Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Computational Geometry

Winter term 2016/17

Point Localization

or

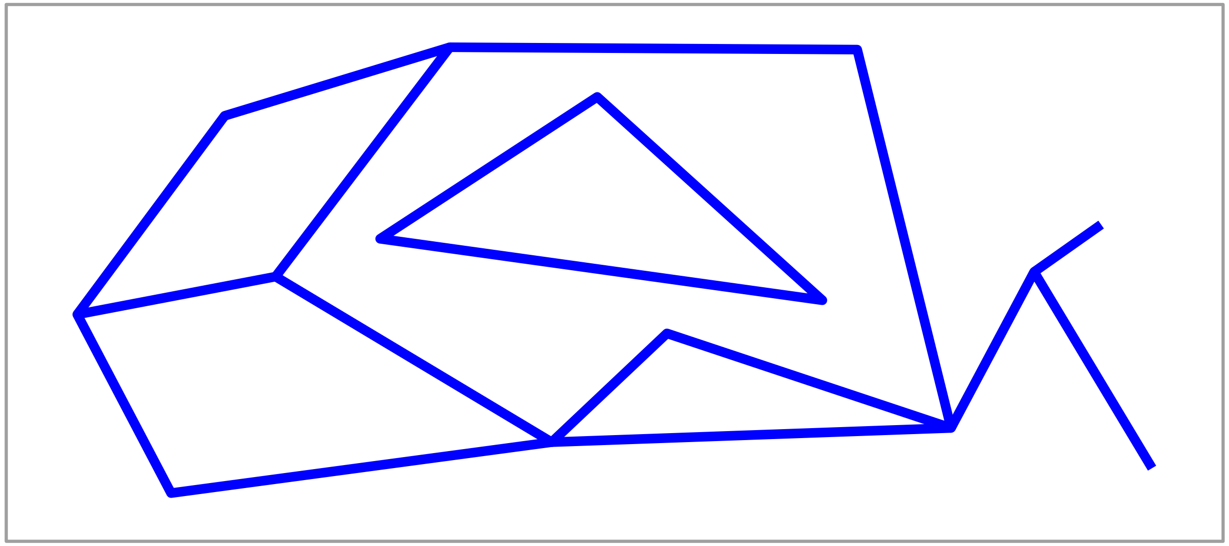
Where am I?

Lecture #6

Joachim Spoerhase

Chair for Computer Science I

What's the Problem?



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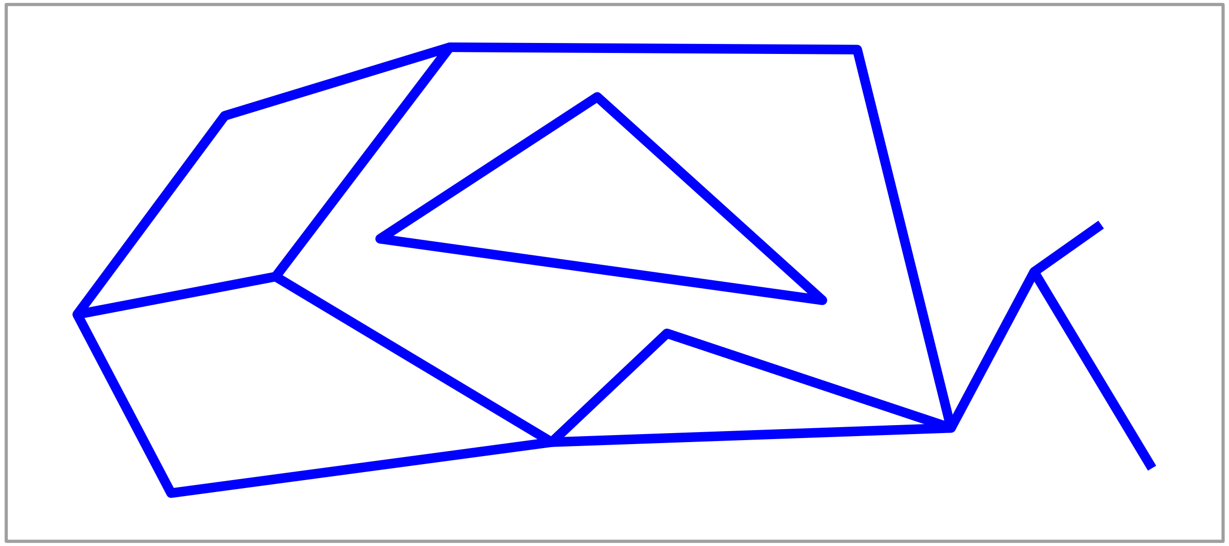

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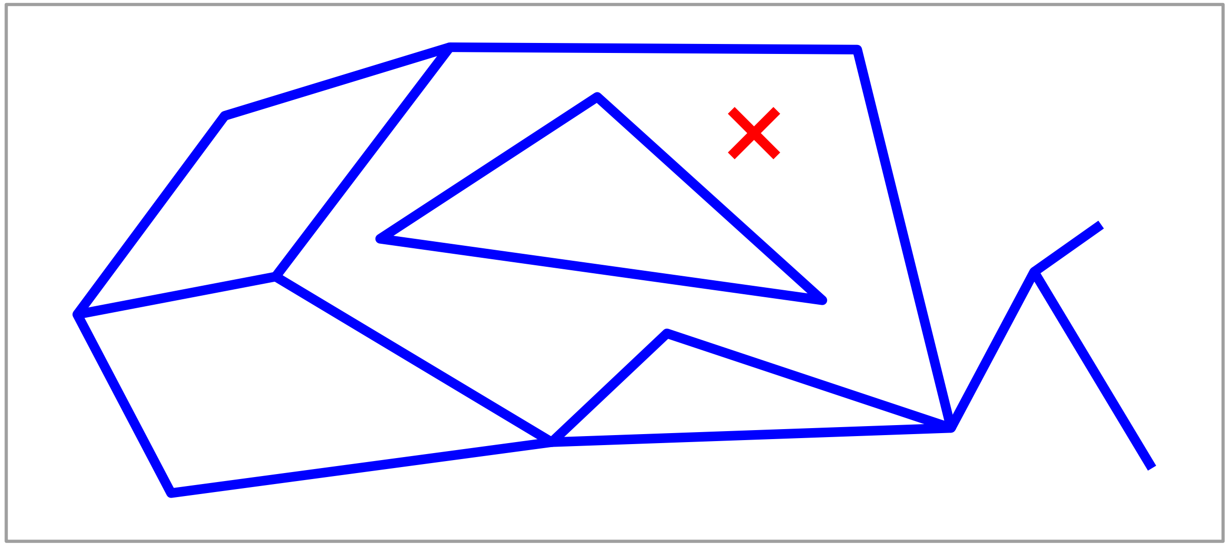

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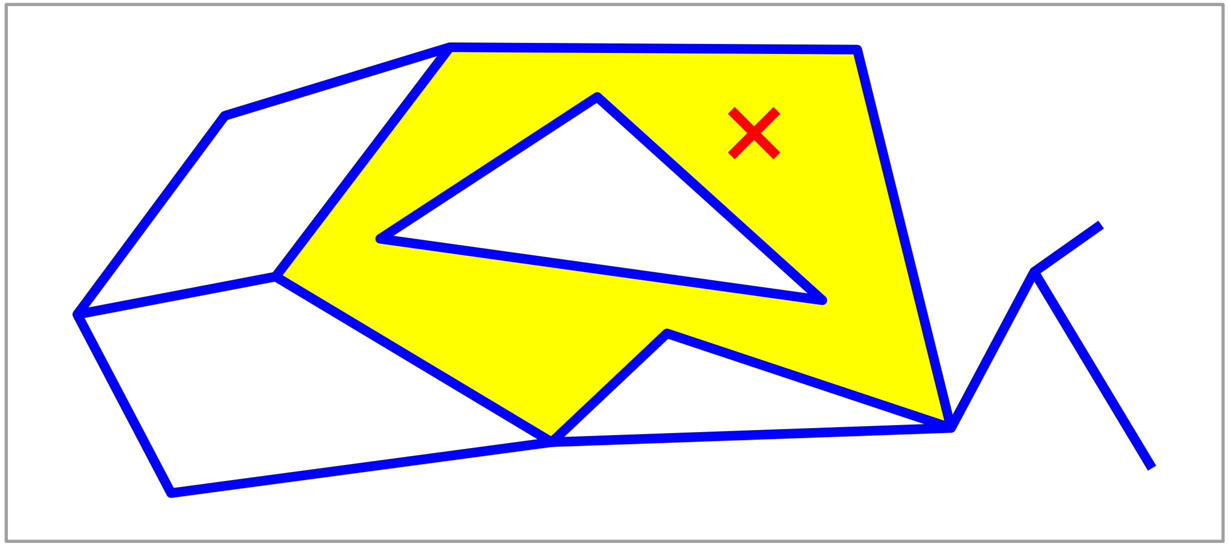

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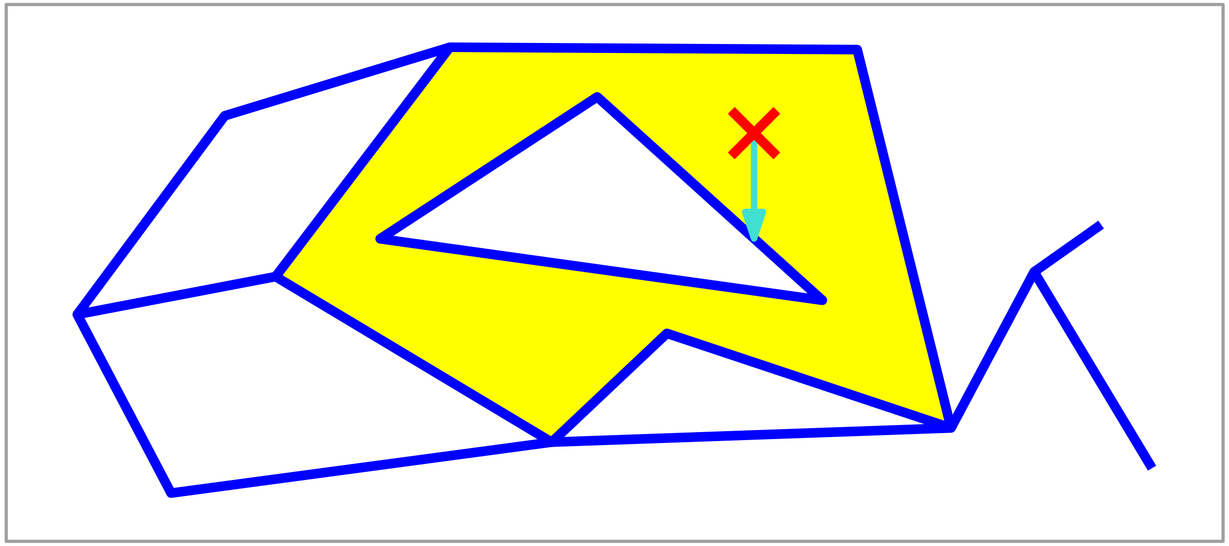

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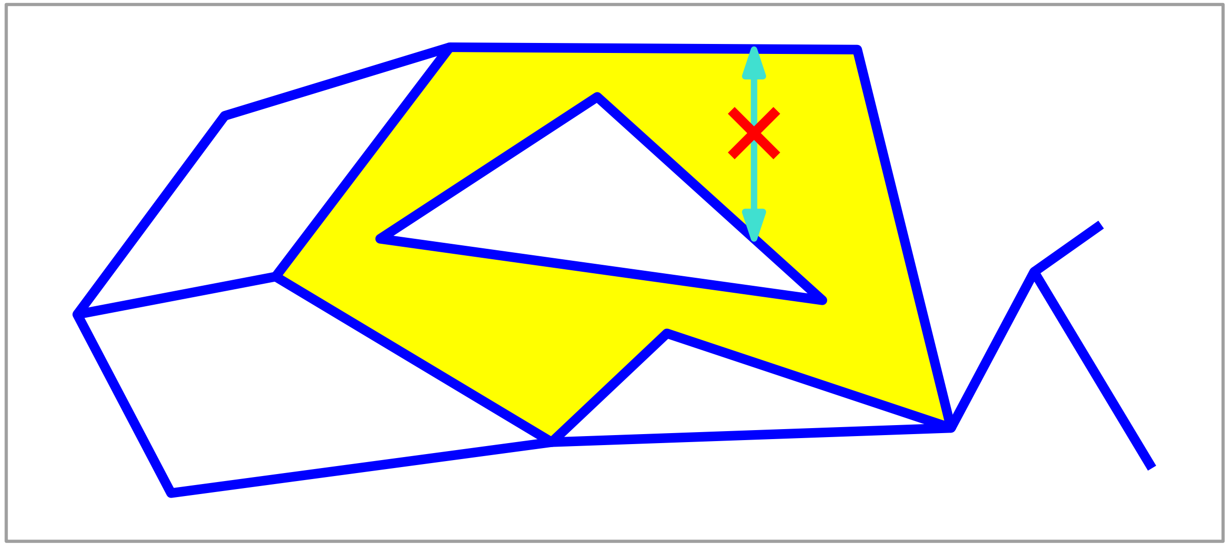

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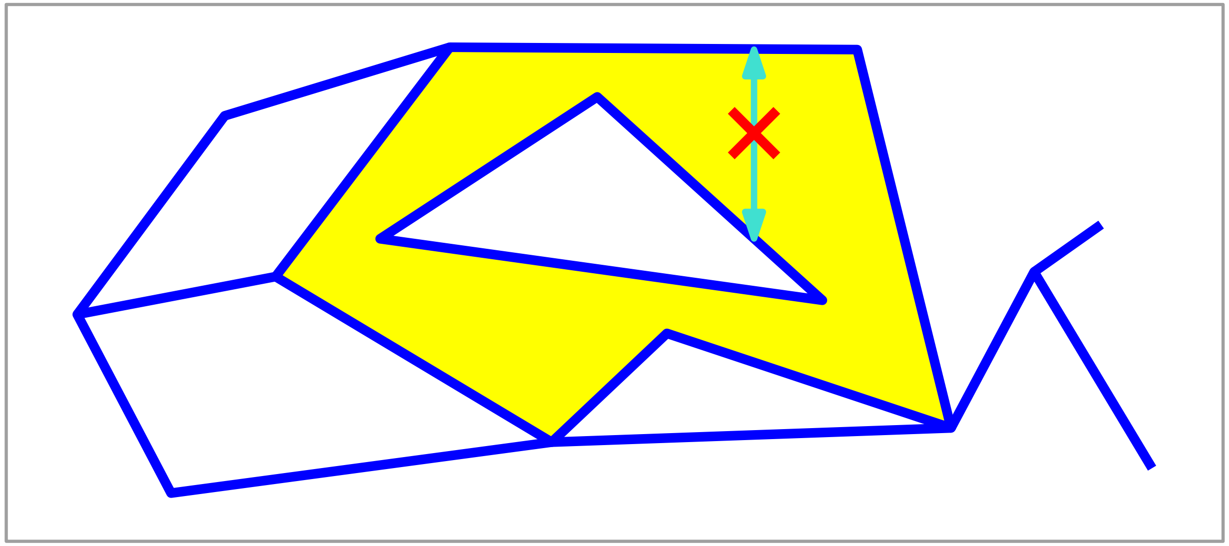

[2 min]

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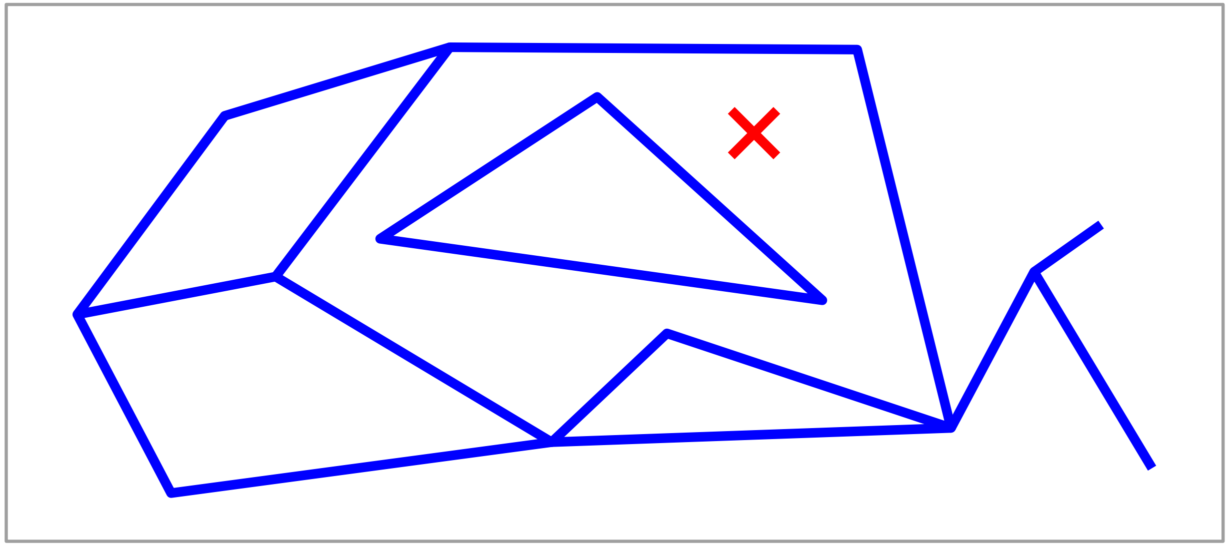

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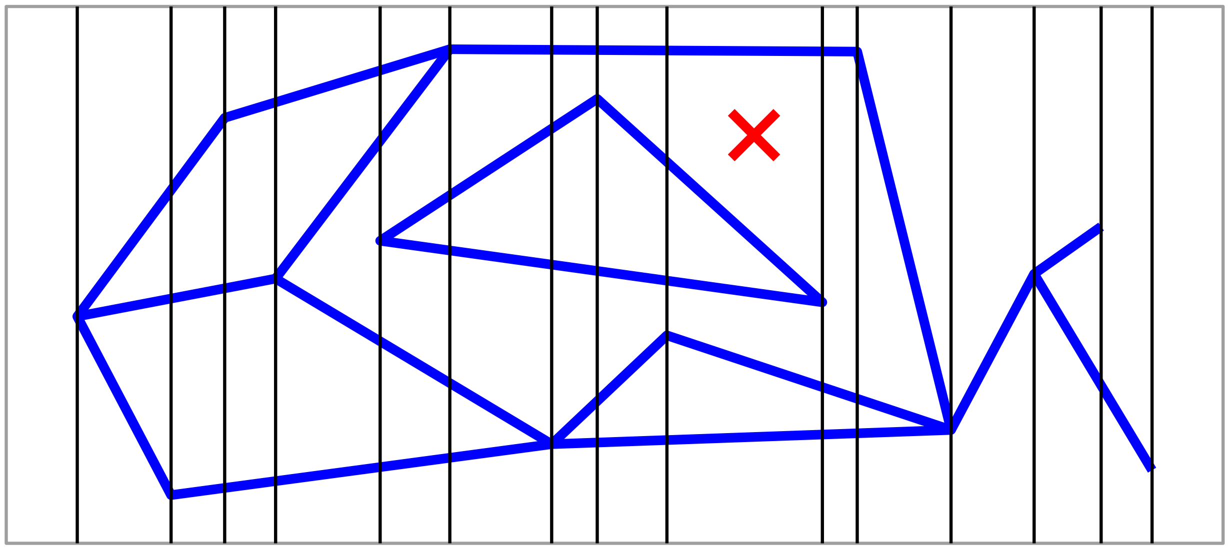

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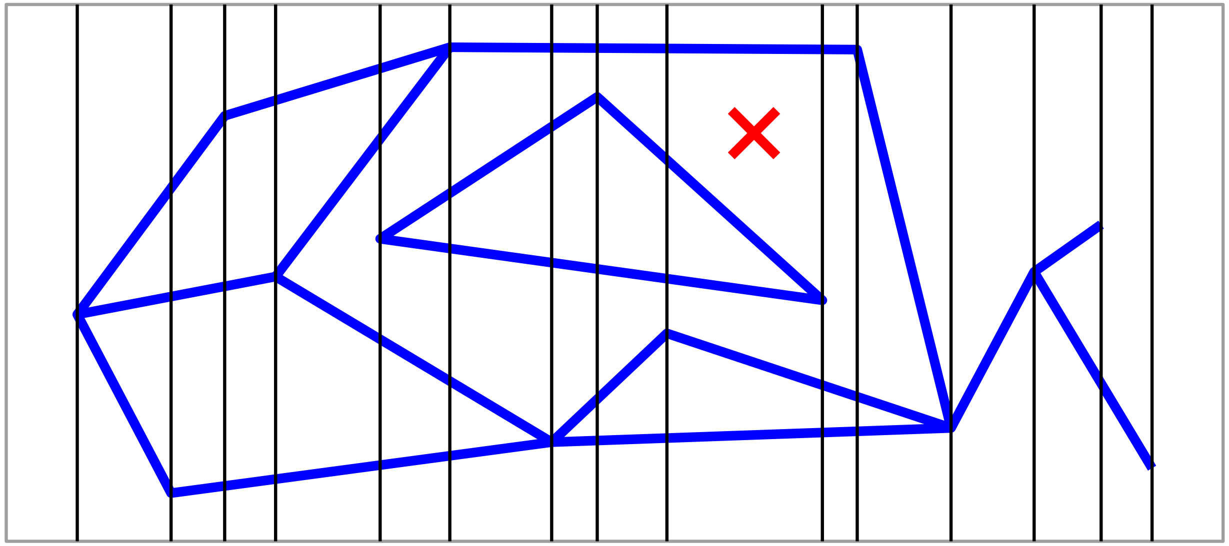

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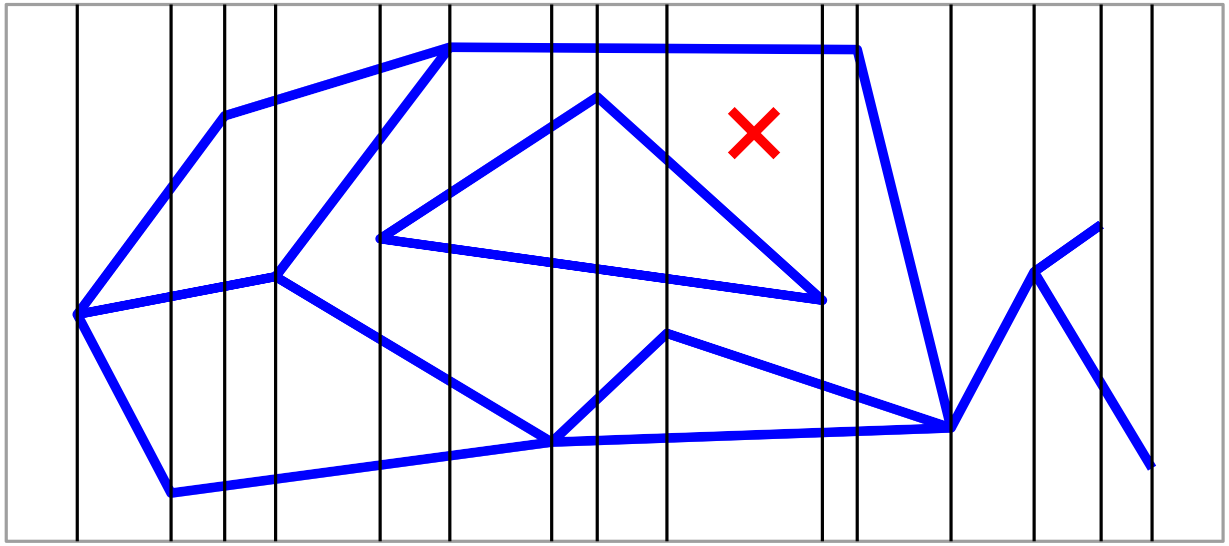

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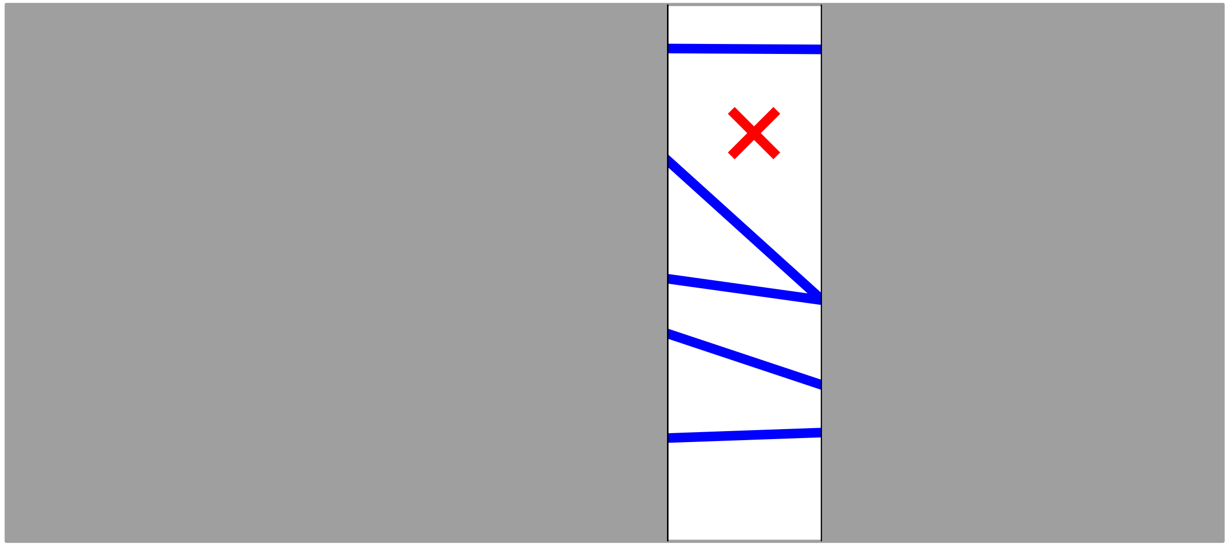

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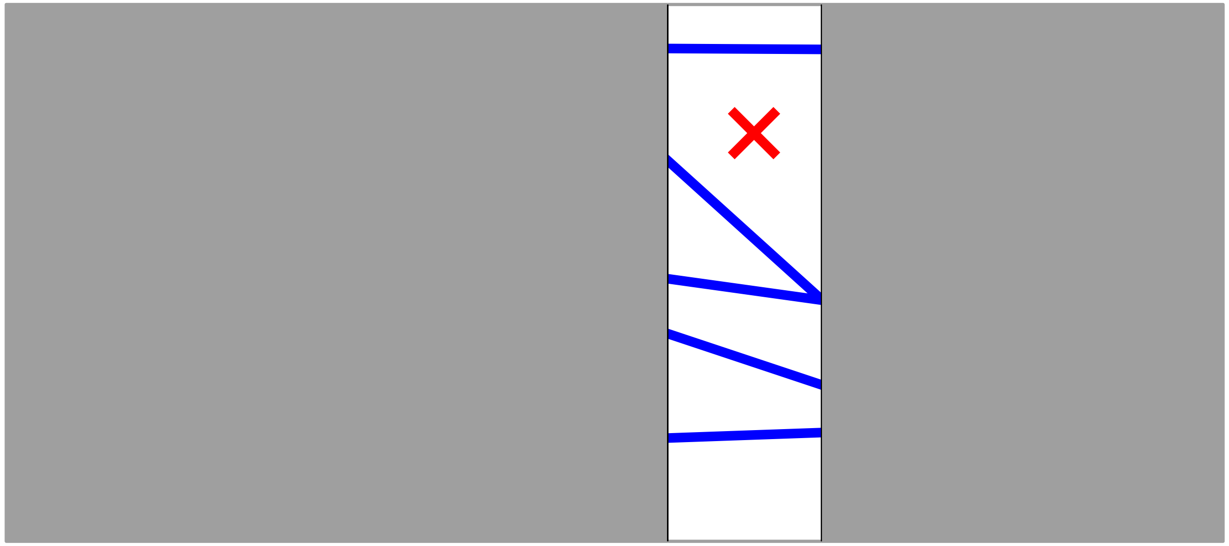

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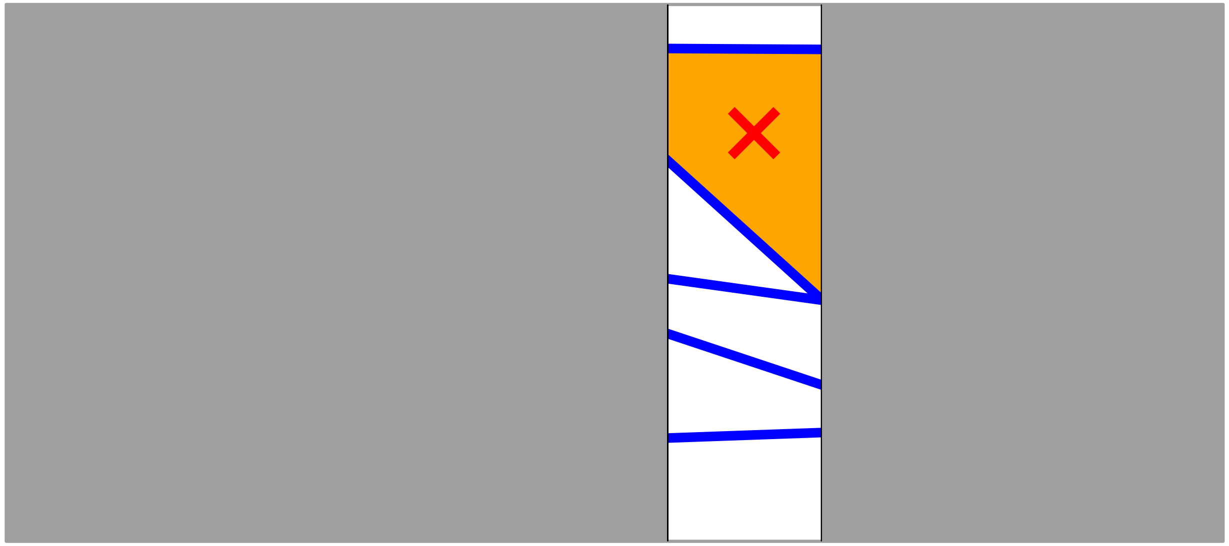

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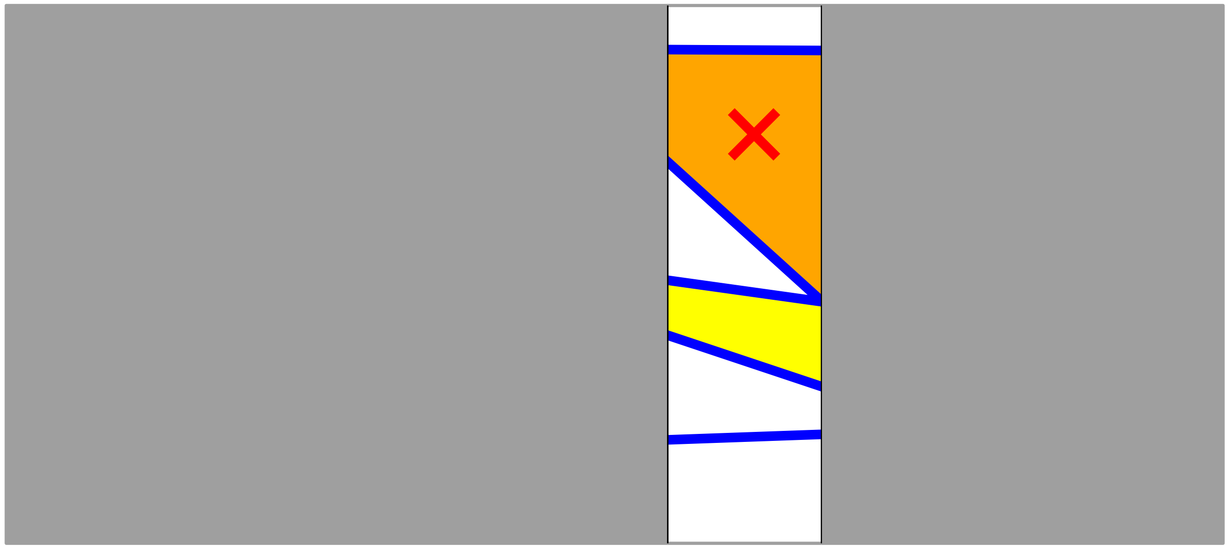

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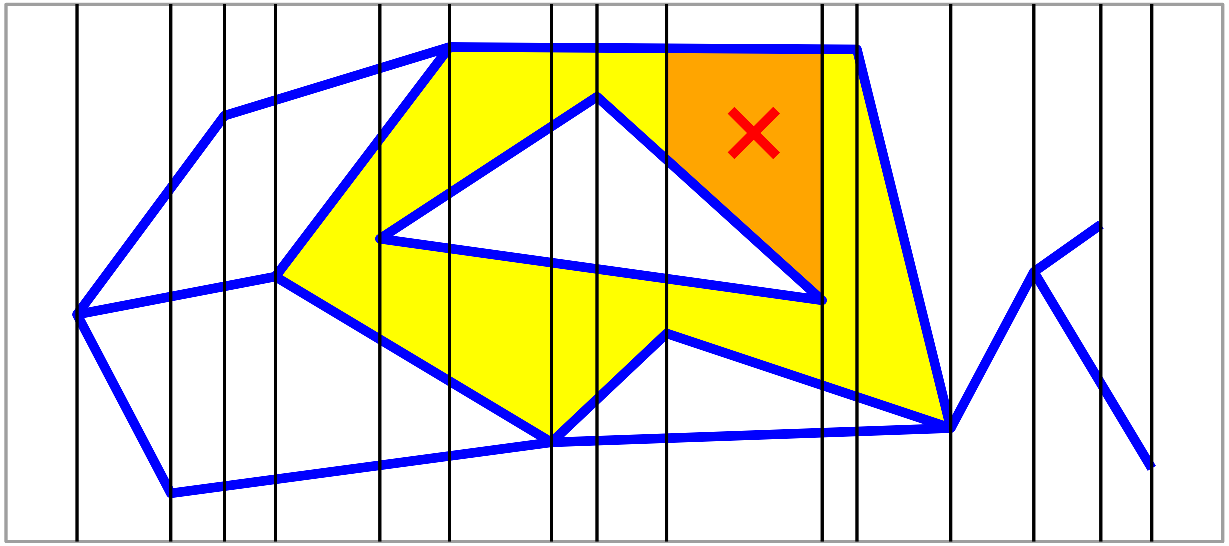

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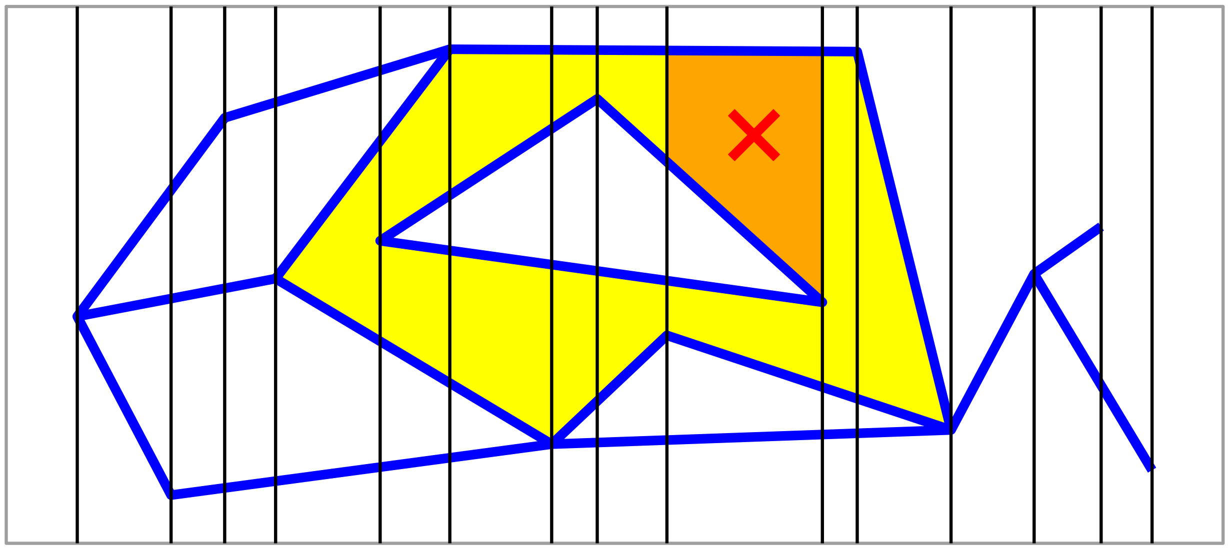

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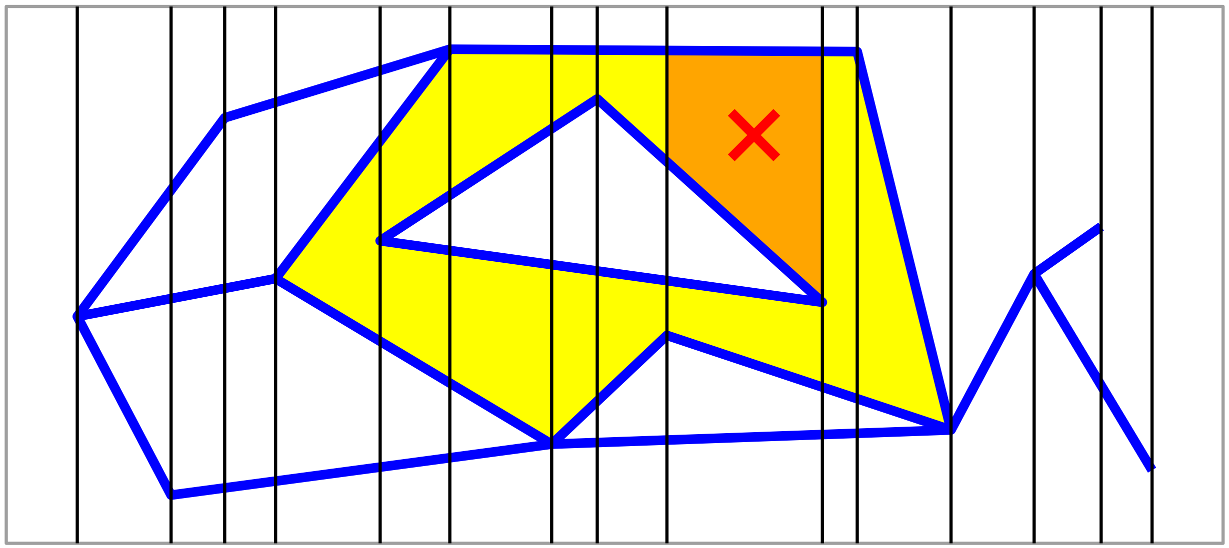

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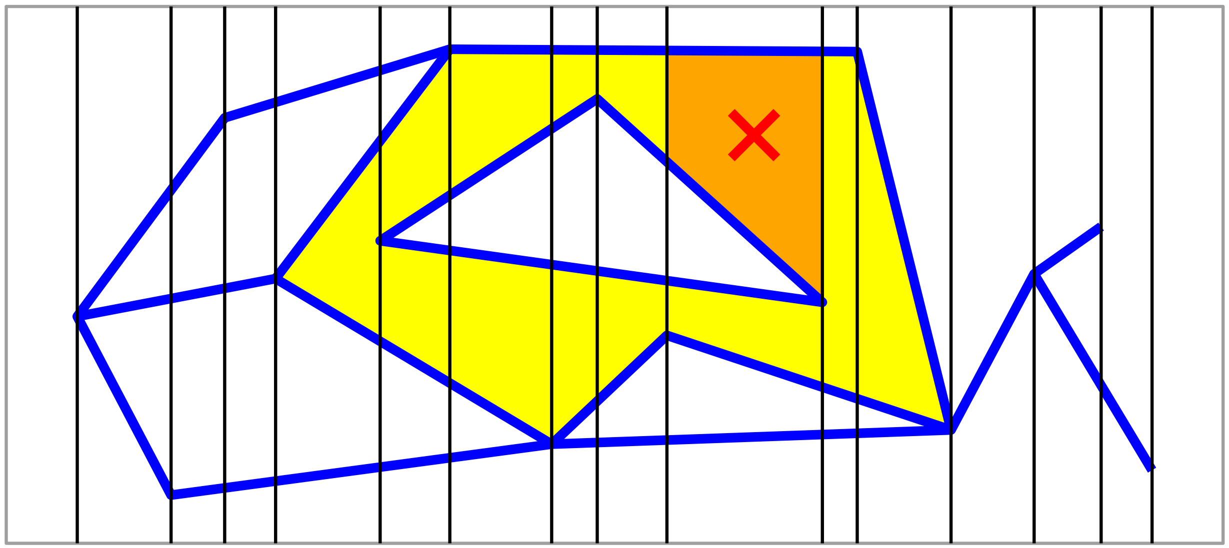

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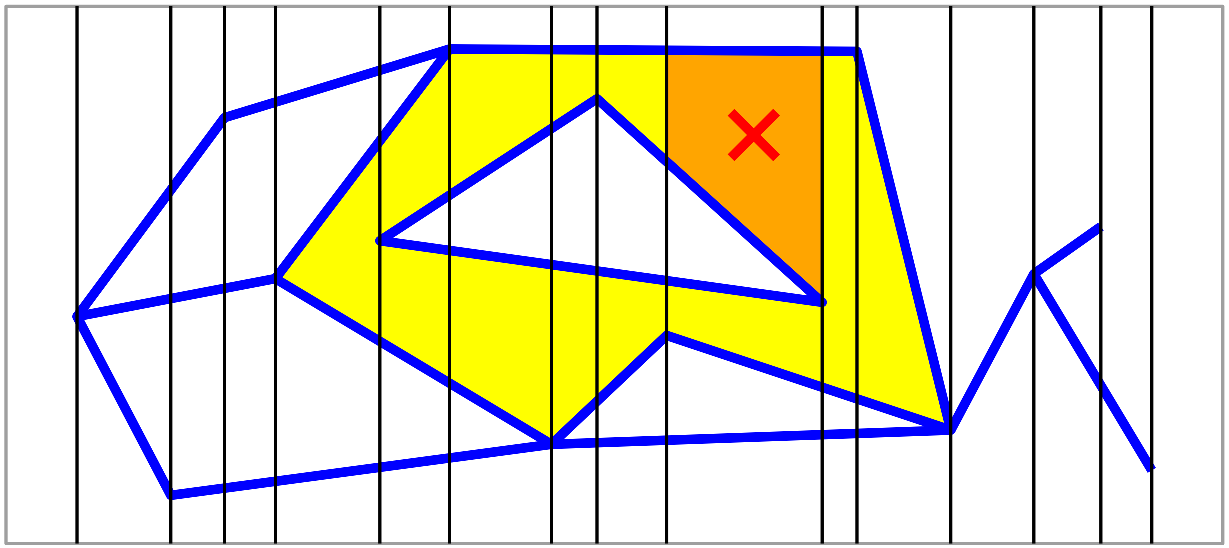

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But: Space? $\Theta(n^2)$ **Task:** Give lower-bound example!

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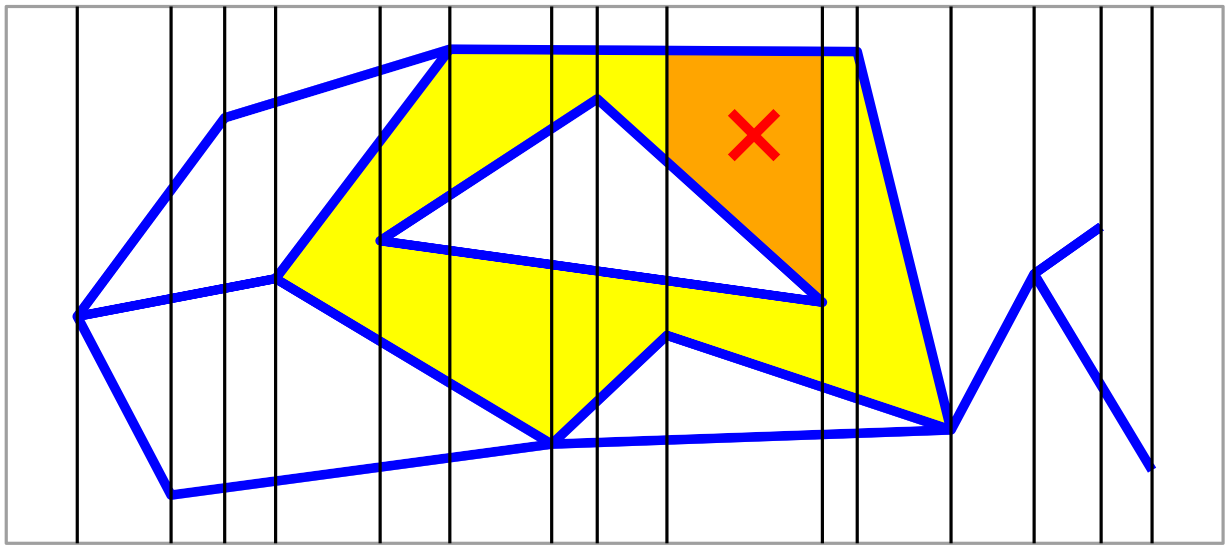

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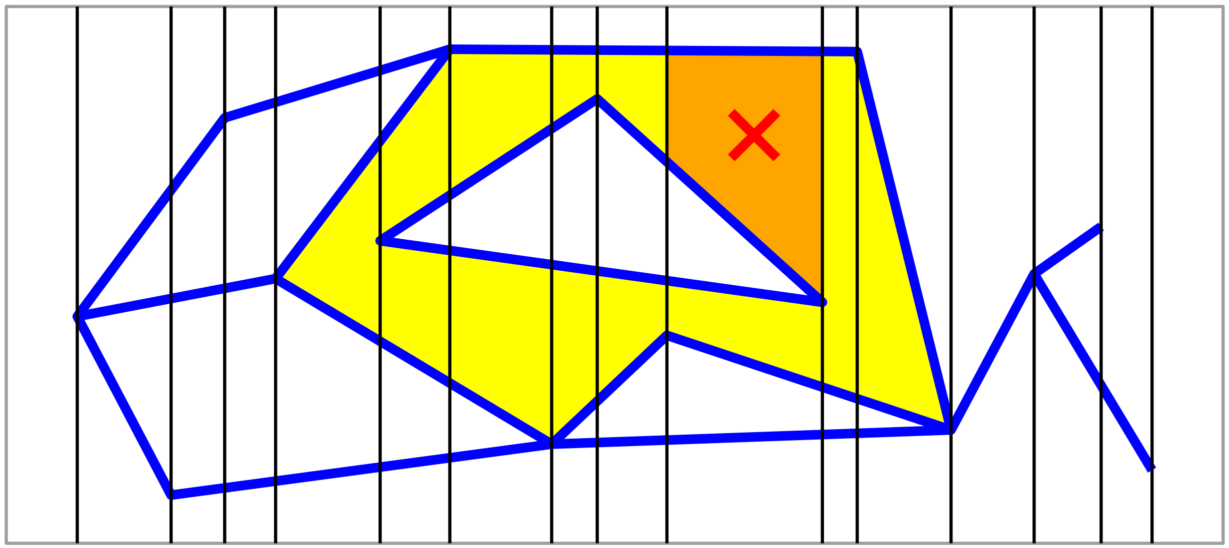

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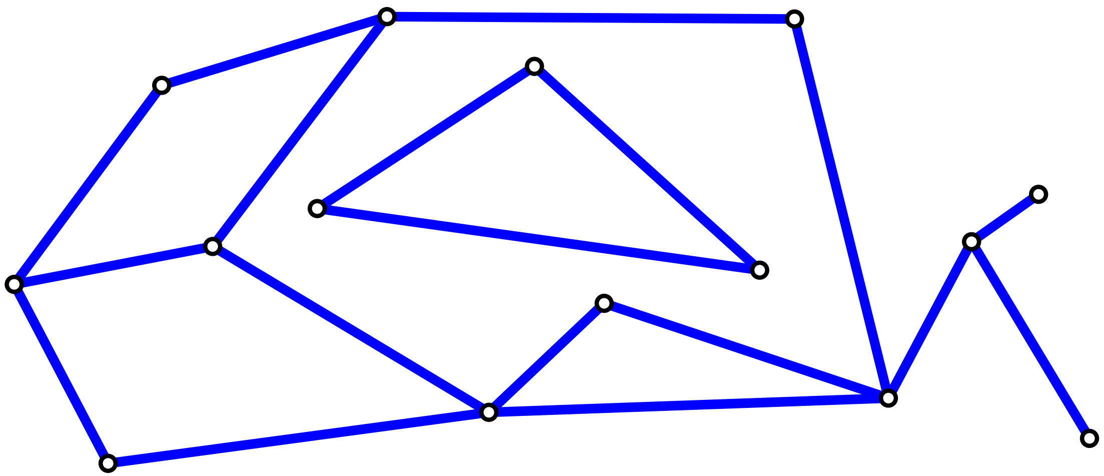

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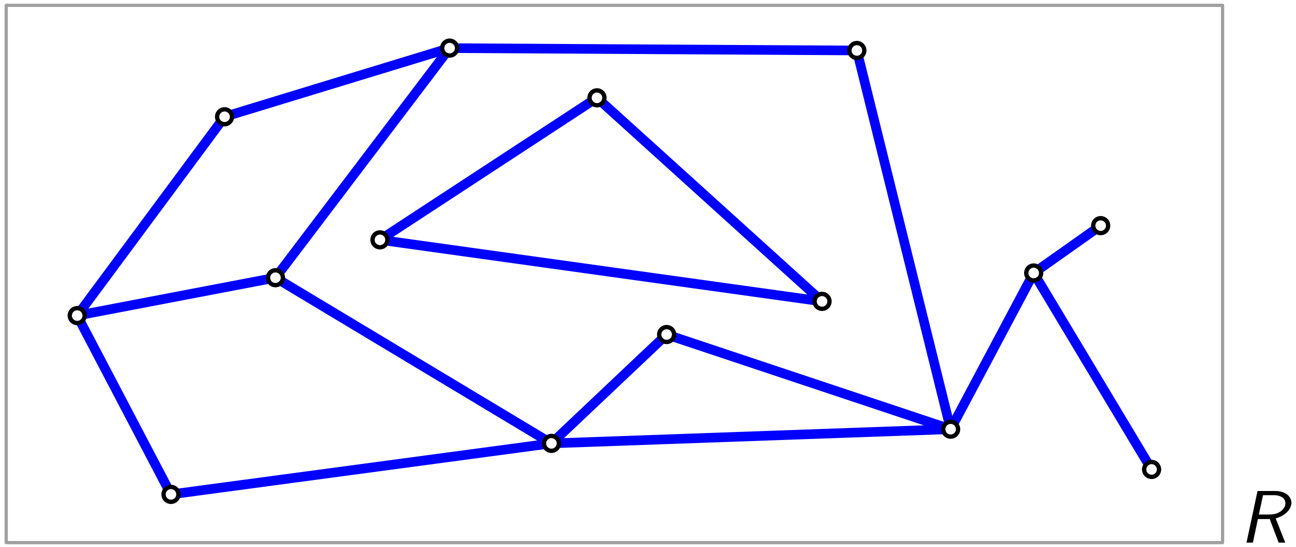

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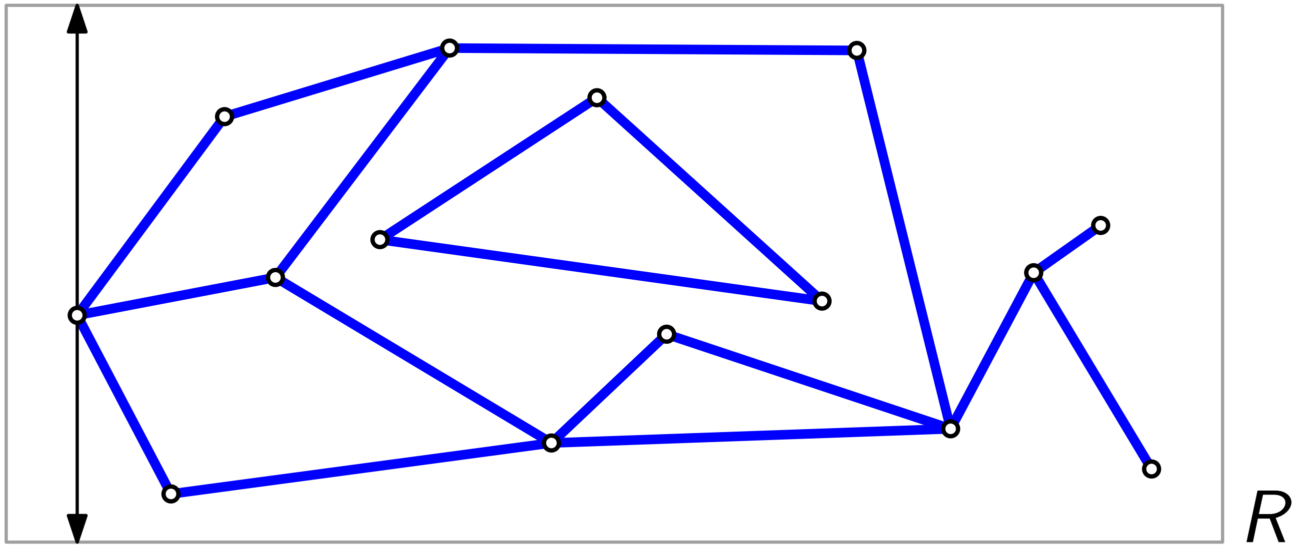

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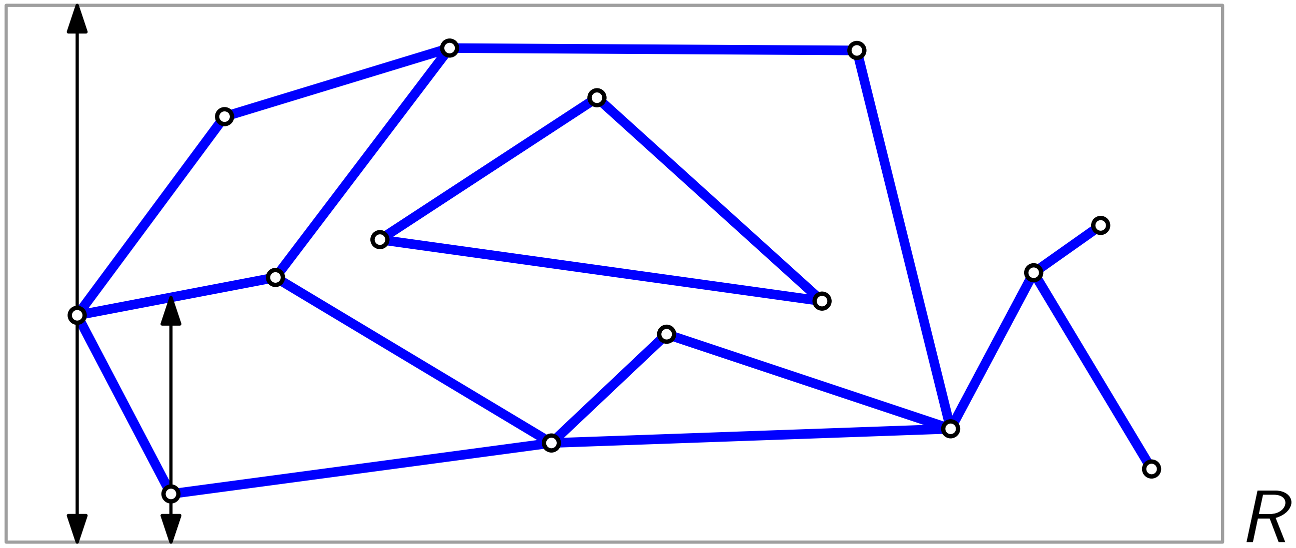

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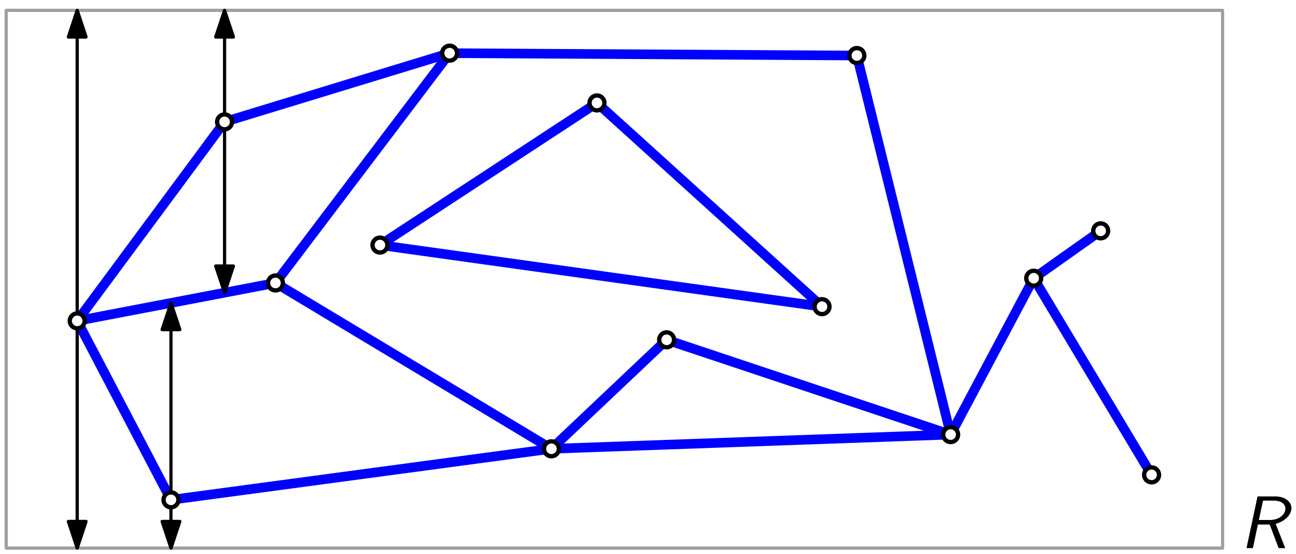

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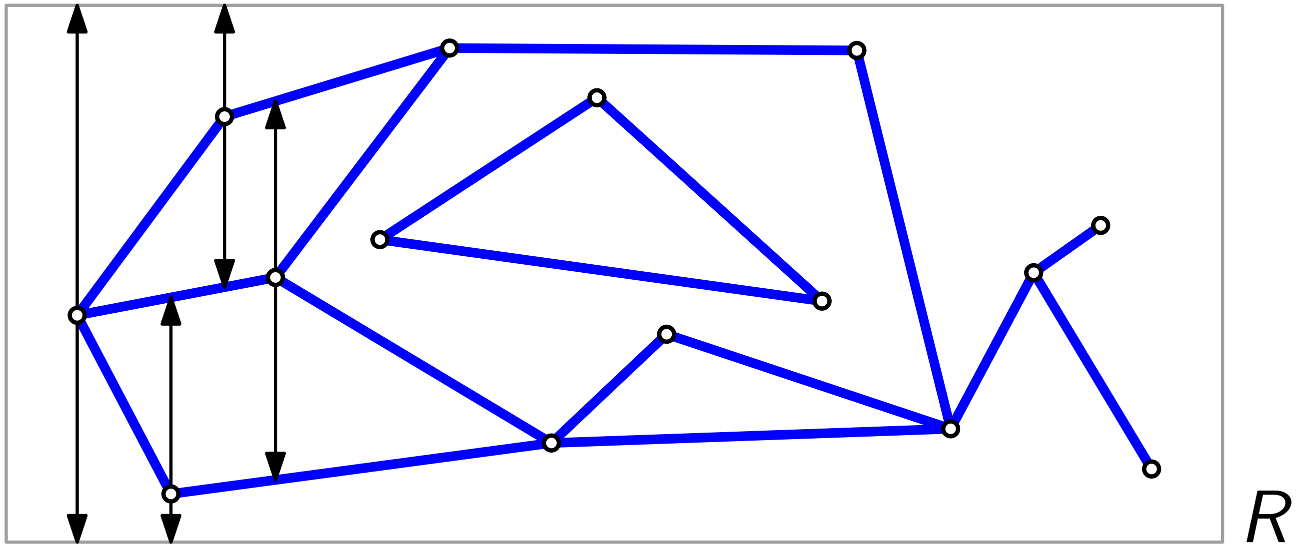

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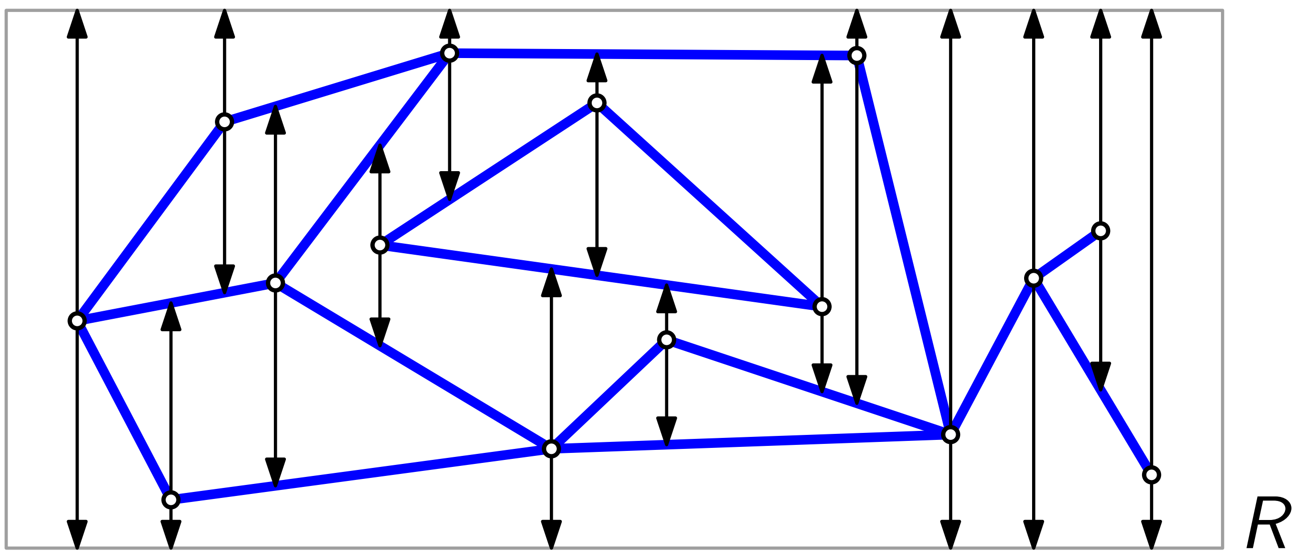

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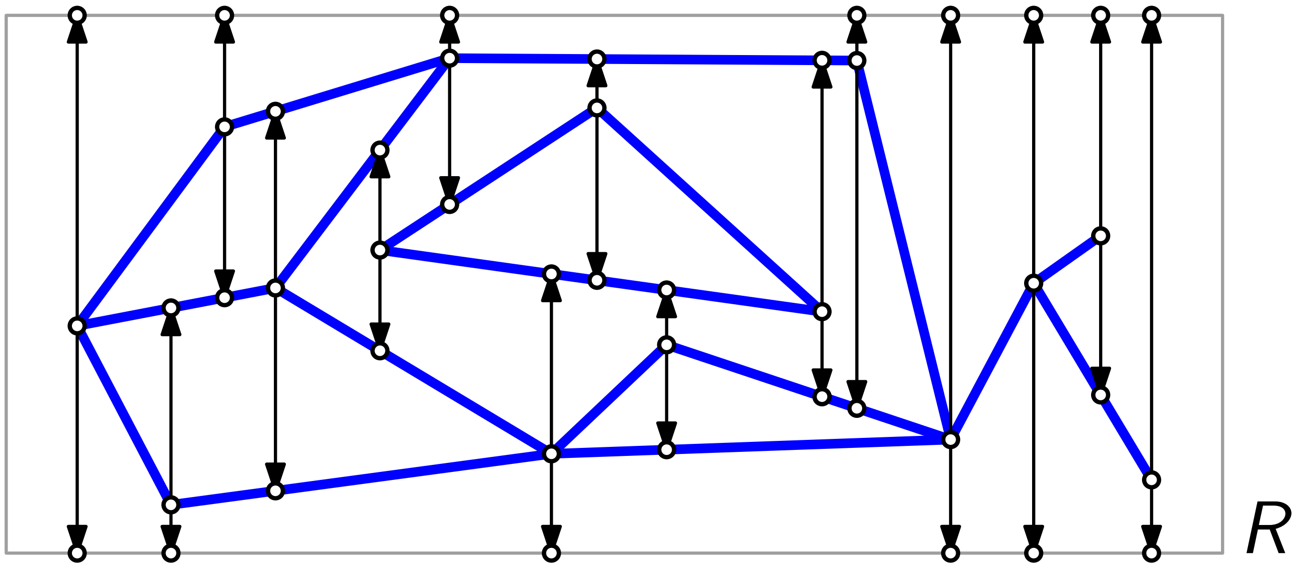

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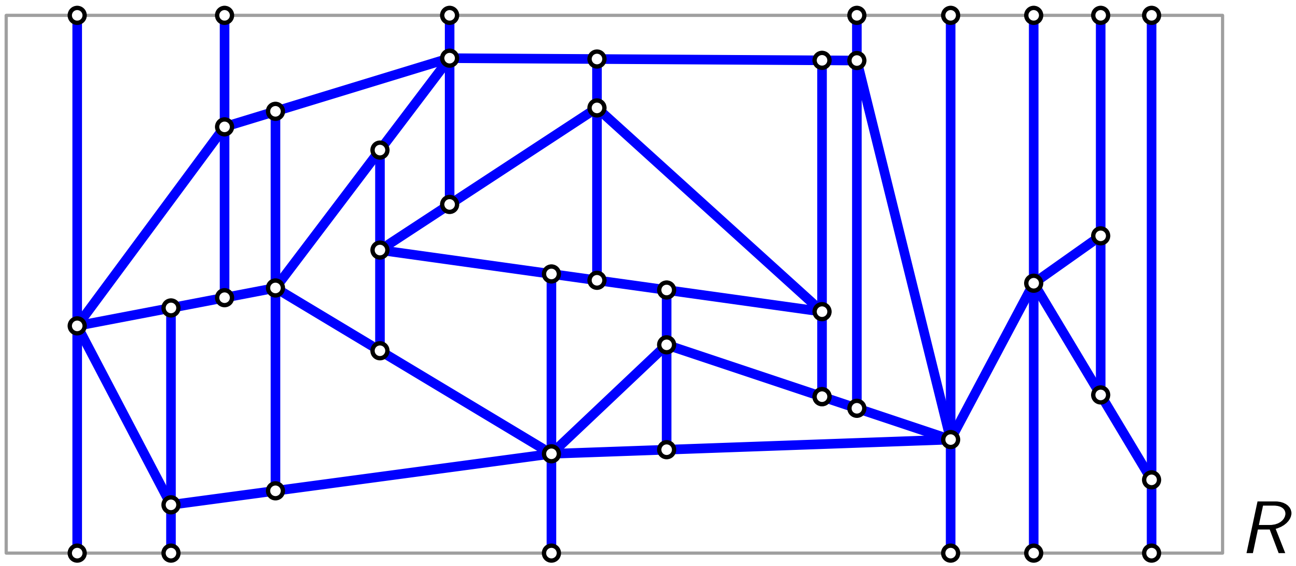

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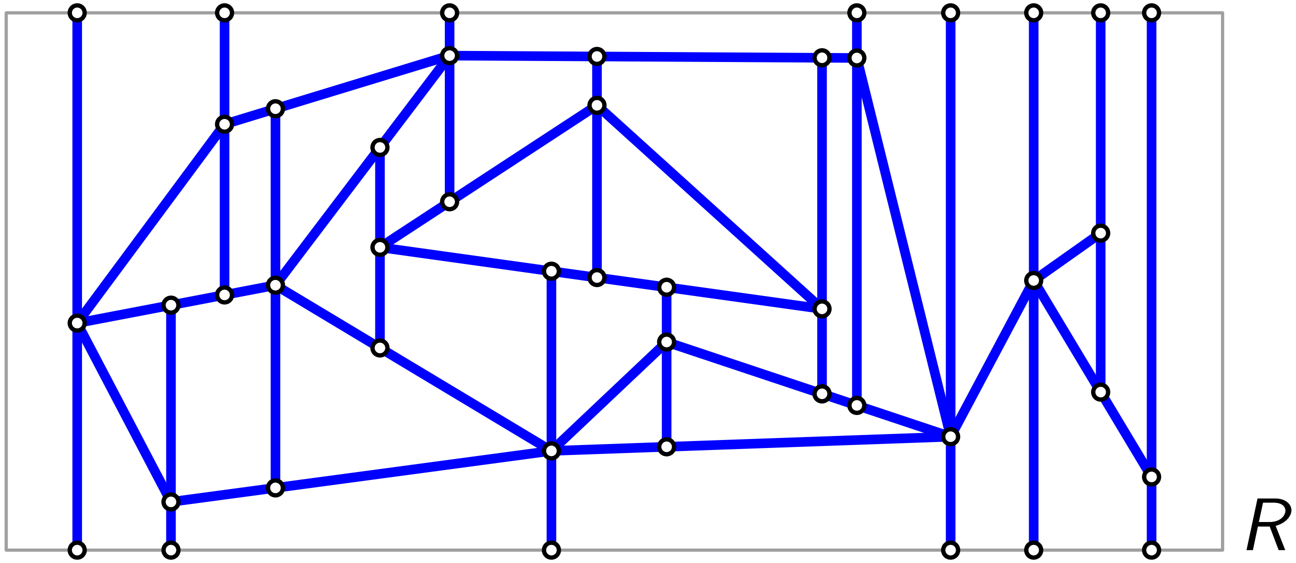
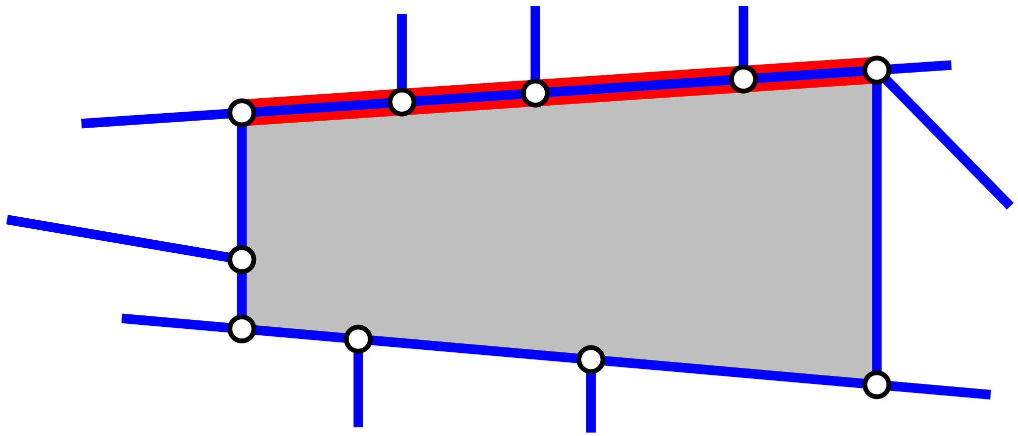


Assumption: $\mathcal{S}$ is in *general position*, that is, no two vertices have the same x-coordinates.

Notation

Definition: A *side* of a face of $\mathcal{T}(\mathcal{S})$ is a segment of maximum length contained in the boundary of the face.



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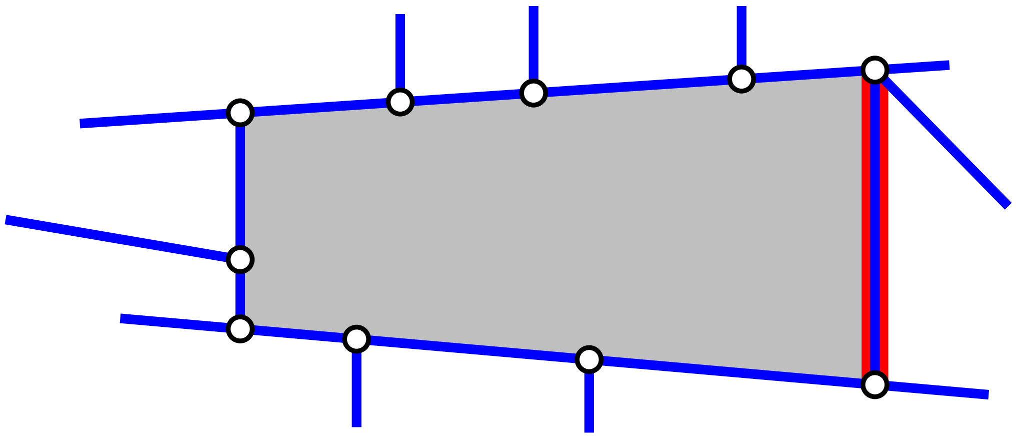

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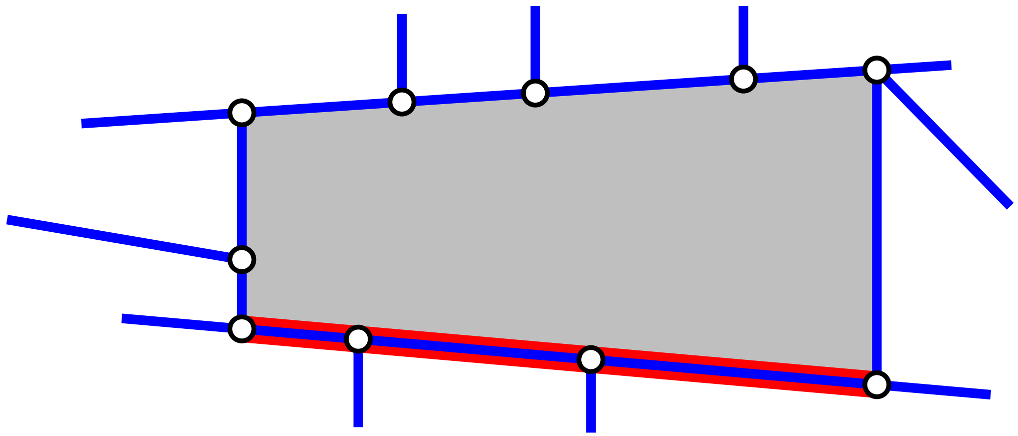

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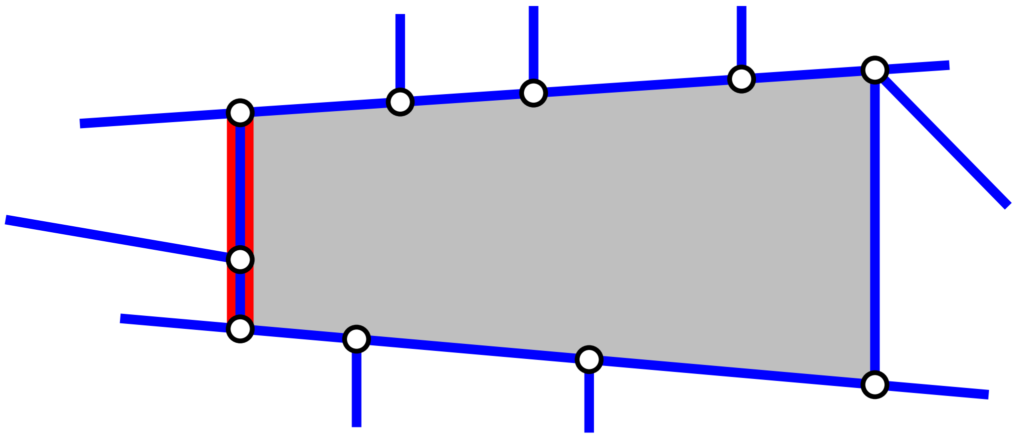

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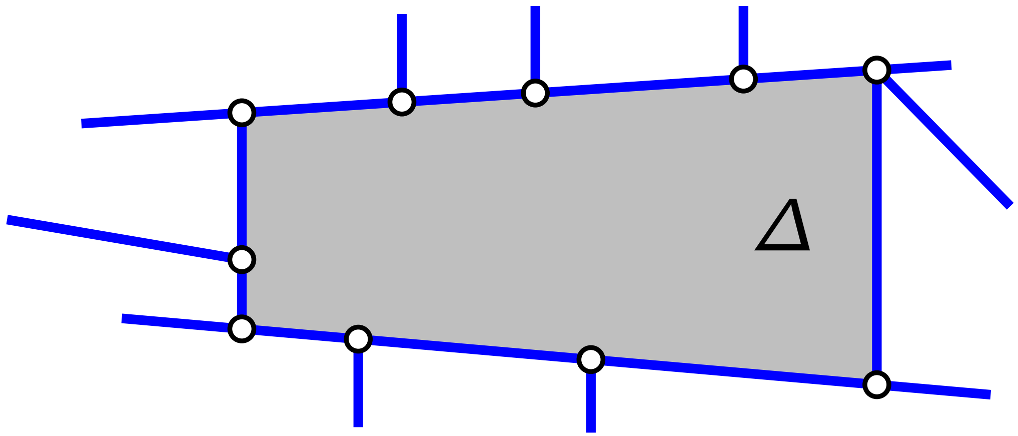

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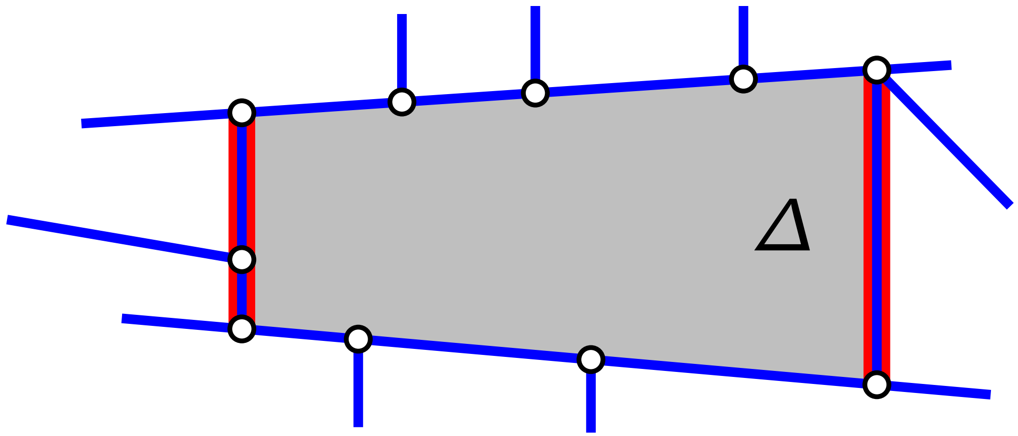

Observation: $\mathcal{S}$ in gen. pos. $\Rightarrow$ each face Δ of $\mathcal{T}(\mathcal{S})$ has:

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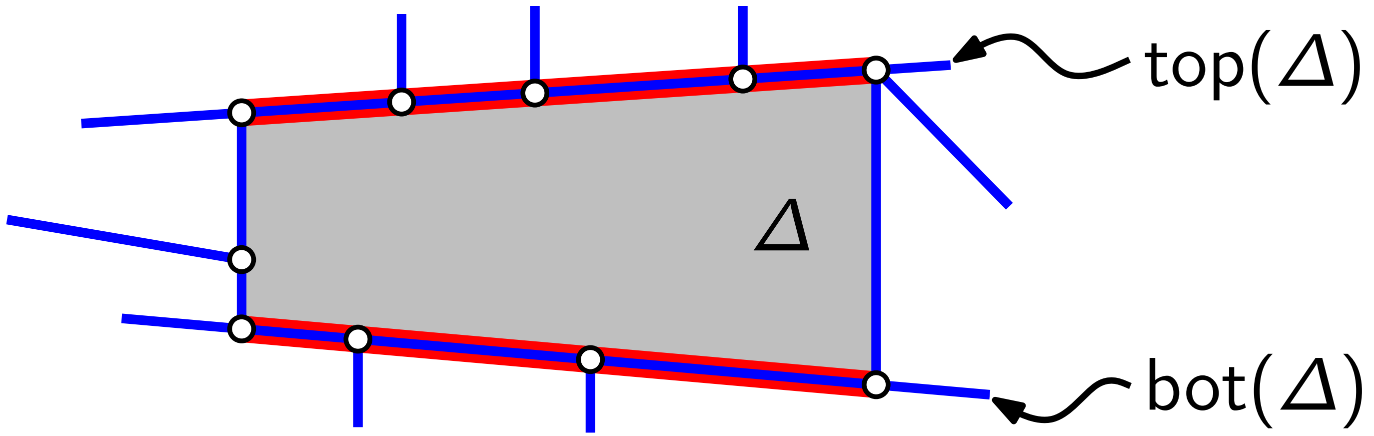

Observation: $\mathcal{S}$ in gen. pos. $\Rightarrow$ each face Δ of $\mathcal{T}(\mathcal{S})$ has:
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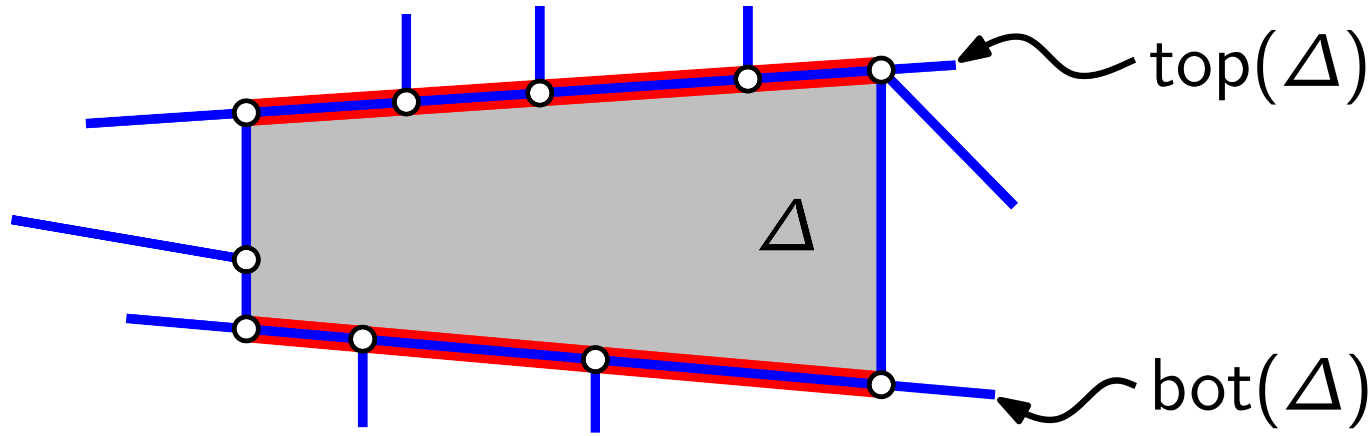

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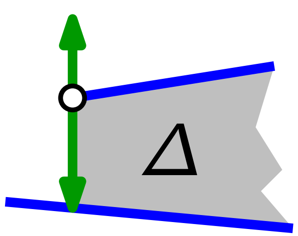

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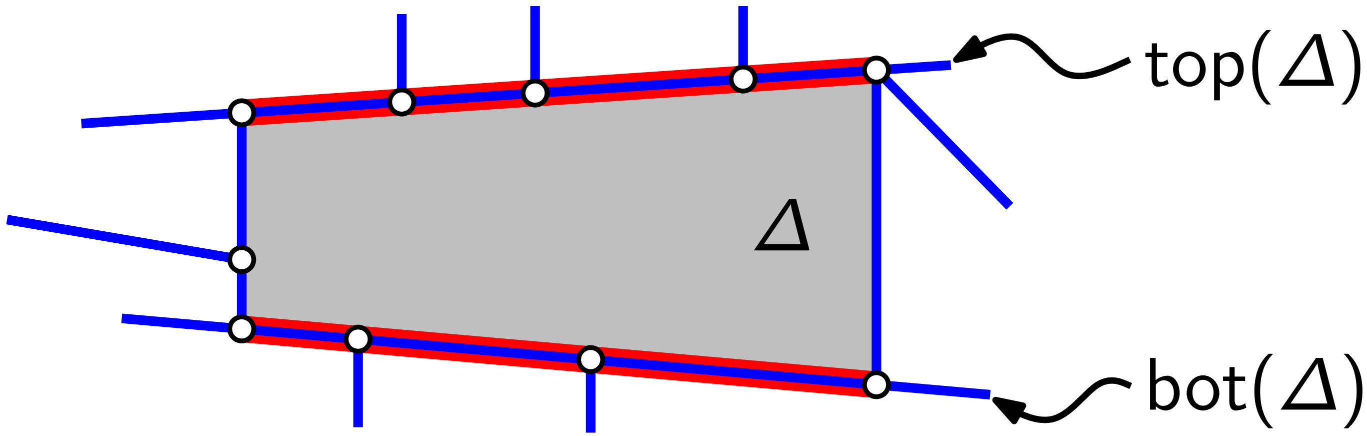

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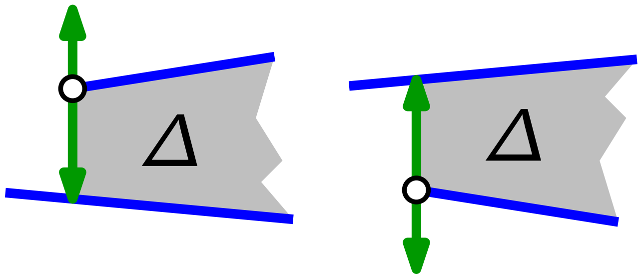

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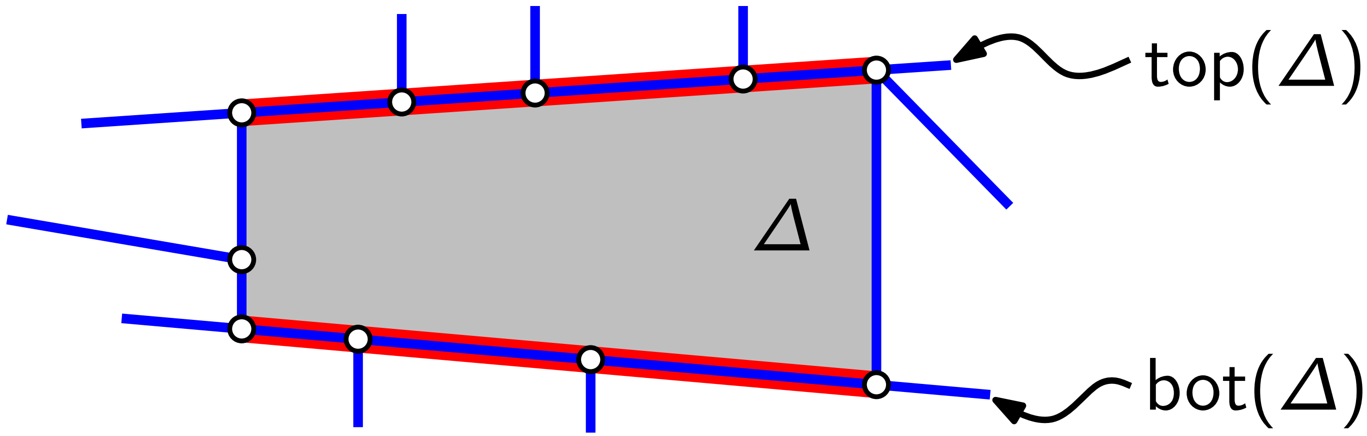

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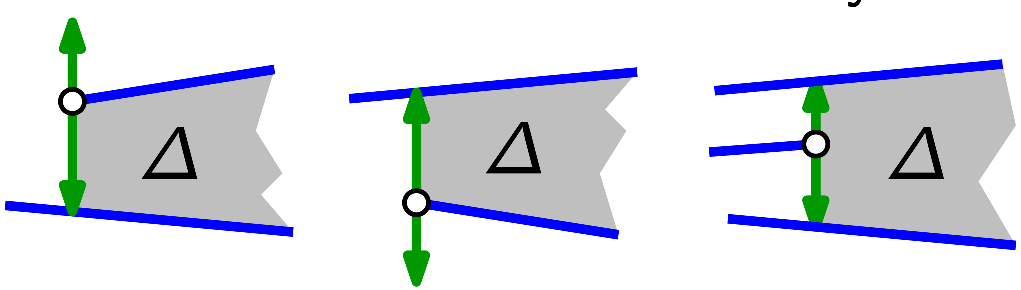
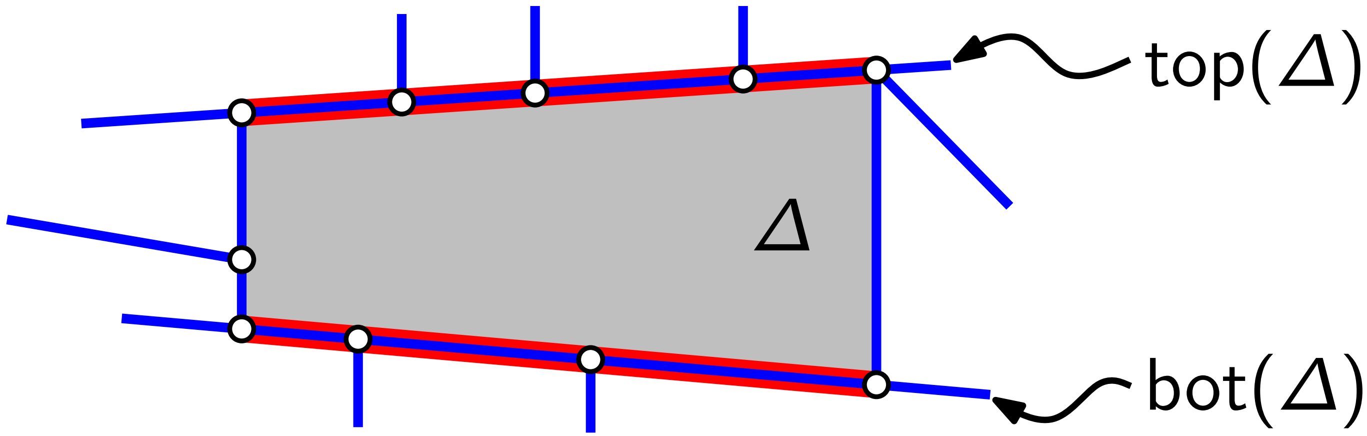


Notation

Definition: A *side* of a face of $\mathcal{T}(\mathcal{S})$ is a segment of maximum length contained in the boundary of the face.



Observation: $\mathcal{S}$ in gen. pos. $\Rightarrow$ each face Δ of $\mathcal{T}(\mathcal{S})$ has:
– one or two vertical sides
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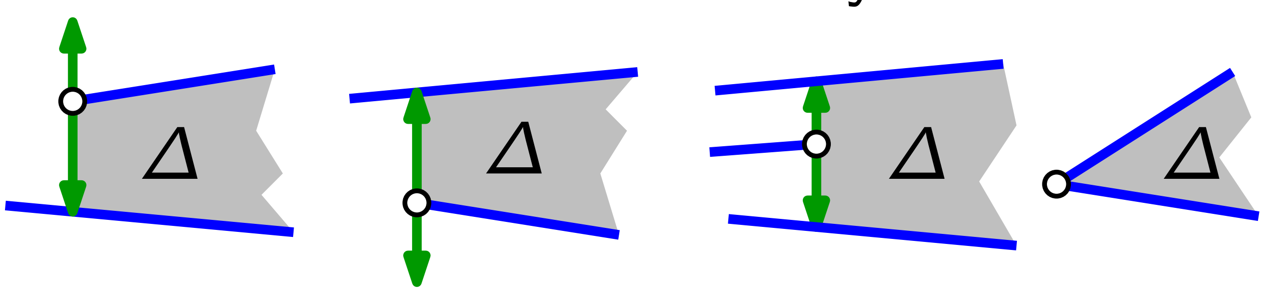

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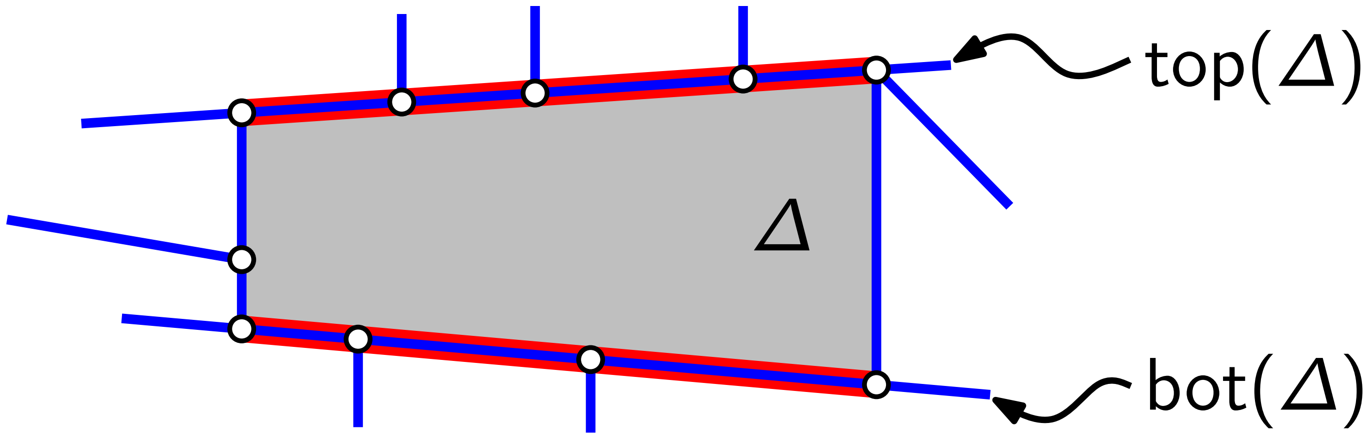

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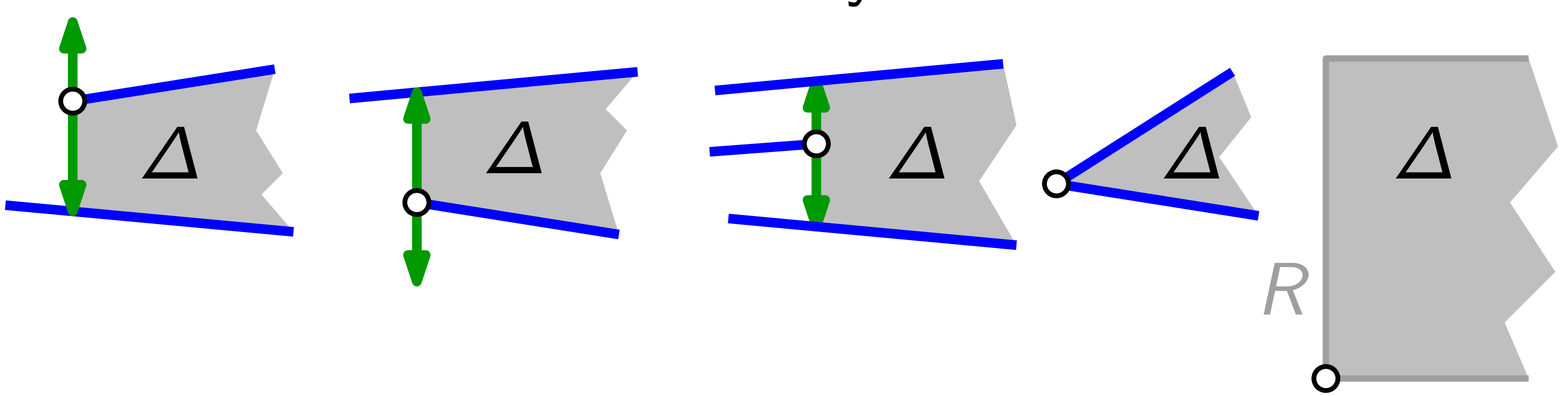

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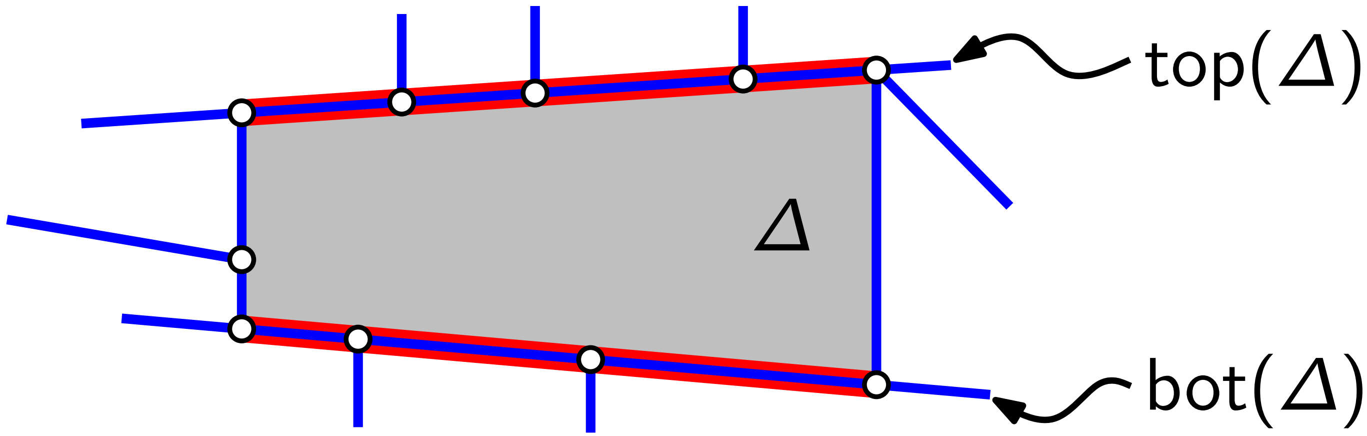

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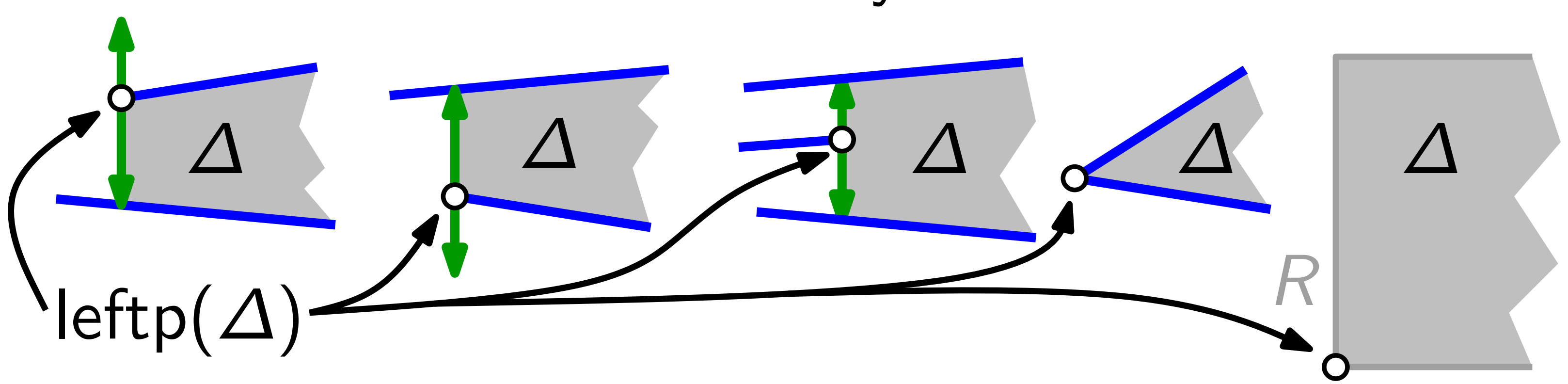

Complexity of $\mathcal{T}(\mathcal{S})$

Observe: A face Δ of $\mathcal{T}(\mathcal{S})$ is uniquely defined by top(Δ), bot(Δ), leftp(Δ), and rightp(Δ).

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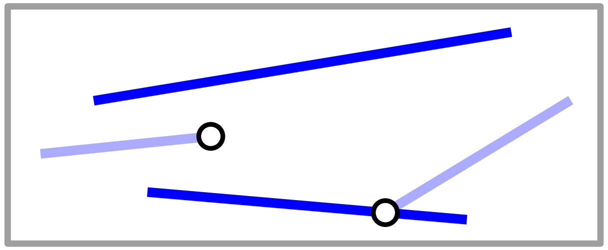

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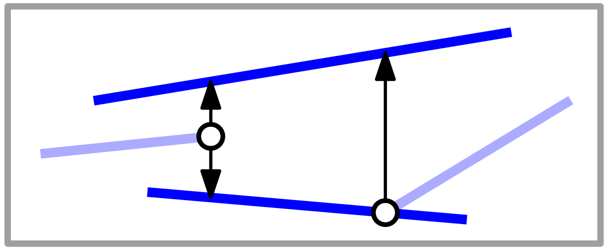

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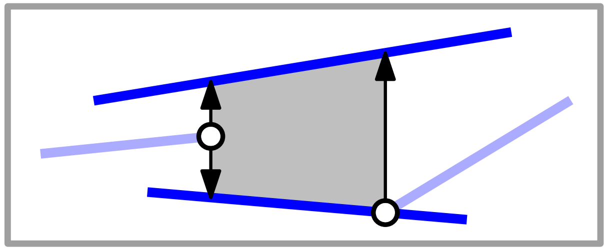

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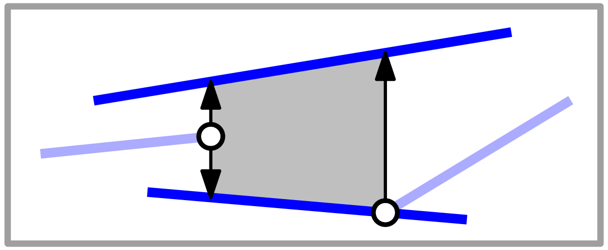

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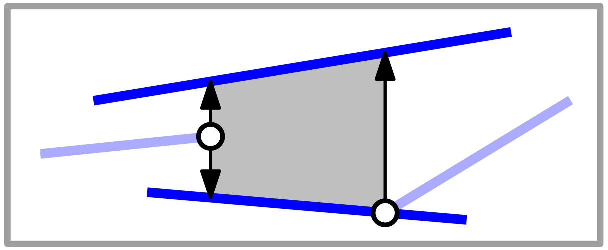

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– endpts of segments in $\mathcal{S}$

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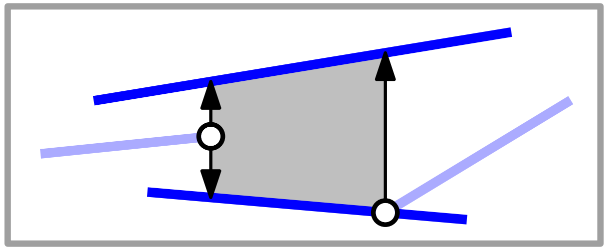

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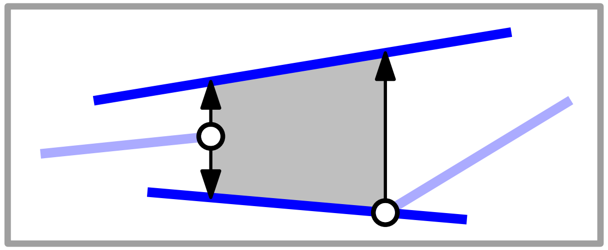

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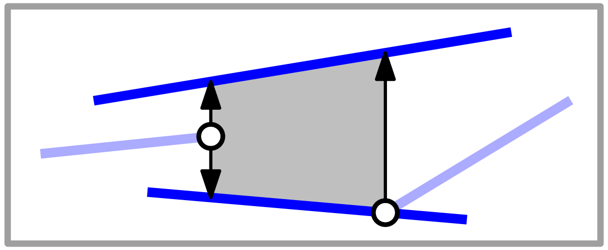

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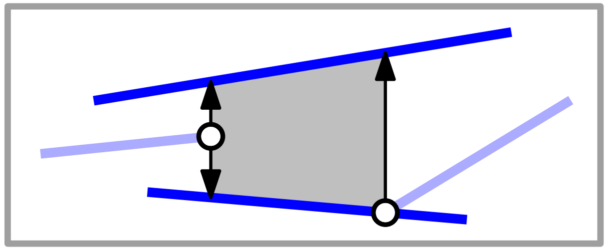

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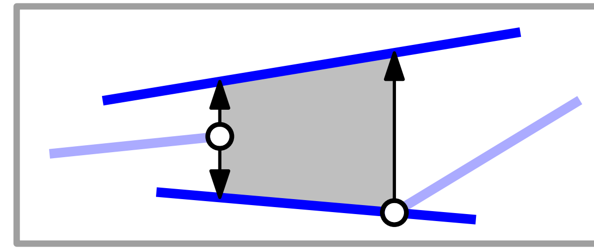
Complexity of $\mathcal{T}(\mathcal{S})$



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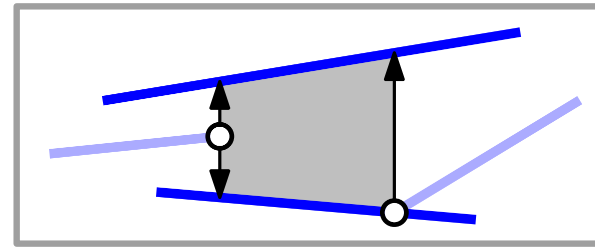
– endpts of segments in $\mathcal{S}$	$\leq 2n$
– endpts of vertical extensions	$\leq 2 \cdot 2n$
– vertices of R	4

Complexity of $\mathcal{T}(\mathcal{S})$



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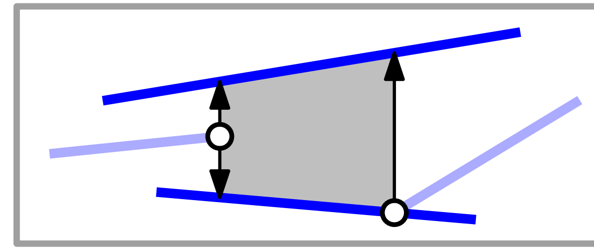

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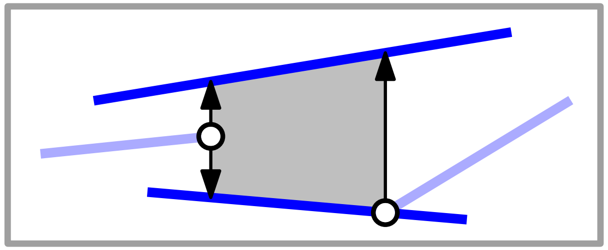

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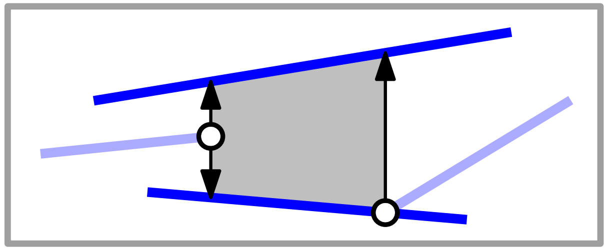

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Bound #trapezoids via Euler or directly (segments/leftp).

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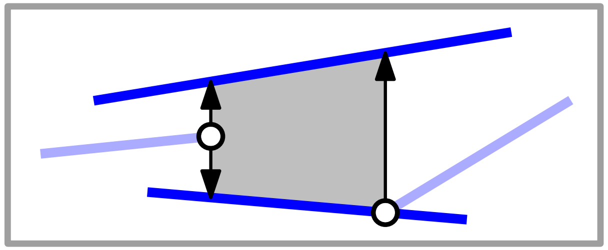

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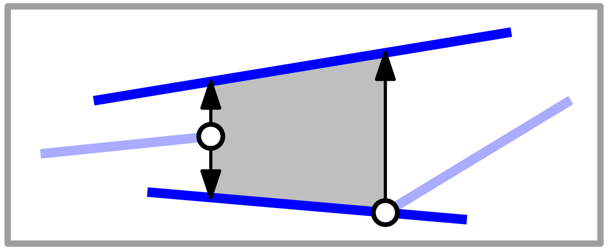

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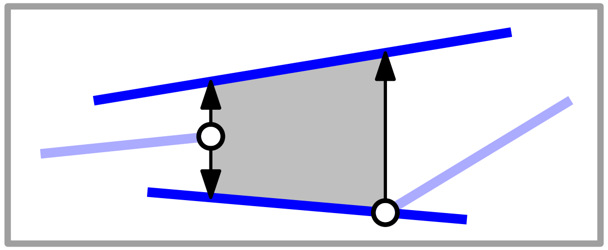
Complexity of $\mathcal{T}(\mathcal{S})$



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algorithm-design paradigm!

The 1d-Problem

Given a set S of n real numbers...

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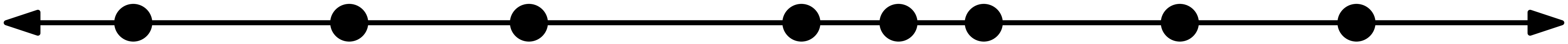

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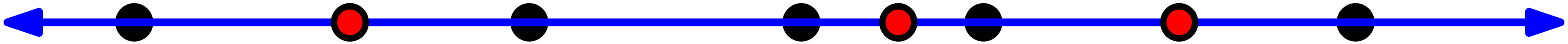
Given a set S of n real numbers... $i \in \{1, \ldots, n\}$



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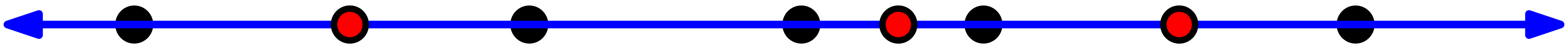

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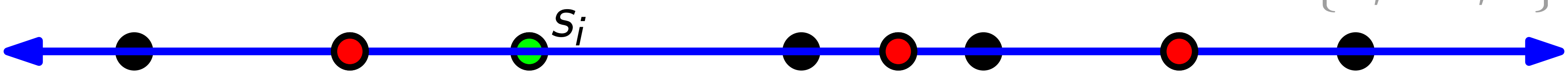

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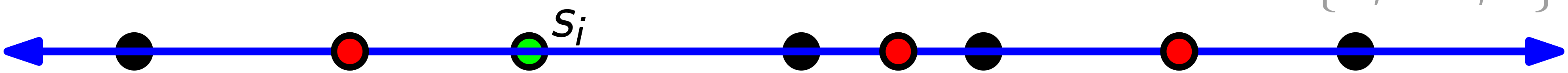

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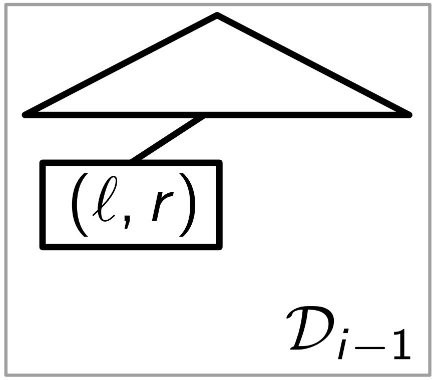

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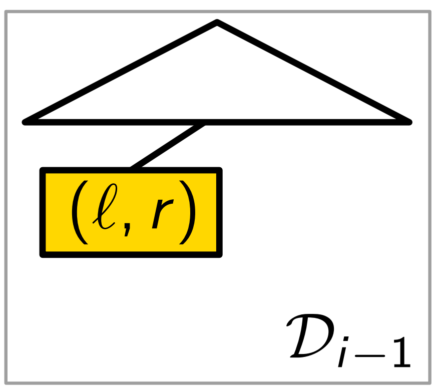

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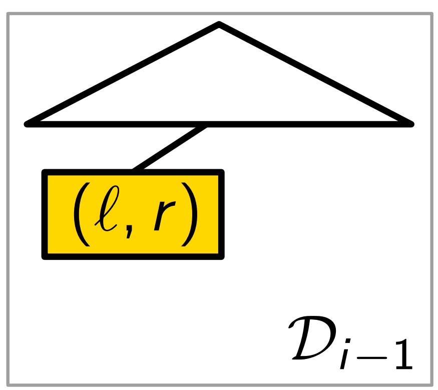

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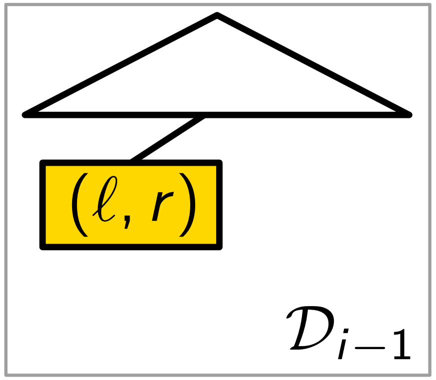

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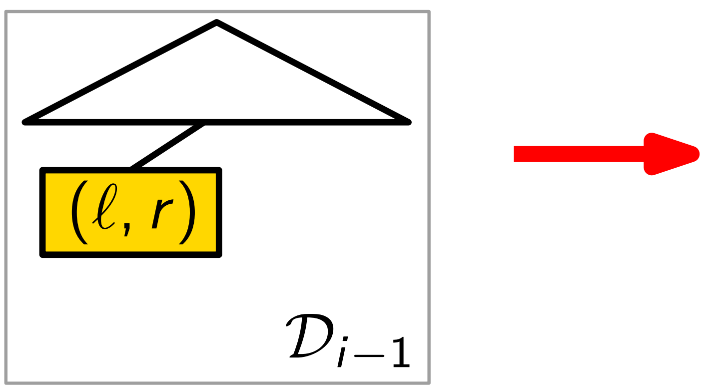

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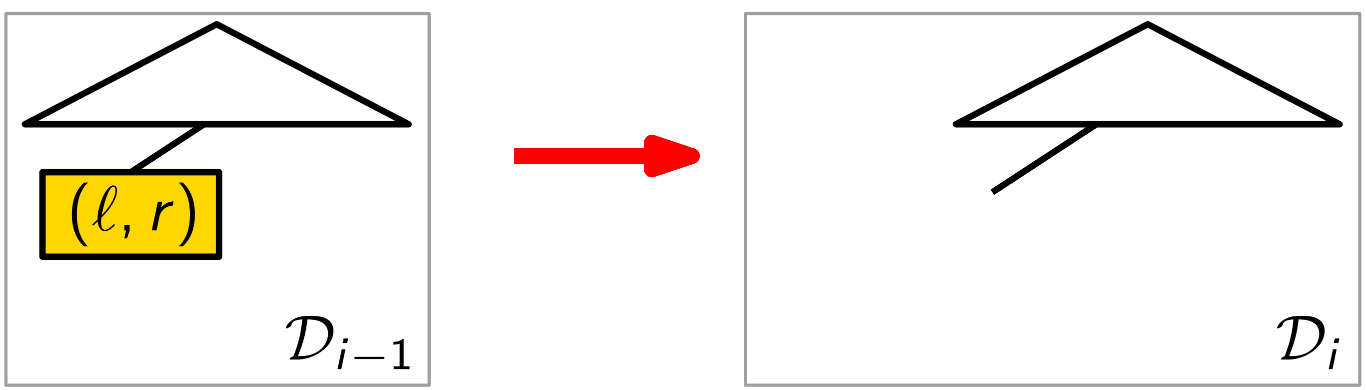

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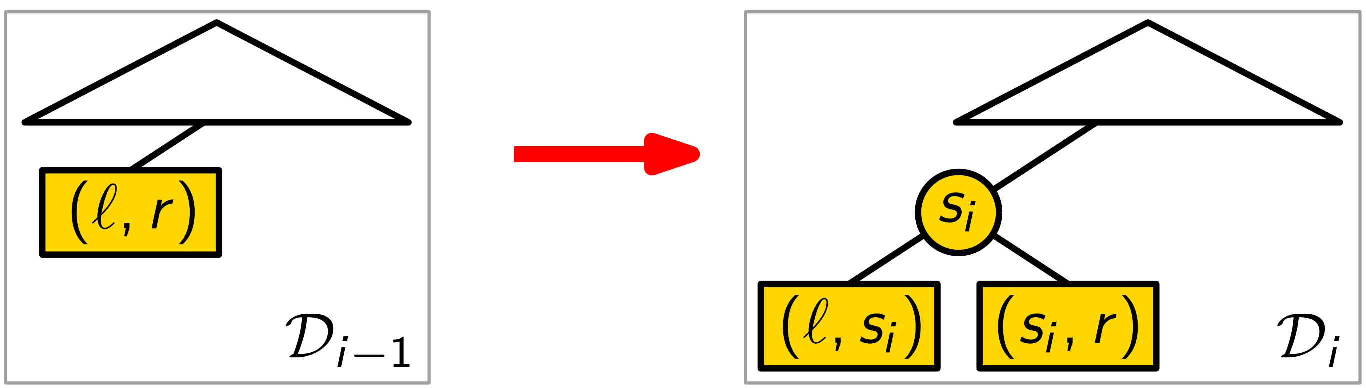

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$S_{i-1} := \{s_1, \dots, s_{i-1}\}$, $I_{i-1} :=$ set of conn. comp. of $\mathbb{R} \setminus S_{i-1}$

- pick an arbitrary point s_i from $S \setminus S_{i-1}$
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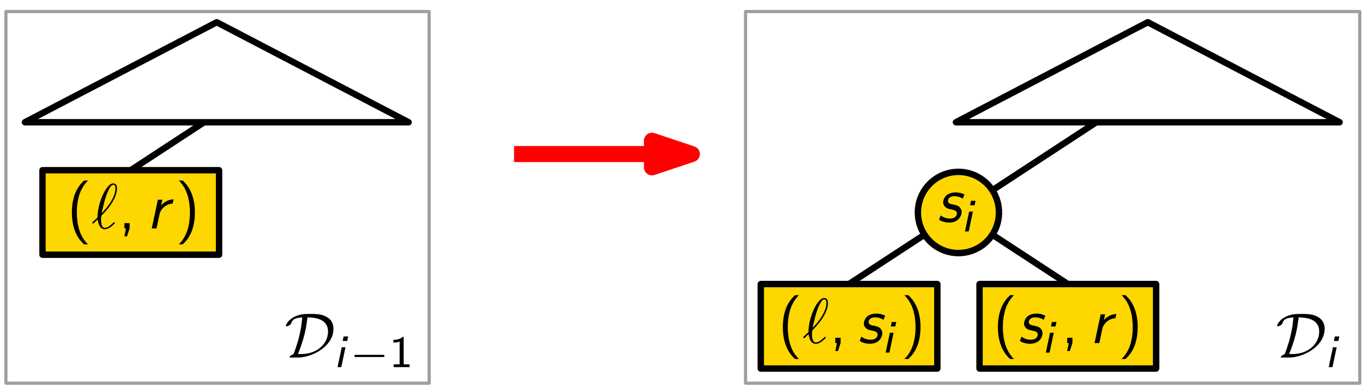
- split interval (ℓ, r) of I_{i-1} containing s_i
- build $\mathcal{D}_i$:



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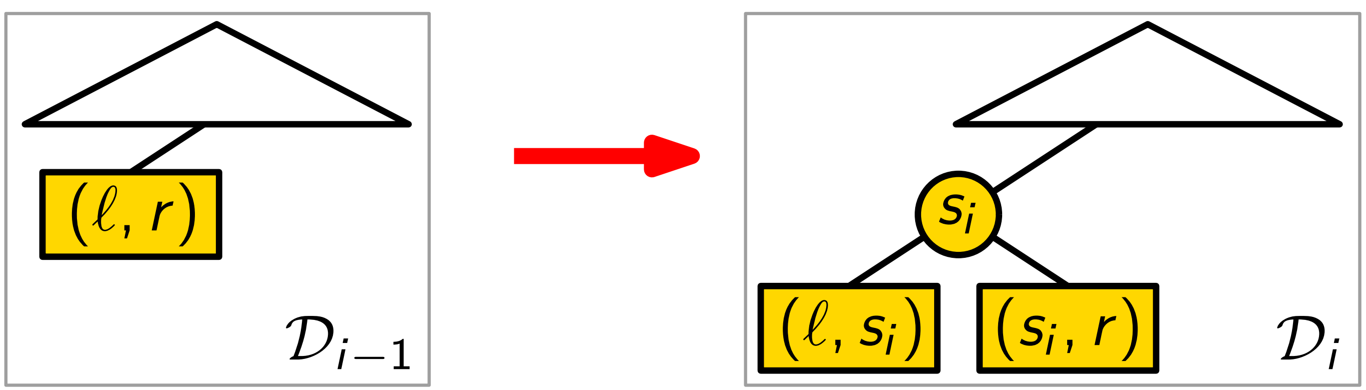

Problem: *looong* search paths!

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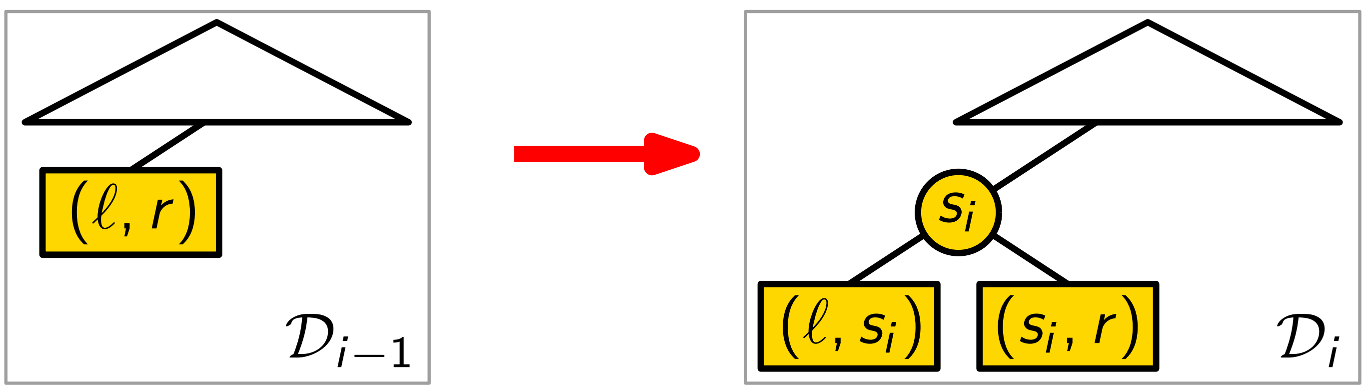

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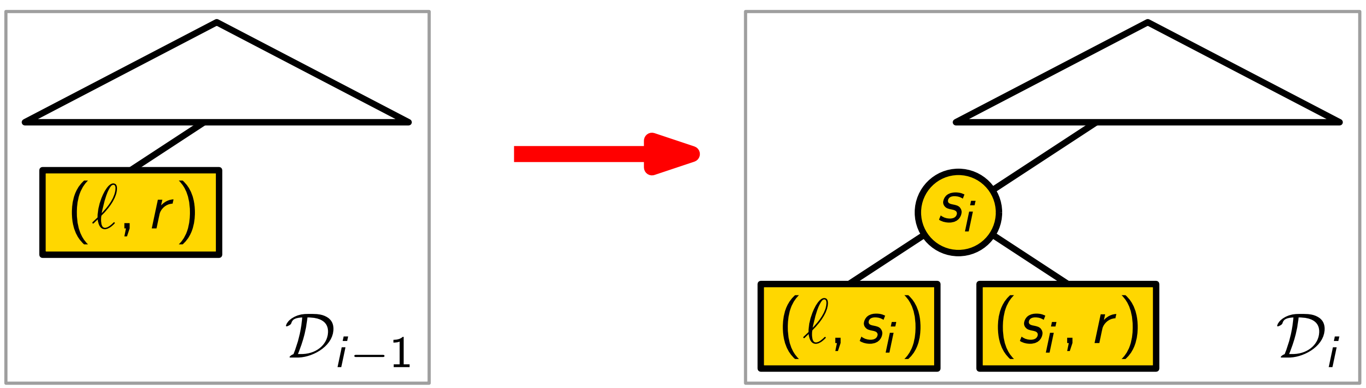

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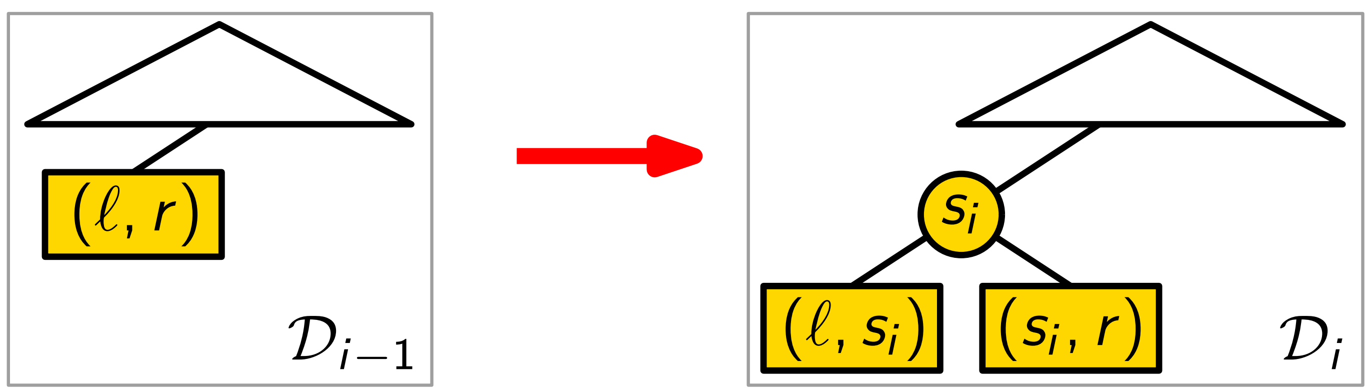

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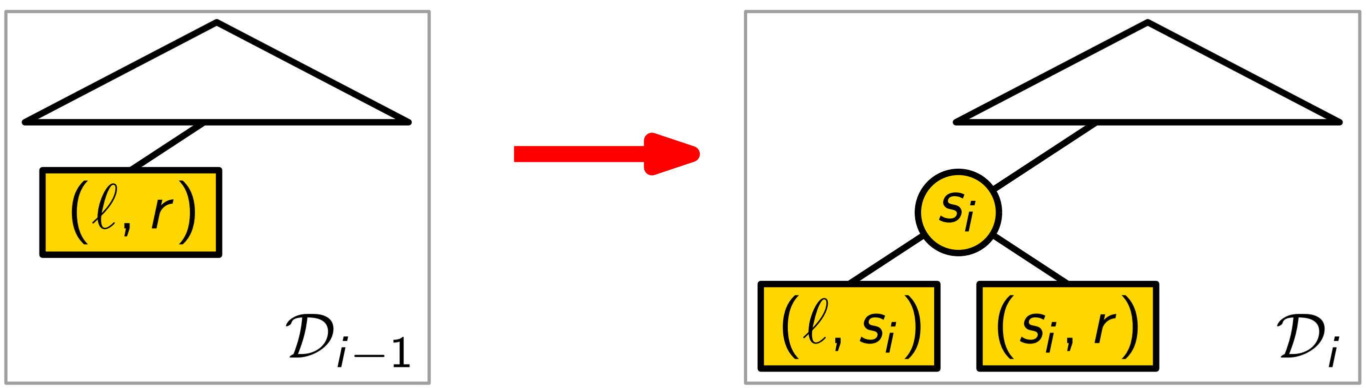

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Thm. The randomized-incremental algorithm preprocesses a set S of n reals in $O(n \log n)$ expected time such that a query takes $O(\log n)$ expected time.

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$O(\log n)$

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The 2d-Problem

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

The 2d-Problem

trapezoidal map

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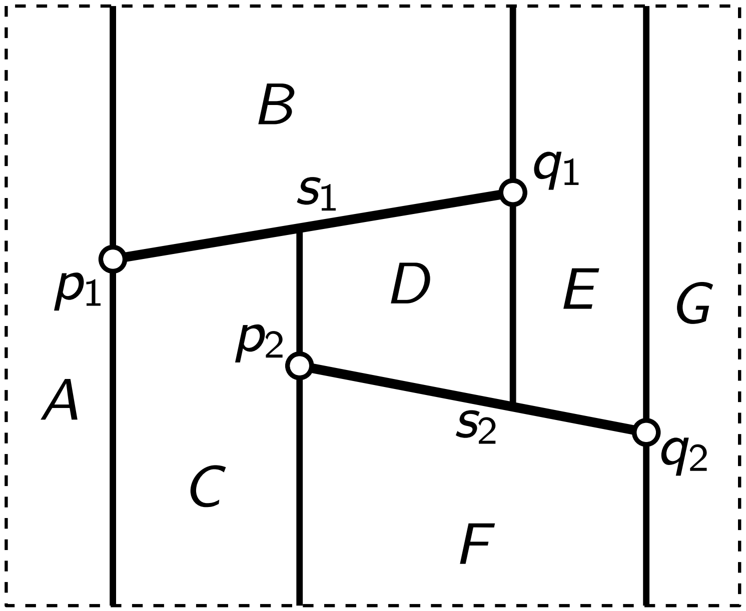
trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

$\mathcal{T}$

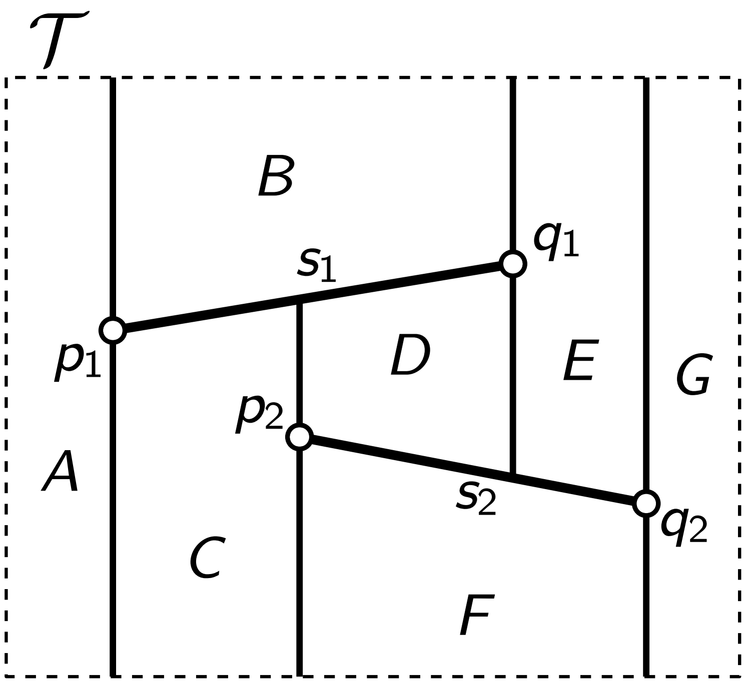


The 2d-Problem

point-location data structure (DAG)

trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$



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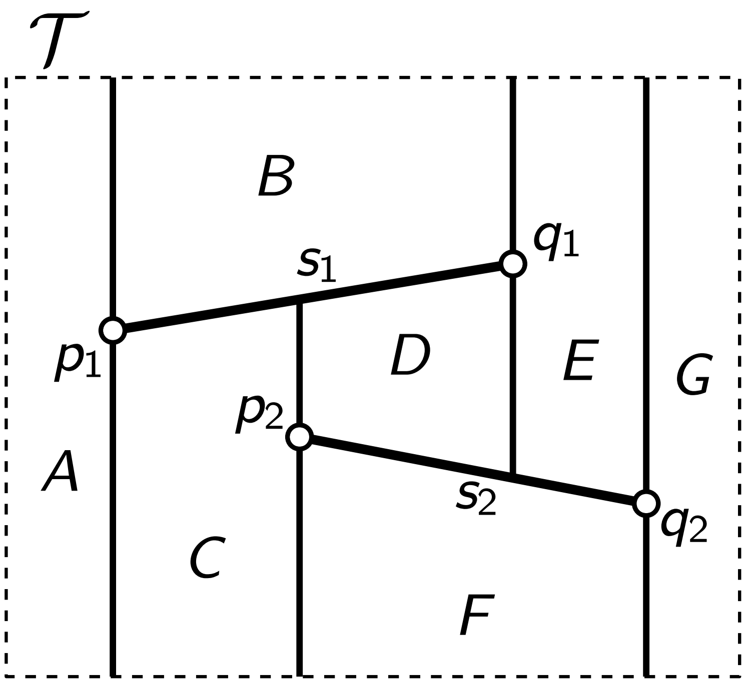

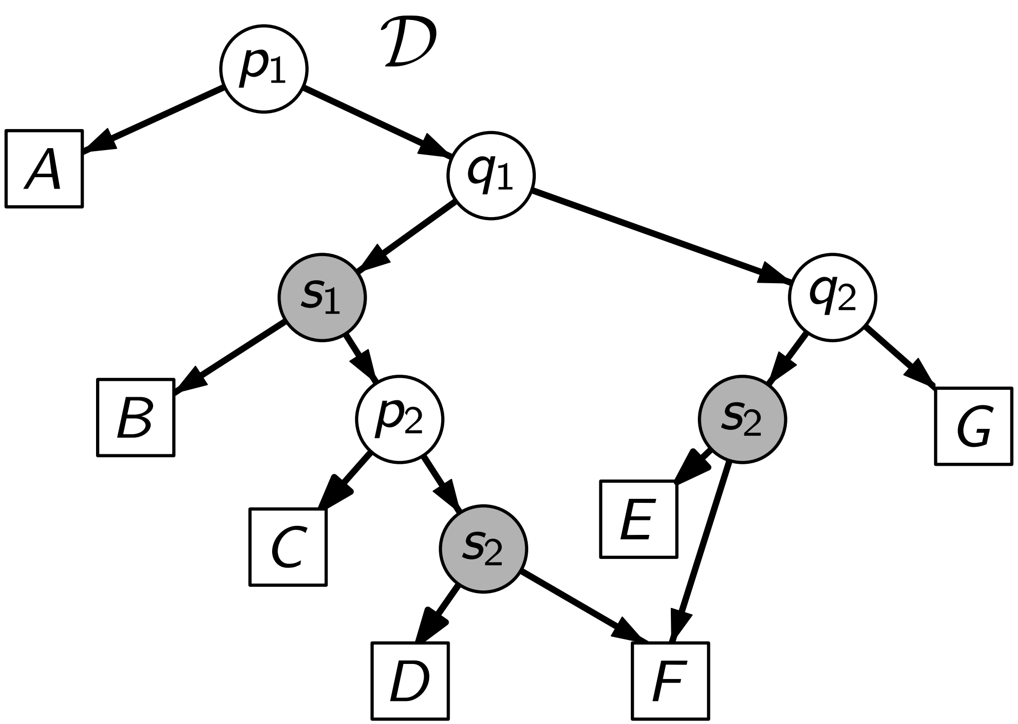

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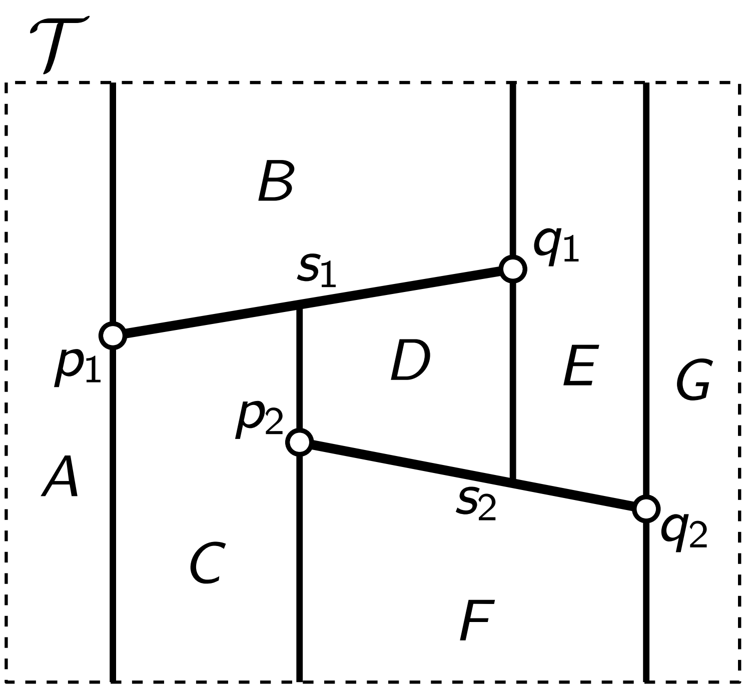
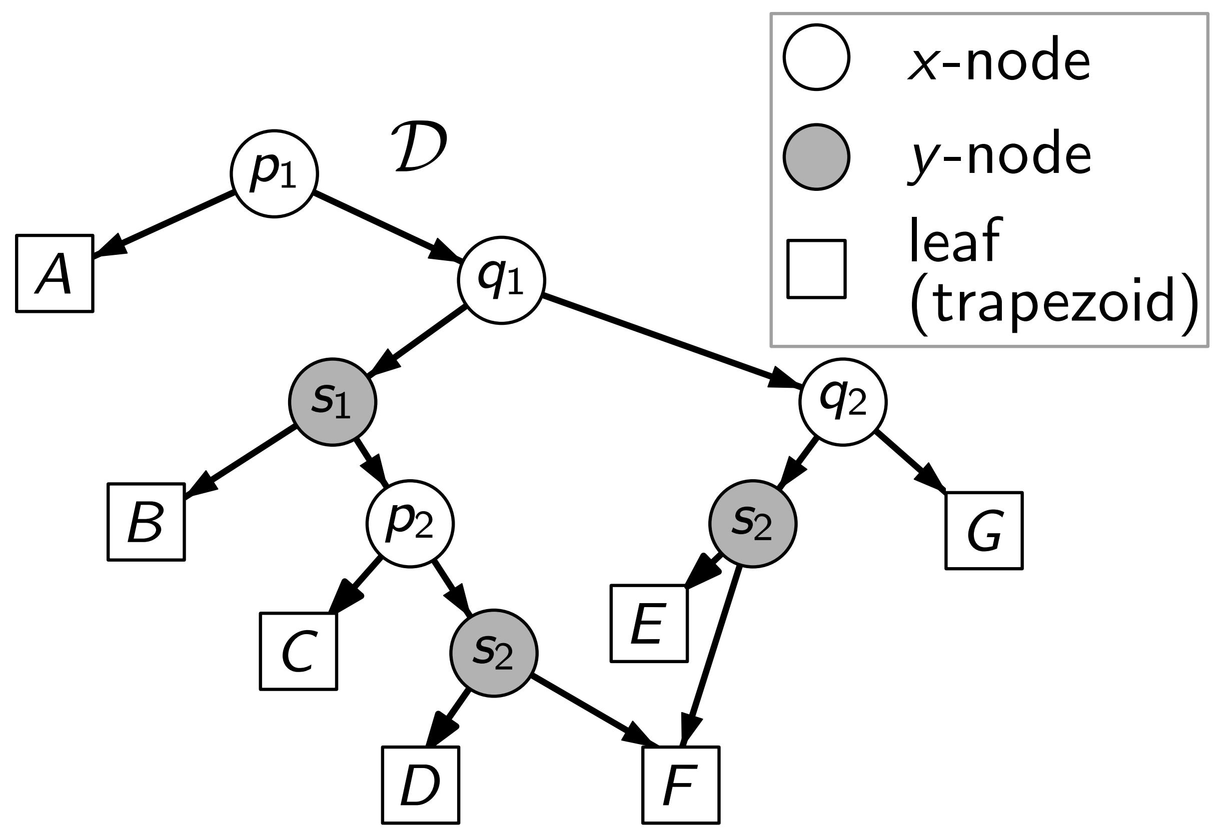




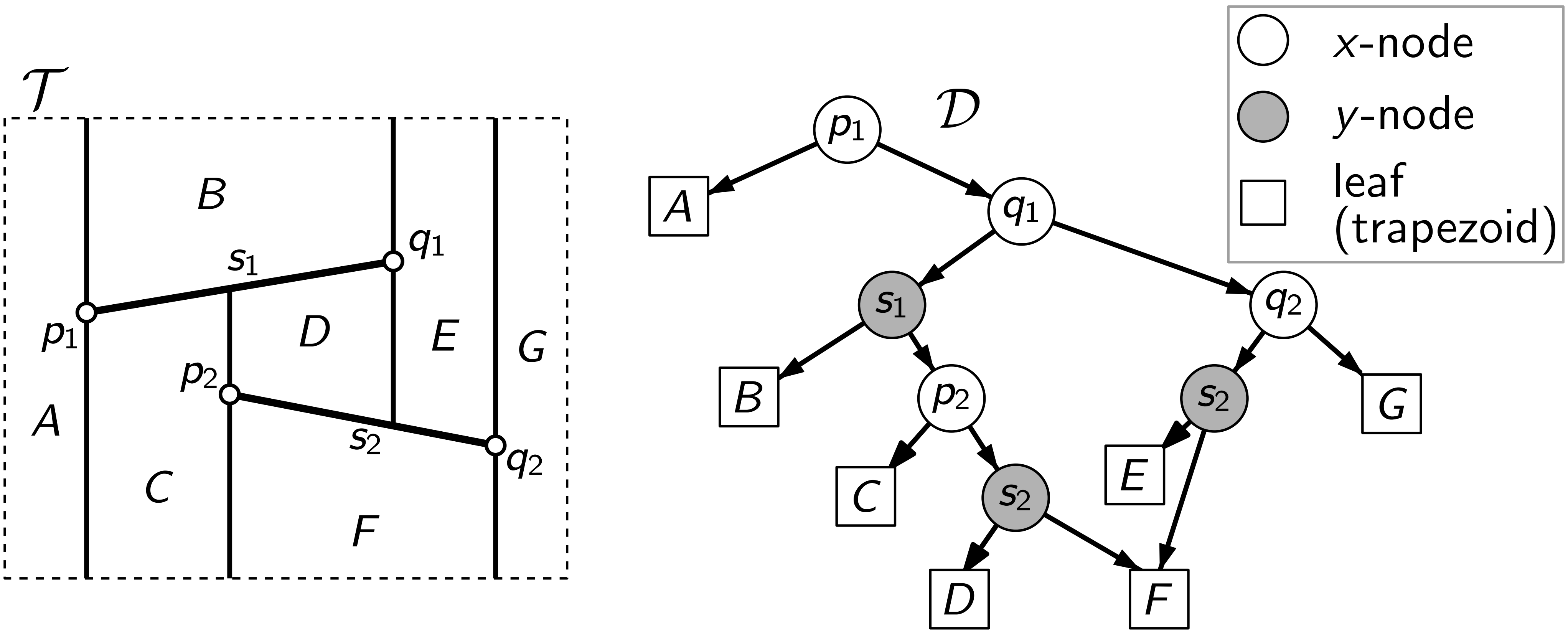
The 2d-Problem

point-location data structure (DAG)

trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

– use $\mathcal{D}$ to locate left endpoint of next segment s



The 2d-Problem

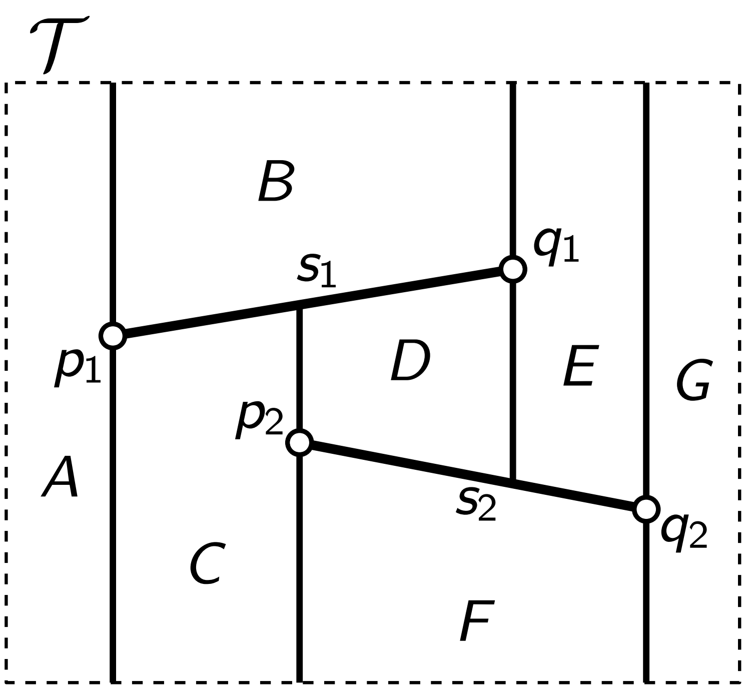
point-location data structure (DAG)

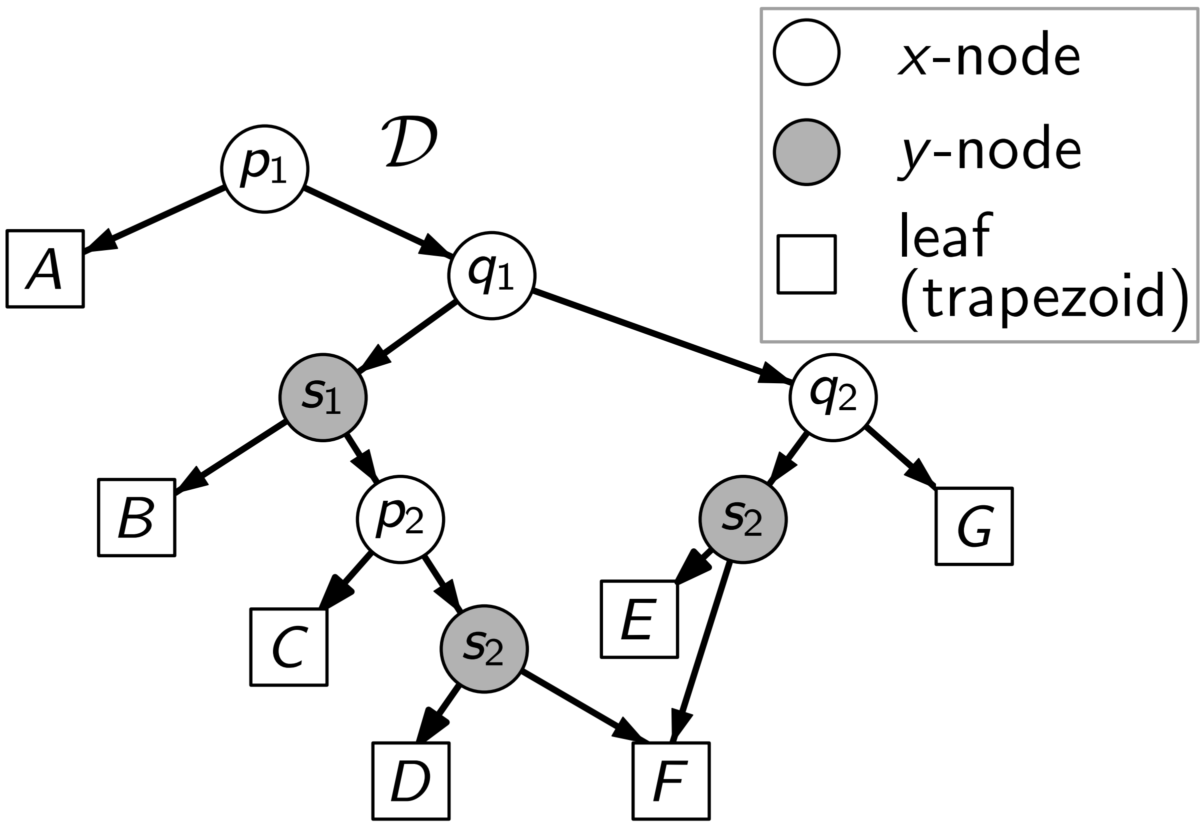
trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

– use $\mathcal{D}$ to locate left endpoint of next segment s

– "walk" along s through $\mathcal{T}$





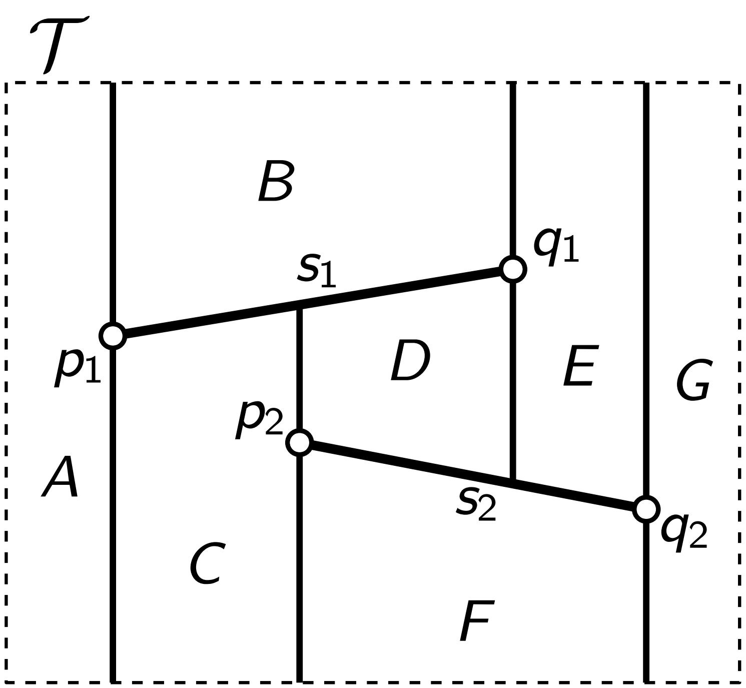
The 2d-Problem

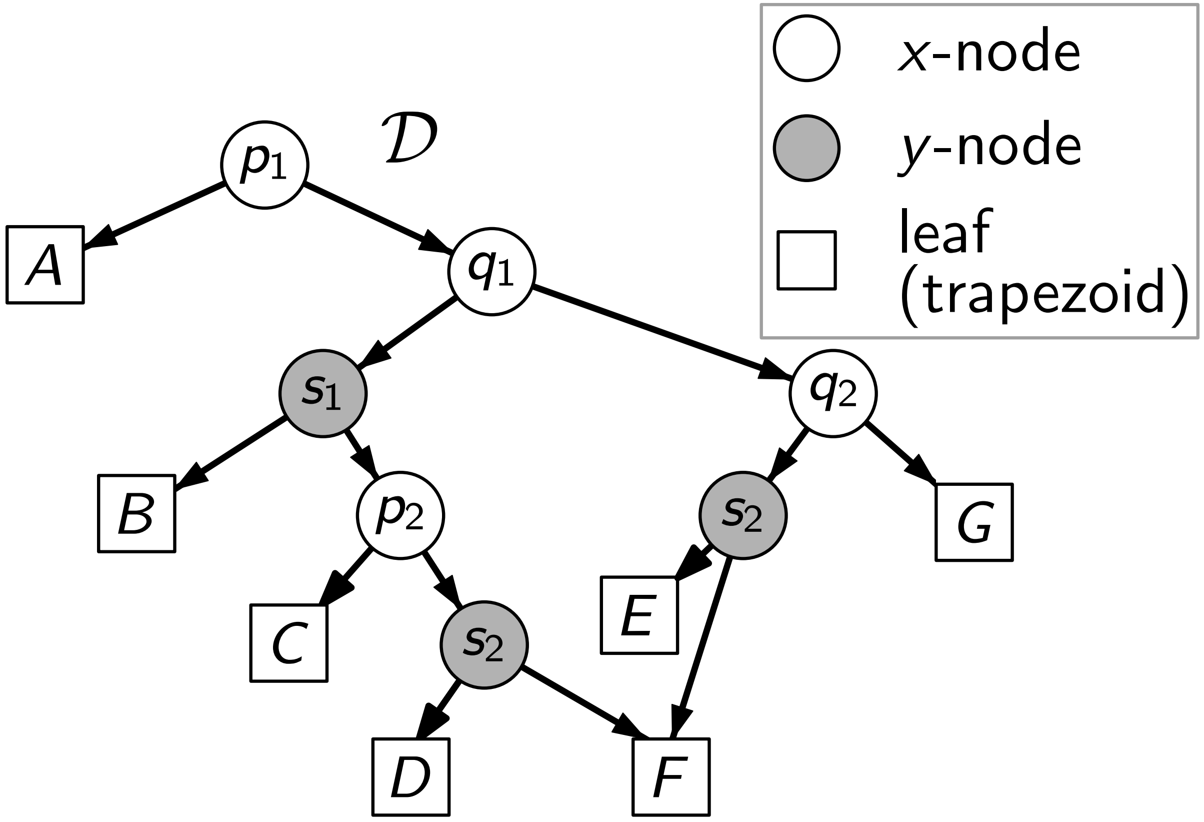
point-location data structure (DAG)

trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

- use $\mathcal{D}$ to locate left endpoint of next segment s
- "walk" along s through $\mathcal{T}$
- destroy all trapezoids of $\mathcal{T}$ intersecting s





The 2d-Problem

point-location data structure (DAG)

trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

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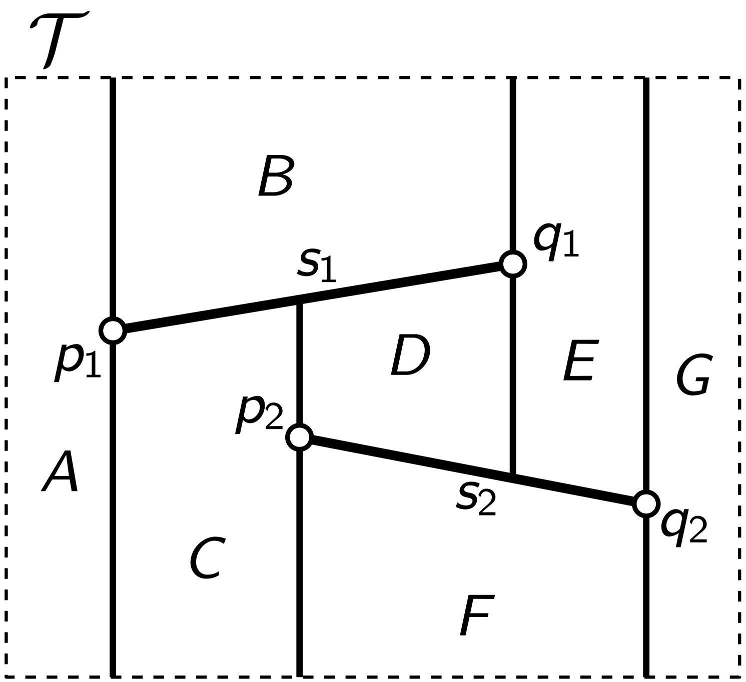

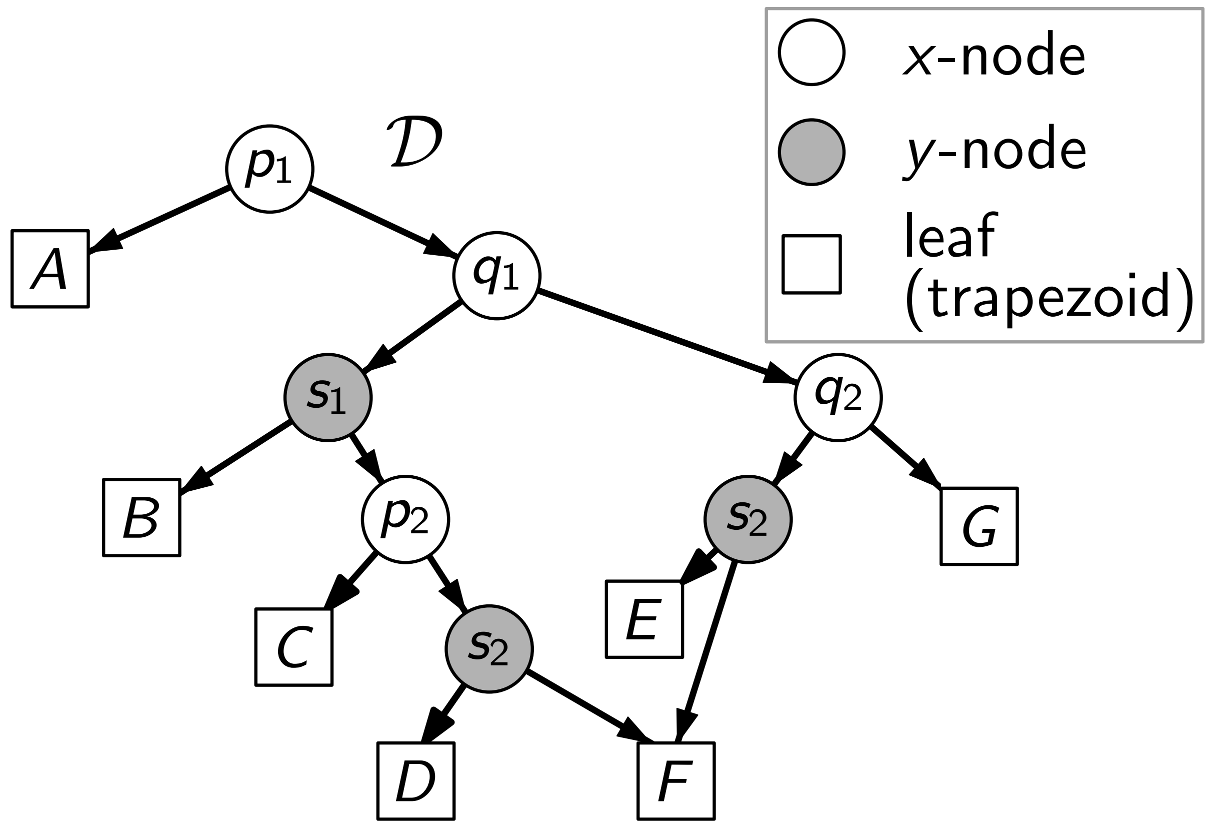

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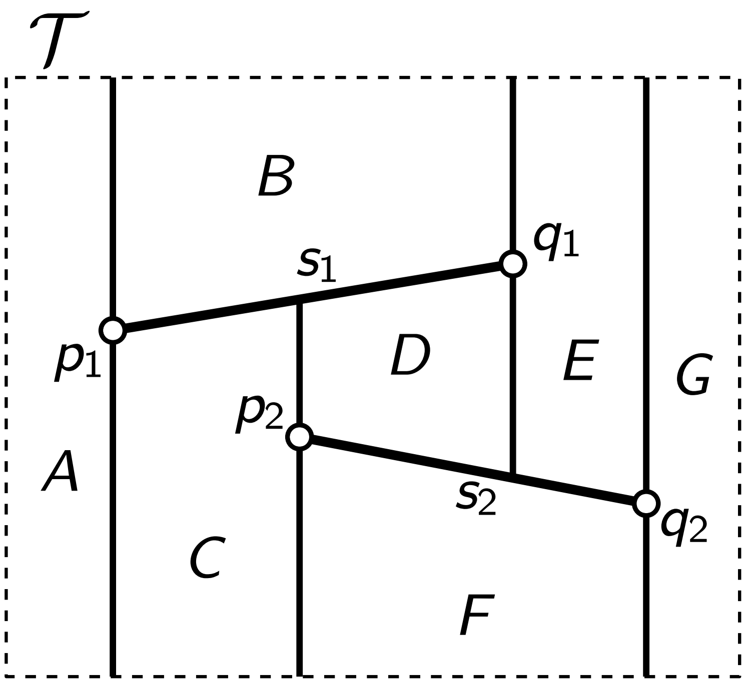

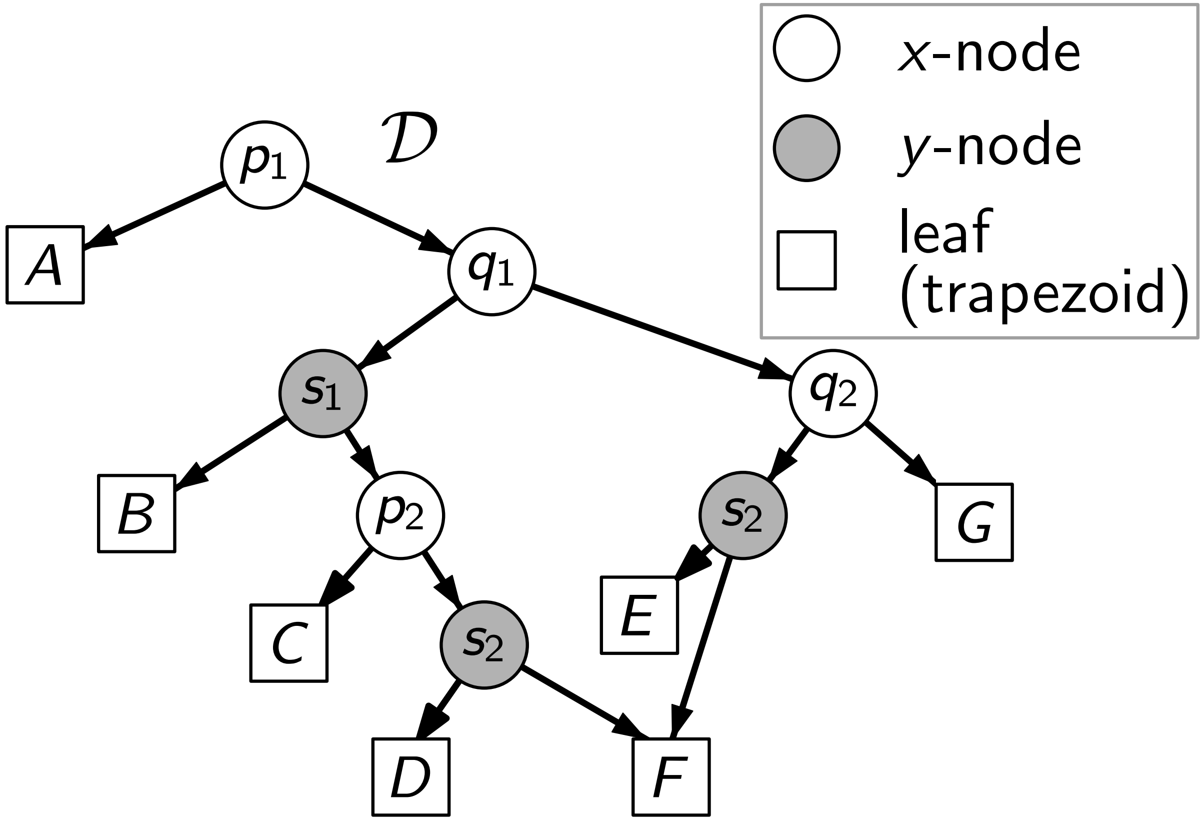

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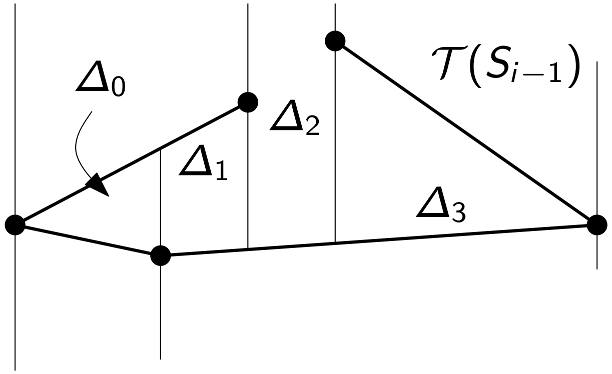

TrapezoidalMap(set S of n non-crossing segments)

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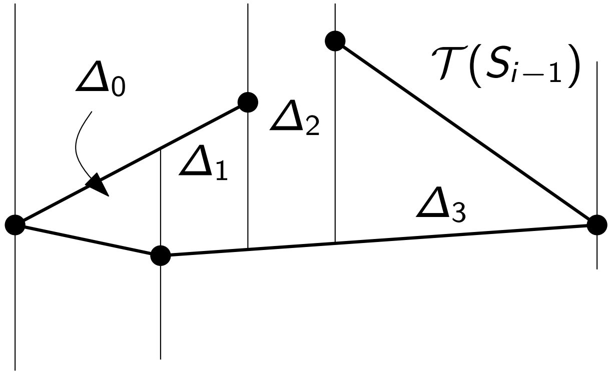

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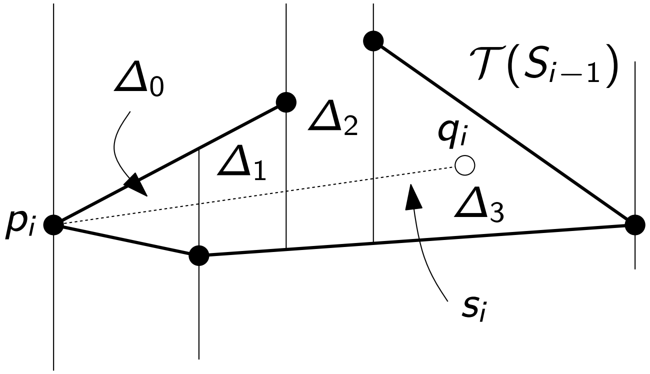

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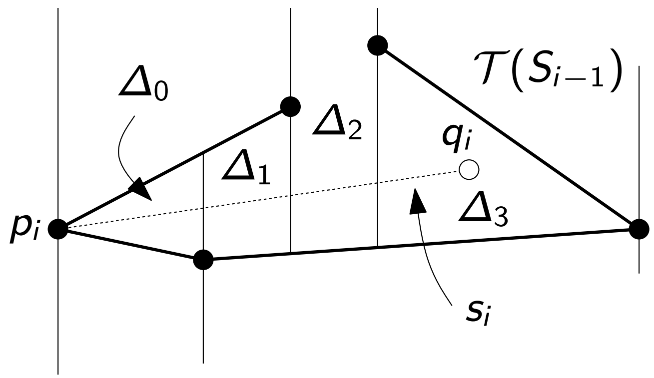

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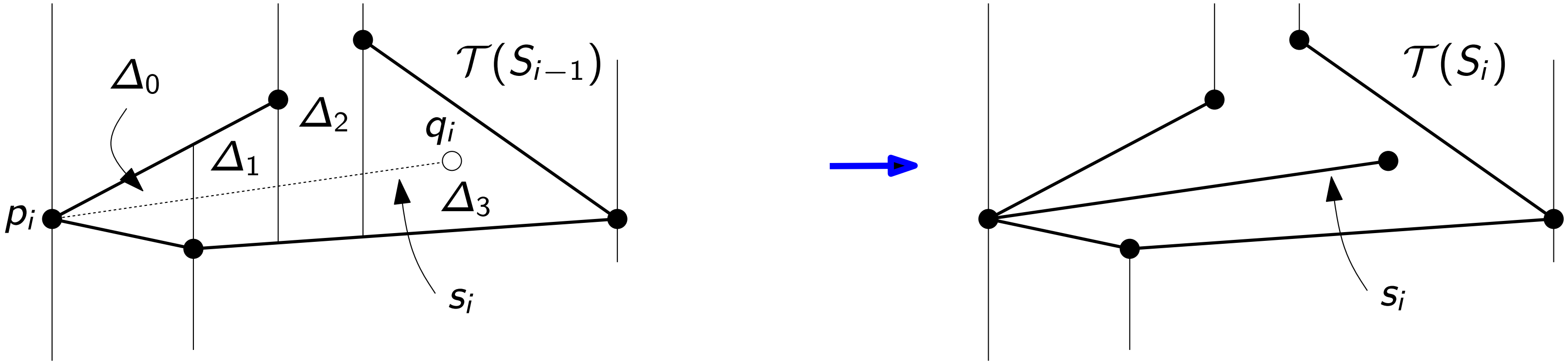

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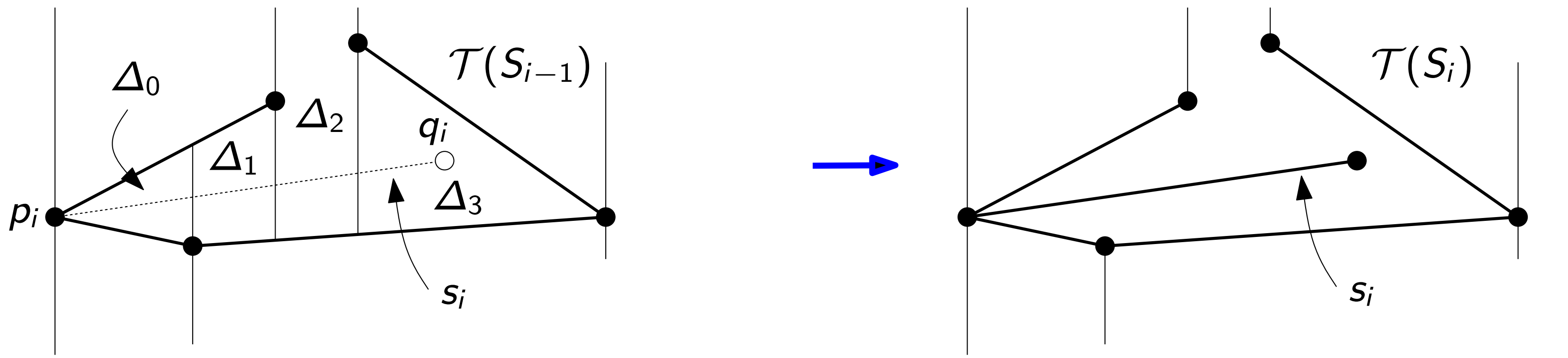

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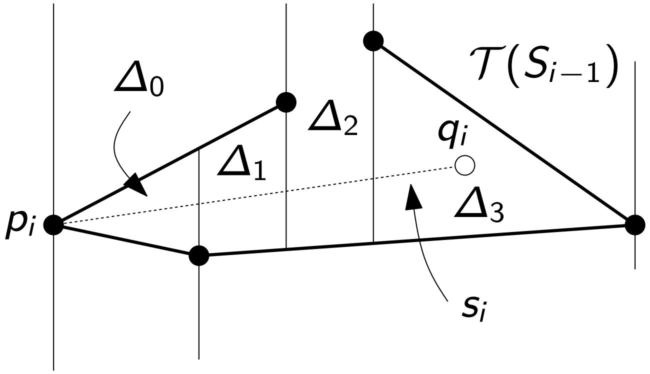

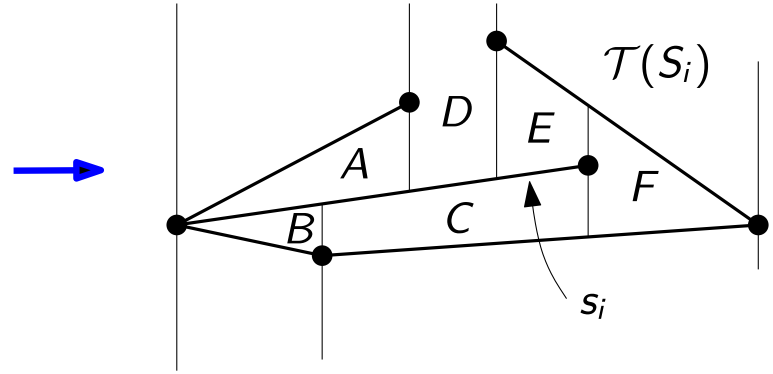

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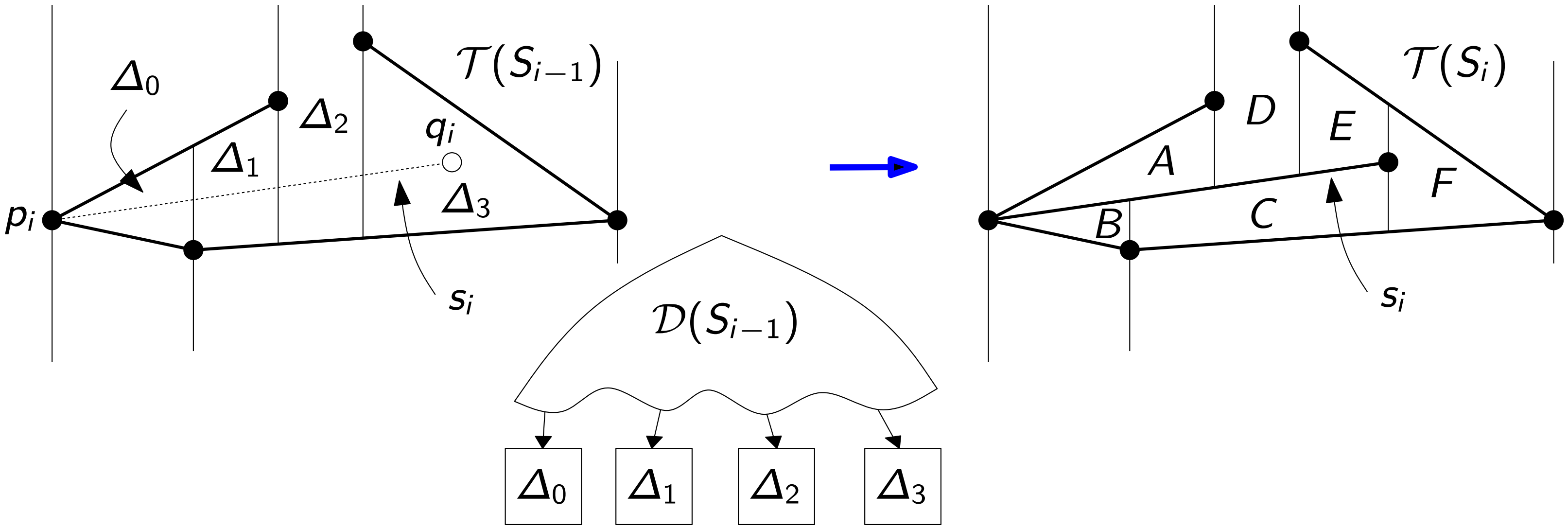

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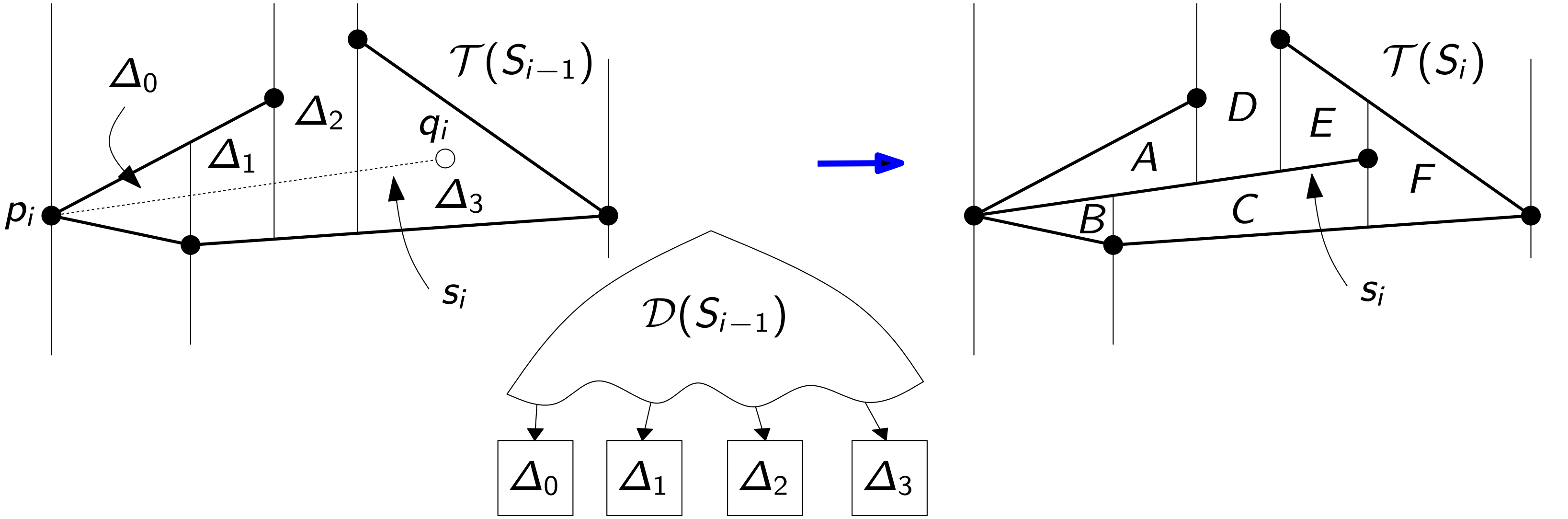

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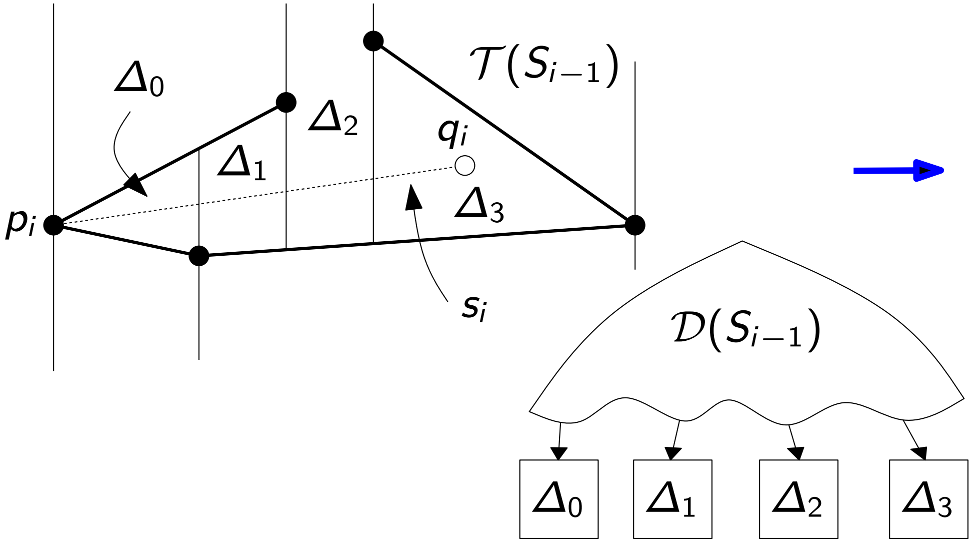

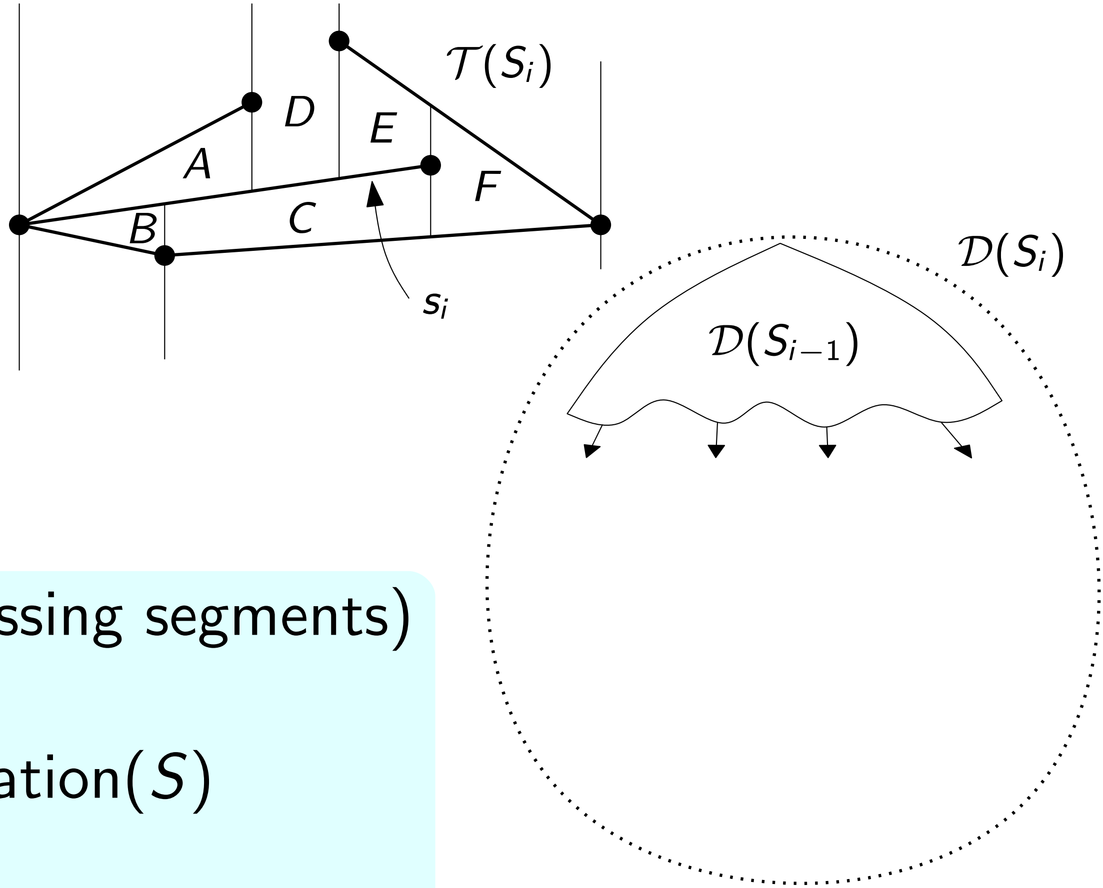

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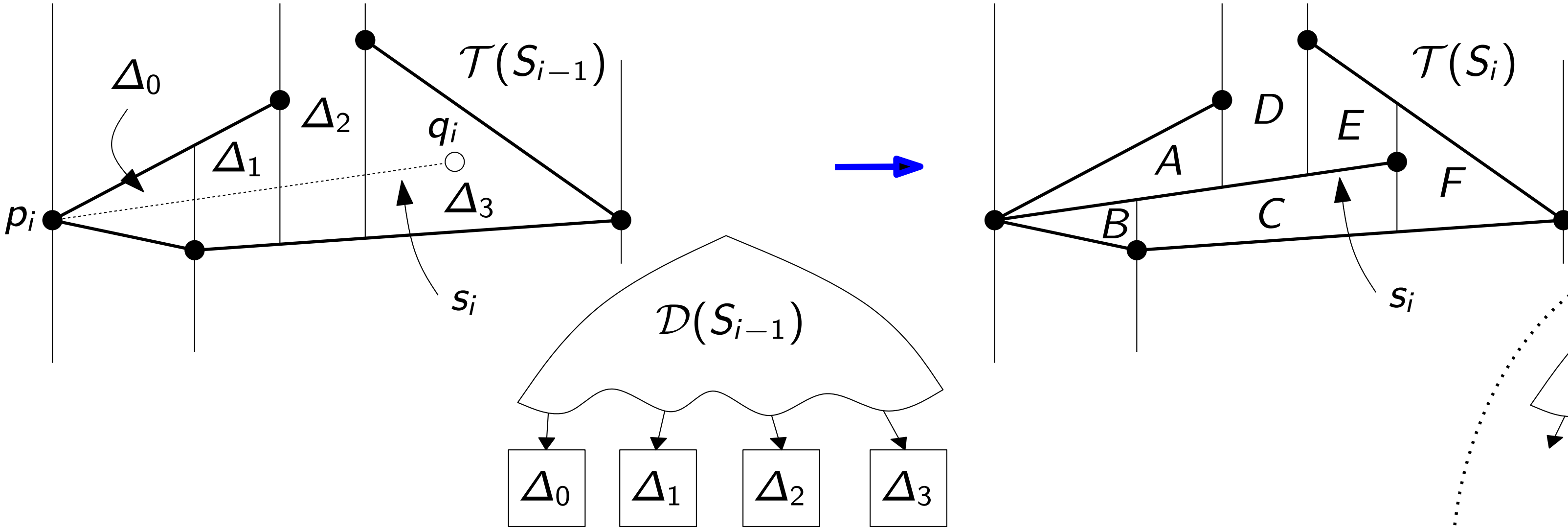

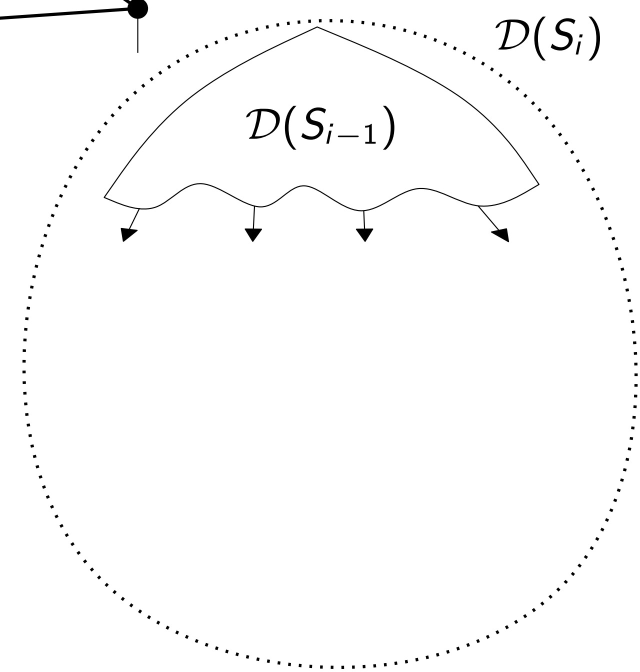

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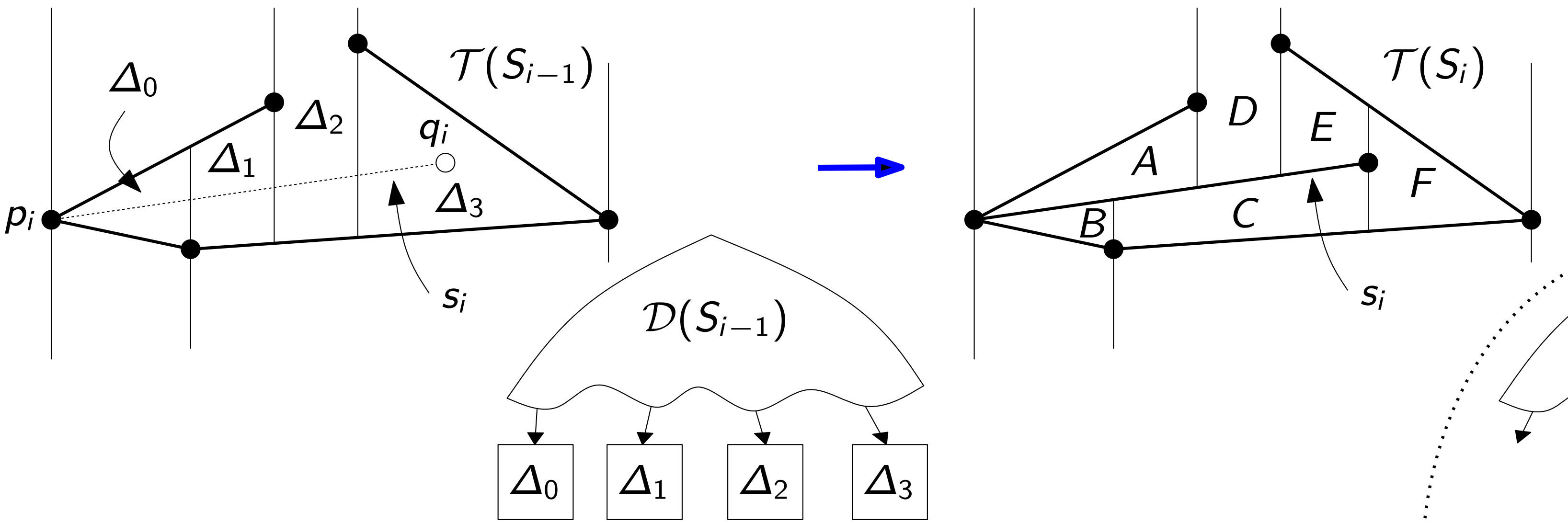

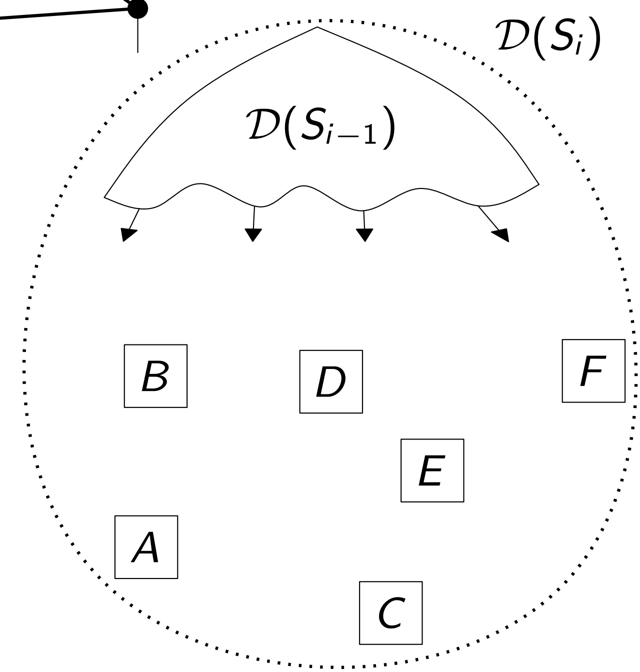

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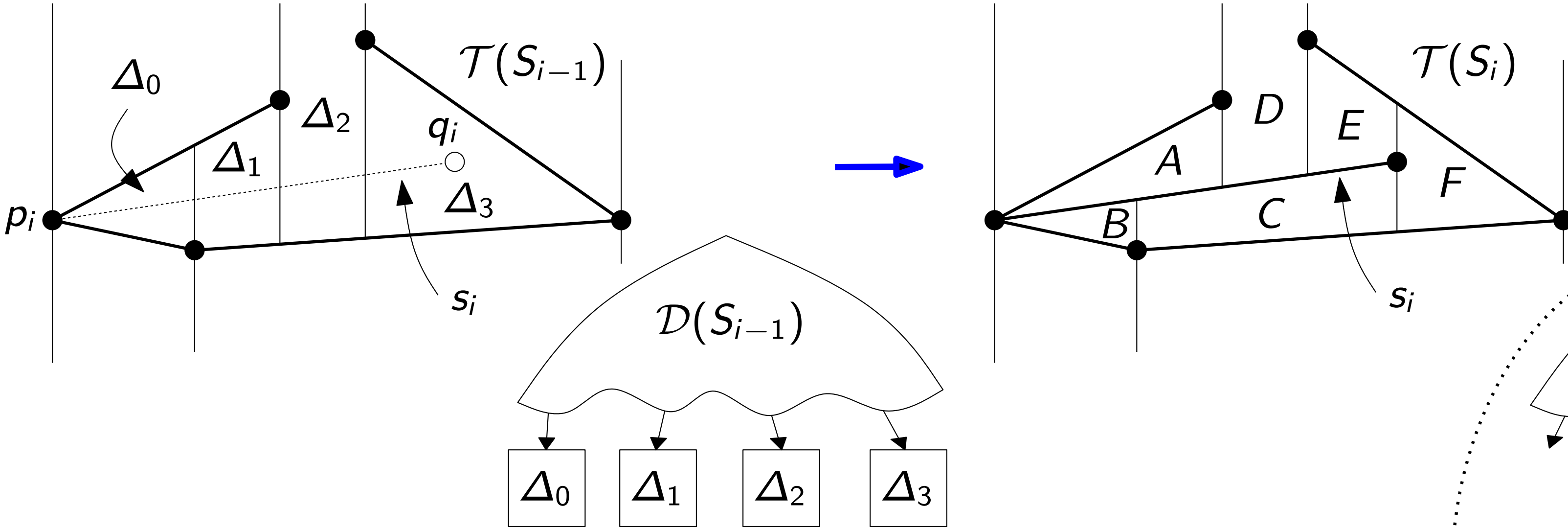

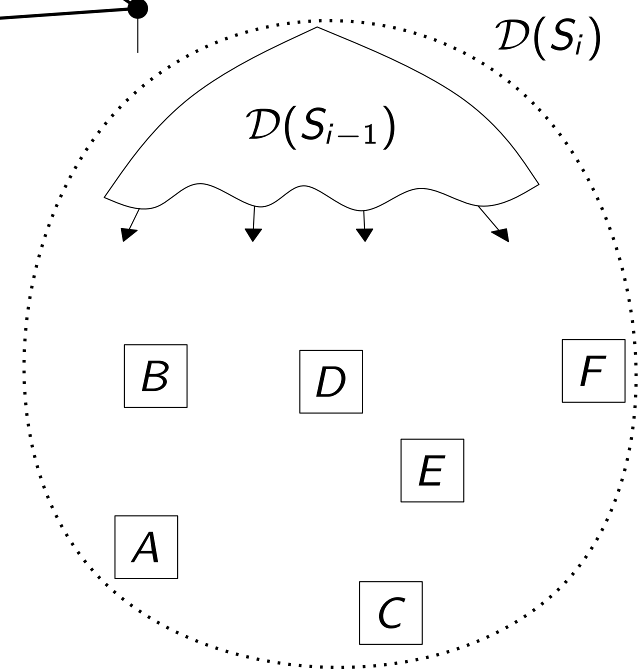

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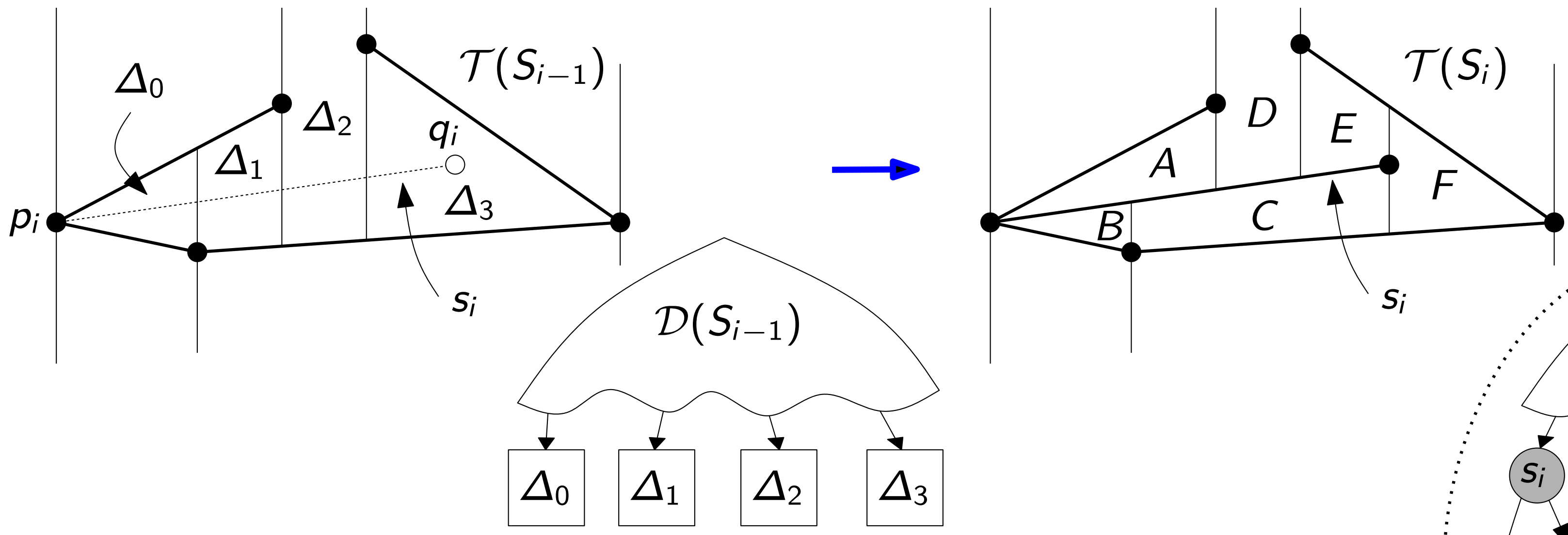
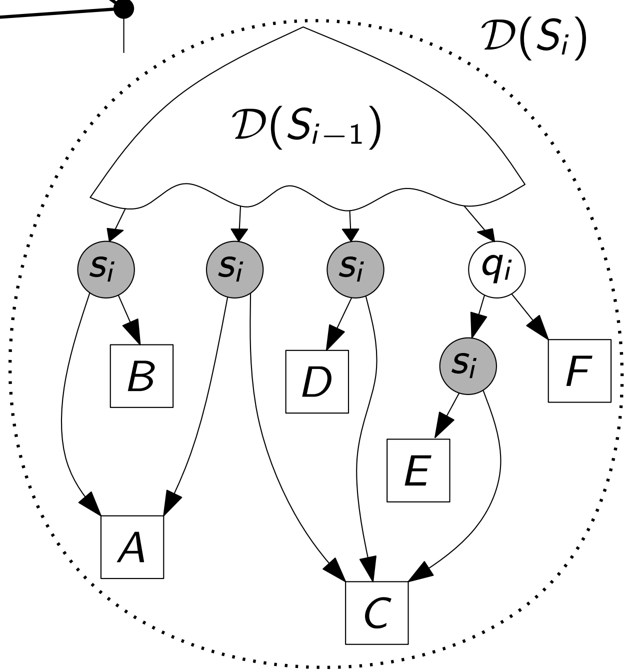
Walking through $\mathcal{T}$ and Updating $\mathcal{D}$





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The 2d-Result

Theorem. TrapezoidalMap(S) computes $\mathcal{T}(S)$ for a set of n line segments in general position and a search structure $\mathcal{D}$ for $\mathcal{T}(S)$ in $O(n \log n)$ expected time.

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Invariant: Before step i, $\mathcal{T}$ is a trapezoidal map for S_{i-1} and $\mathcal{D}$ is a valid search structure for $\mathcal{T}$.

Proof.
– Correctness by loop invariant.
– Query time similar to 1d analysis.
$\Rightarrow$ construction time

Query Time

Let $T(q)$ be the query time for a fixed query pt q.

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$\Rightarrow \mathbf{E}[X_i] = \sum_{j=0}^{3} j \cdot \mathbf{P}[X_i = j] \leq \sum_{j=0}^{3} 3 \cdot \mathbf{P}[X_i \geq 1] =$

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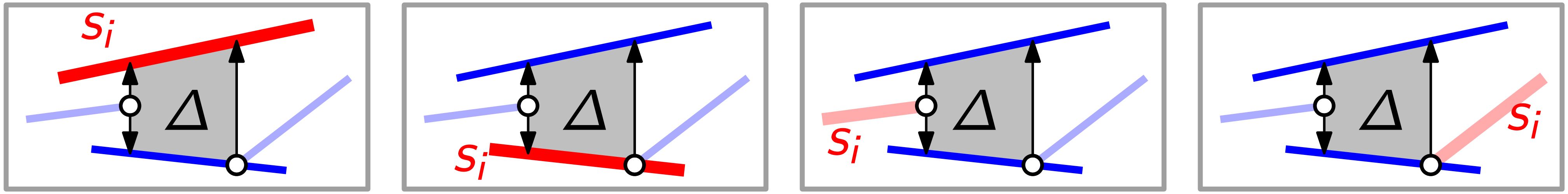
$\Rightarrow \mathrm{top}(\Delta) = s_i$, $\mathrm{bot}(\Delta) = s_i$, $\mathrm{leftp}(\Delta) \in s_i$, or $\mathrm{rightp}(\Delta) \in s_i$.



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Trick: $\mathcal{T}(S_i)$ (and thus Δ) is uniquely determined by S_i.

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Trick: $\mathcal{T}(S_i)$ (and thus Δ) is uniquely determined by S_i.
Consider $S_i \subseteq S$ fixed.
$\Rightarrow \Delta$ does *not* depend on insertion order.

Query Time (cont'd)

$p_i =$ prob. that the search path Π_q of q in $\mathcal{D}$ contains a node that was created in iteration i.

i.e., prob that Δ changes when inserting s_i.

Aim: bound p_i.

Query Time (cont'd)

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Aim: bound p_i.

Tool:

Query Time (cont'd)

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$p_i =$ prob that Δ changes when s_i is removed

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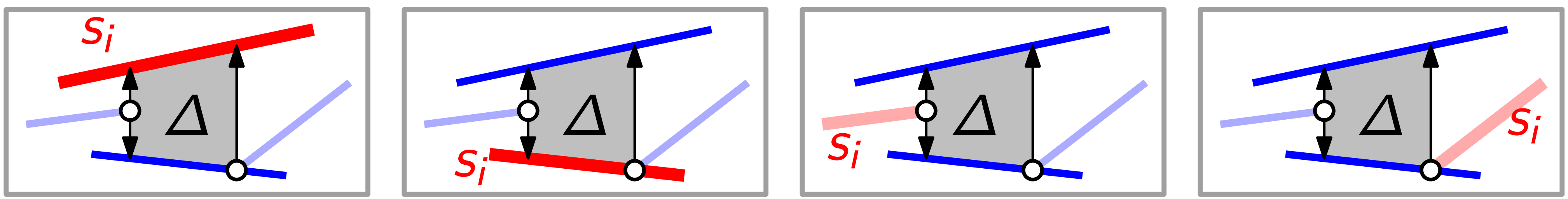
Four cases:



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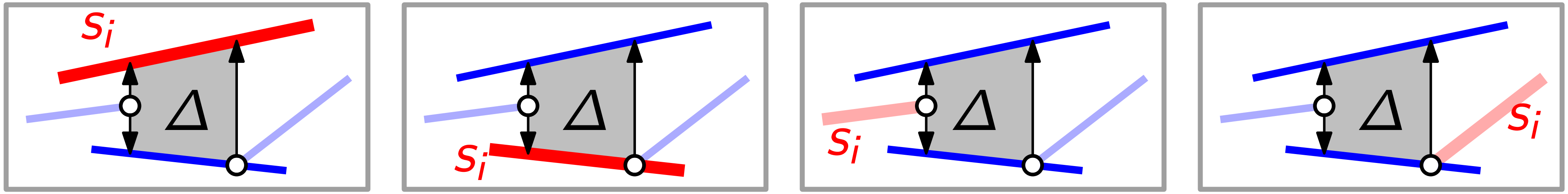

$\mathbf{P}(\text{top}(\Delta) = s_i) =?$

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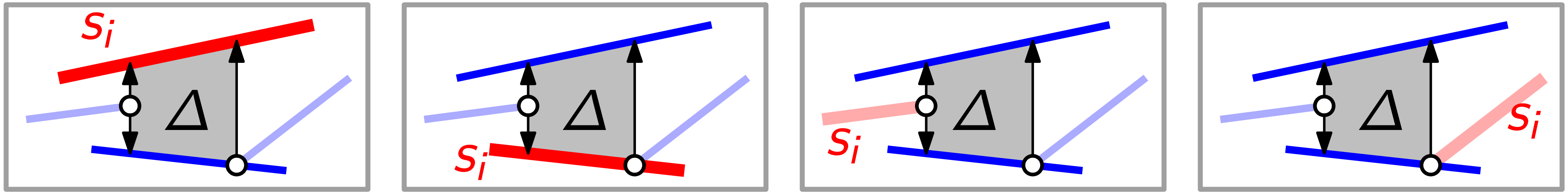

$\mathbf{P}(\mathrm{top}(\Delta) = s_i) = 1/i$ (since exactly one of i segments is top(Δ)).

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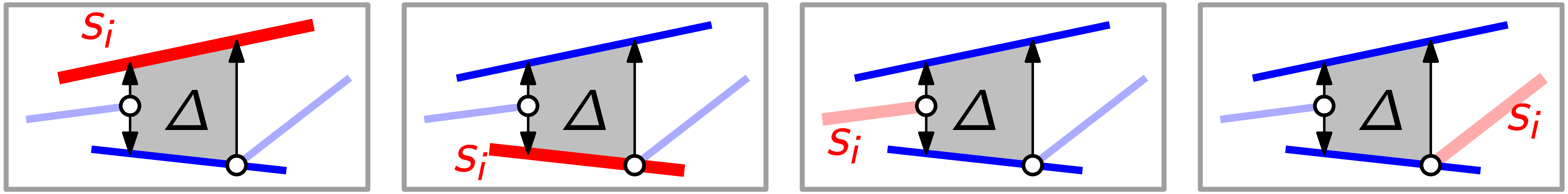

$\mathbf{P}(\text{top}(\Delta) = s_i) = 1/i$ (since exactly one of i segments is top(Δ)).

$\Rightarrow p_i \leq 4/i$

$\Rightarrow \mathbf{E}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathbf{E}[X_i] \leq \sum_{i=1}^{n} 3 \cdot p_i$

$= 12 \sum_{i=1}^{n} 1/i \leq O(log\, n)$

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$p_i =$ prob that Δ changes when s_i is removed

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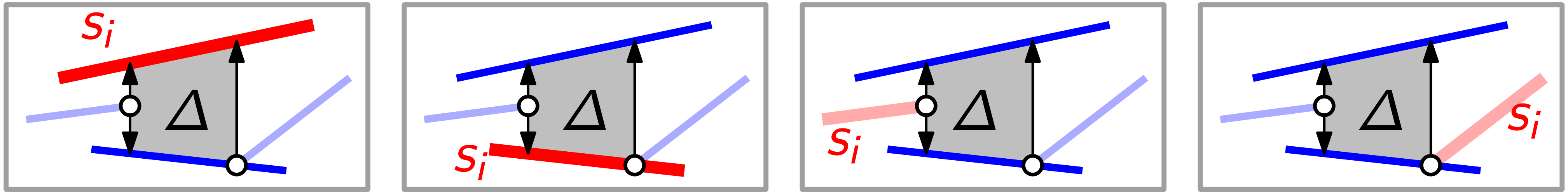

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