEL() // to-be Vor(P)
while not $\mathcal{Q}$.empty() **do**
    $p \leftarrow \mathcal{Q}$.ExtractMax()
    if p site event **then**
        HandleSiteEvent(p)   exactly n such events
    else
        $\alpha \leftarrow$ arc on β that will disappear
        HandleCircleEvent(α)   at most $2n - 5$ such events
treat remaining internal nodes of $\mathcal{T}$ ($\equiv$ unbnd. edges of Vor(P))
return $\mathcal{D}$

Summary

Theorem. Given a set P of n pts in the plane, Fortune's sweep computes $\mathrm{Vor}(P)$ in $O(n \log n)$ time and $O(n)$ space.

Summary

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Steven Fortune
Bell Labs

Steven Fortune. A sweepline algorithm for Voronoi diagrams. *Proc. 2nd Annual ACM Symposium on Computational Geometry.* Yorktown Heights, NY, pp. 313–322. 1986.