Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG
Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems
Institute for Informatics

Computational Geometry

Winter term 2016/17

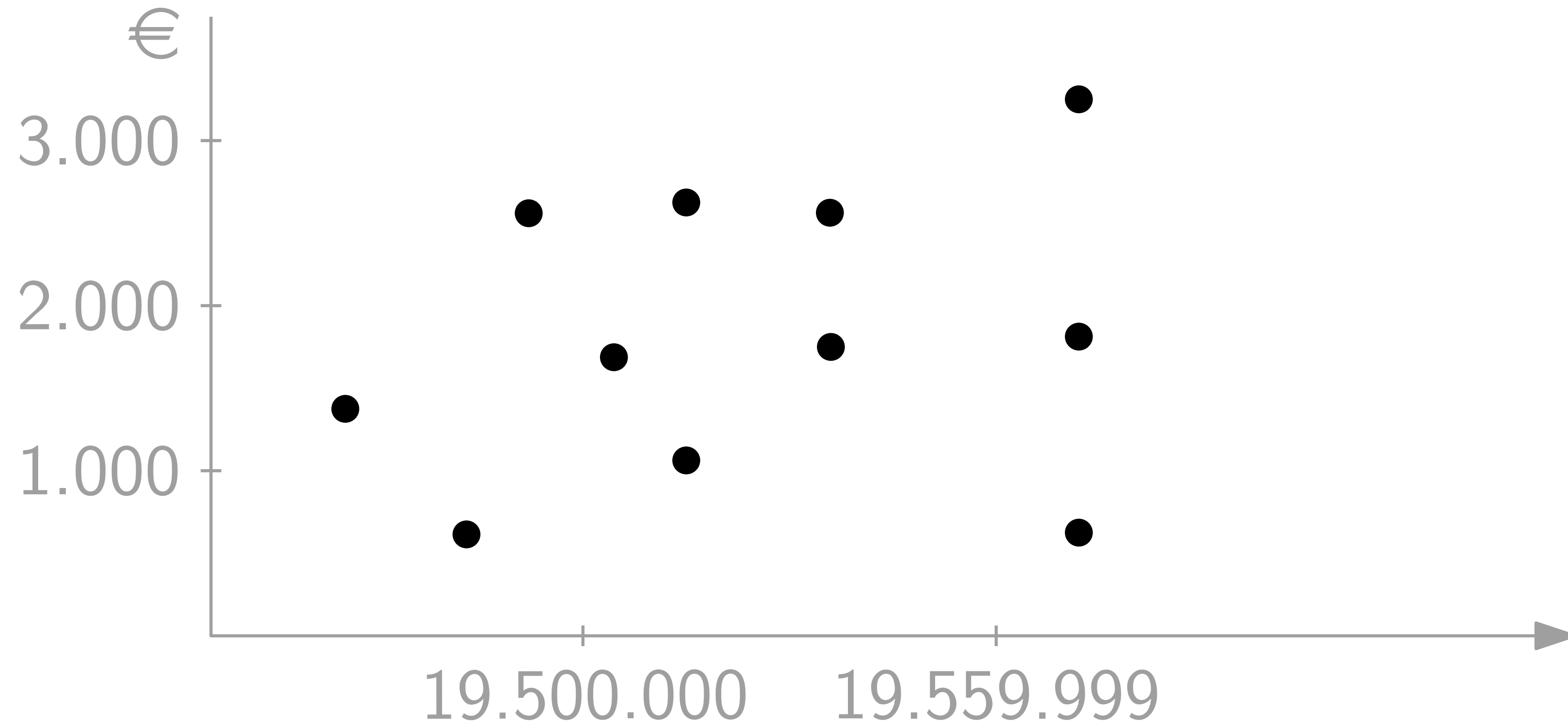
Orthogonal Range Queries
or
Fast Access to Data Bases

Lecture #5

Joachim Spoerhase
Chair for Computer Science I

Orthogonal Range Queries

Example: Personnel management in a company



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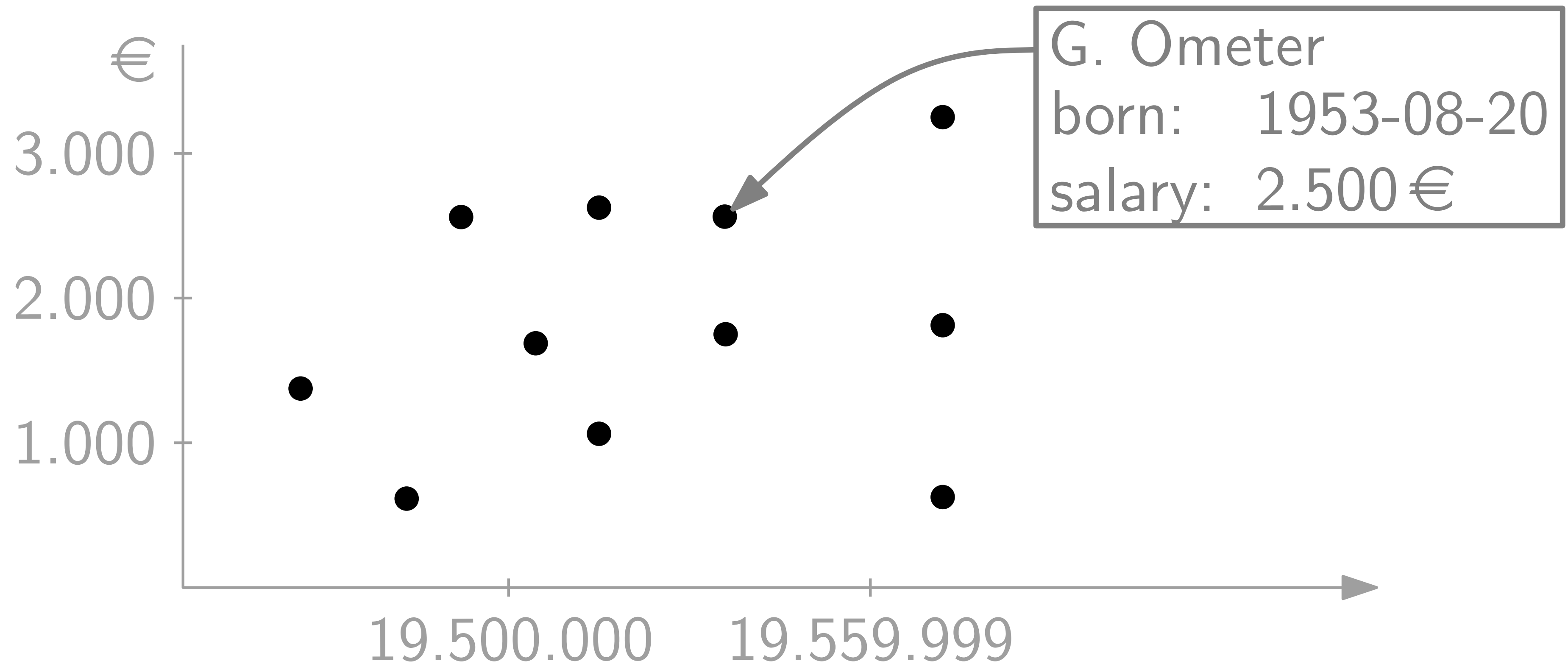

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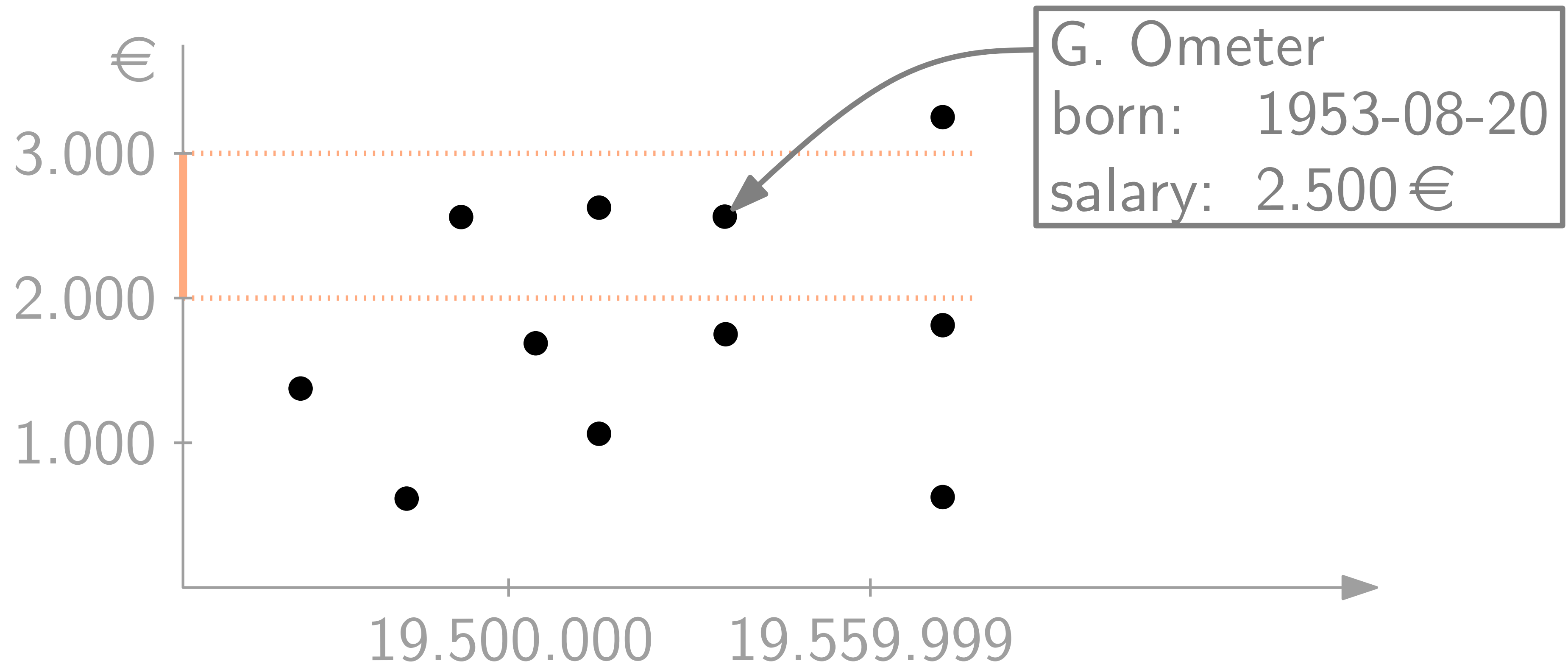

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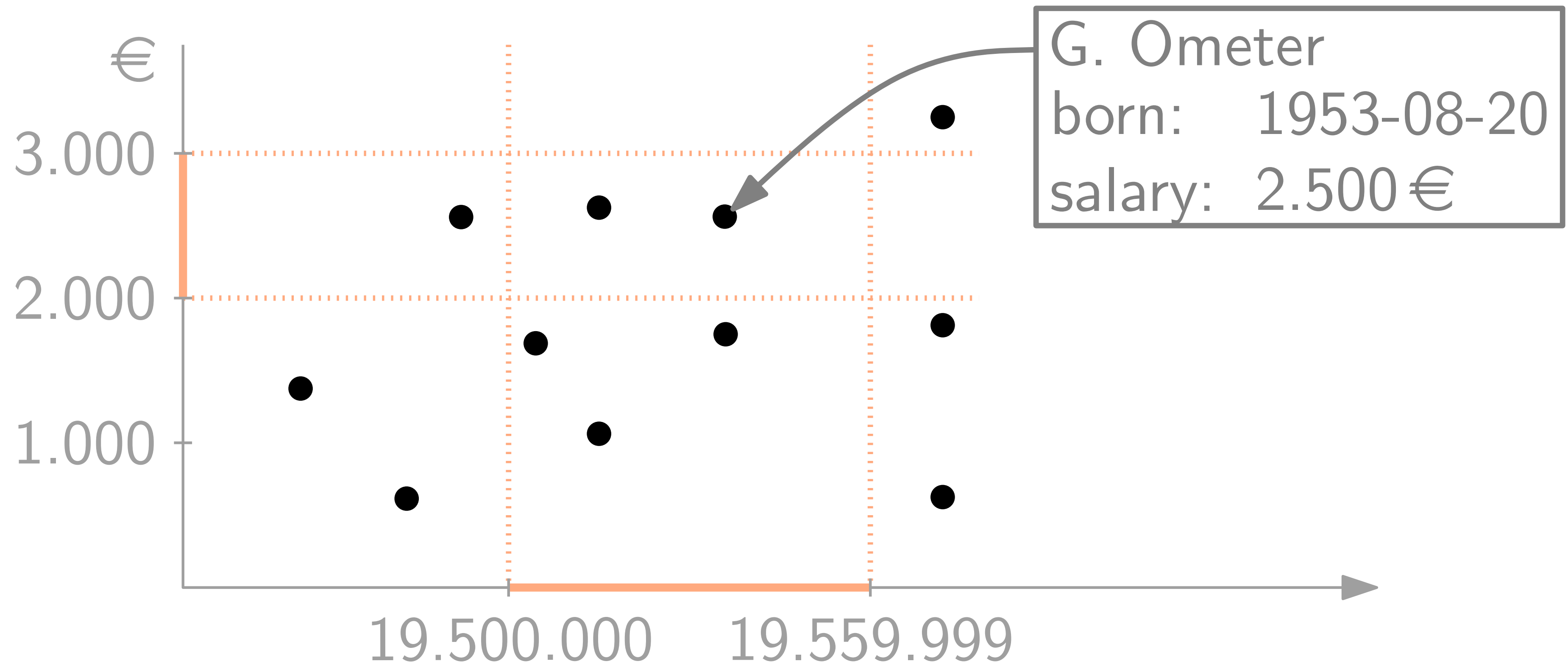

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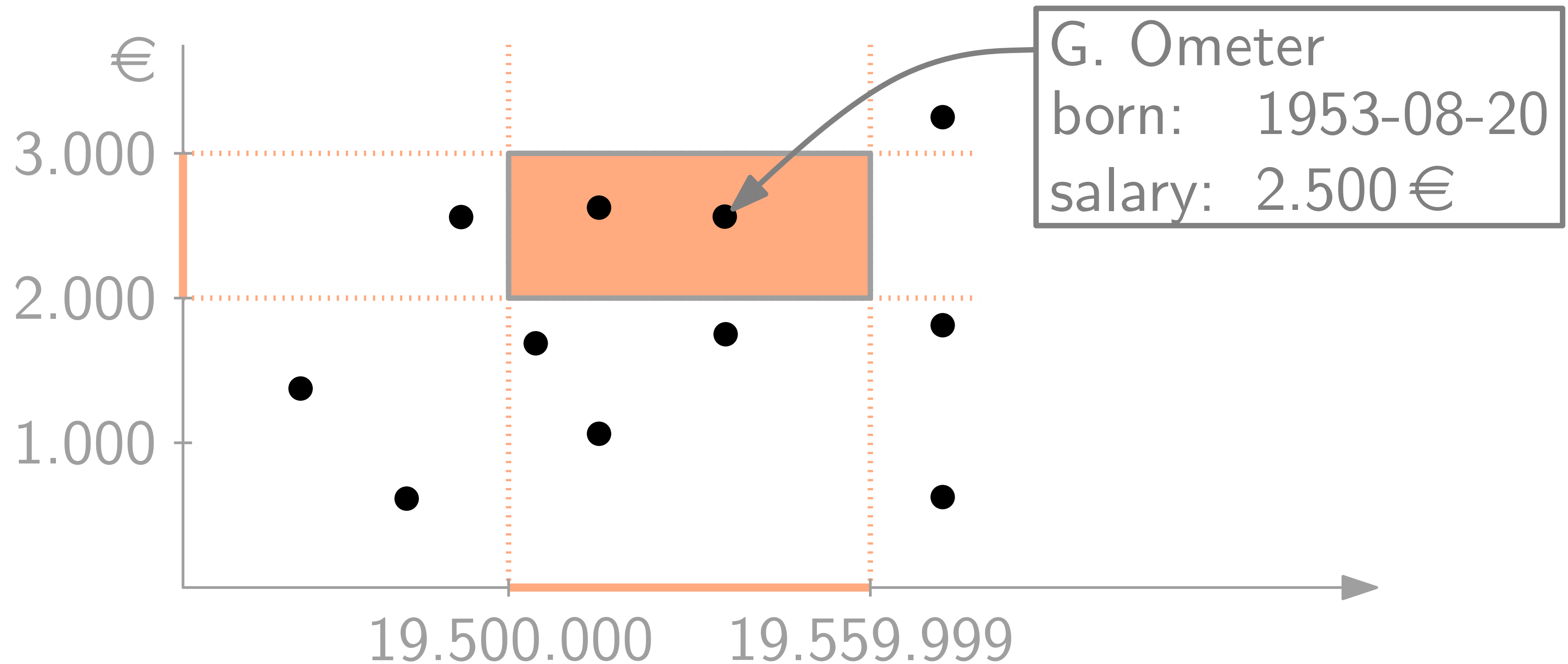

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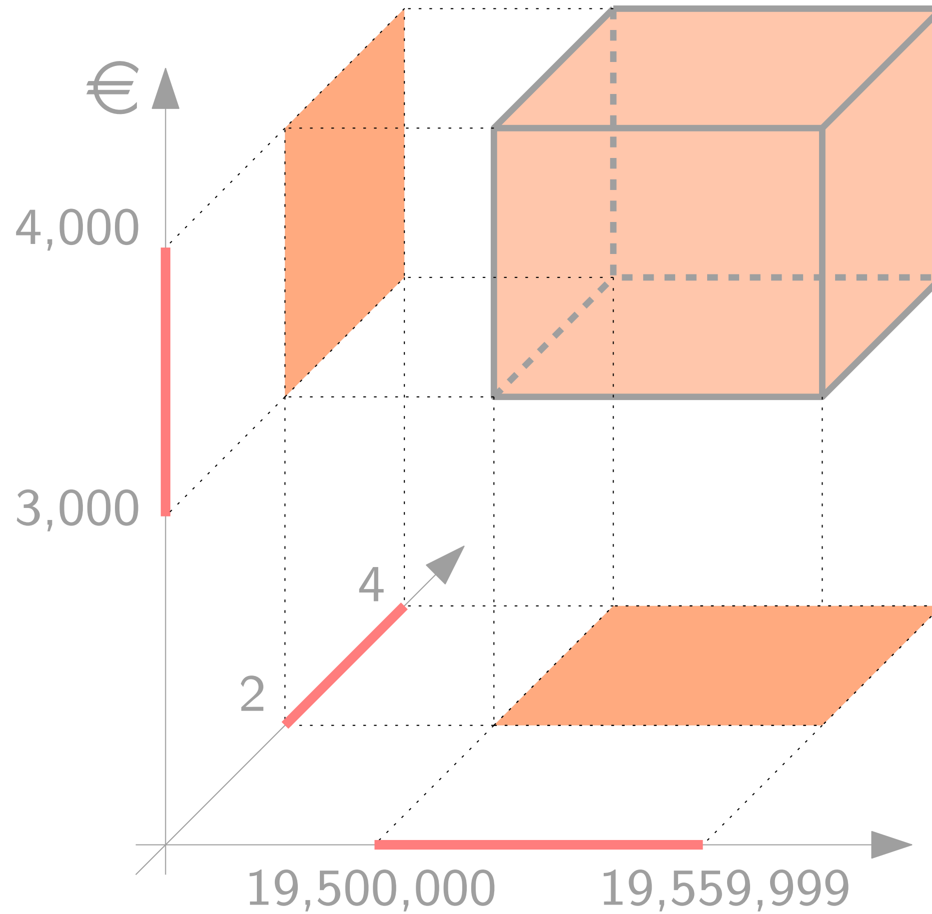

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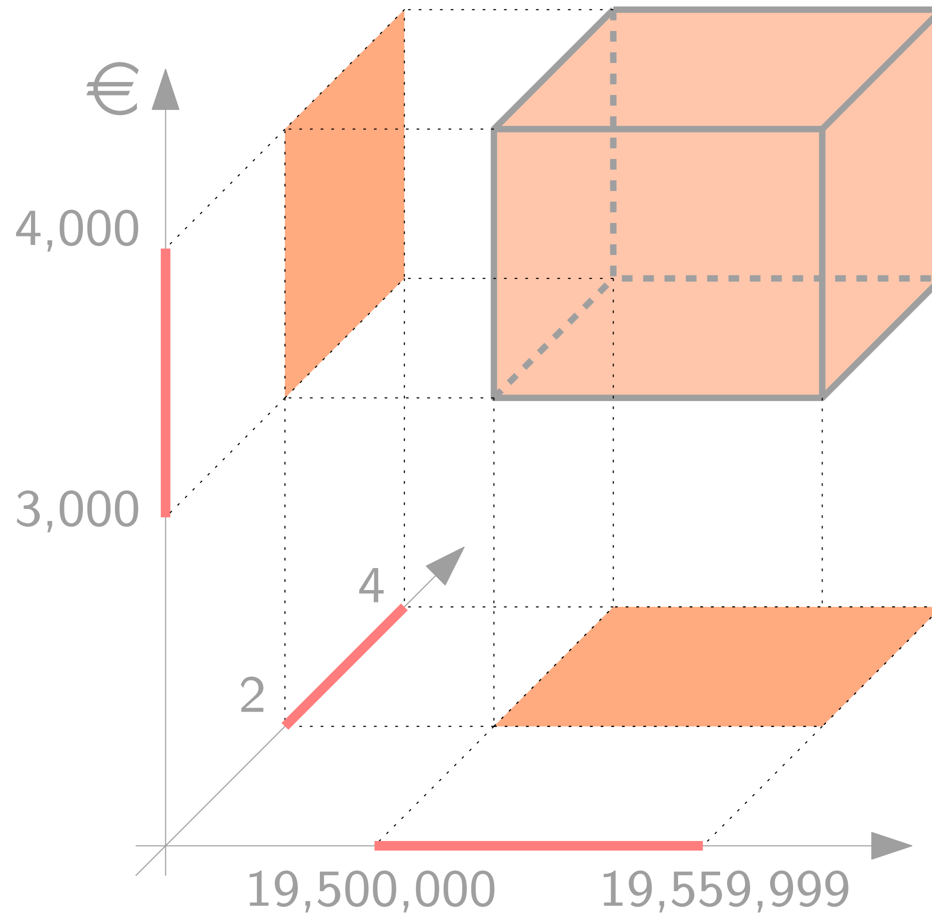

Typical queries for data bases!

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Task: Preprocess a finite set $P \subset \mathbb{R}$ such that for any interval $[x, x']$ the set $P \cap [x, x']$ can be reported quickly. [2 min]

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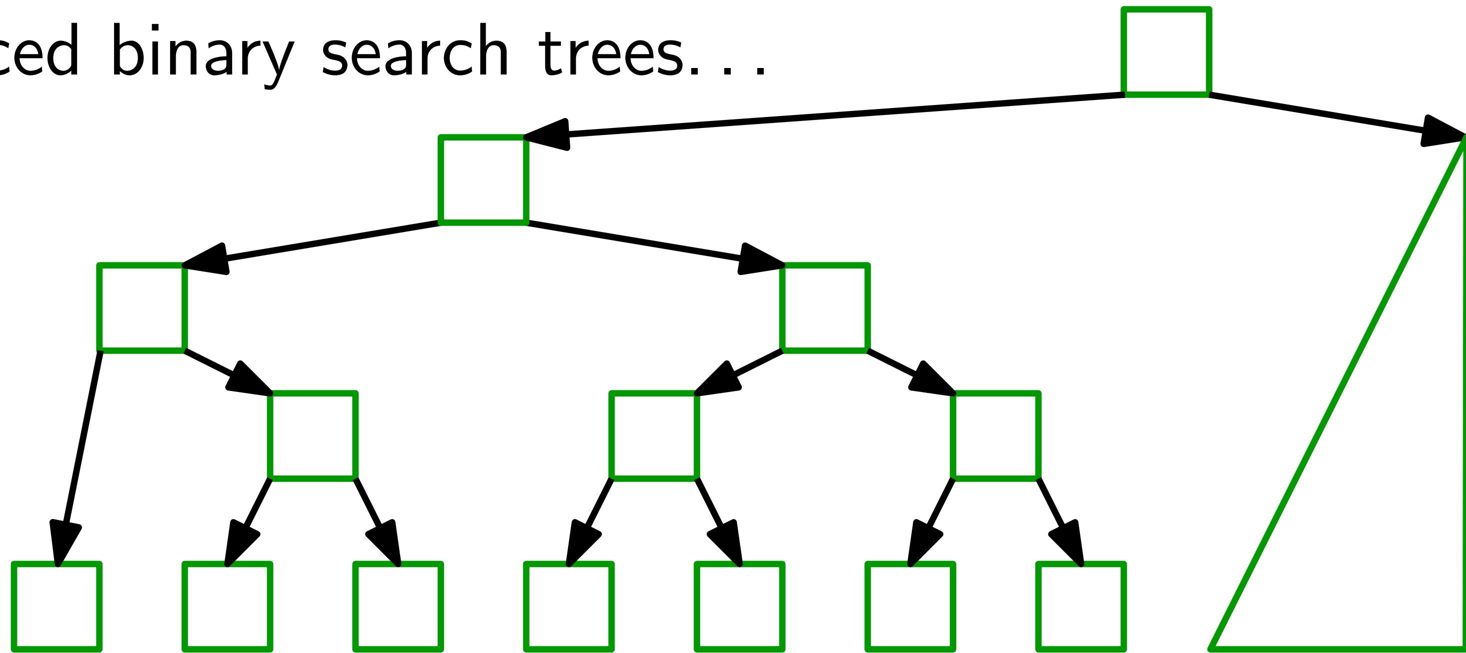

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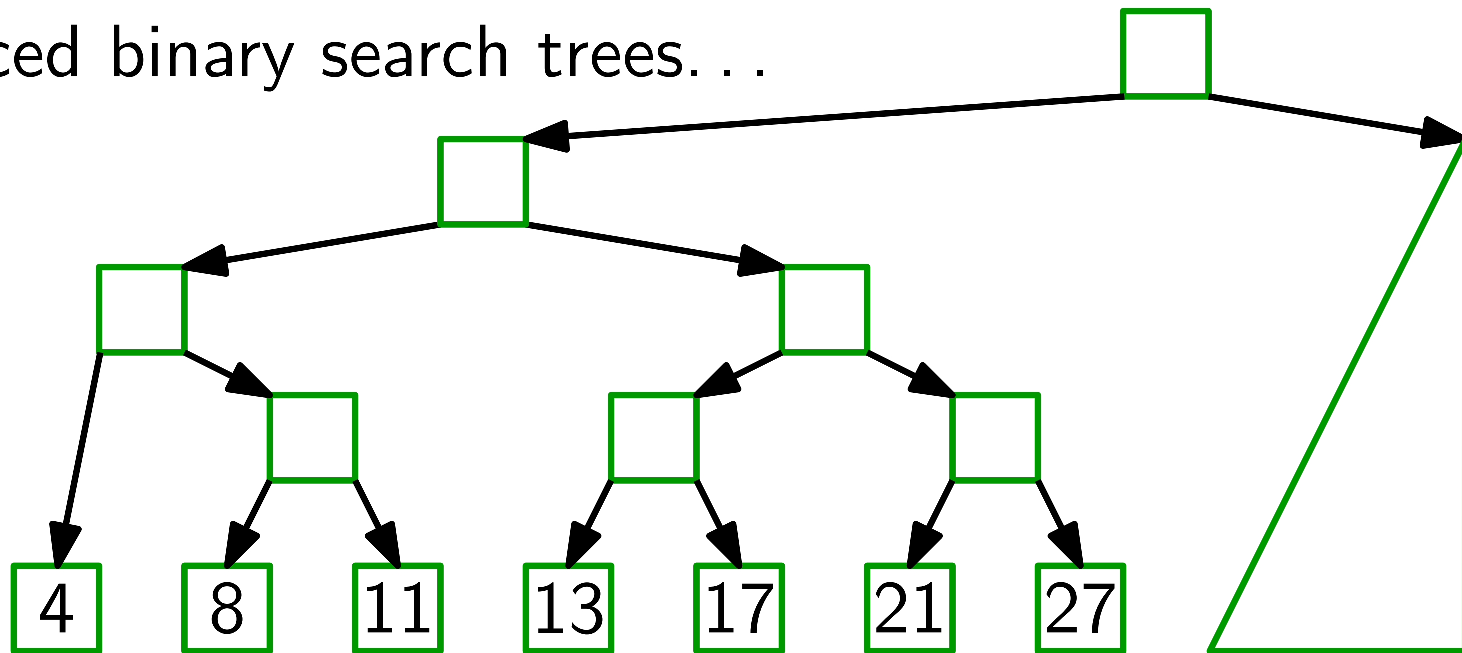

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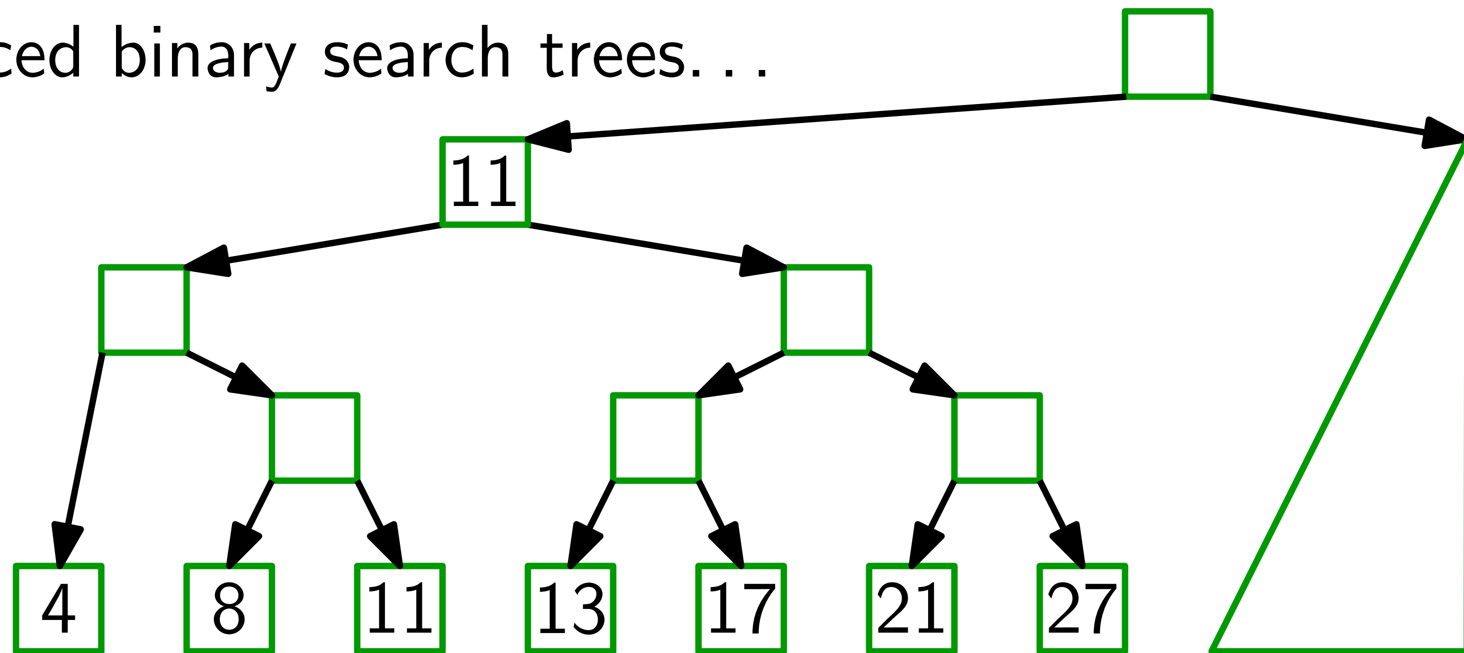

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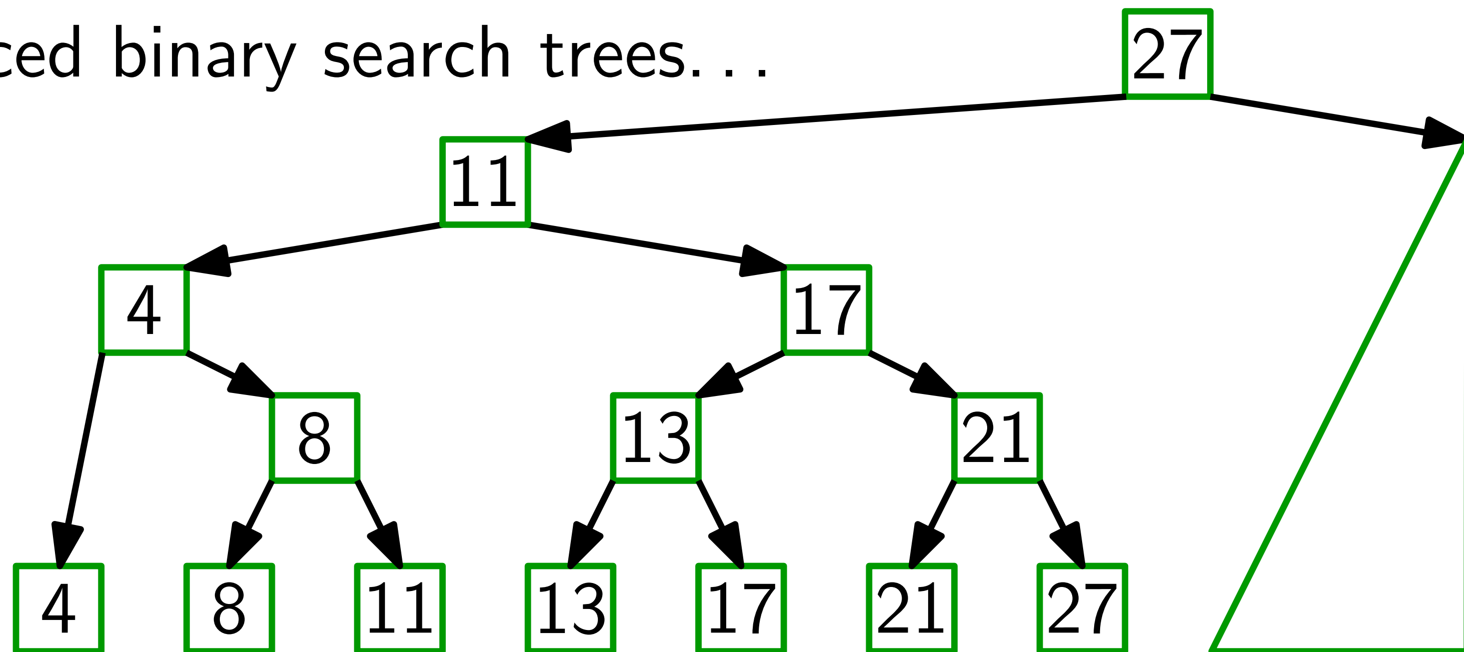

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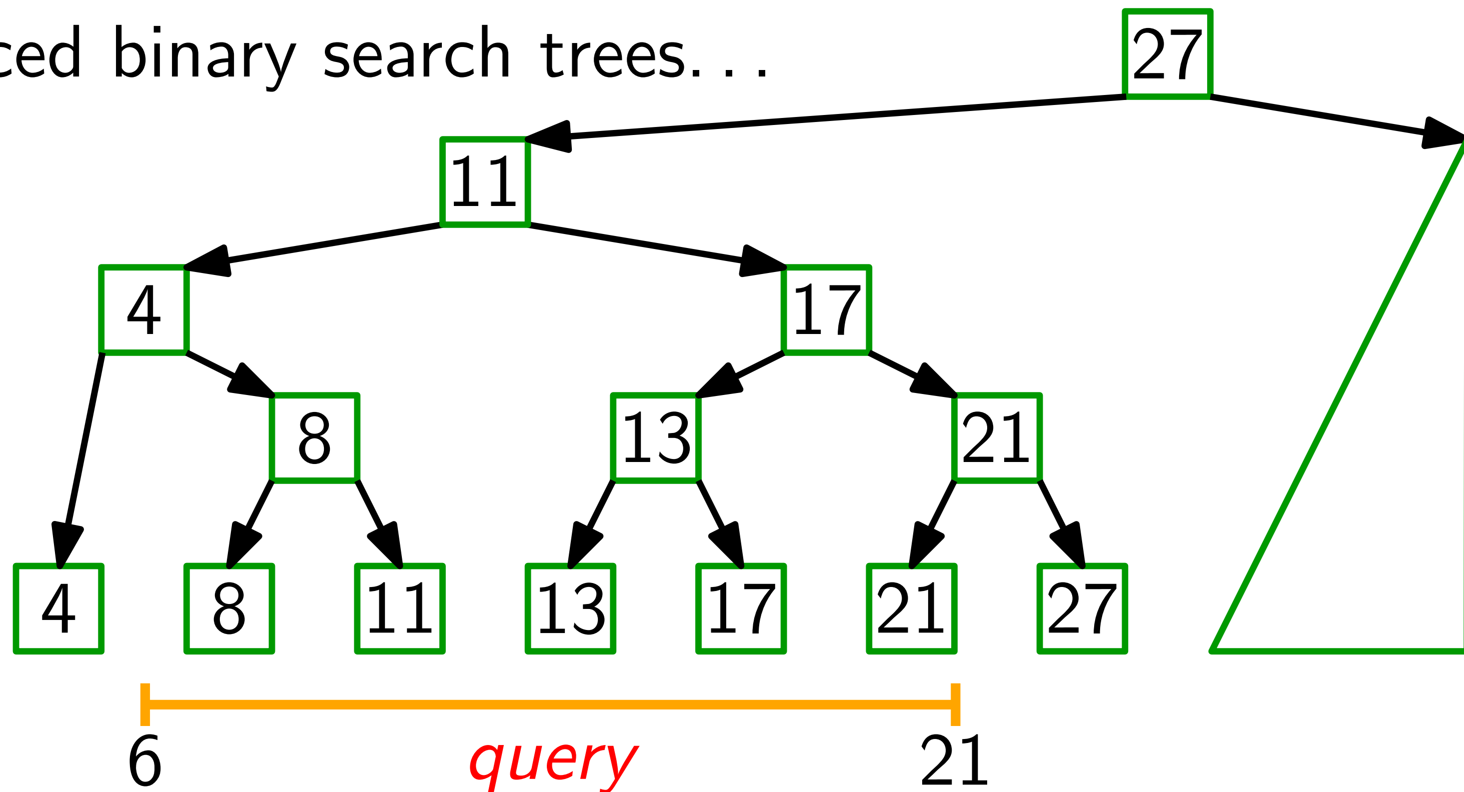

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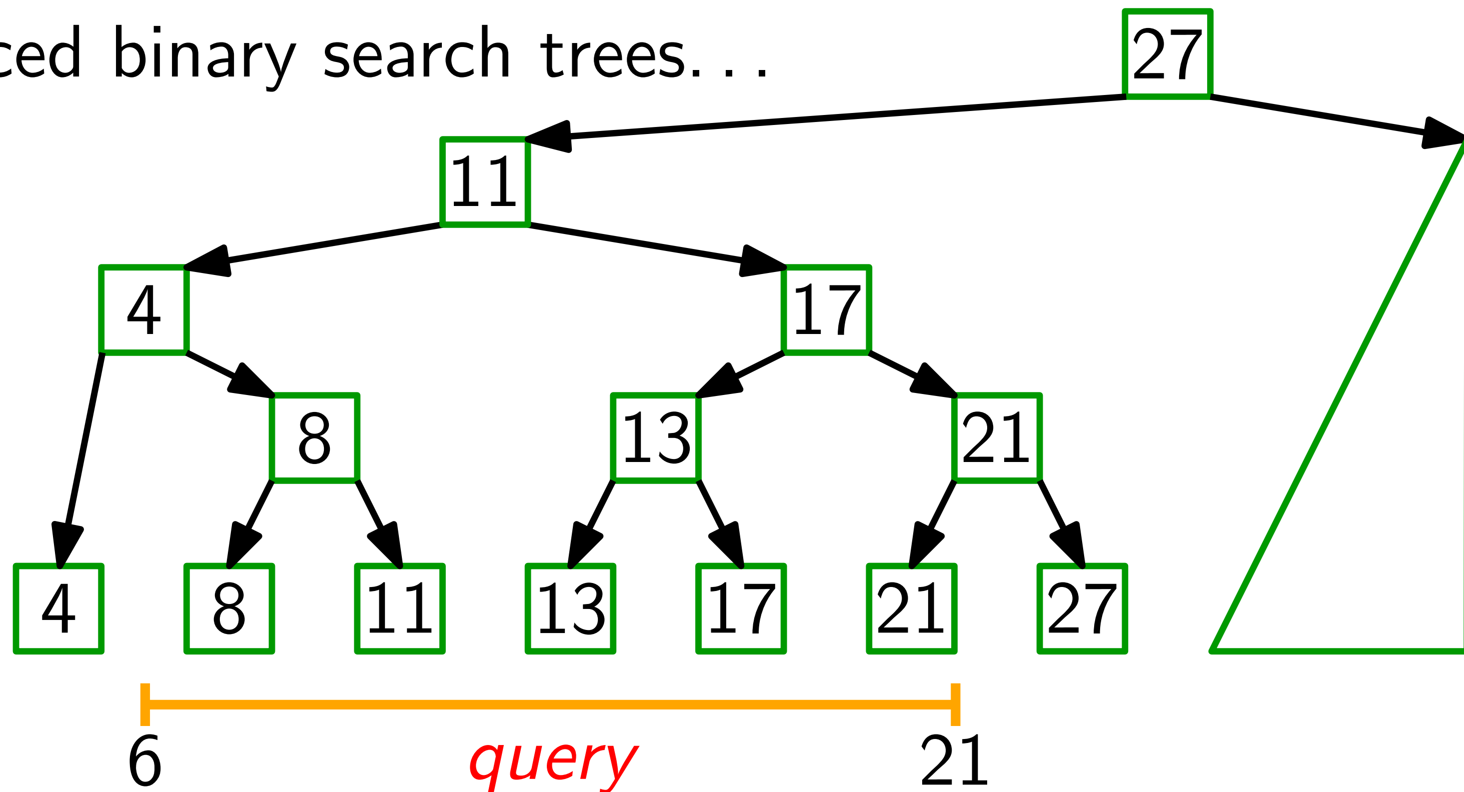

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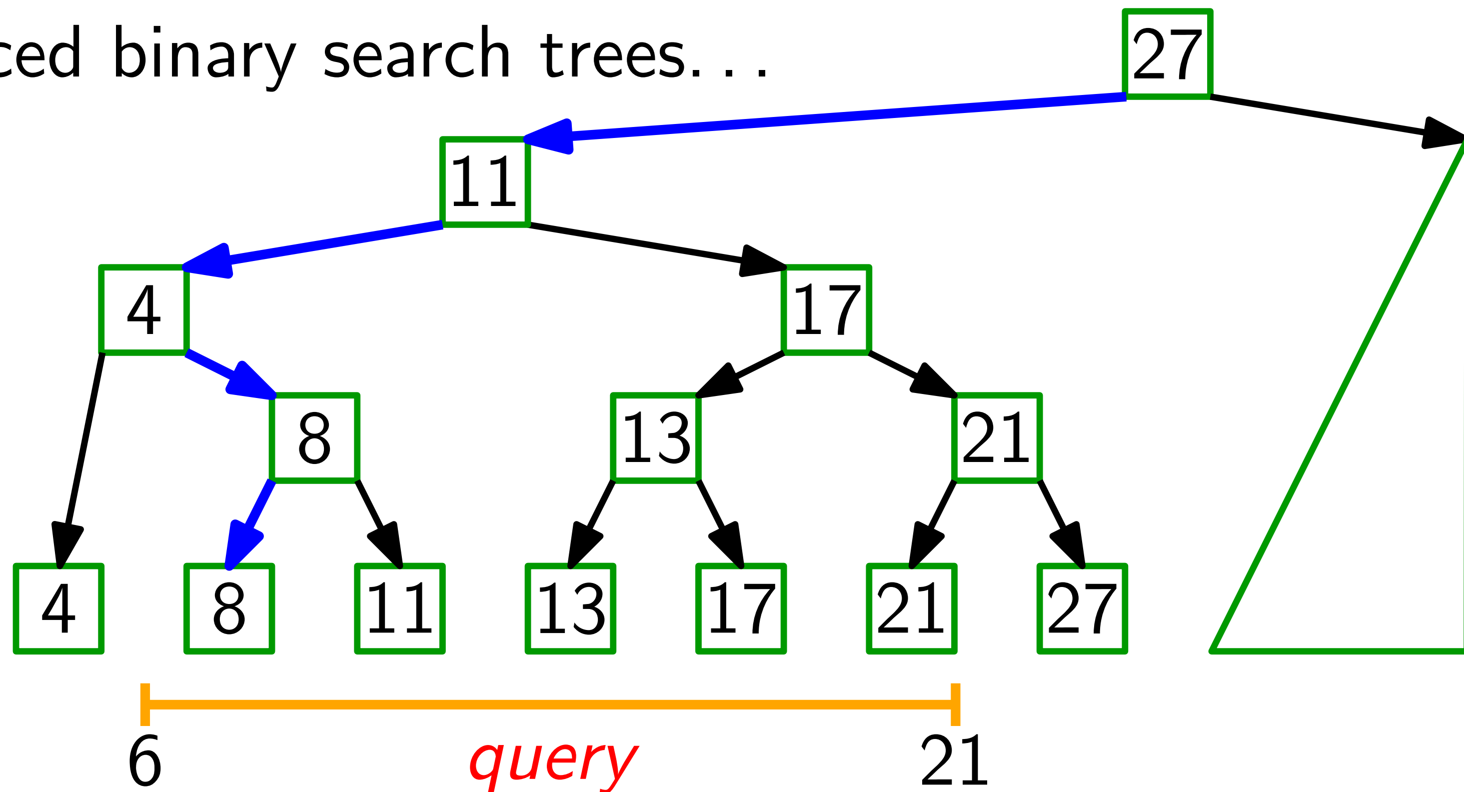

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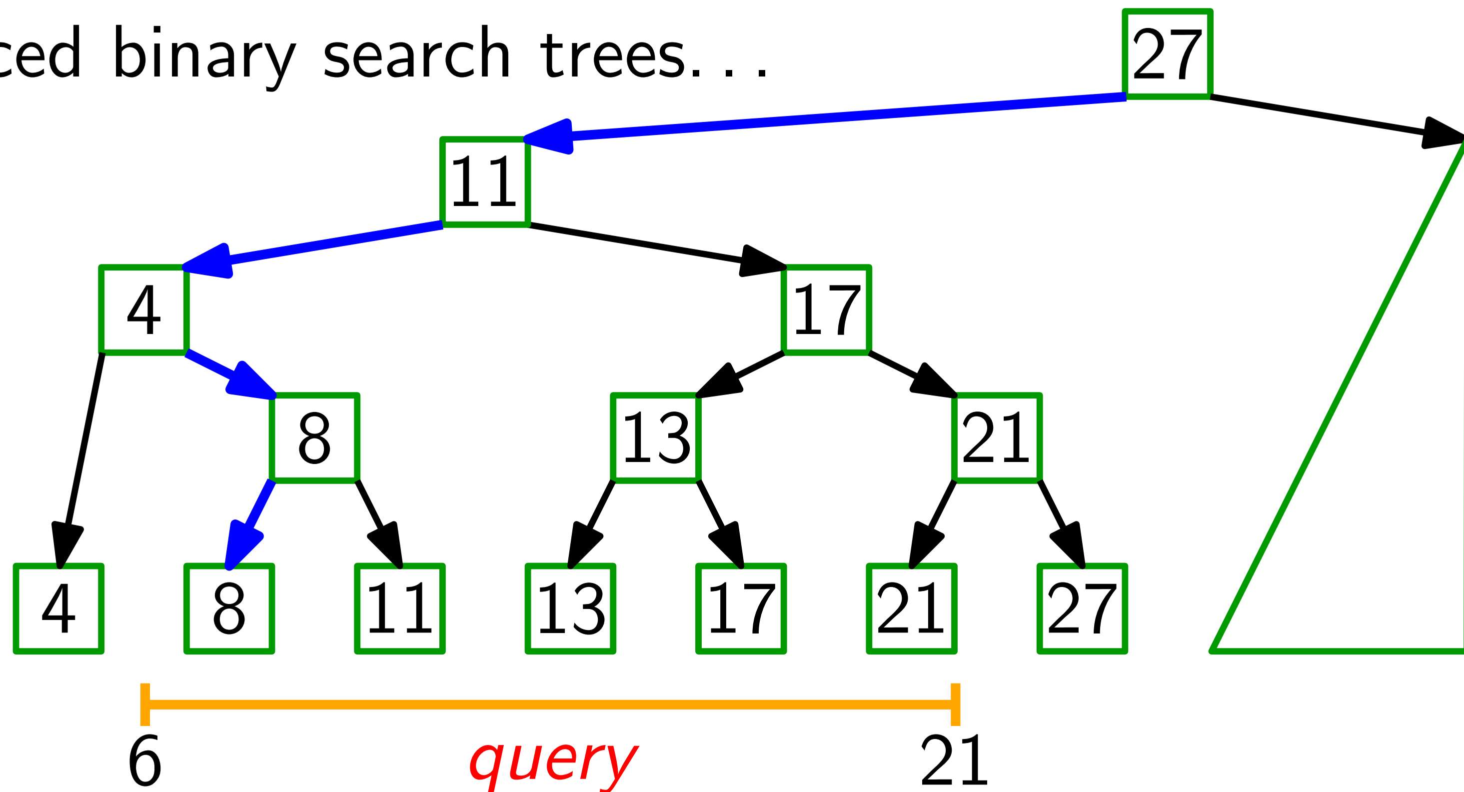

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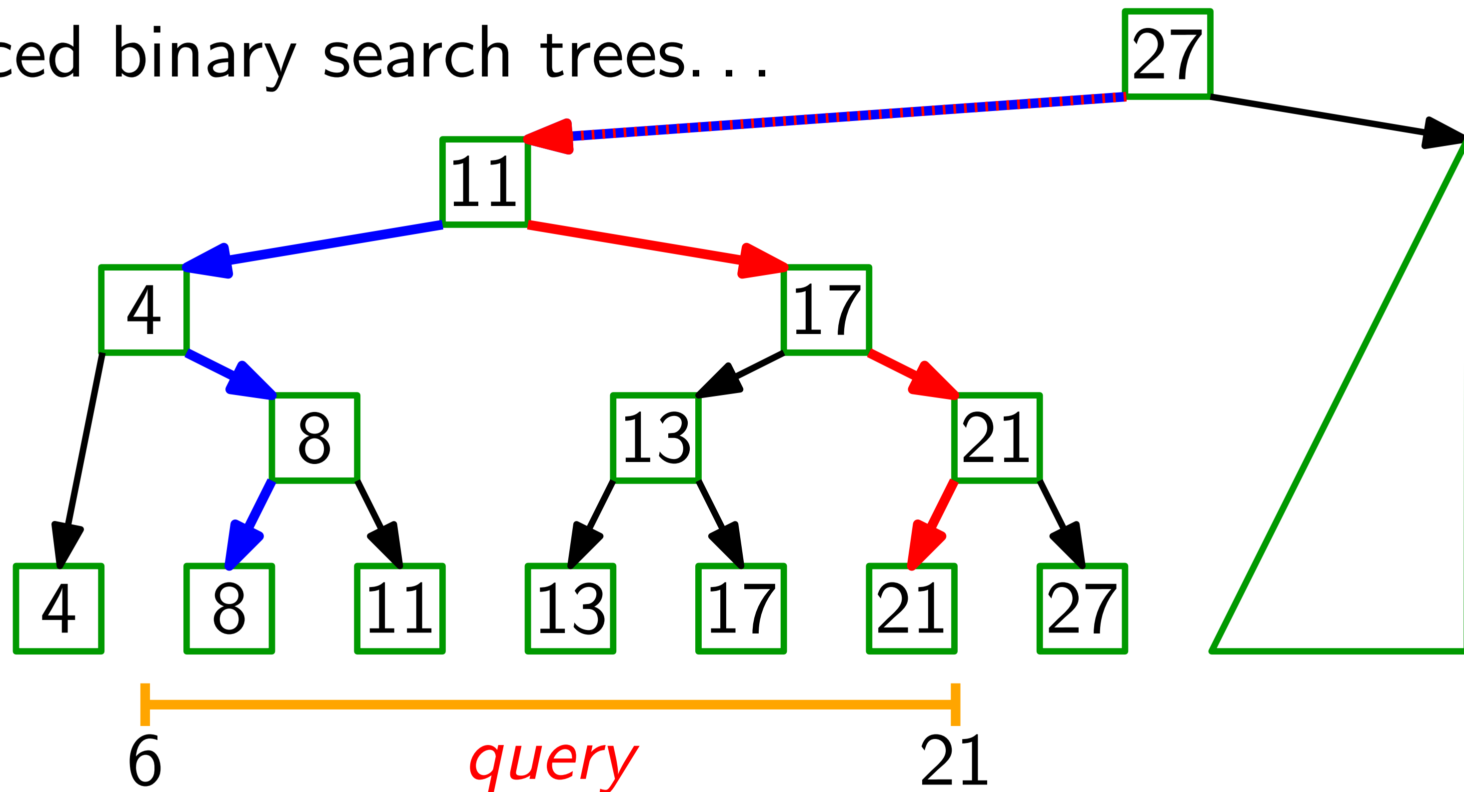

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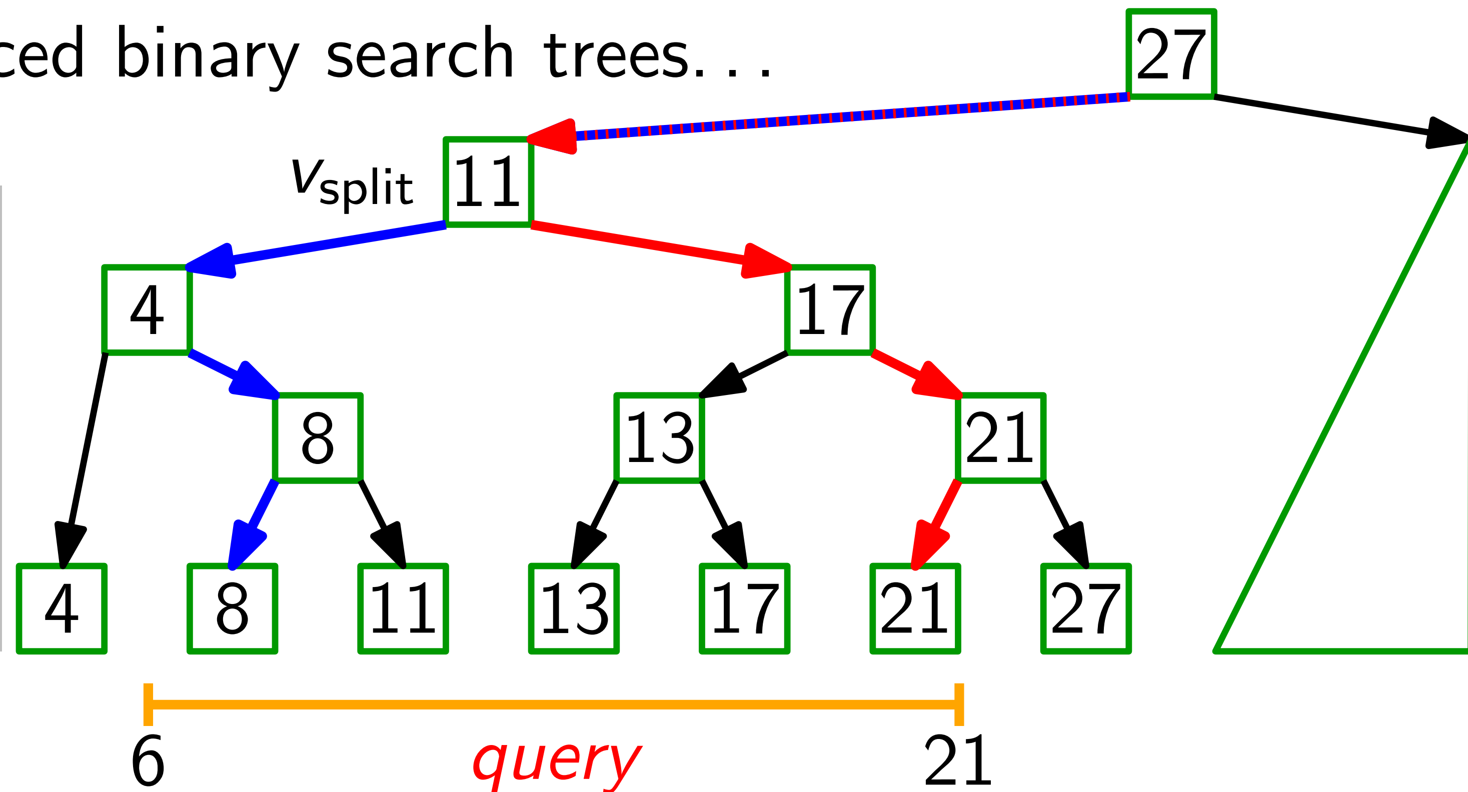
1. Search $x = 6$.
2. Search $x' = 21$.



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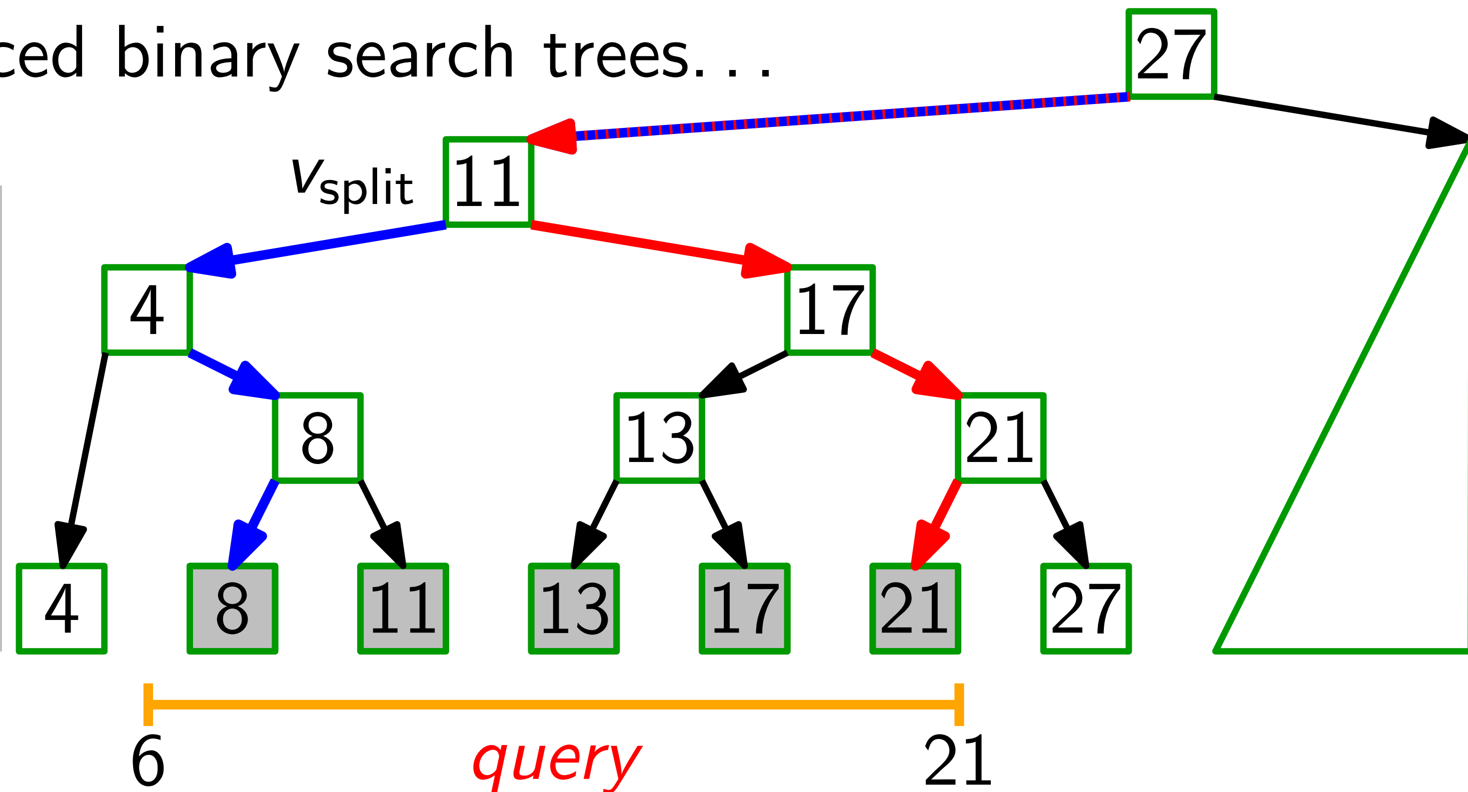

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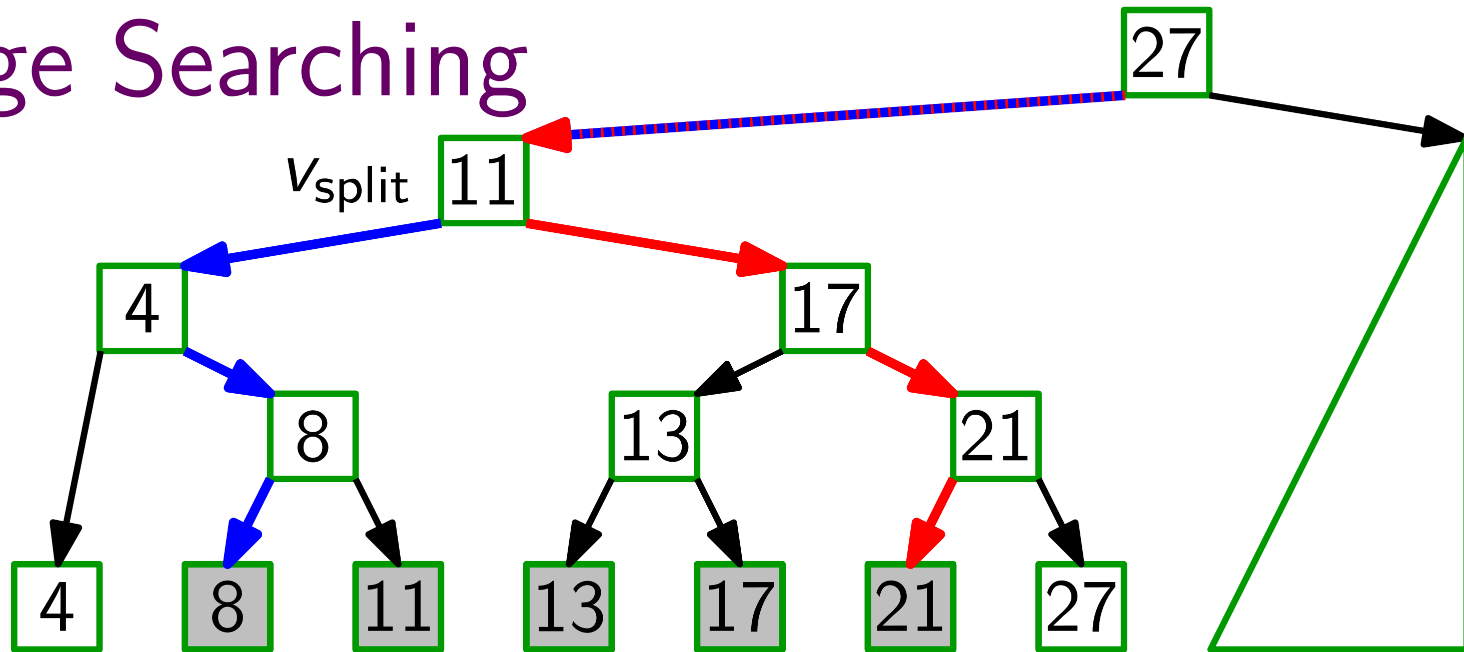

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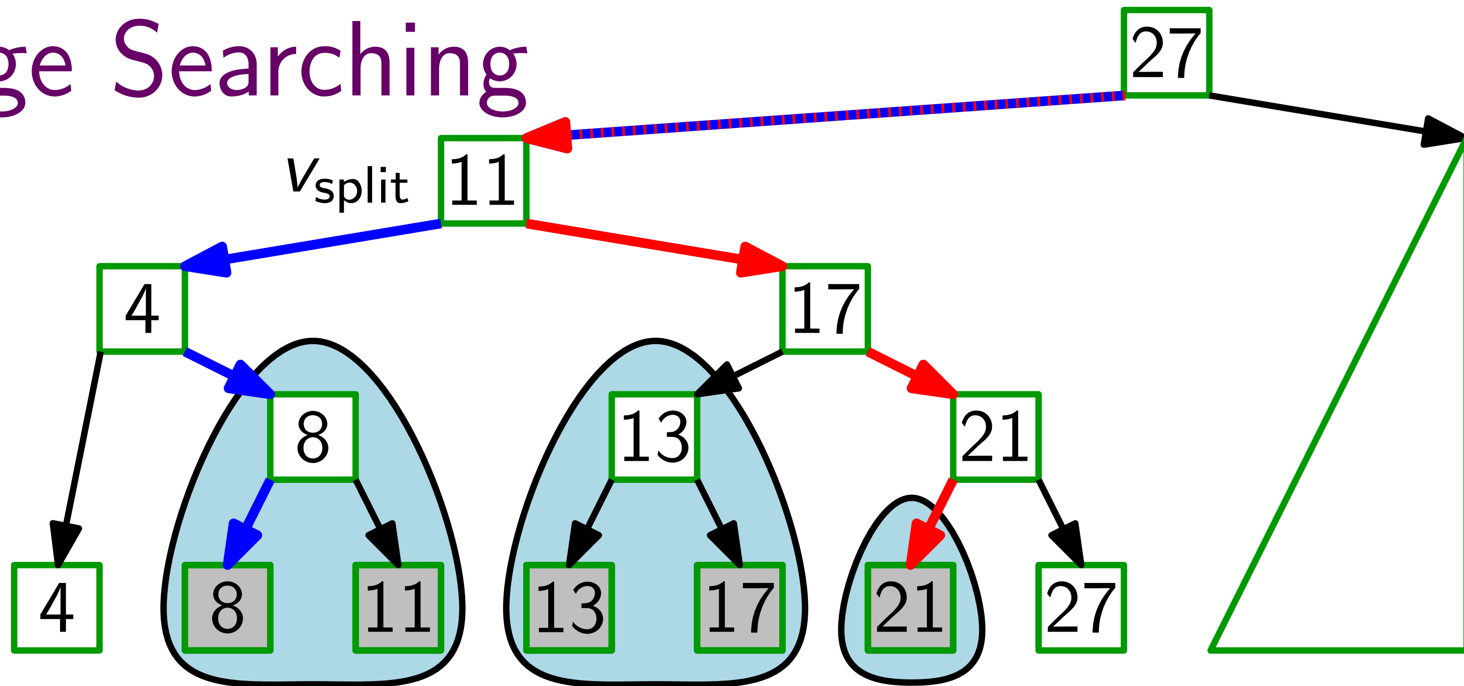

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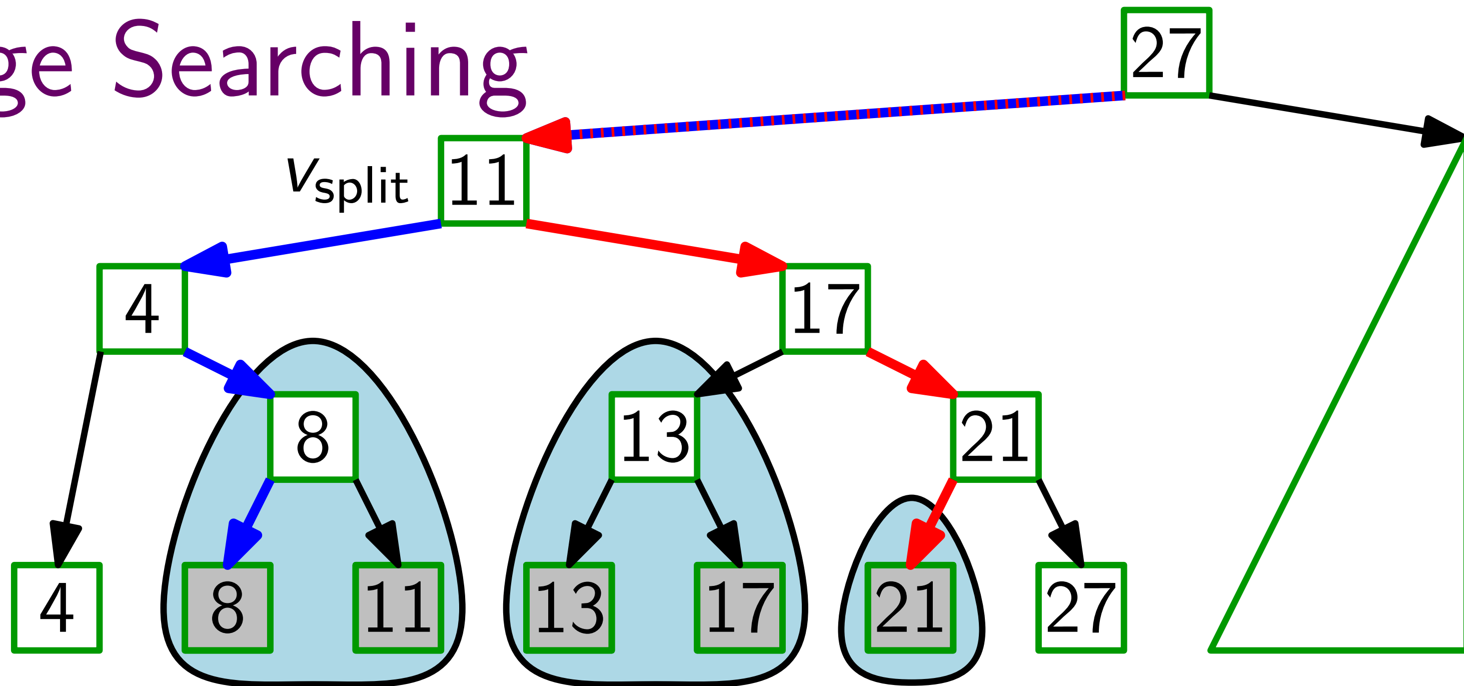

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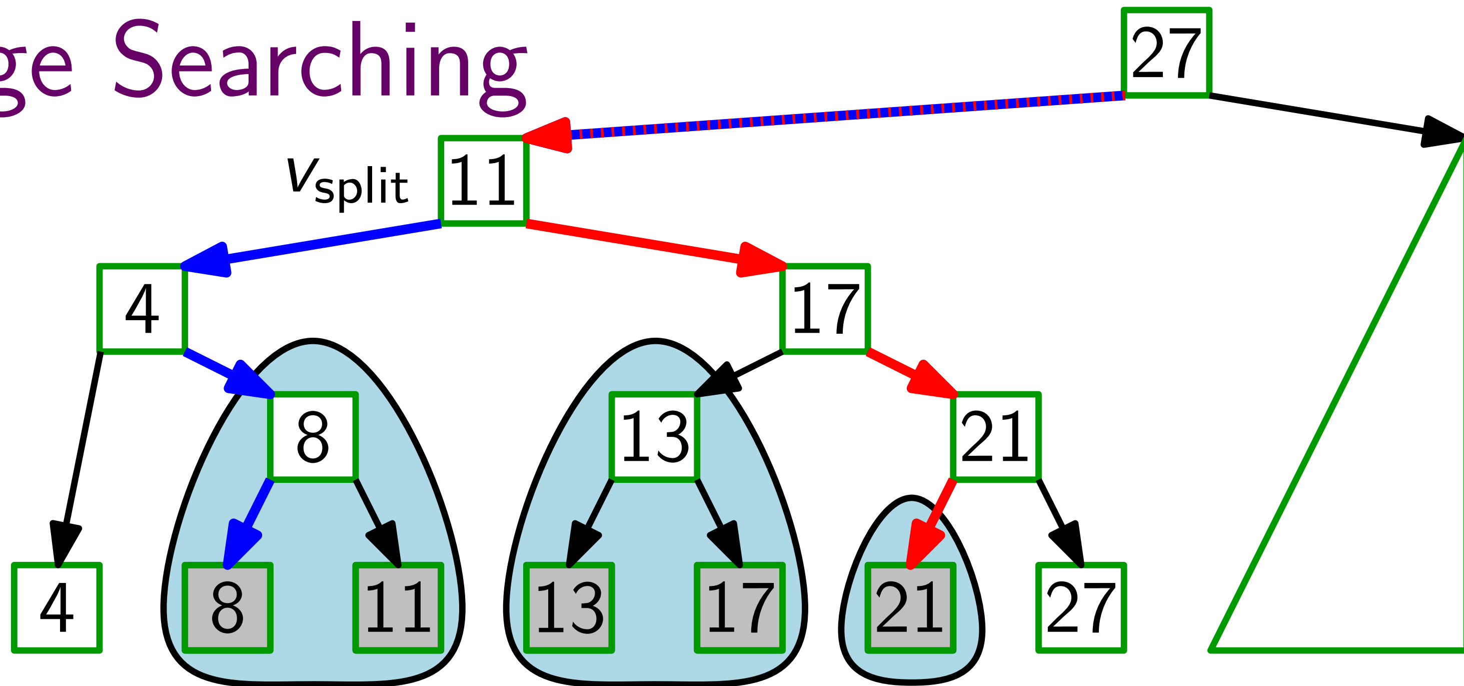

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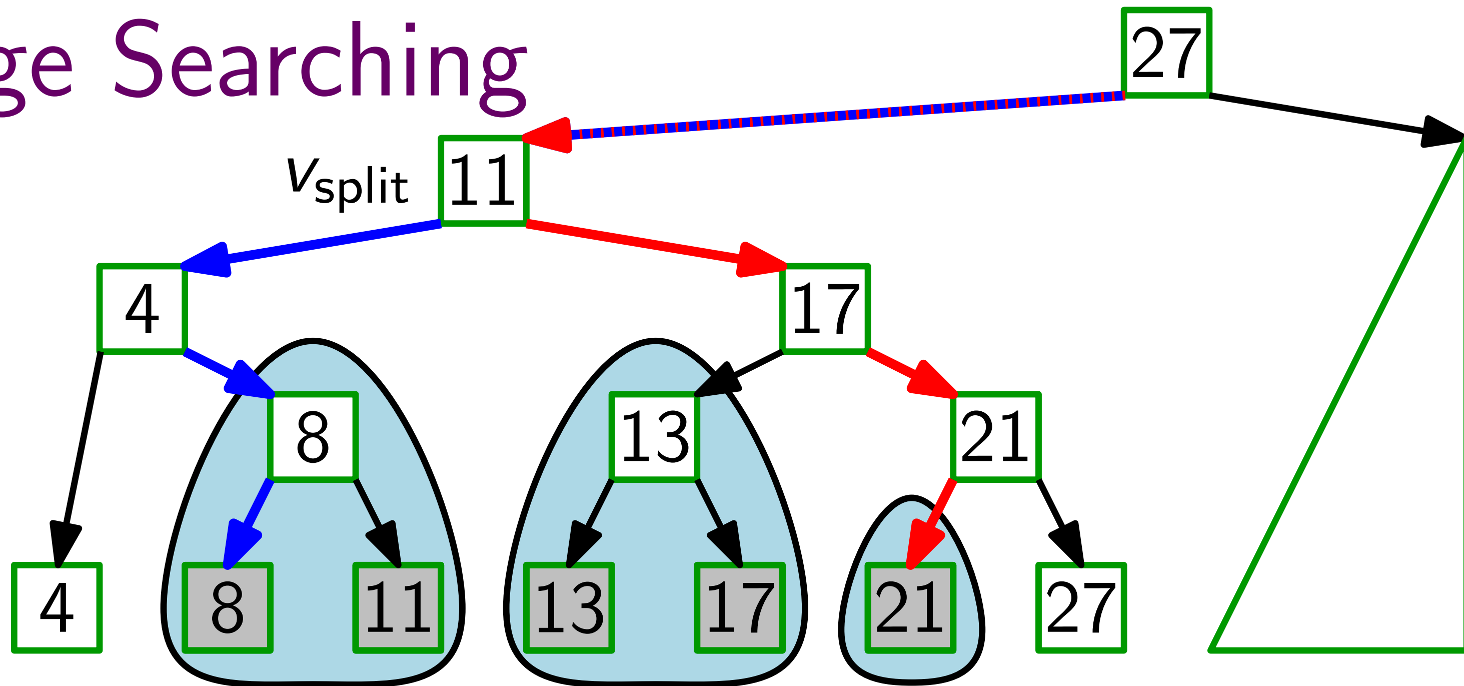

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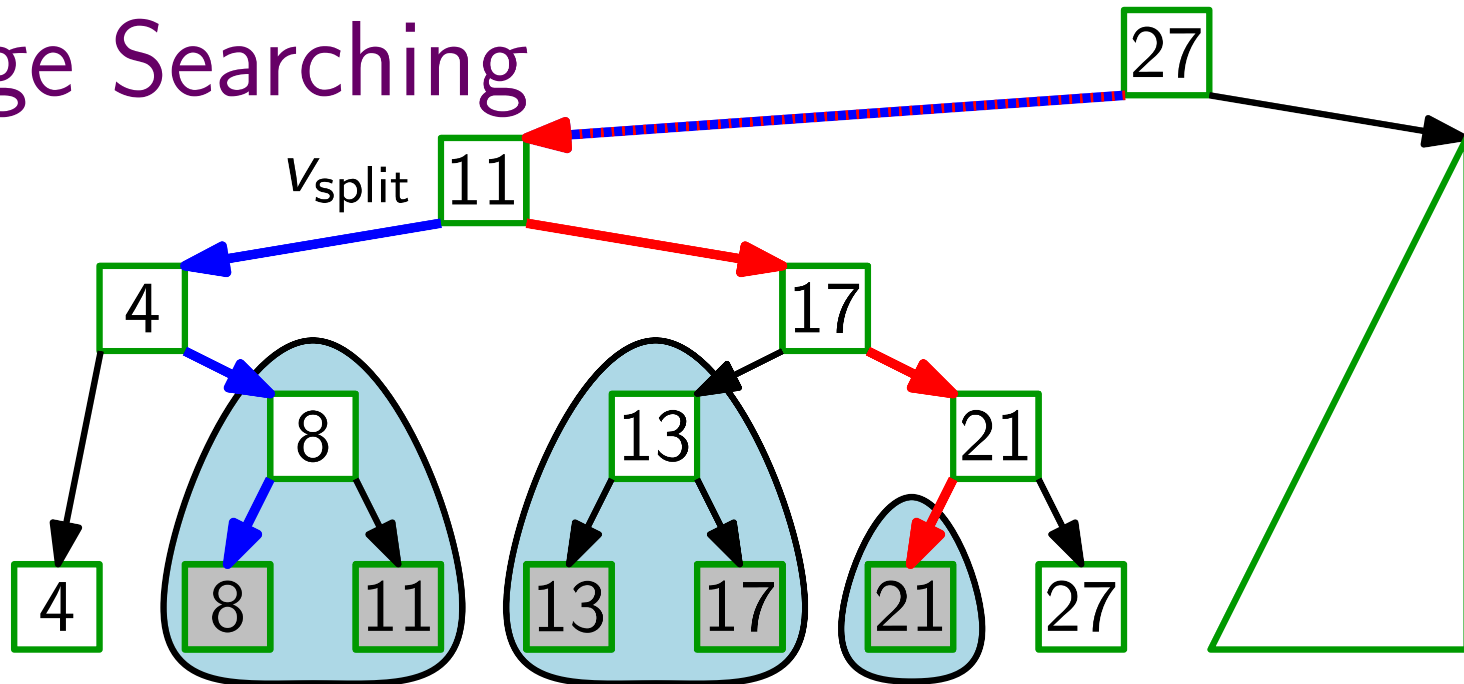

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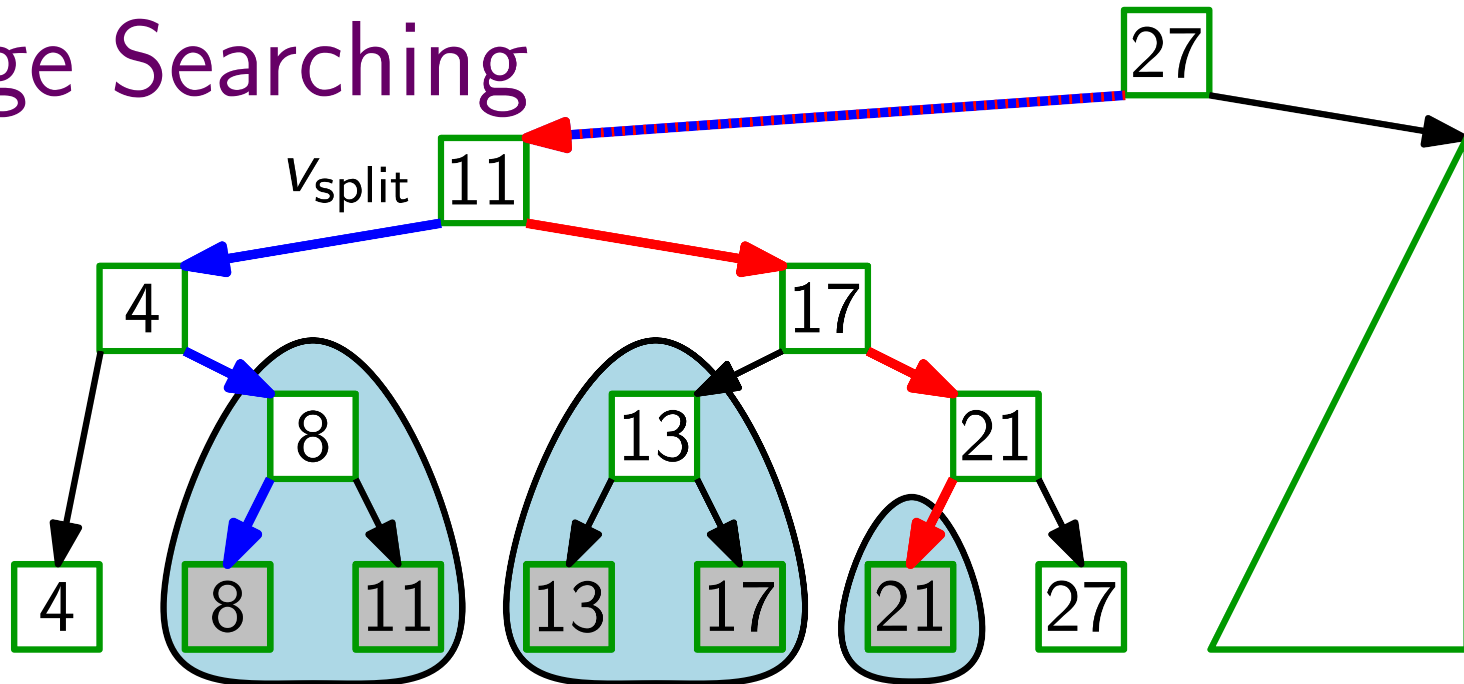

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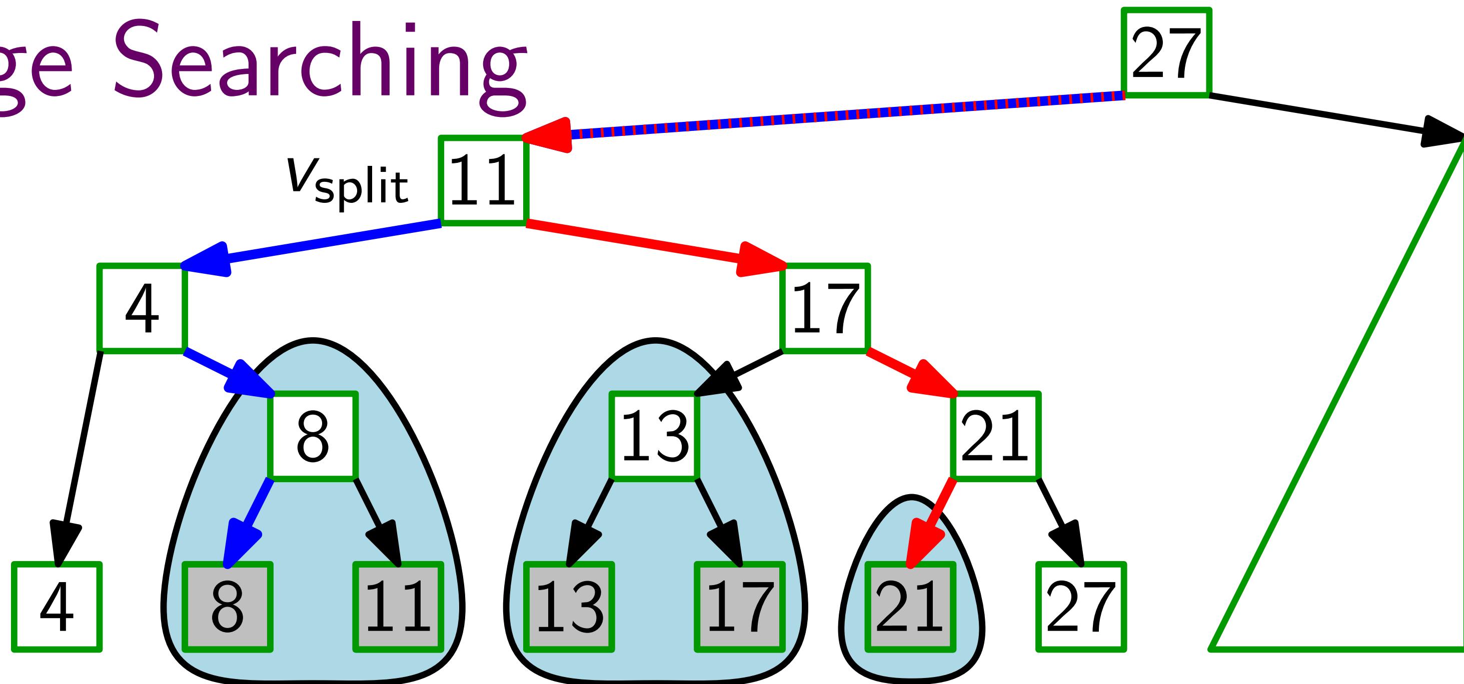

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output sensitive!

Extensions to 2d

Think... [3 min]

Task: Preprocess a finite set $P \subset \mathbb{R}^2$ such that for any range query $R = [x, x'] \times [y, y']$ the set $P \cap R$ can be reported quickly.

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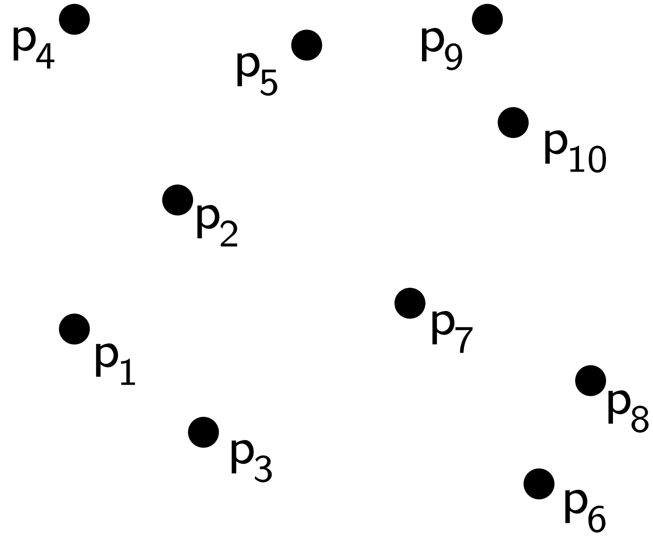
Here: no two points have the same x- or y-coordinate.

Kd-Trees: Example

[dBCvKO'08]



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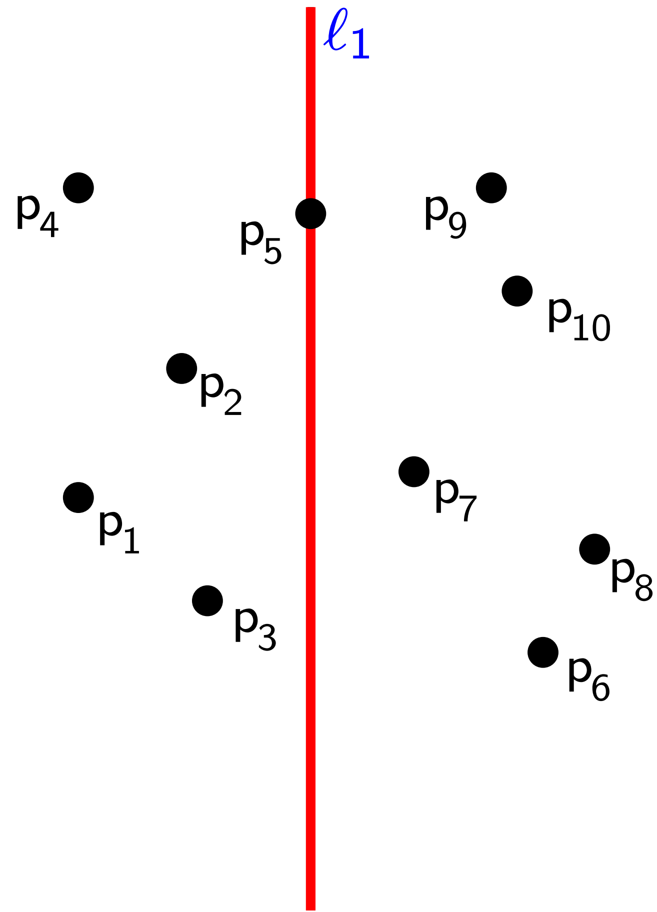

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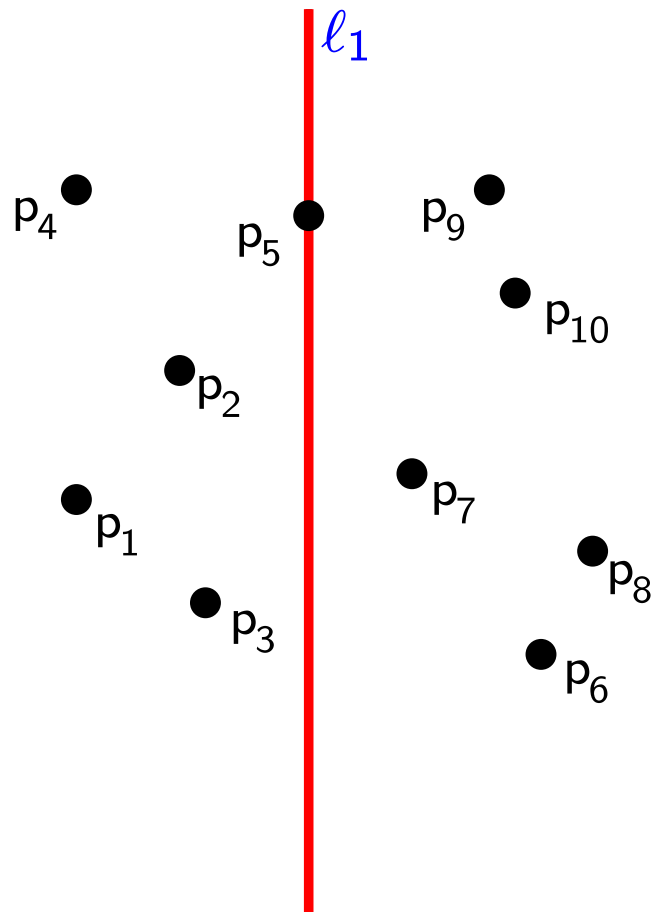

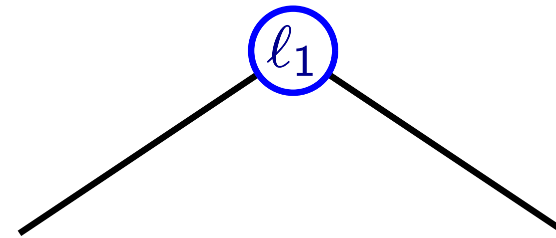

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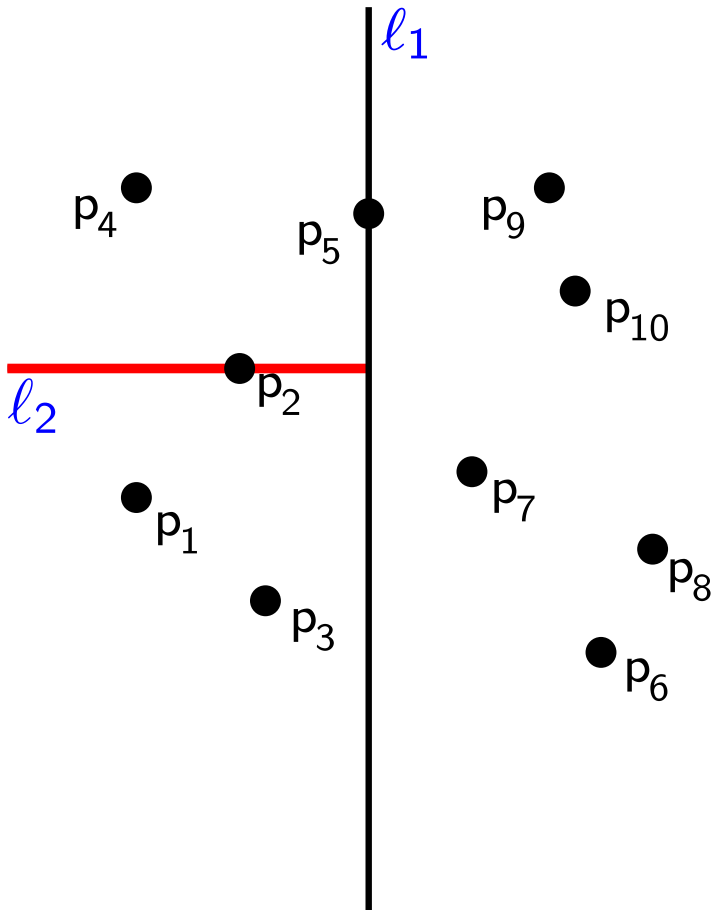

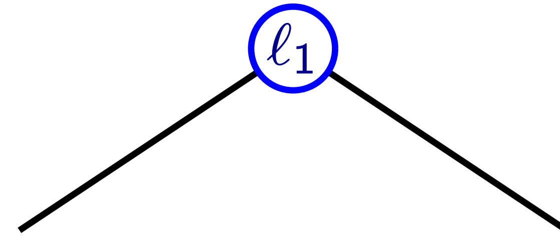

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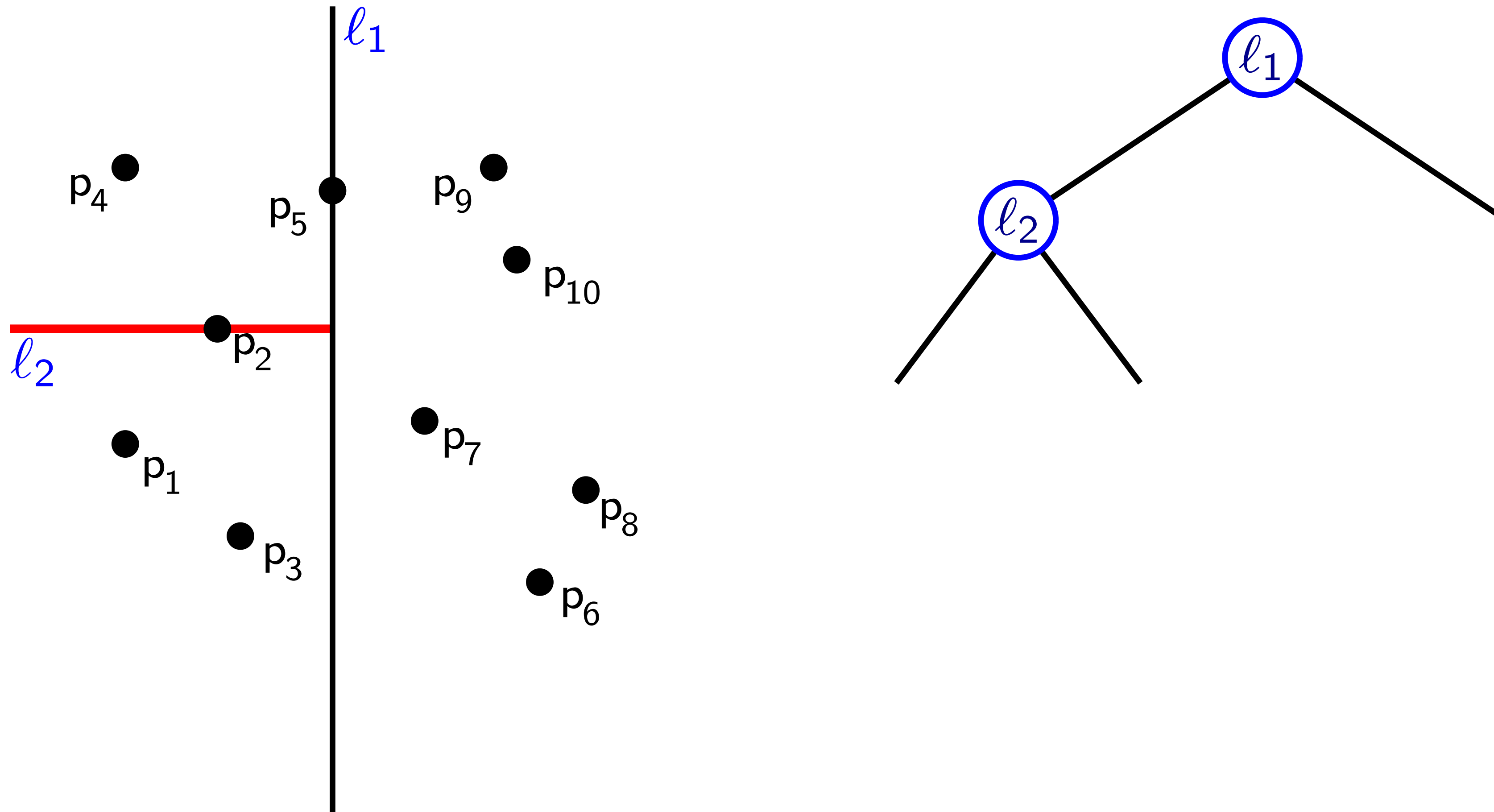

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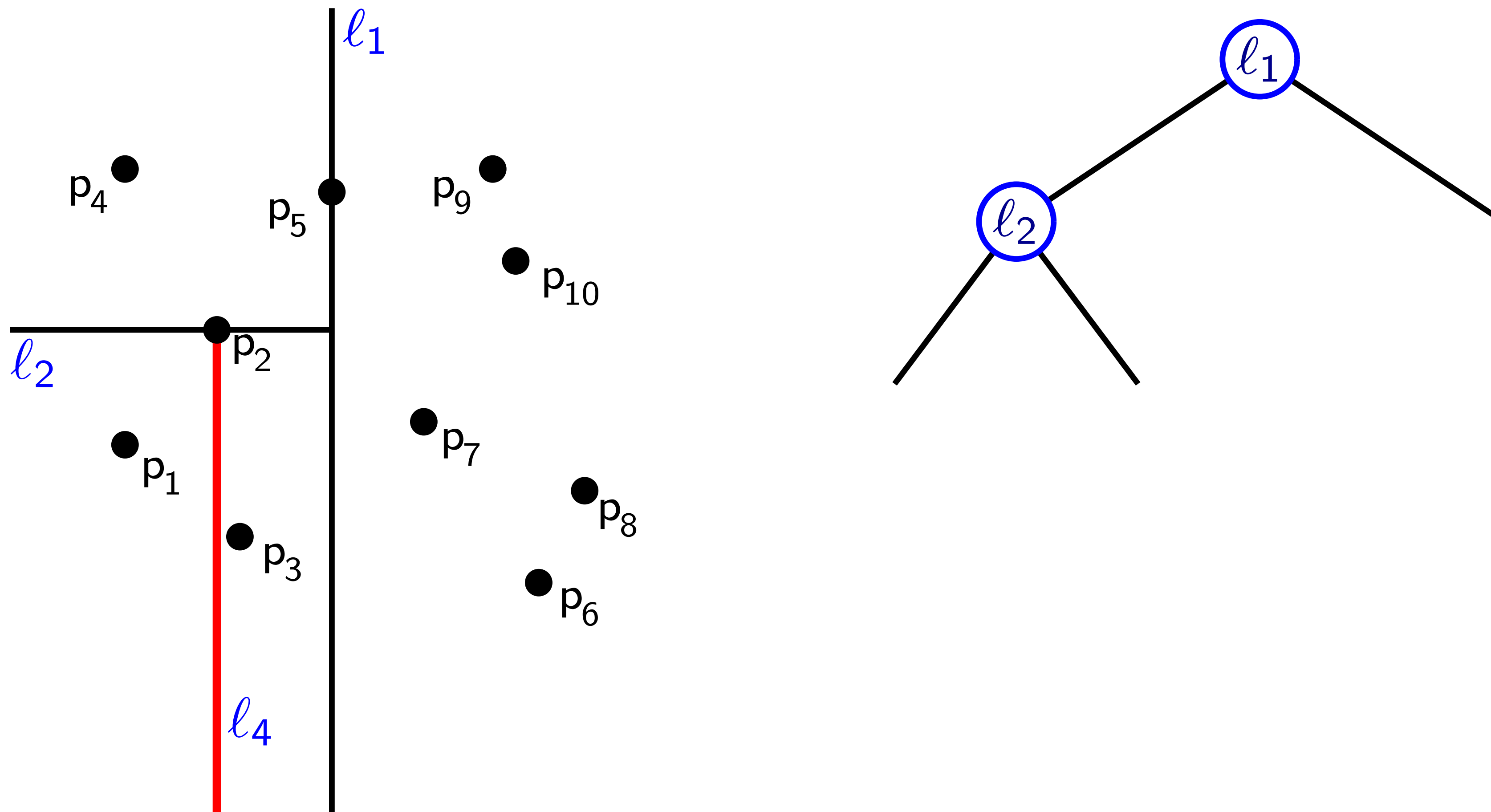

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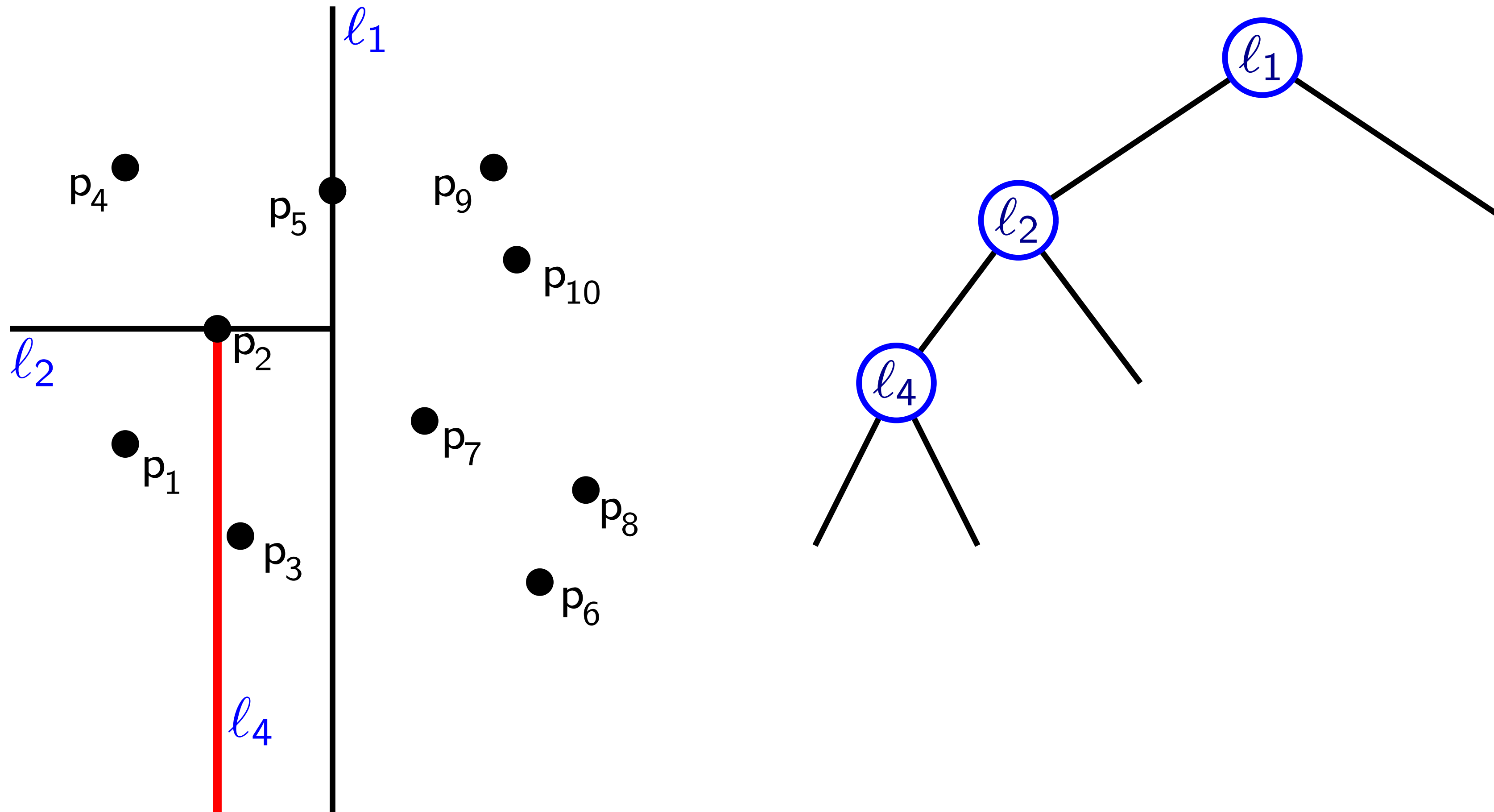

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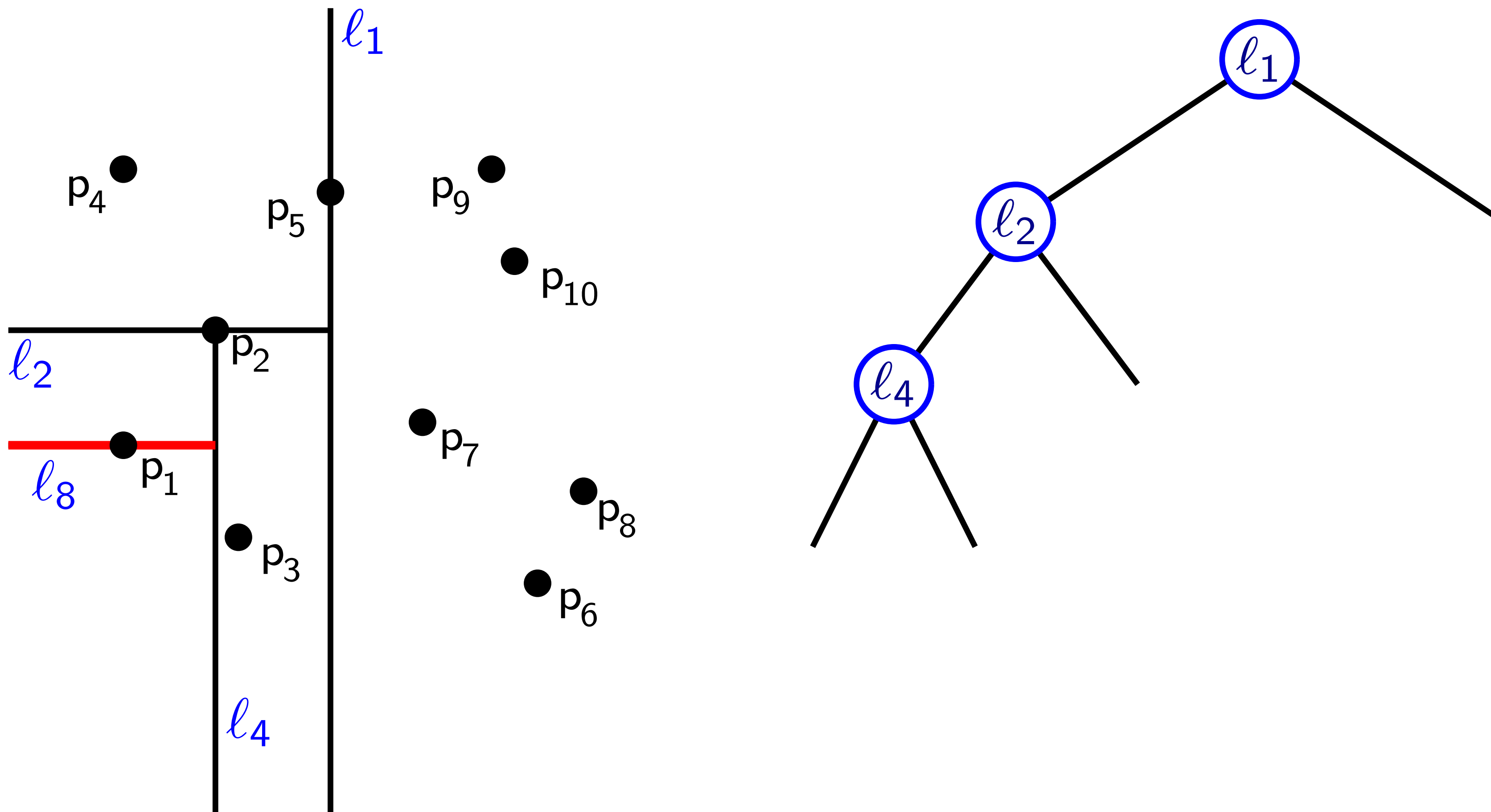

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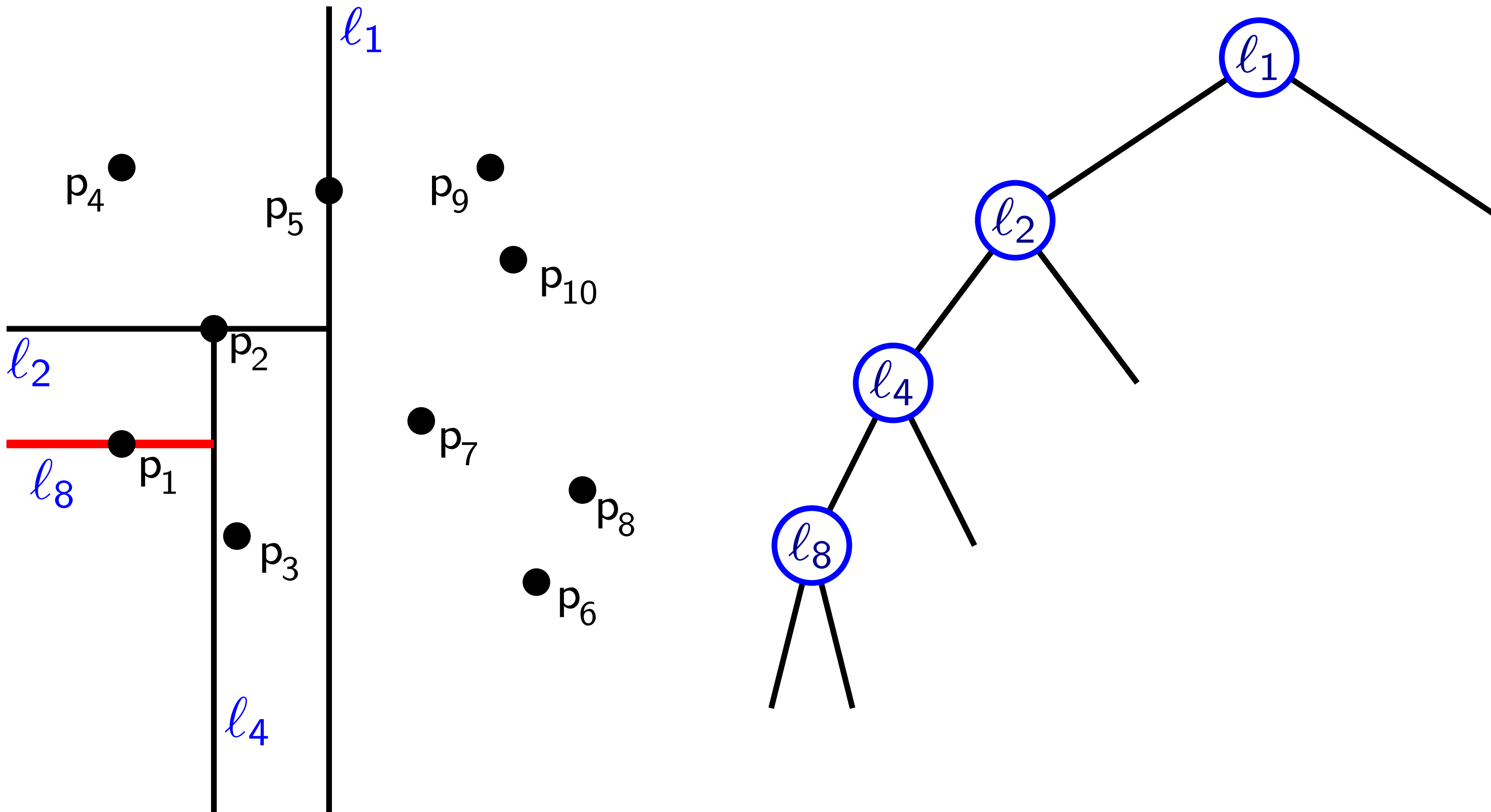

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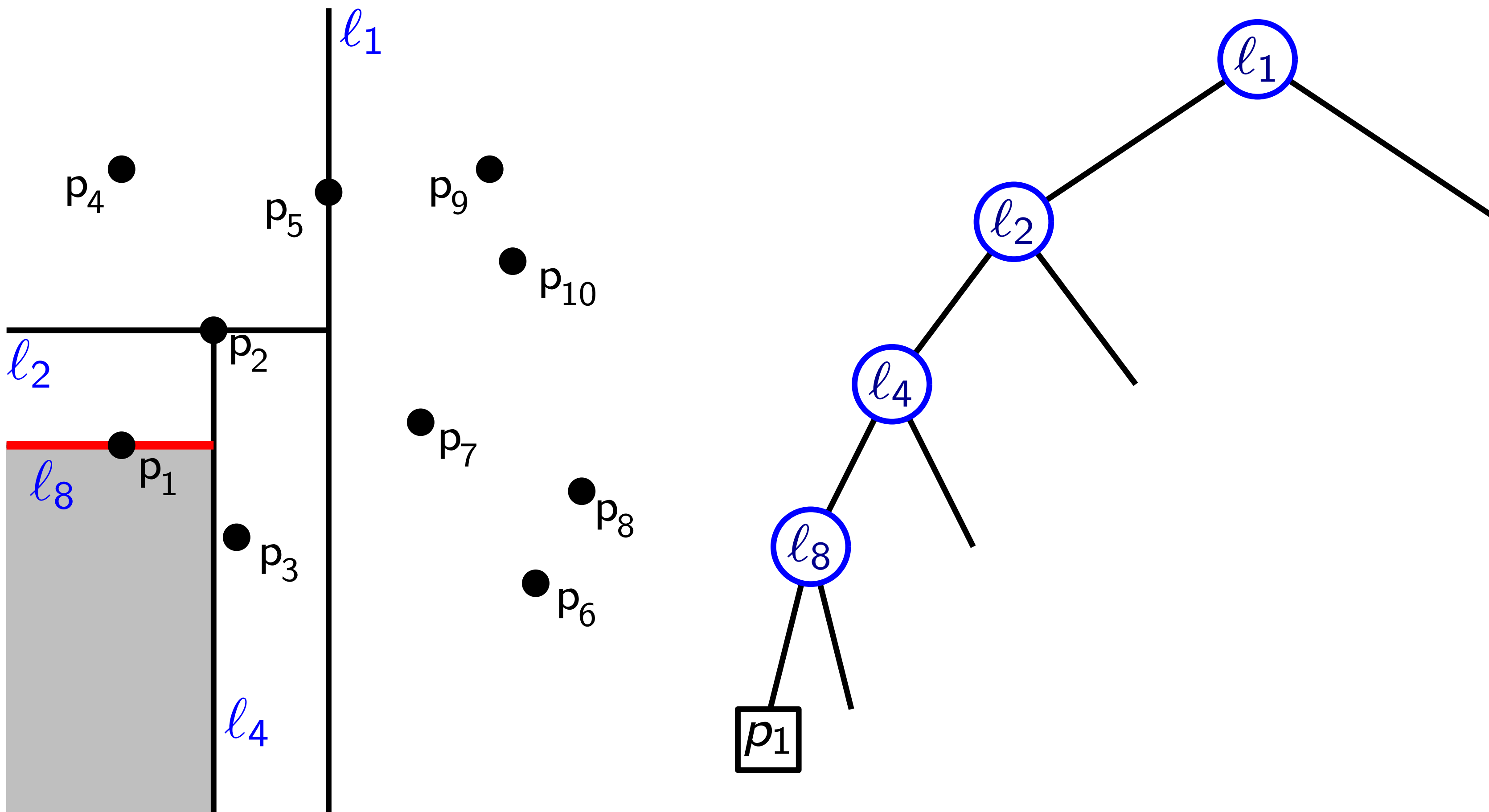

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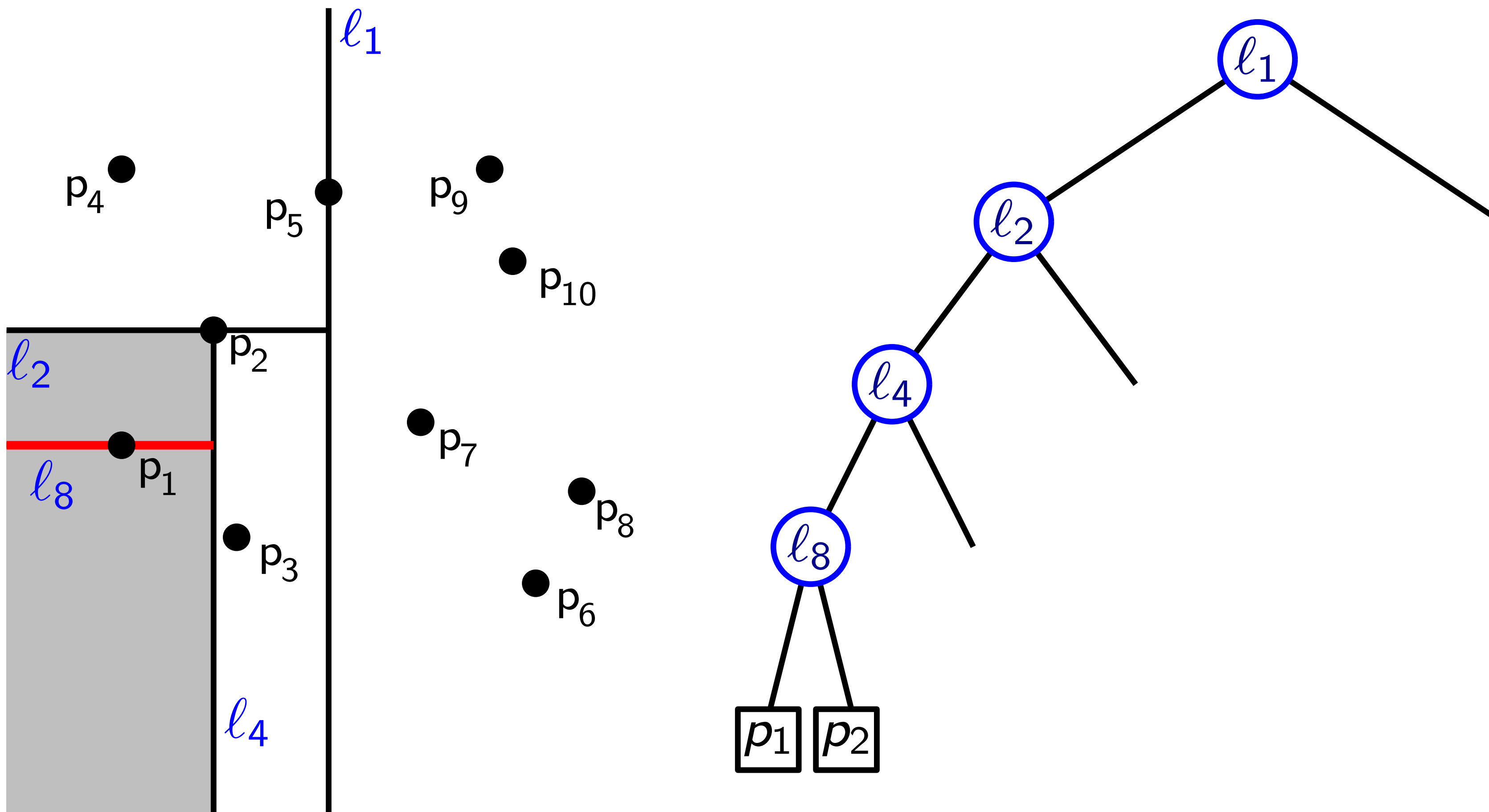

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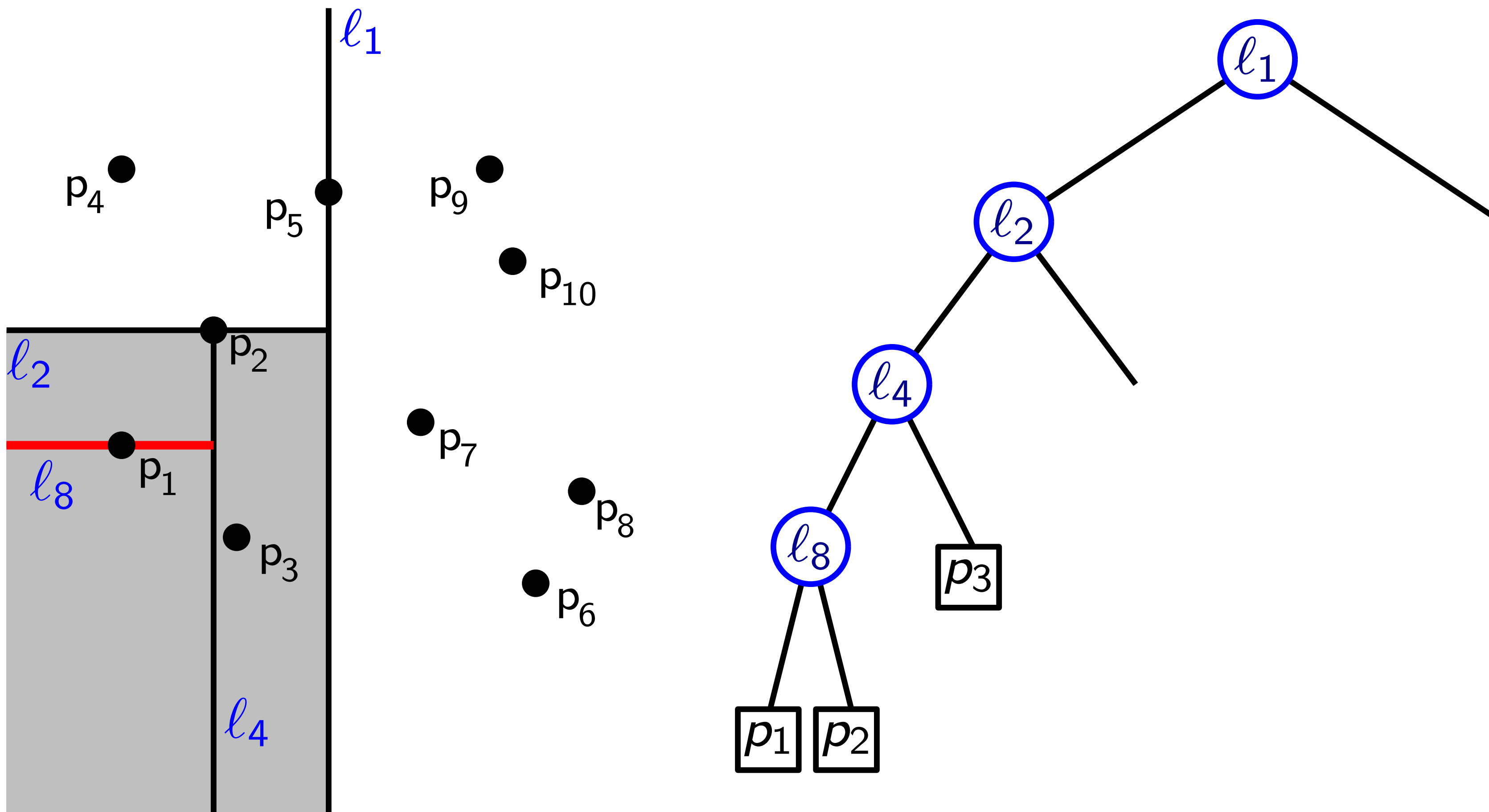

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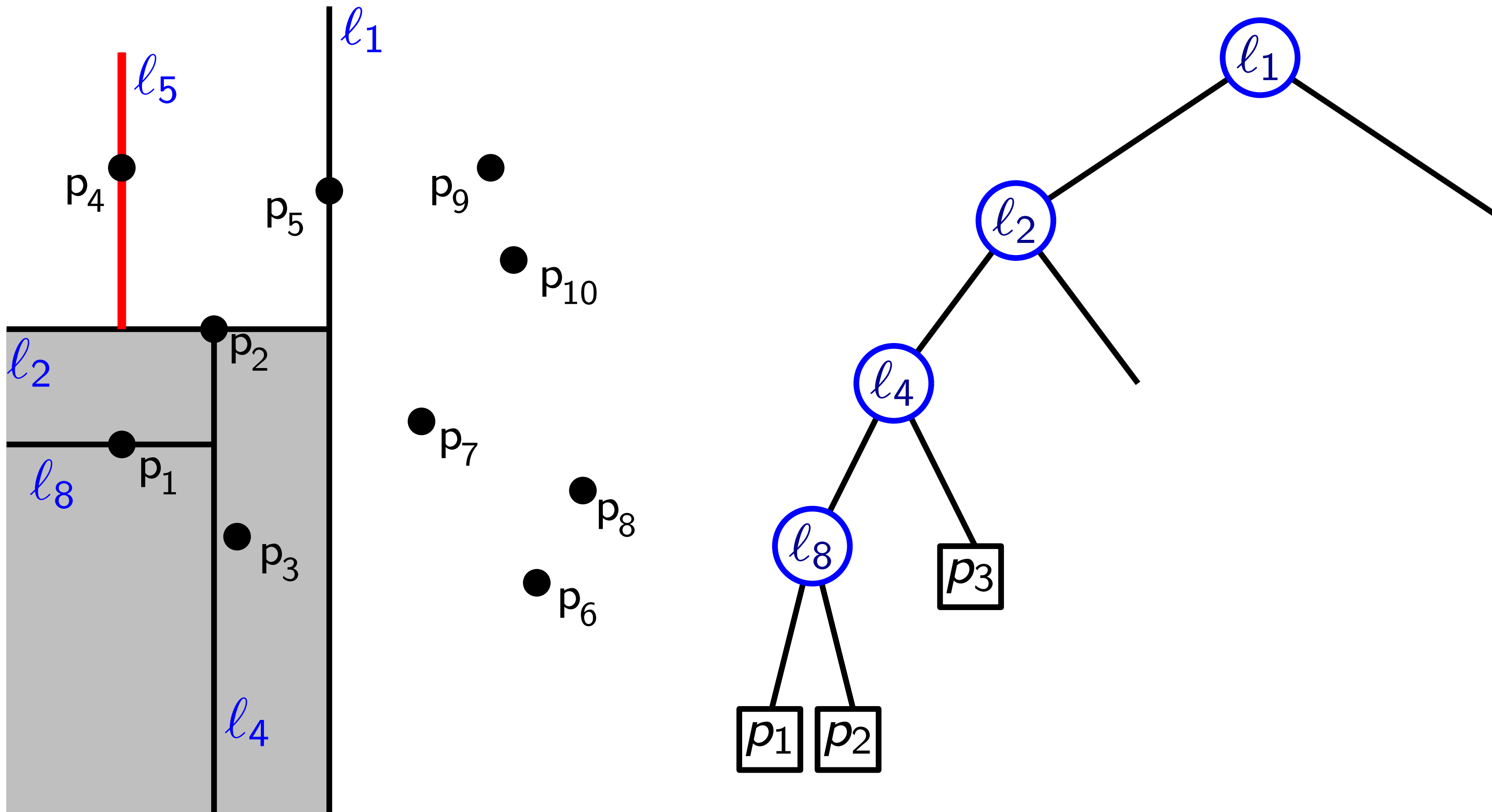

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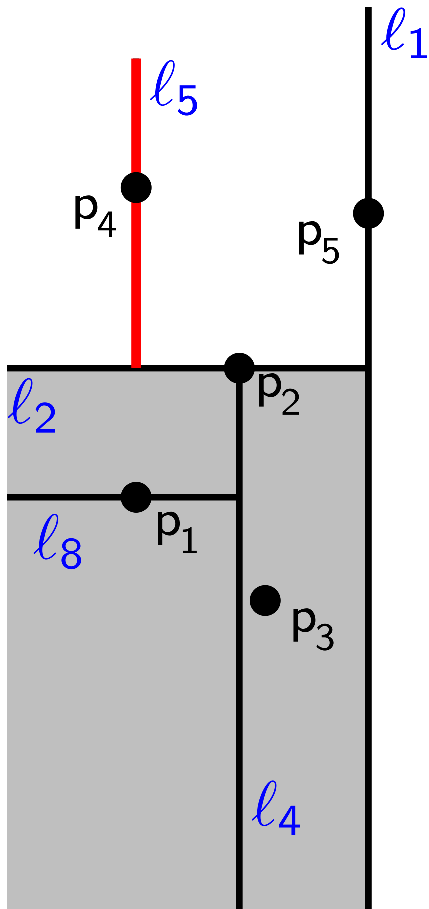

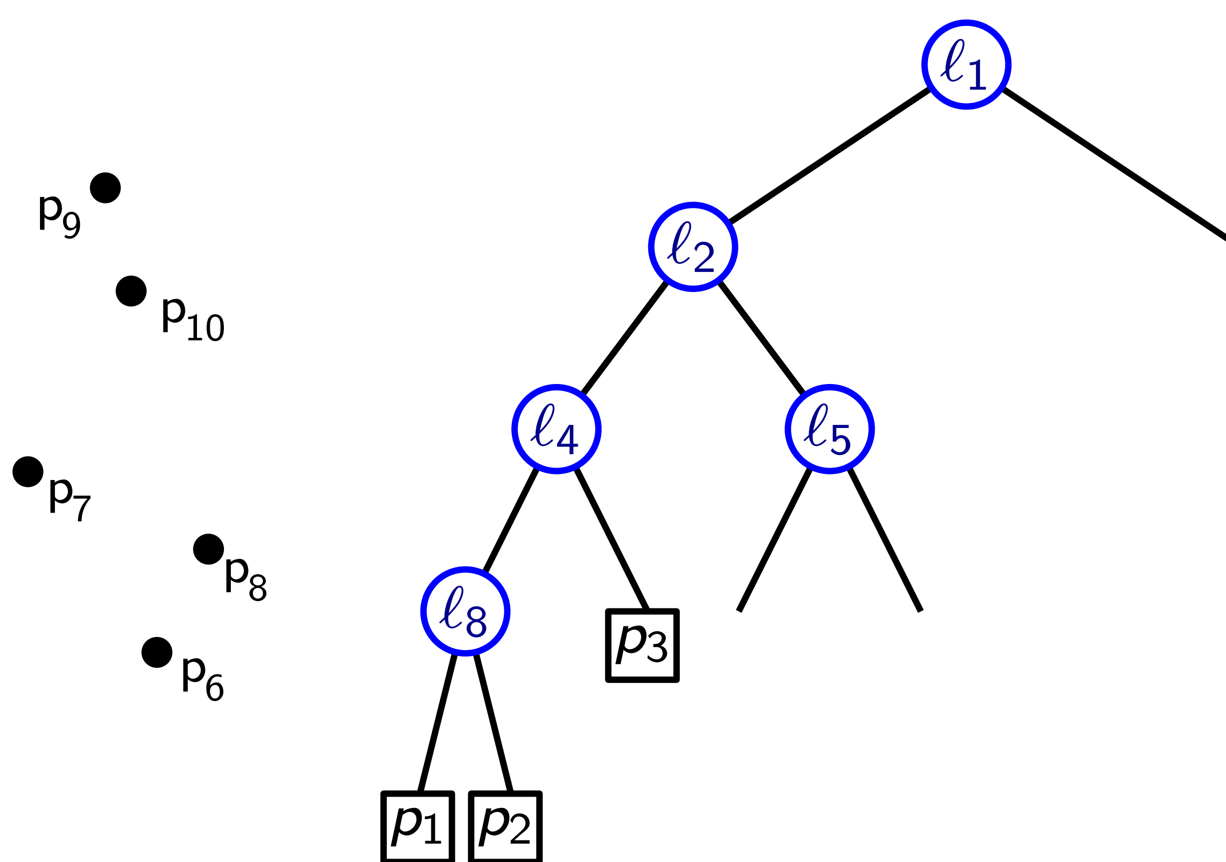

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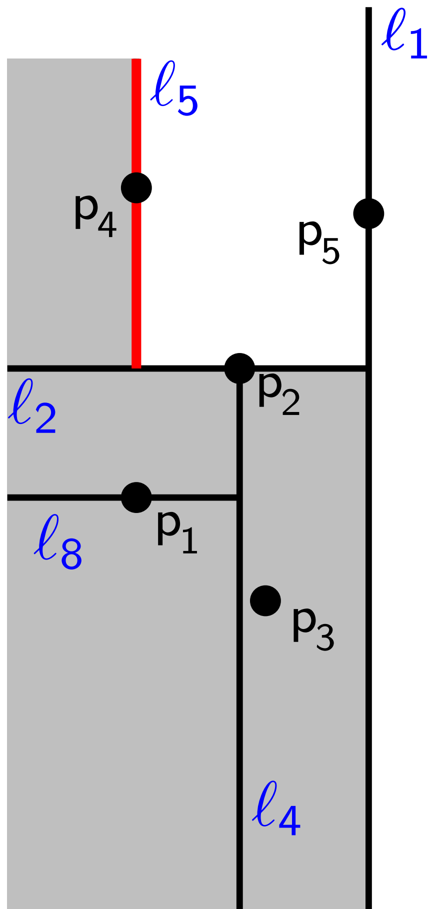

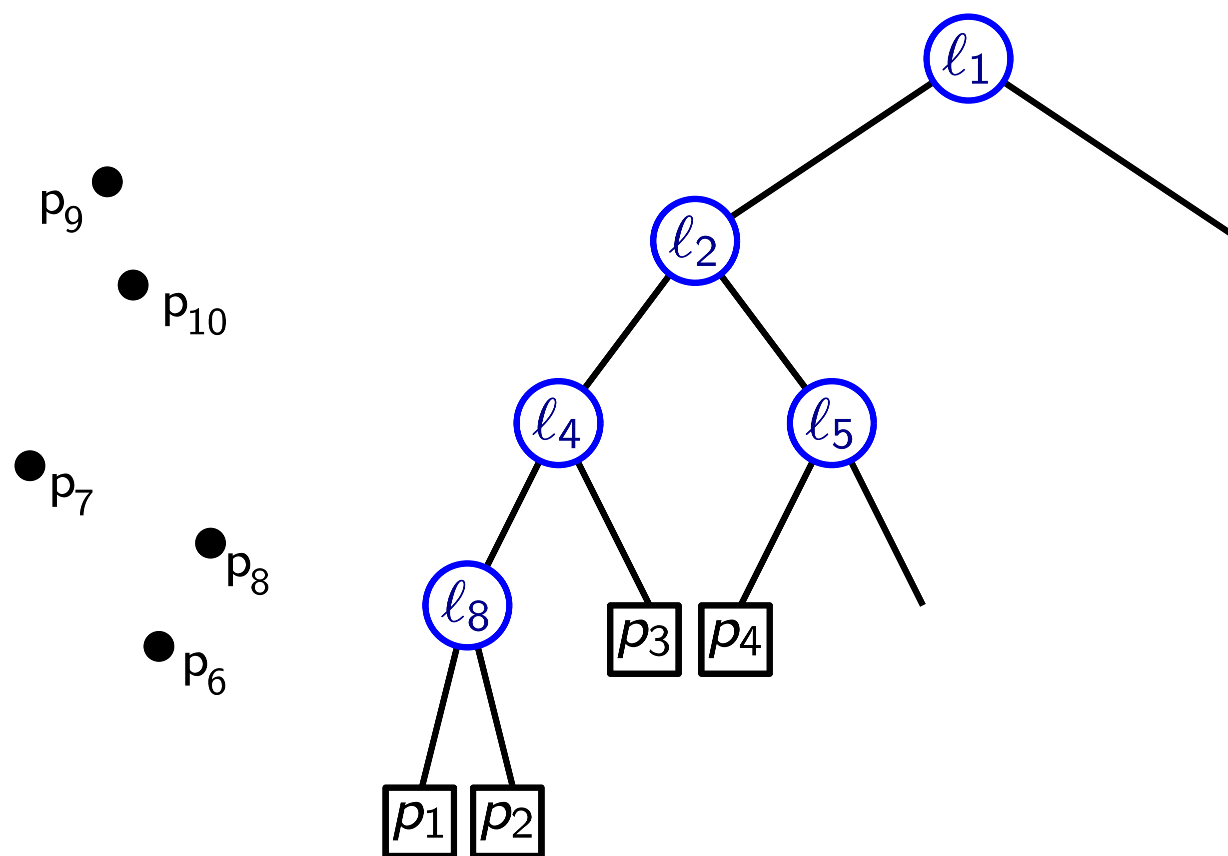

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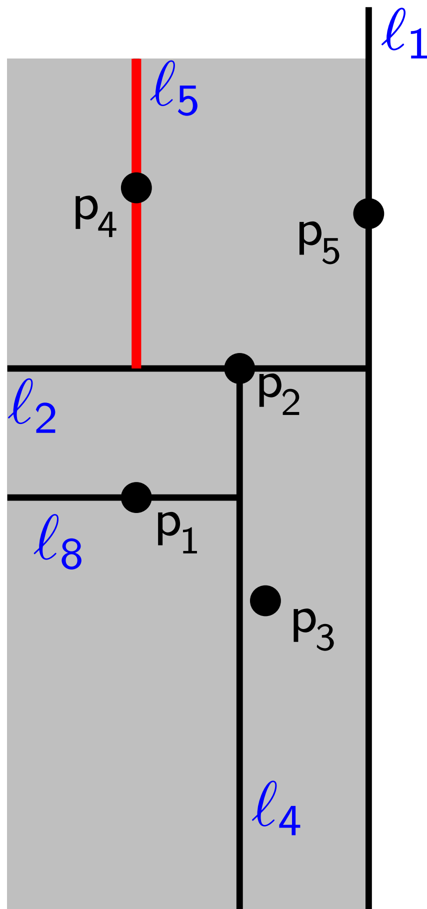

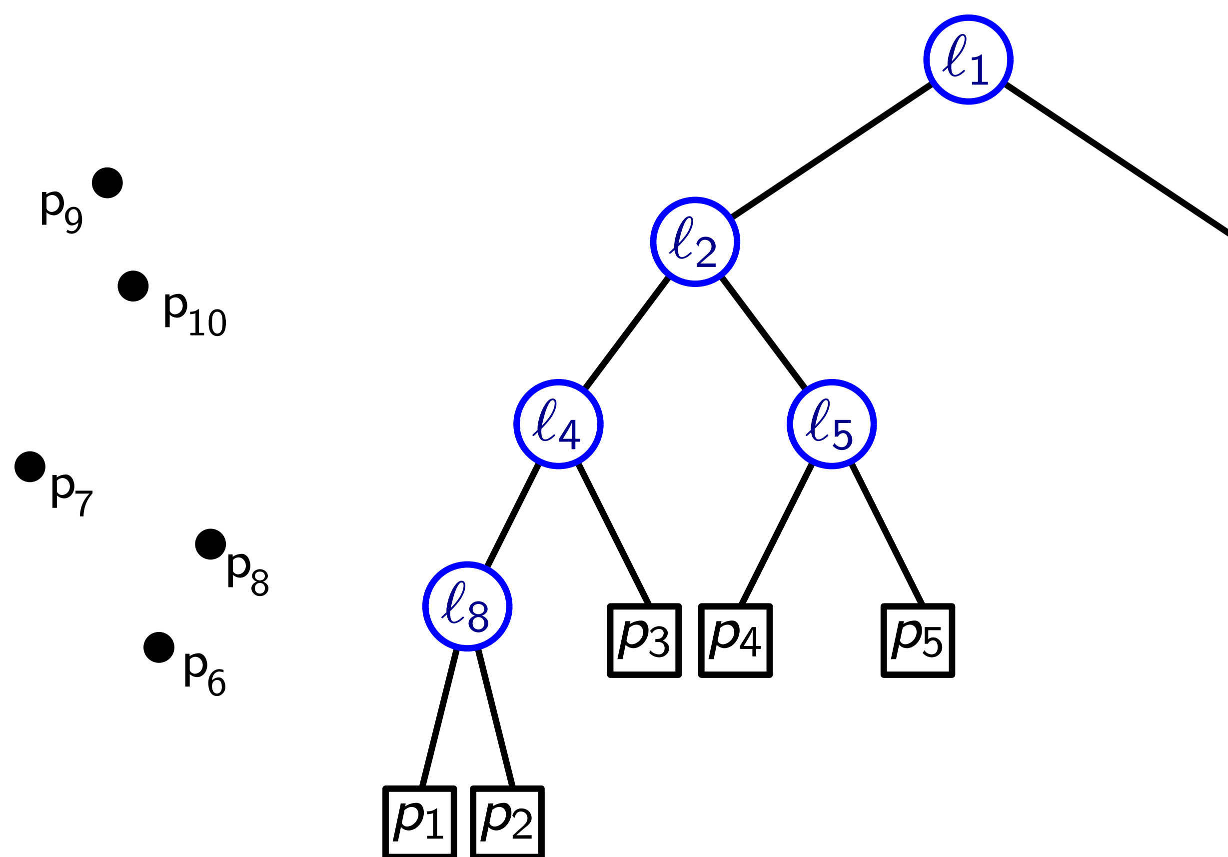

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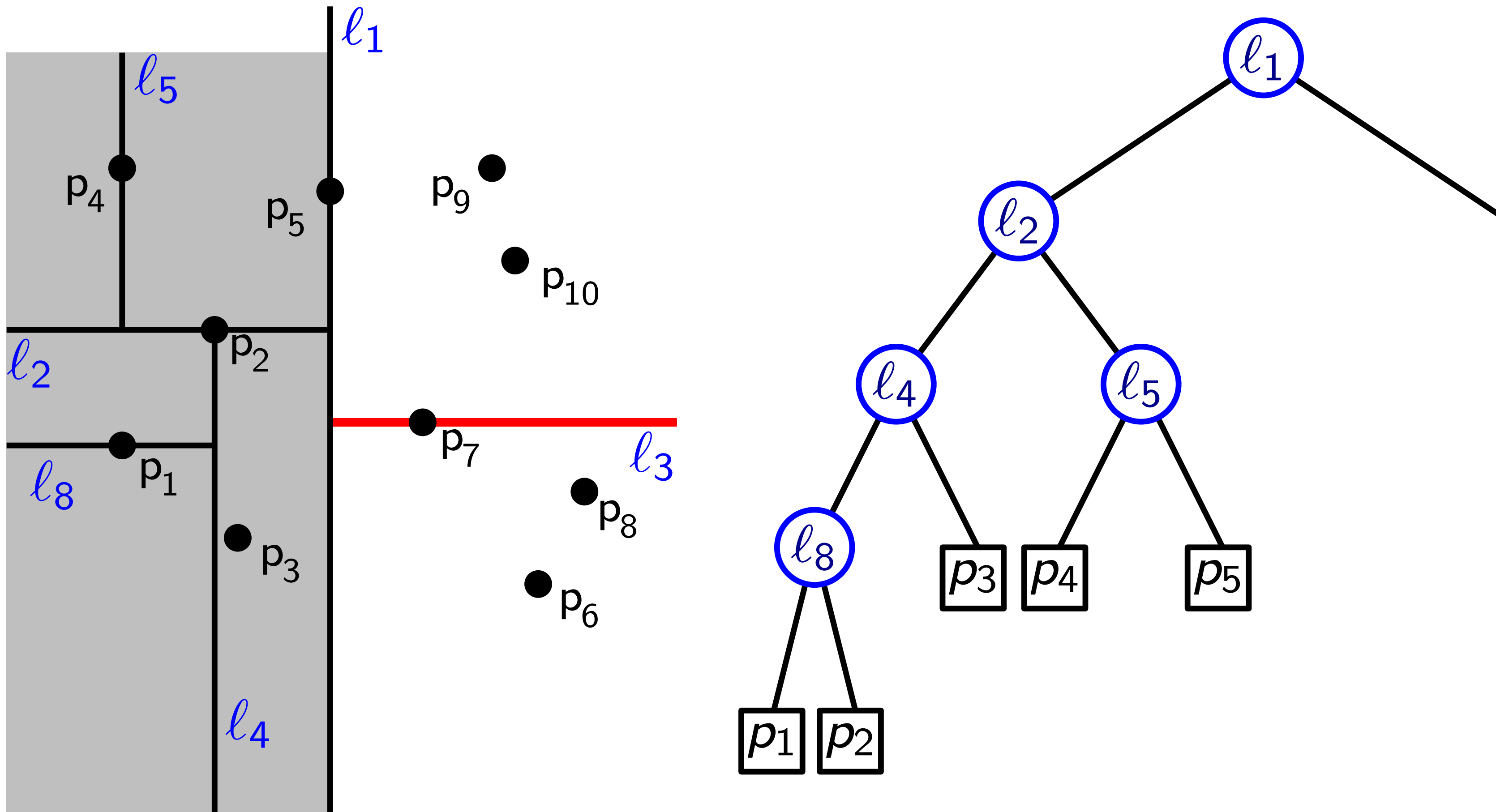

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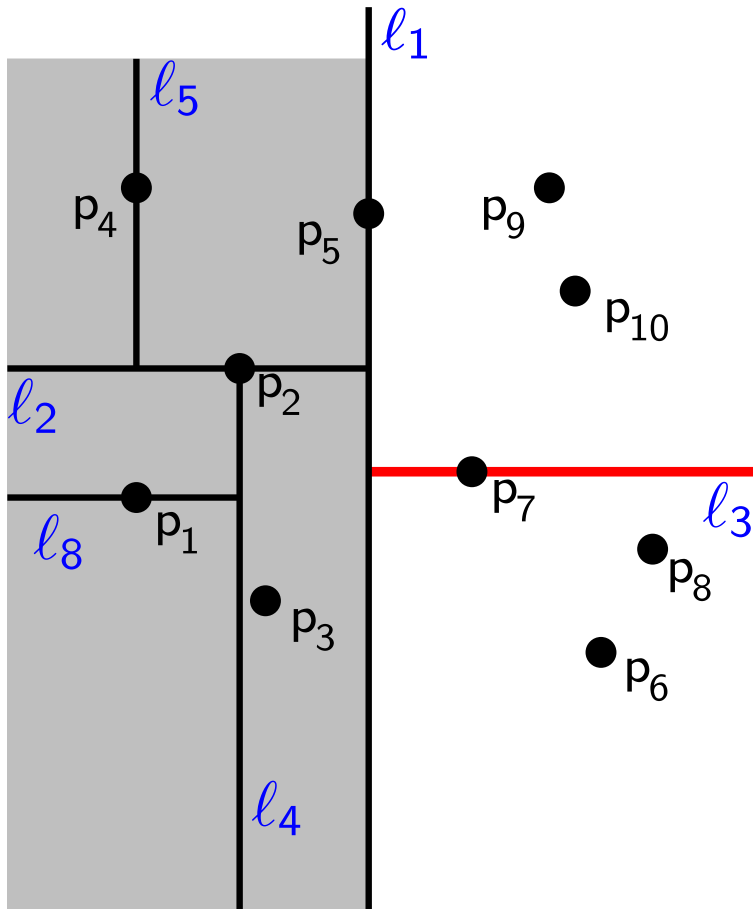
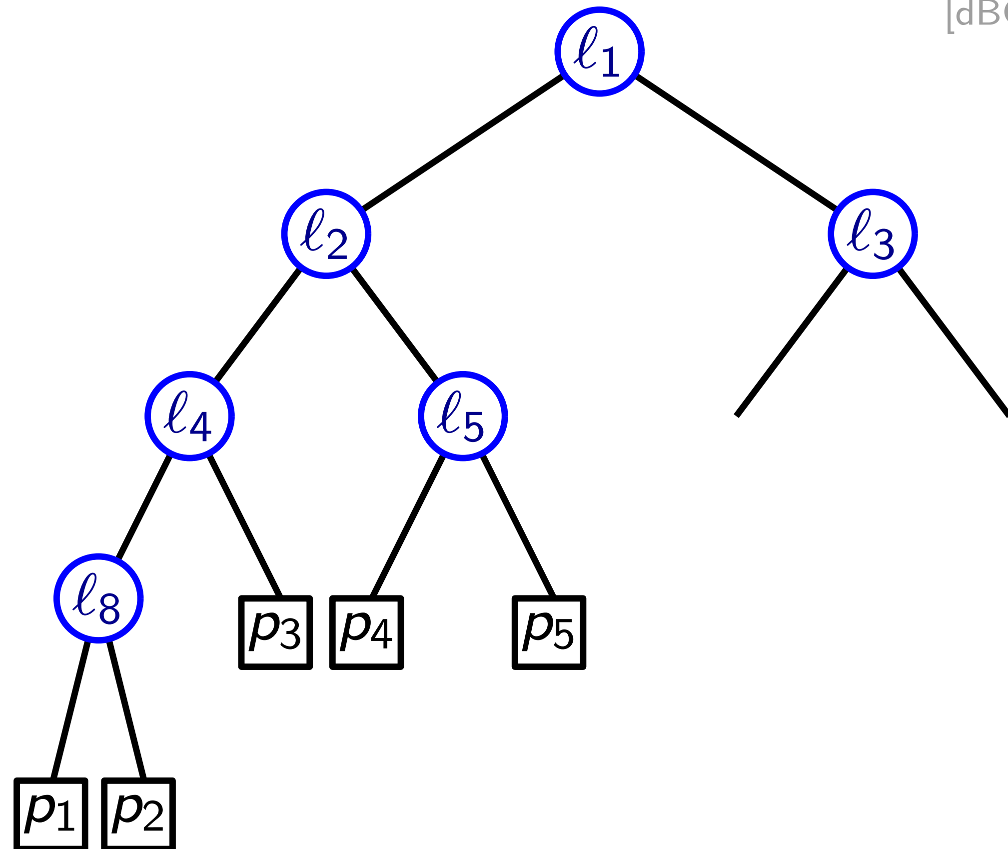


[dBCvKO'08]



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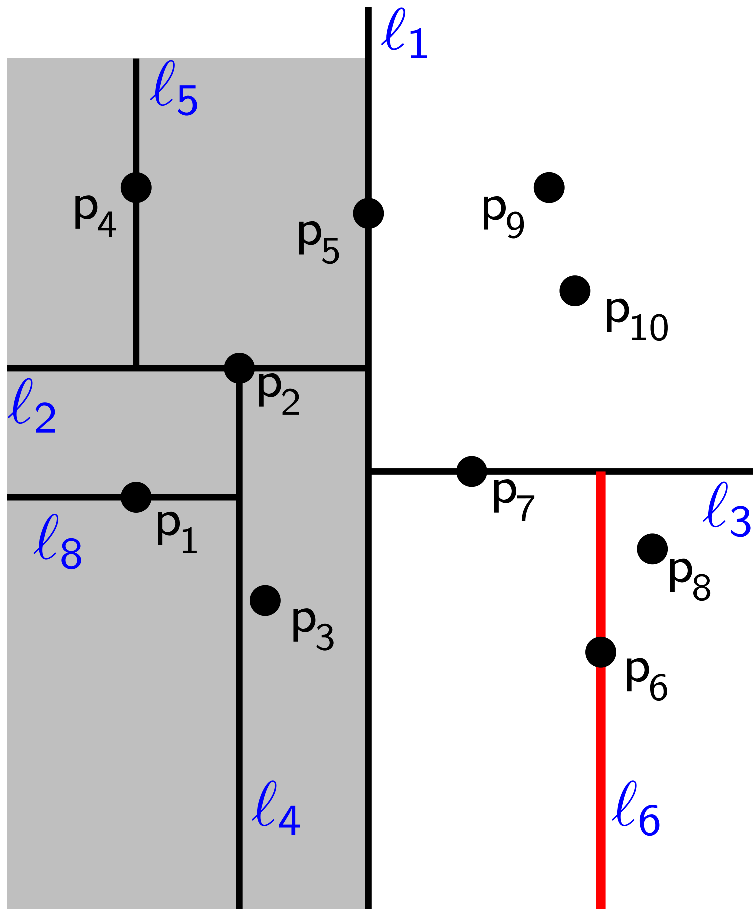

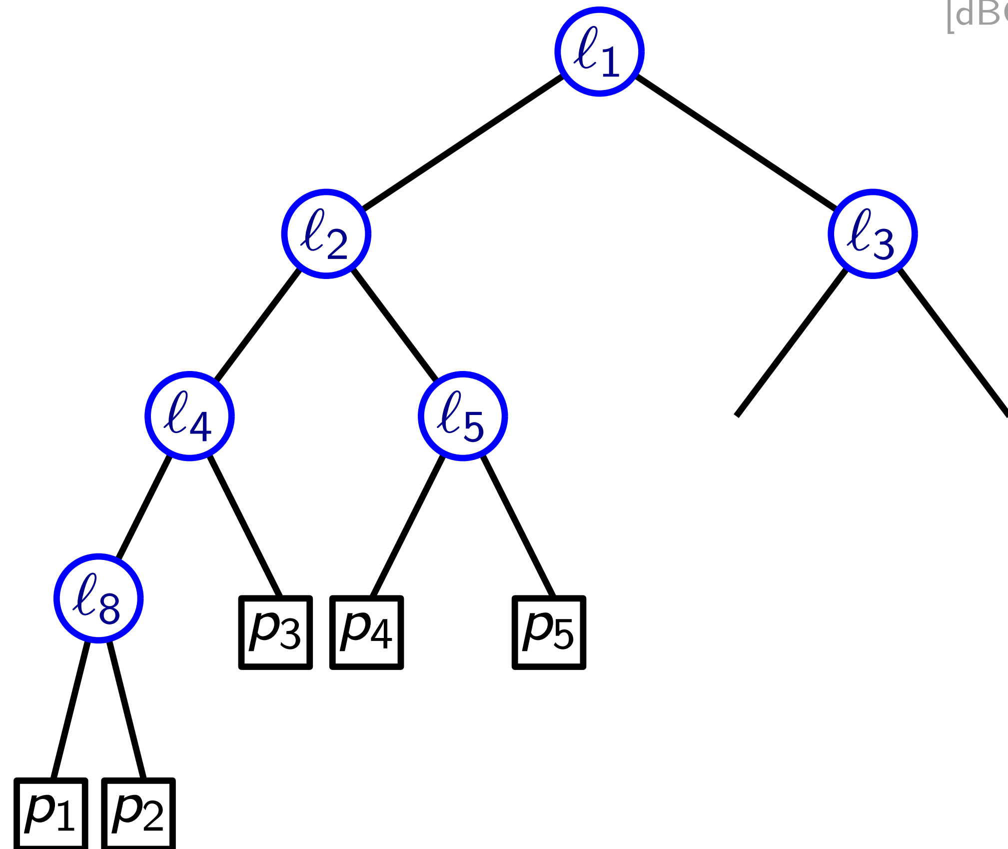

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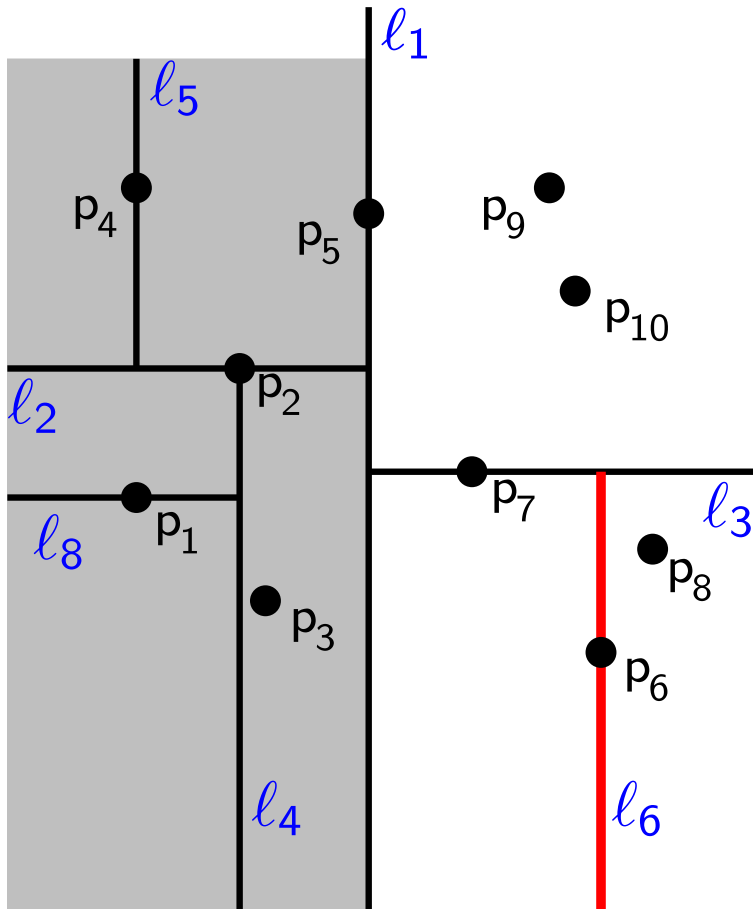

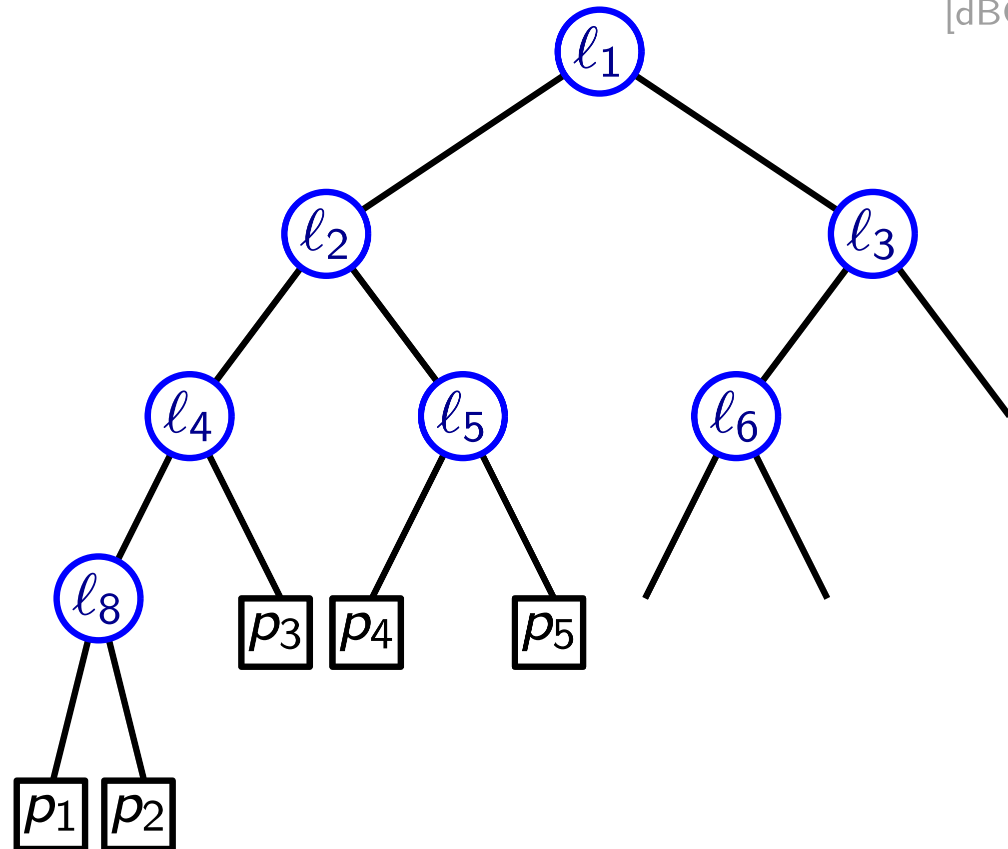

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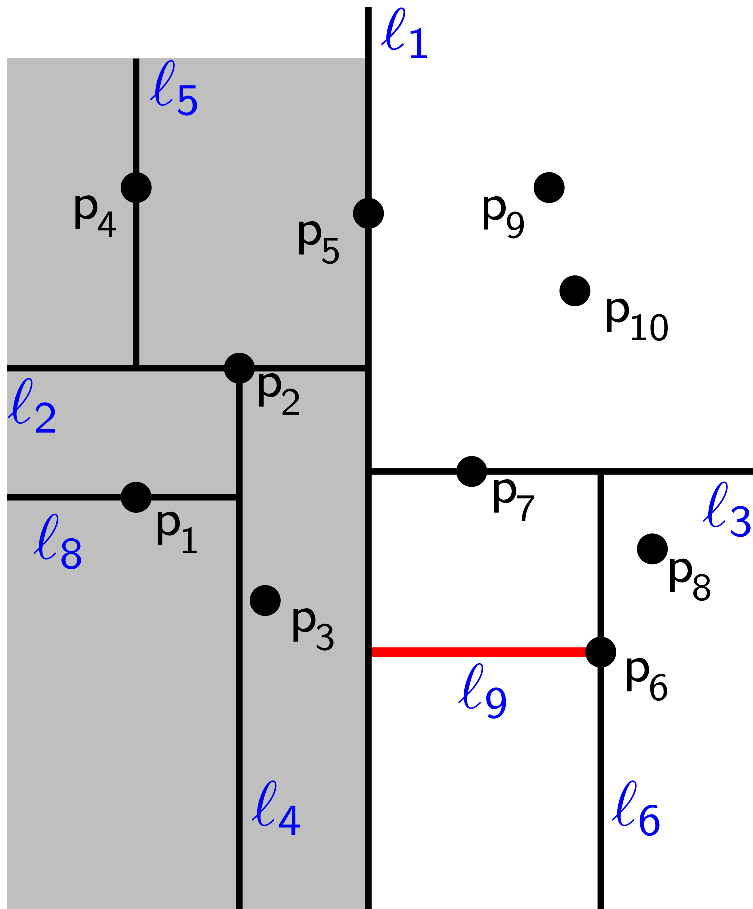

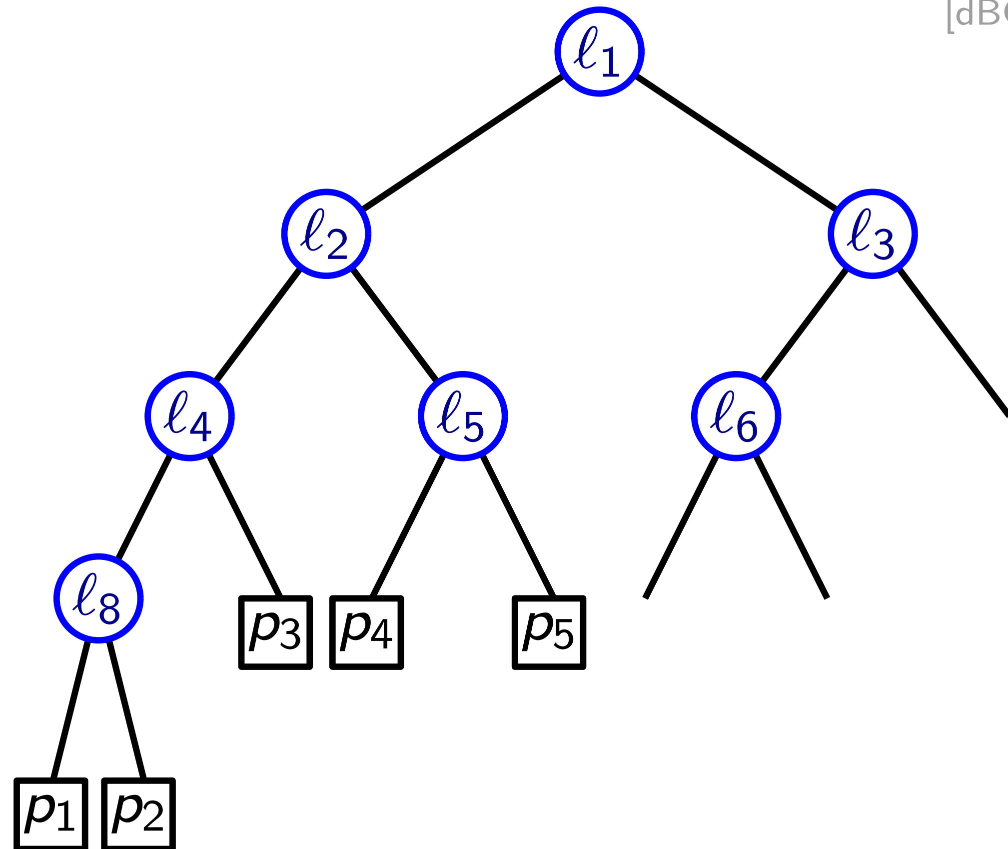

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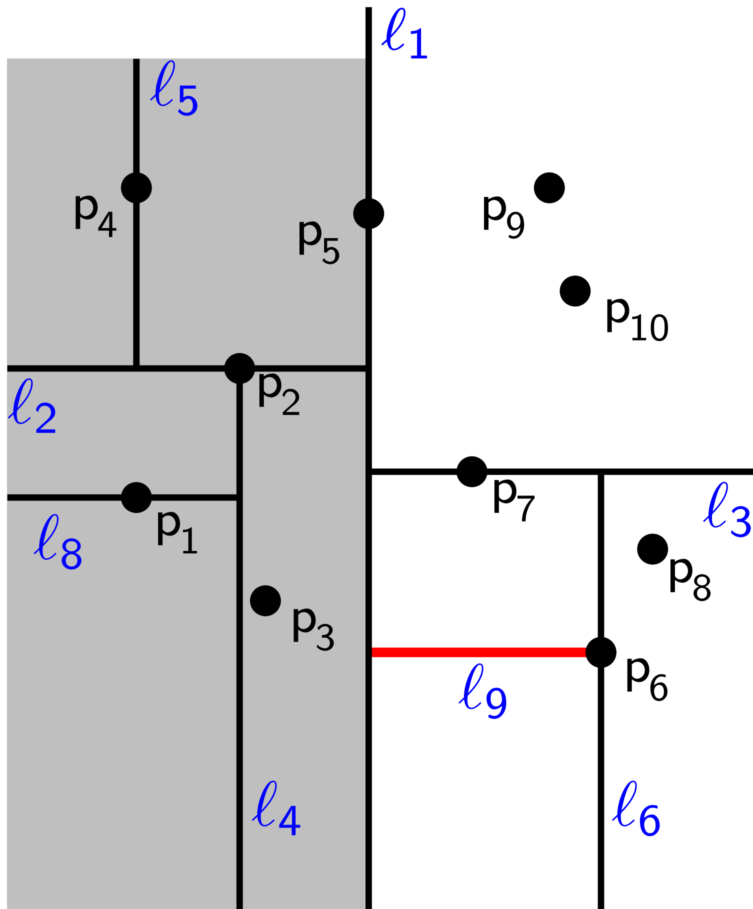
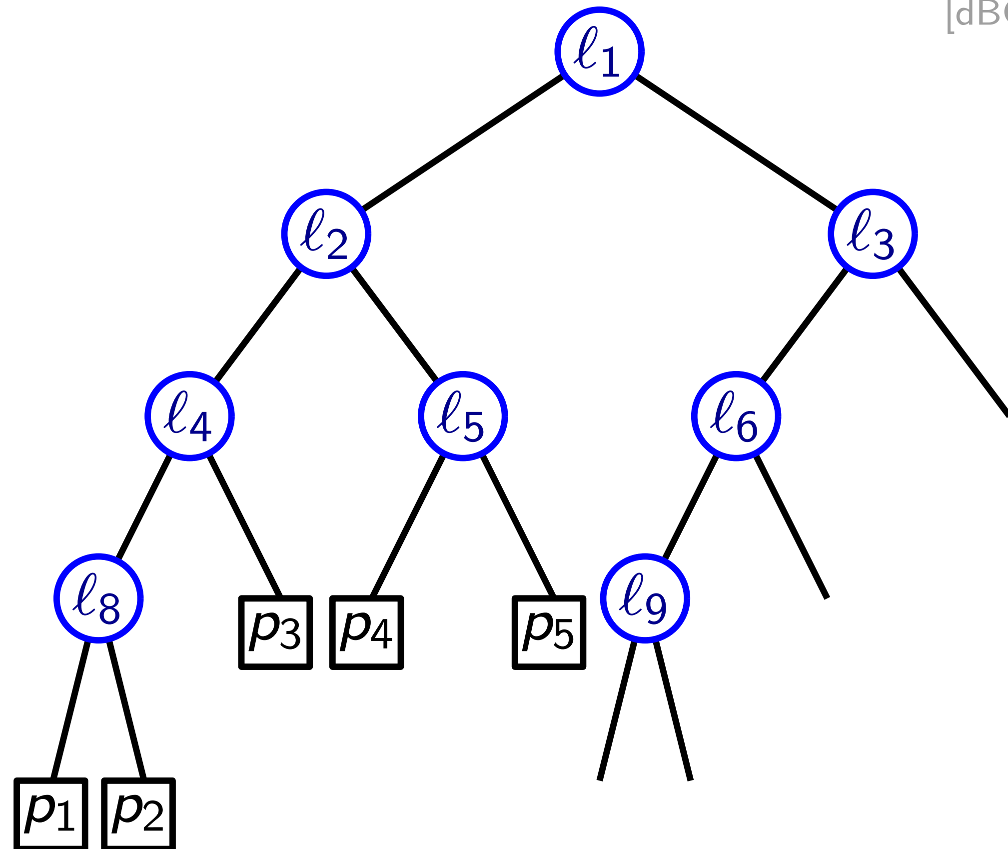


[dBCvKO'08]



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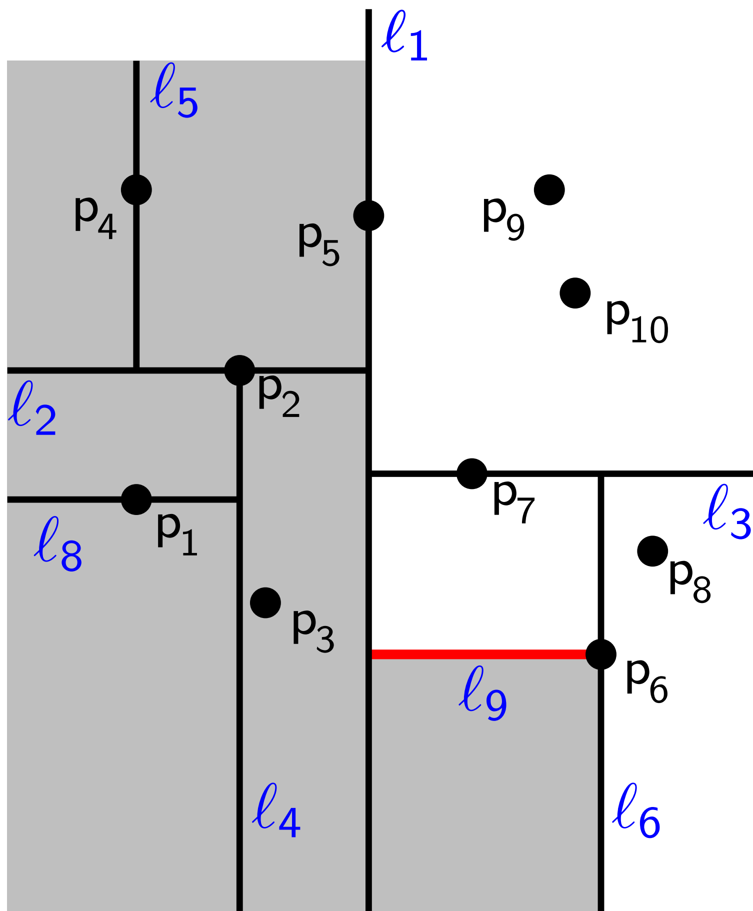

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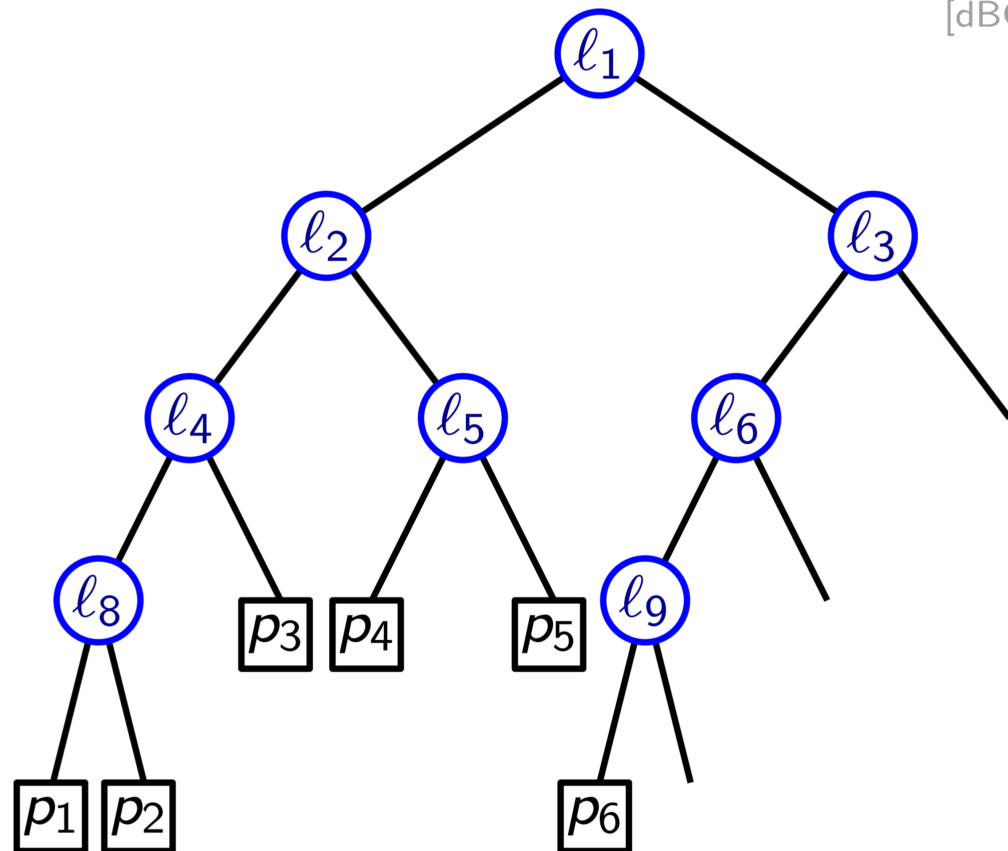

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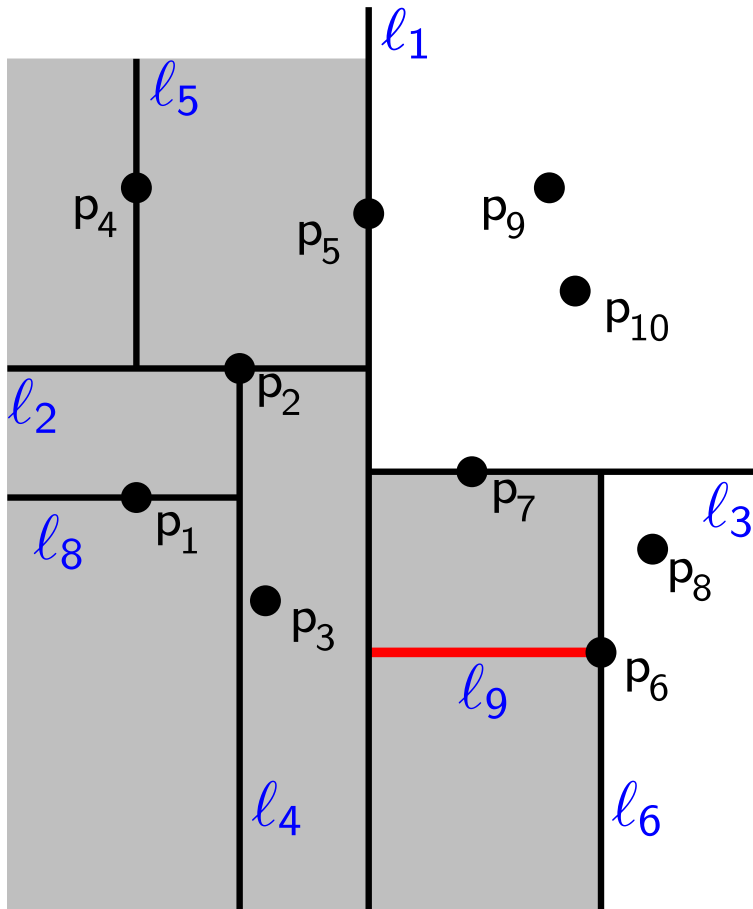

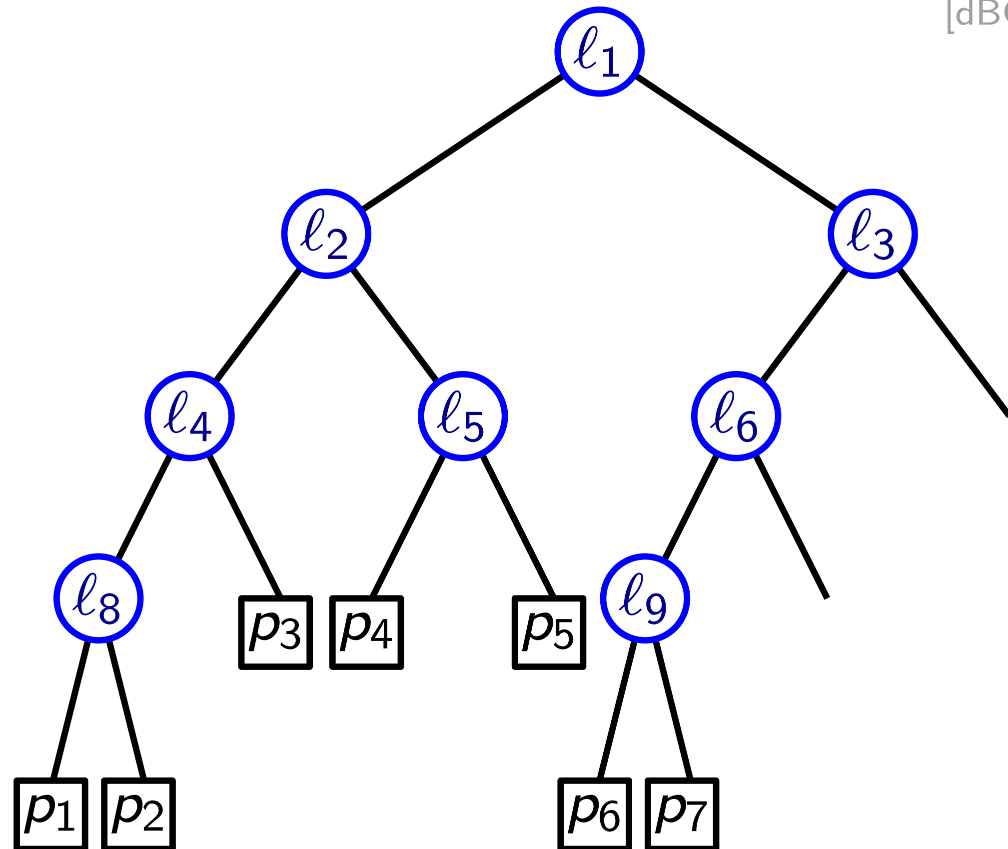

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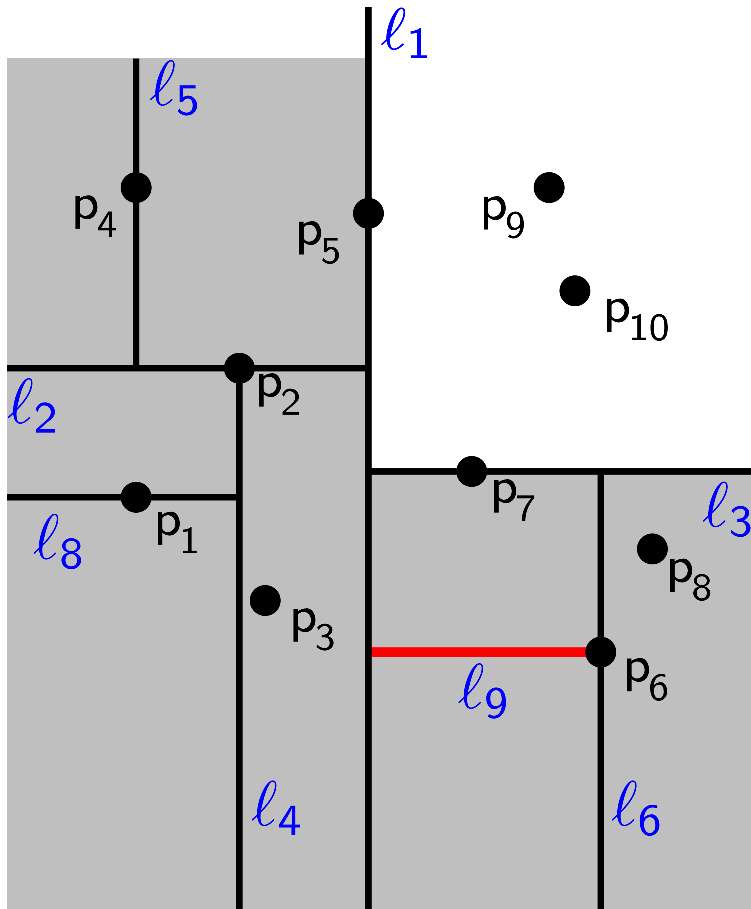

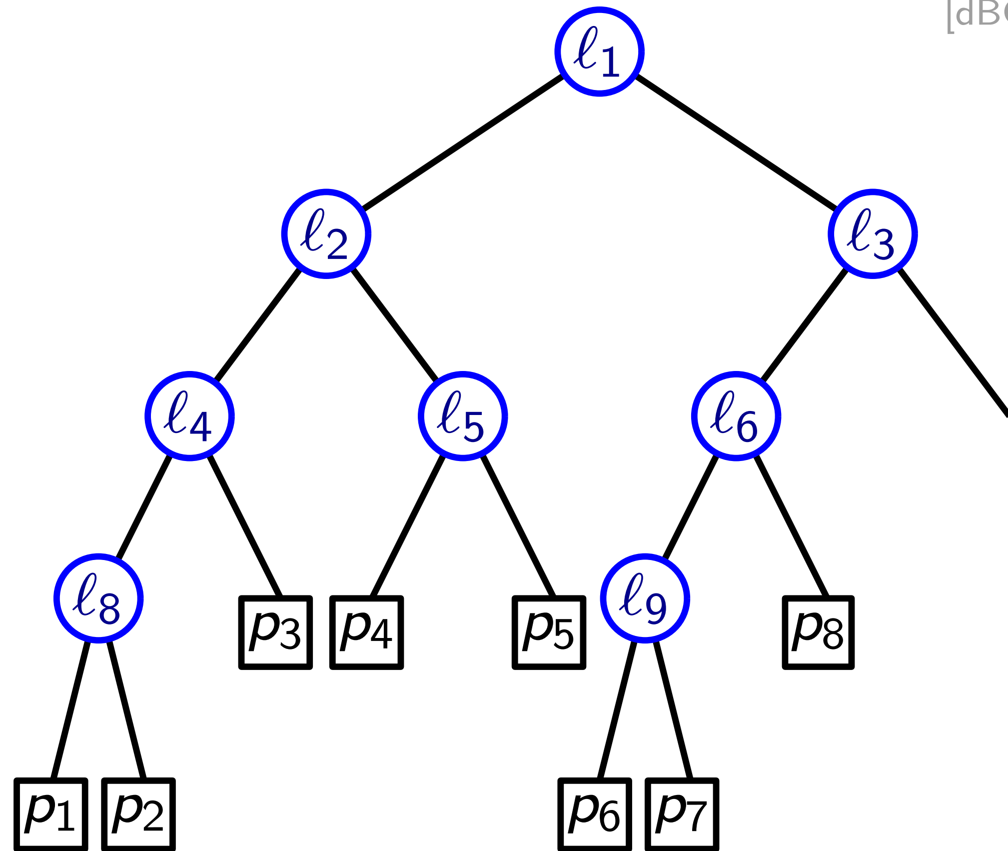

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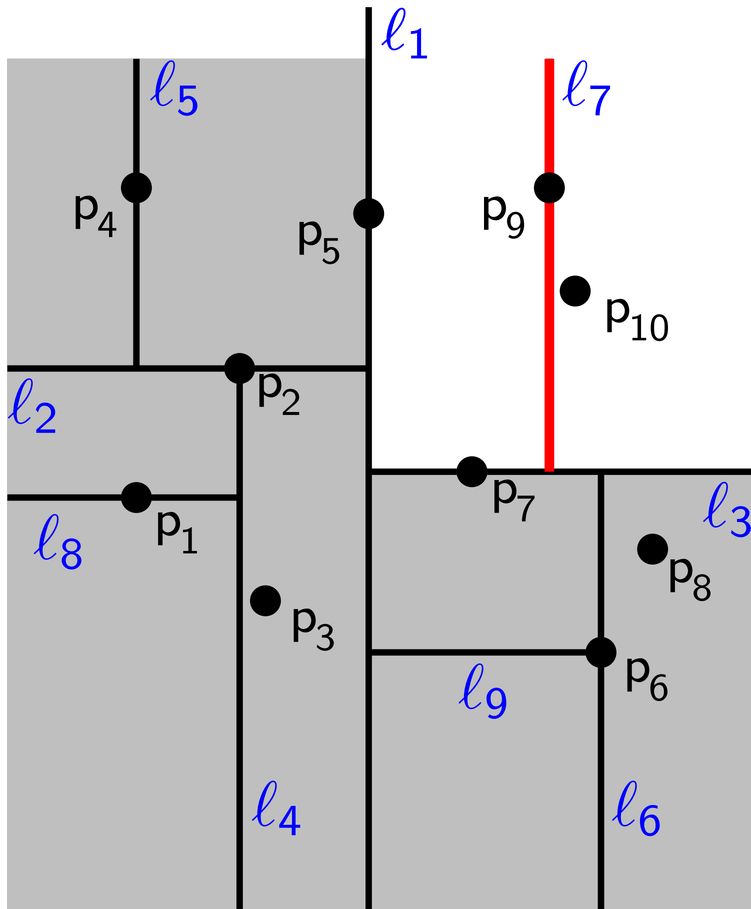

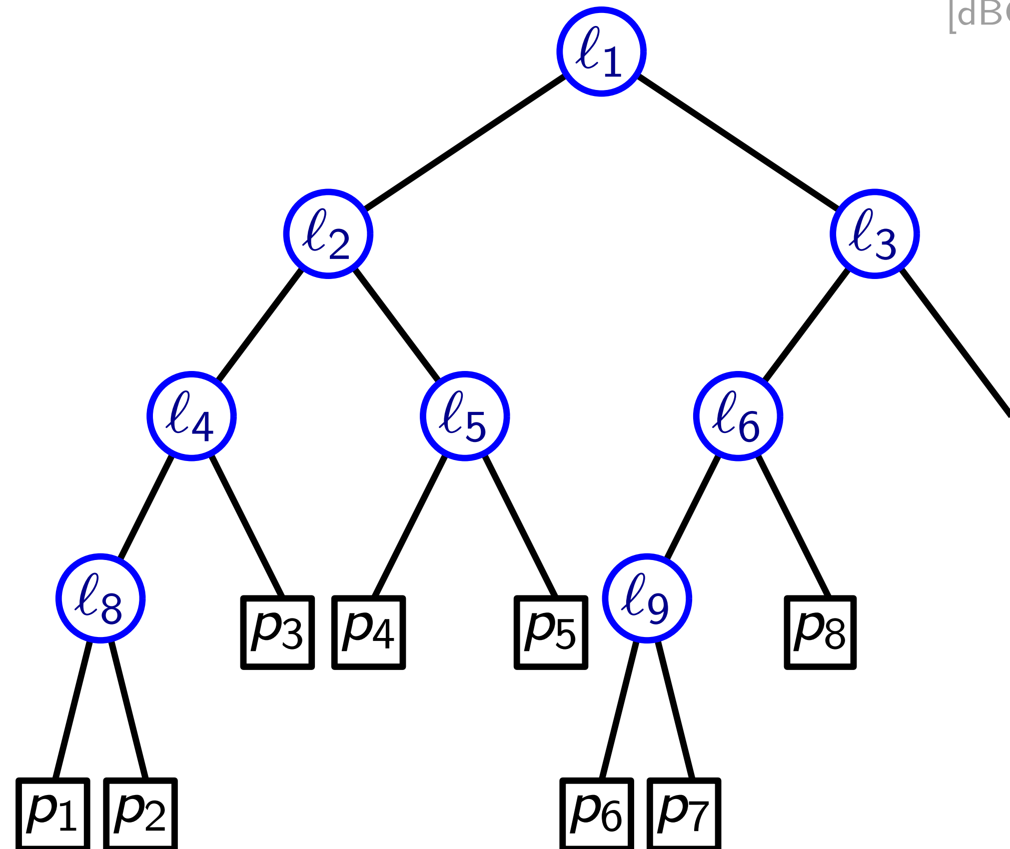

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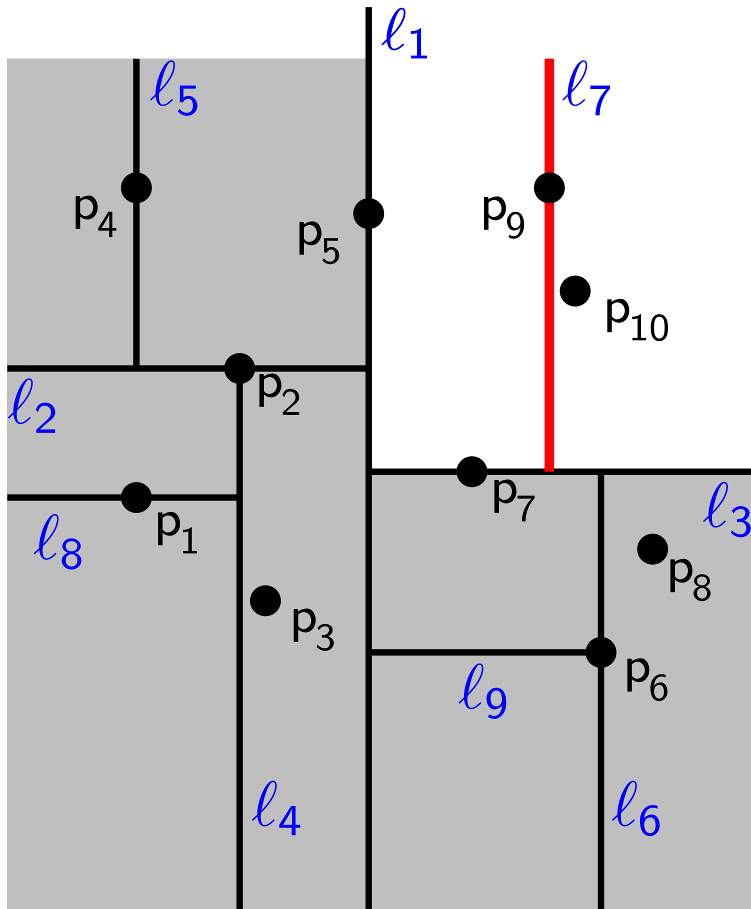

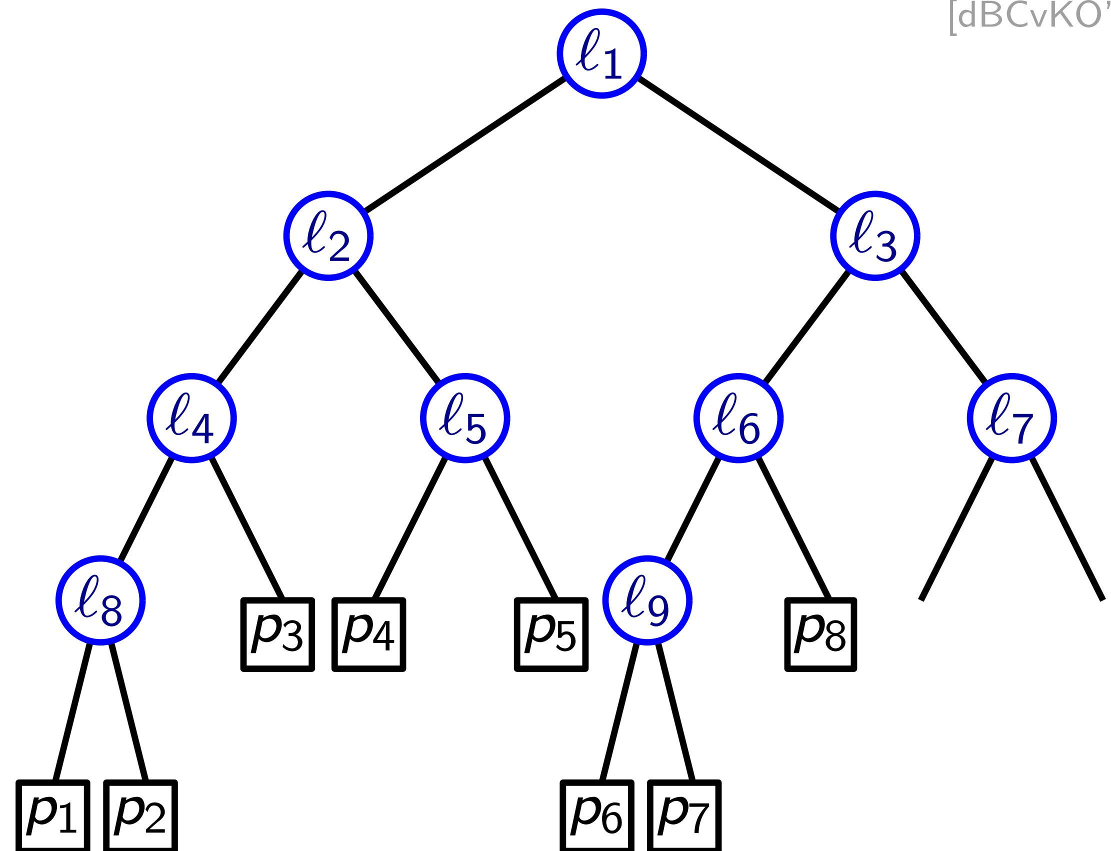

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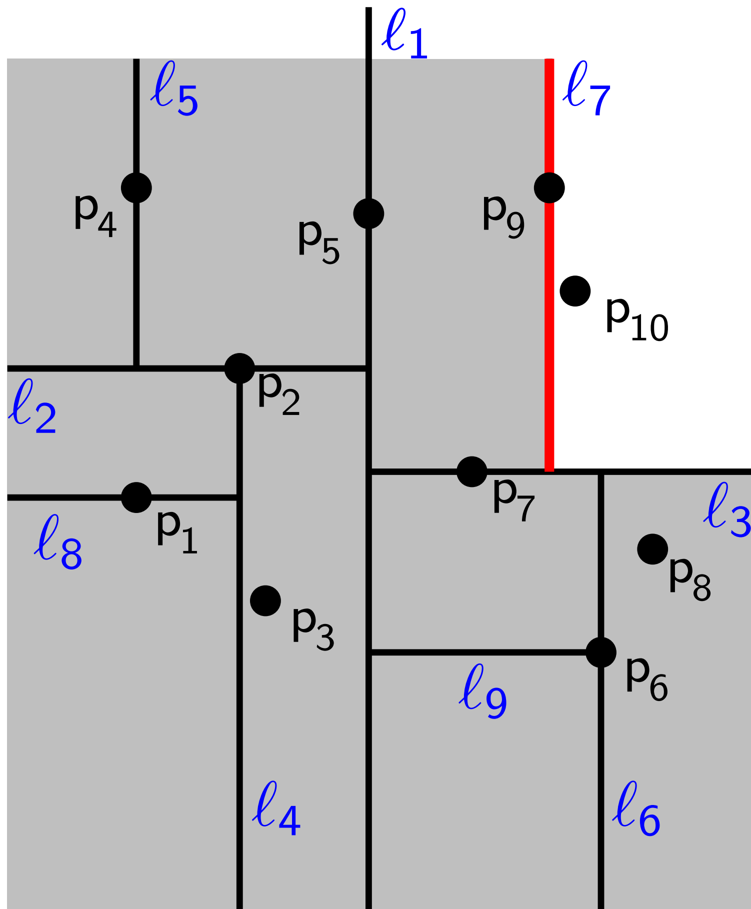

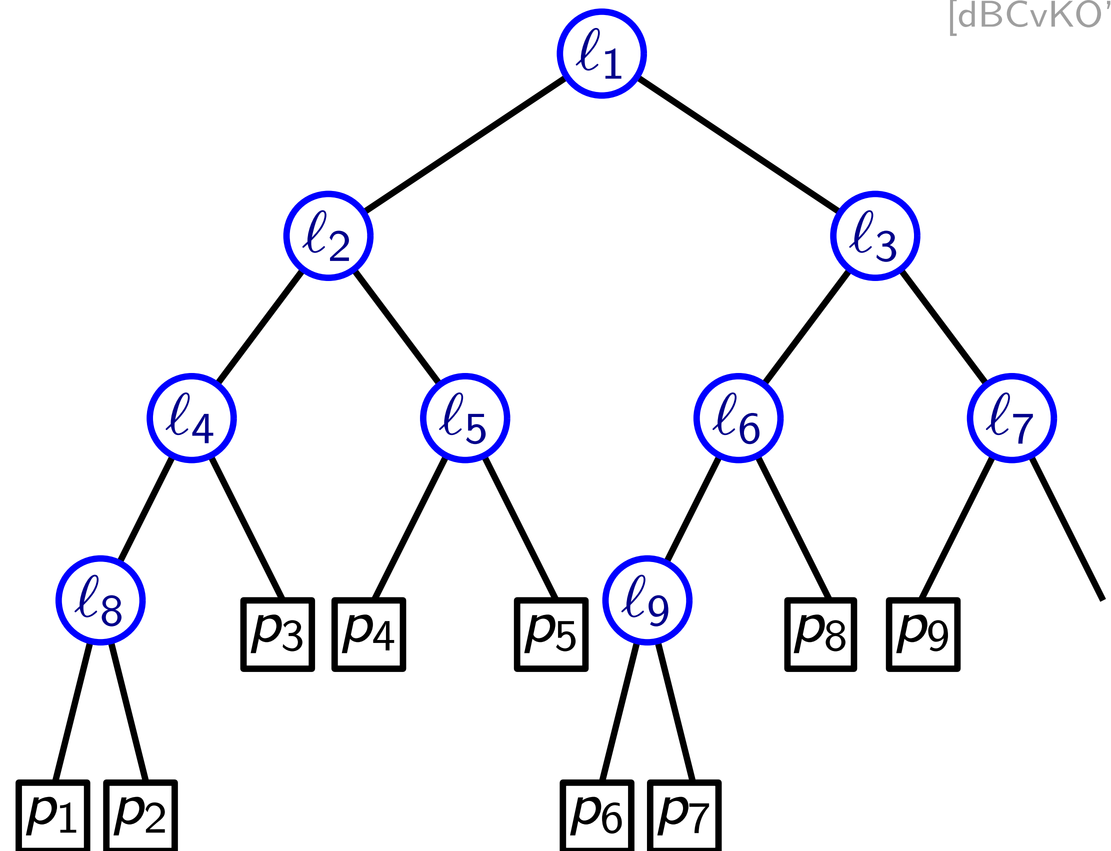

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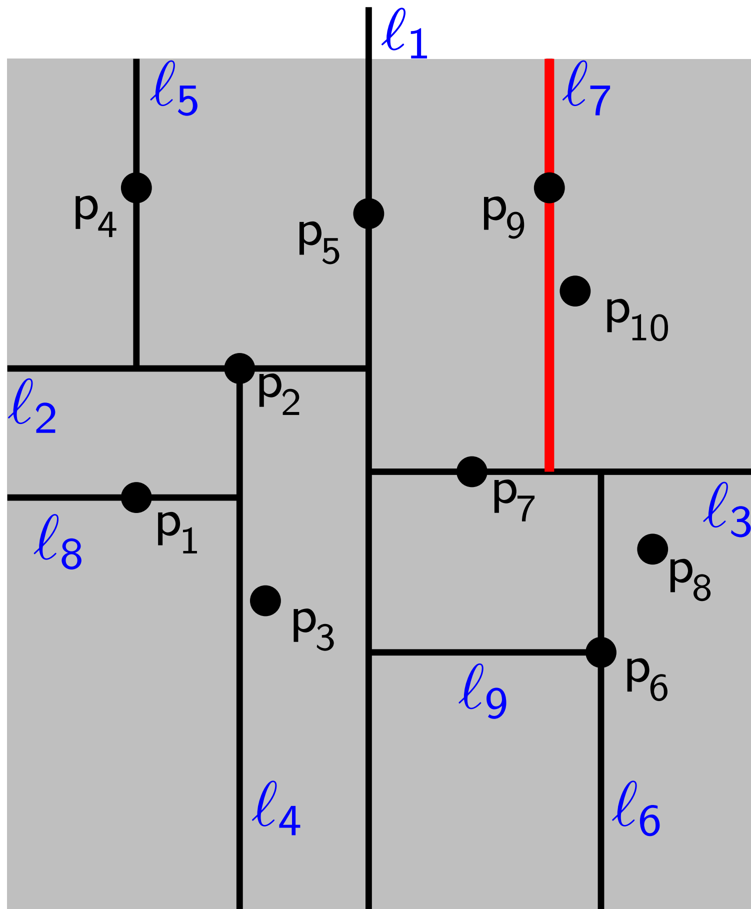
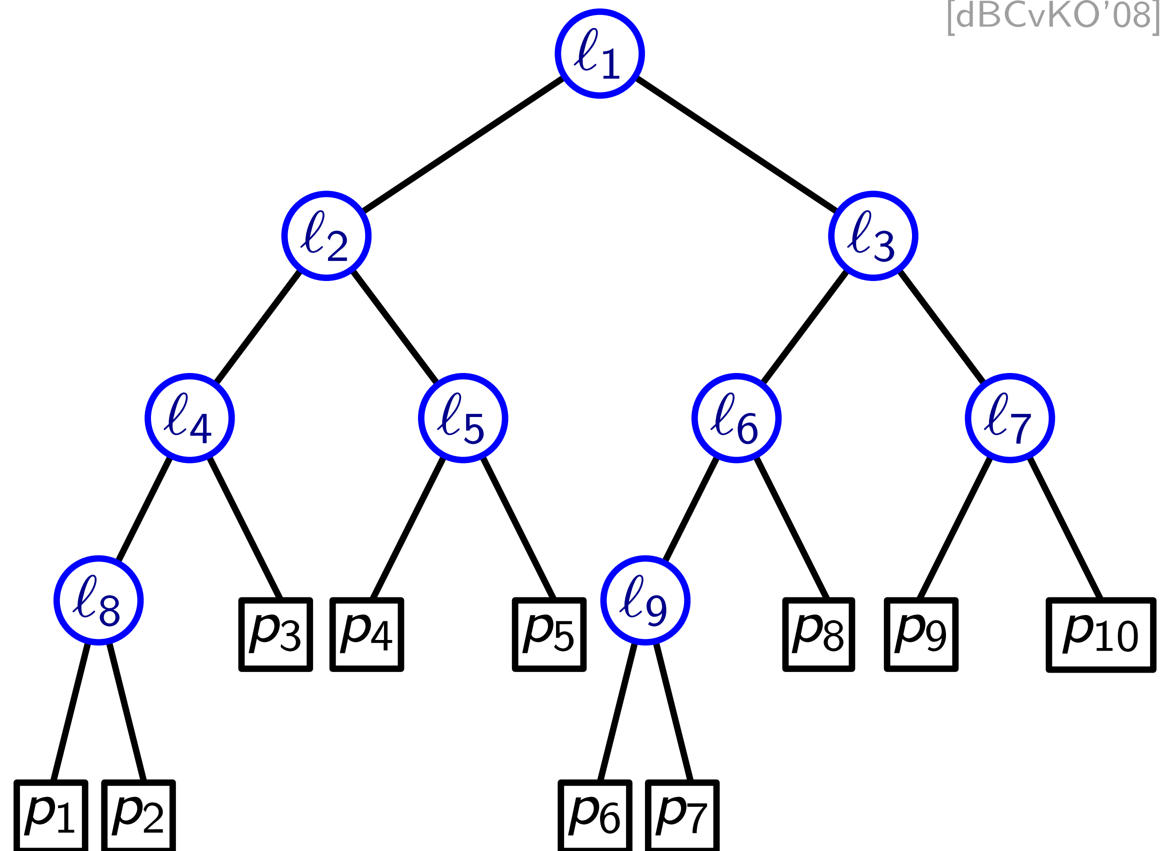


[dBCvKO'08]



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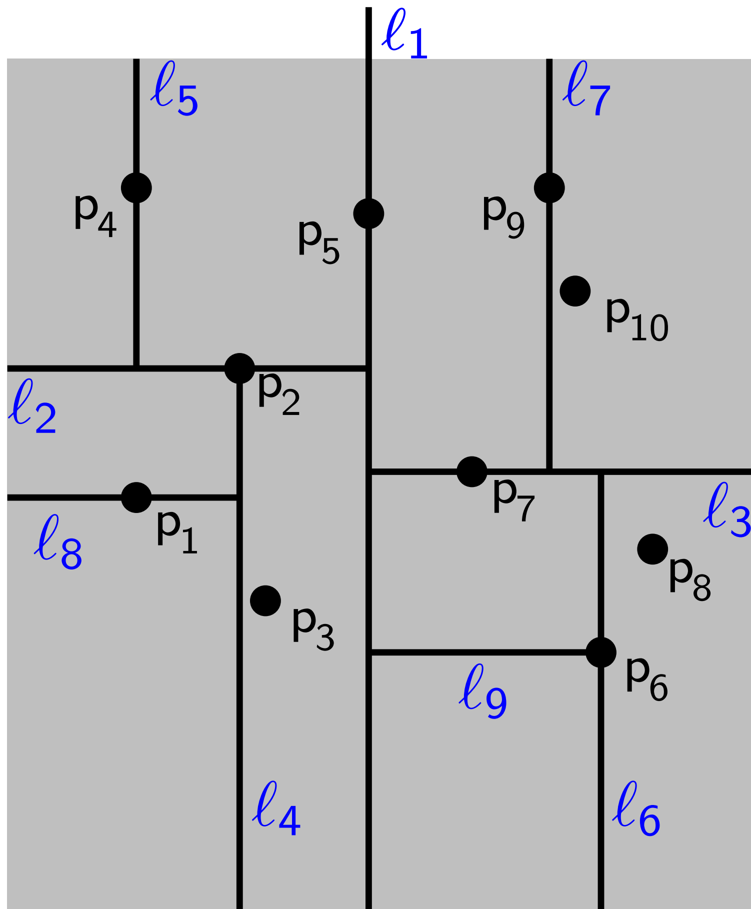

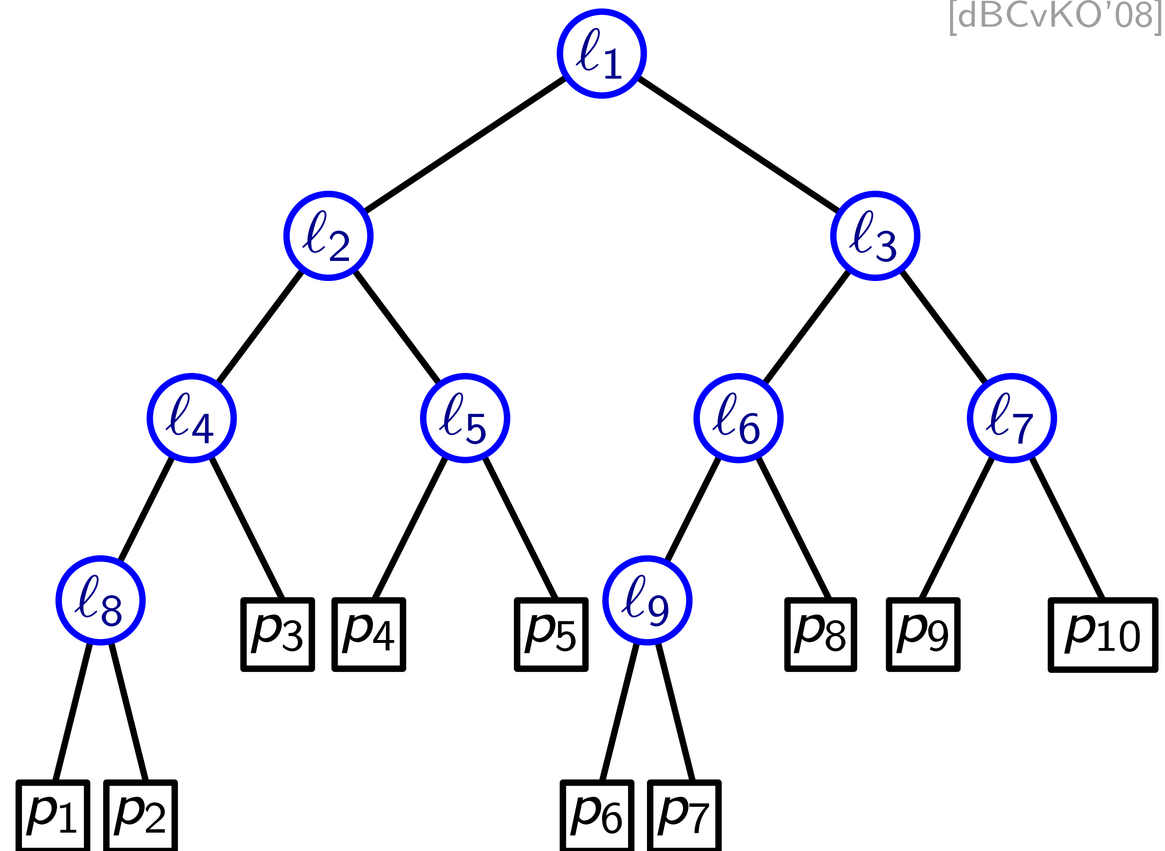

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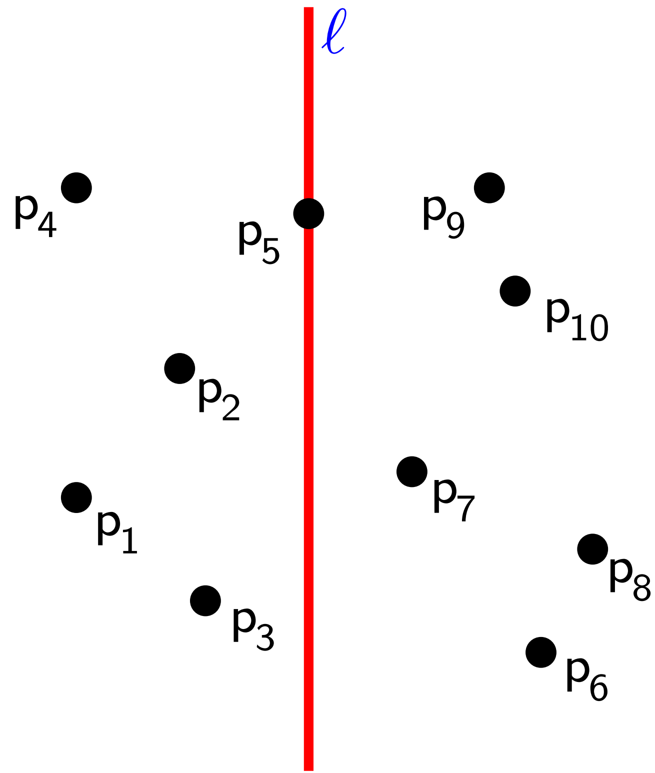
Kd-Trees: Construction



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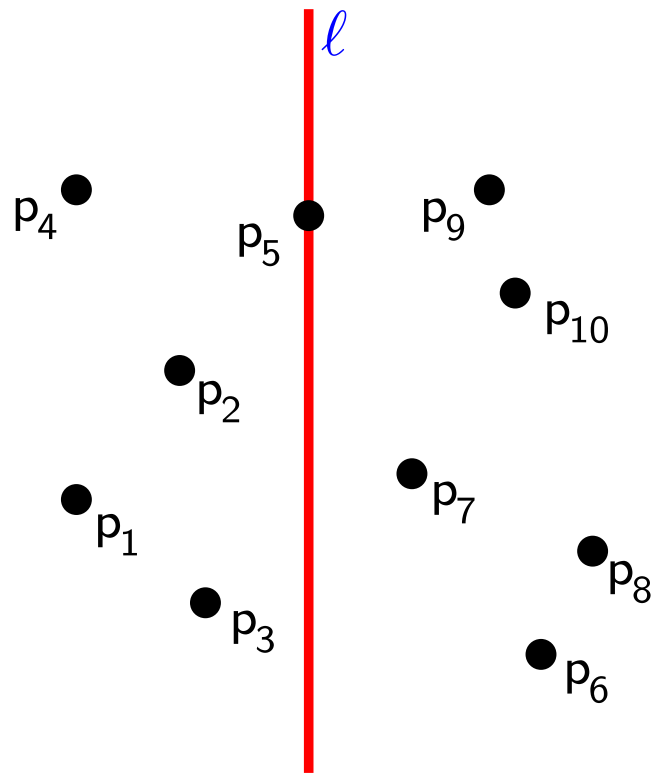

Pseudo-code:

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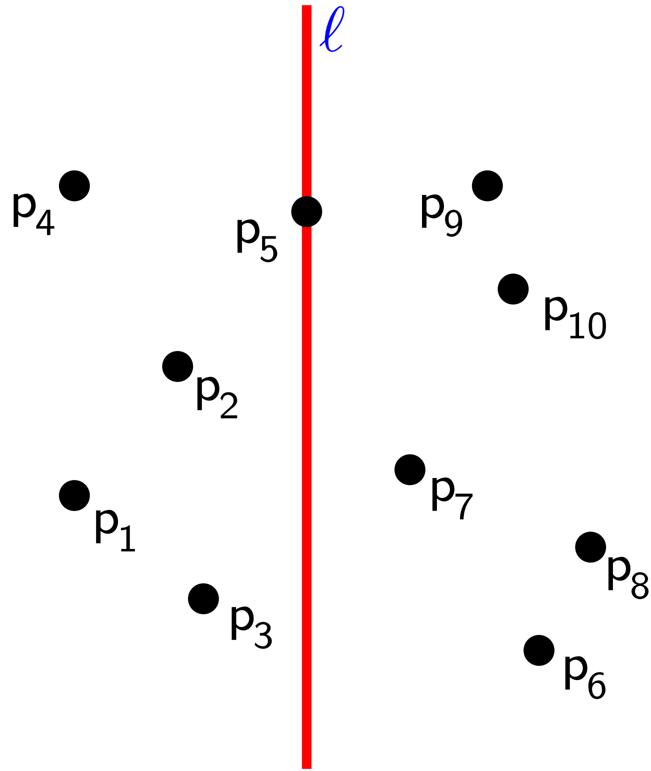

Pseudo-code:

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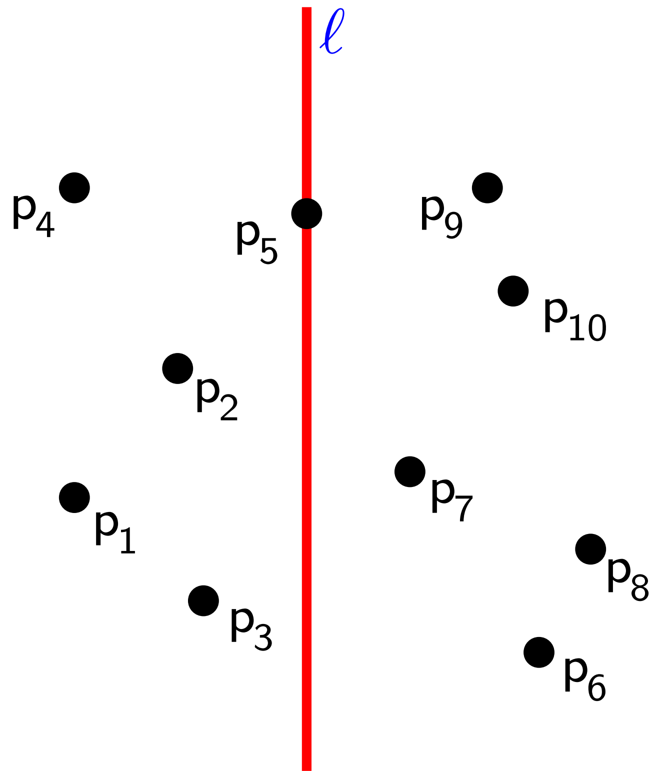

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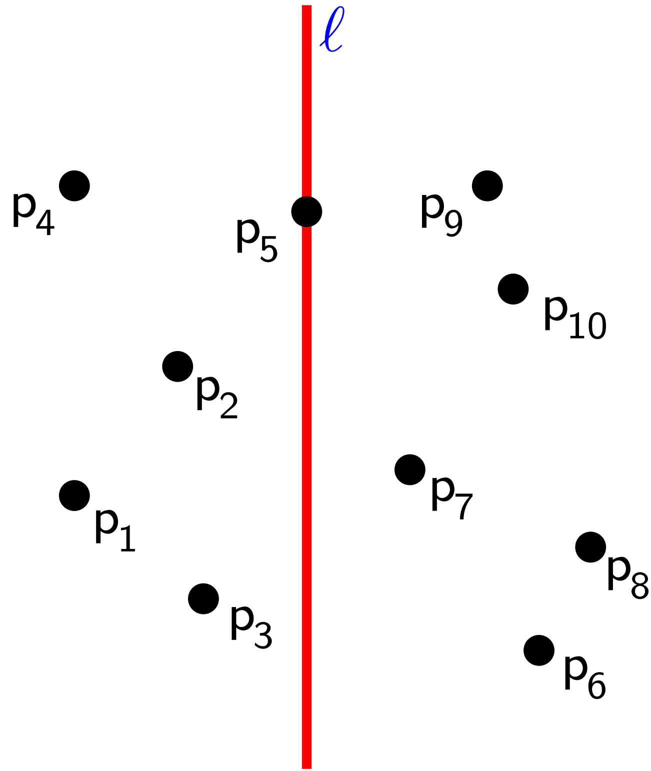

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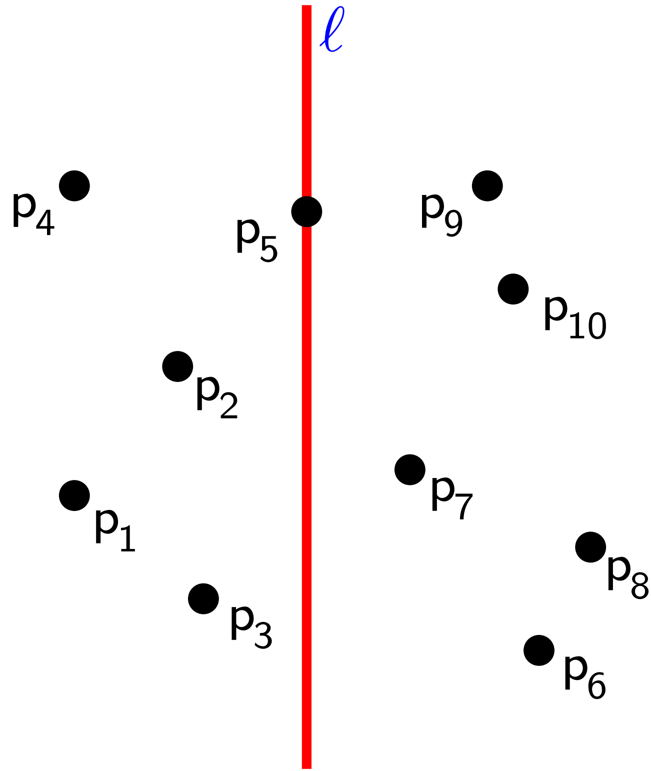

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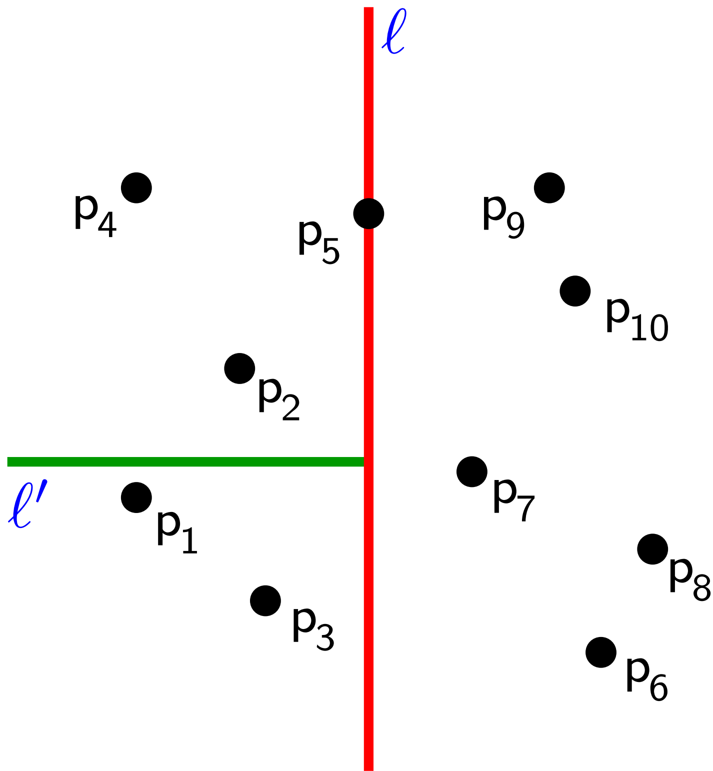

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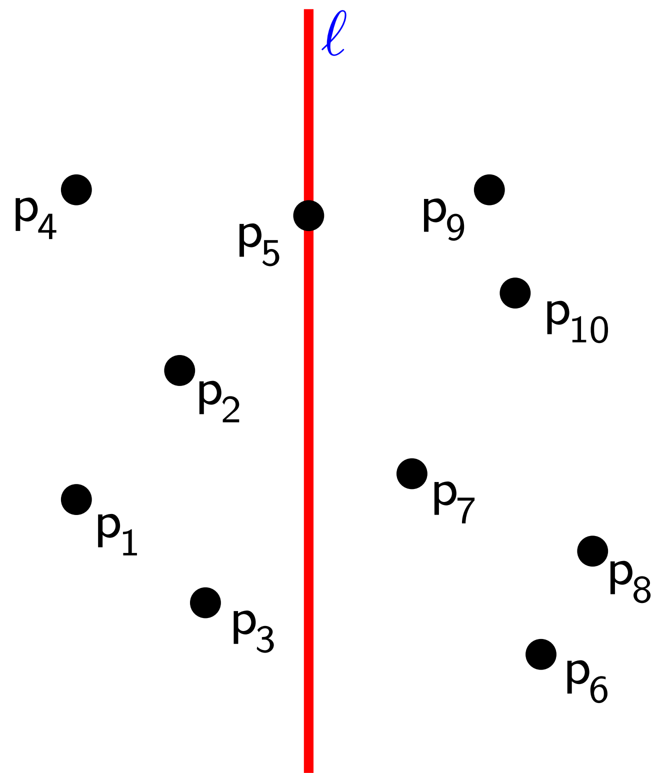

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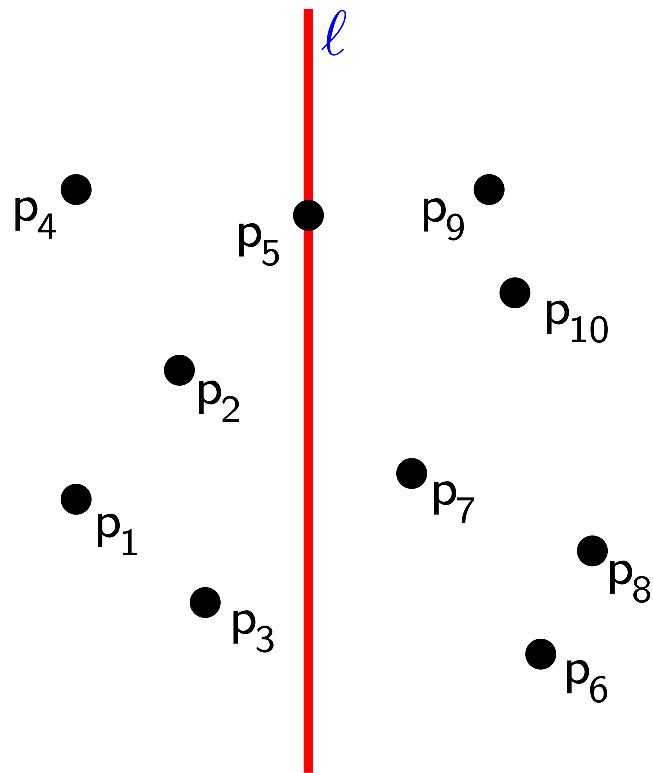

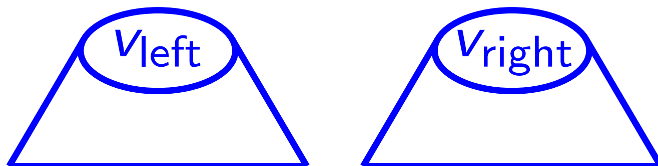

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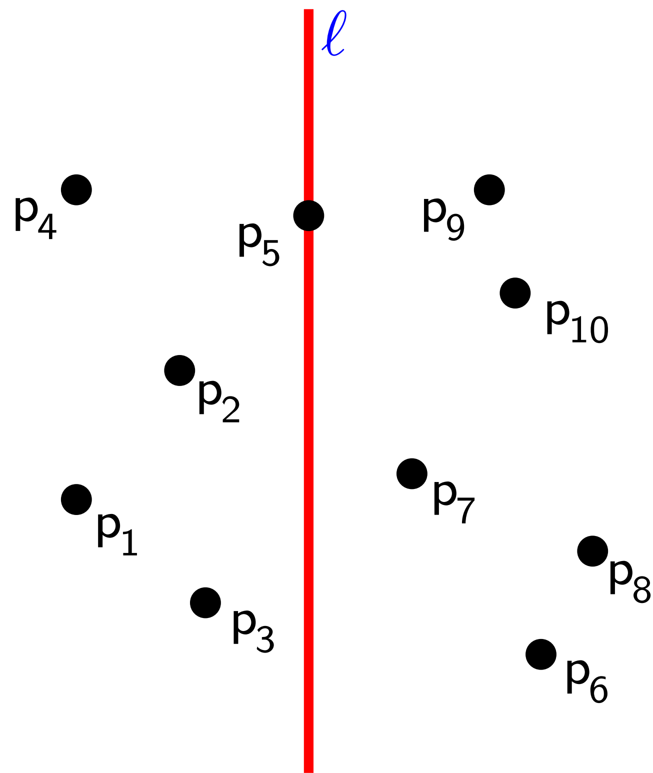

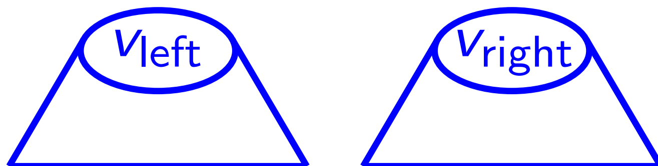

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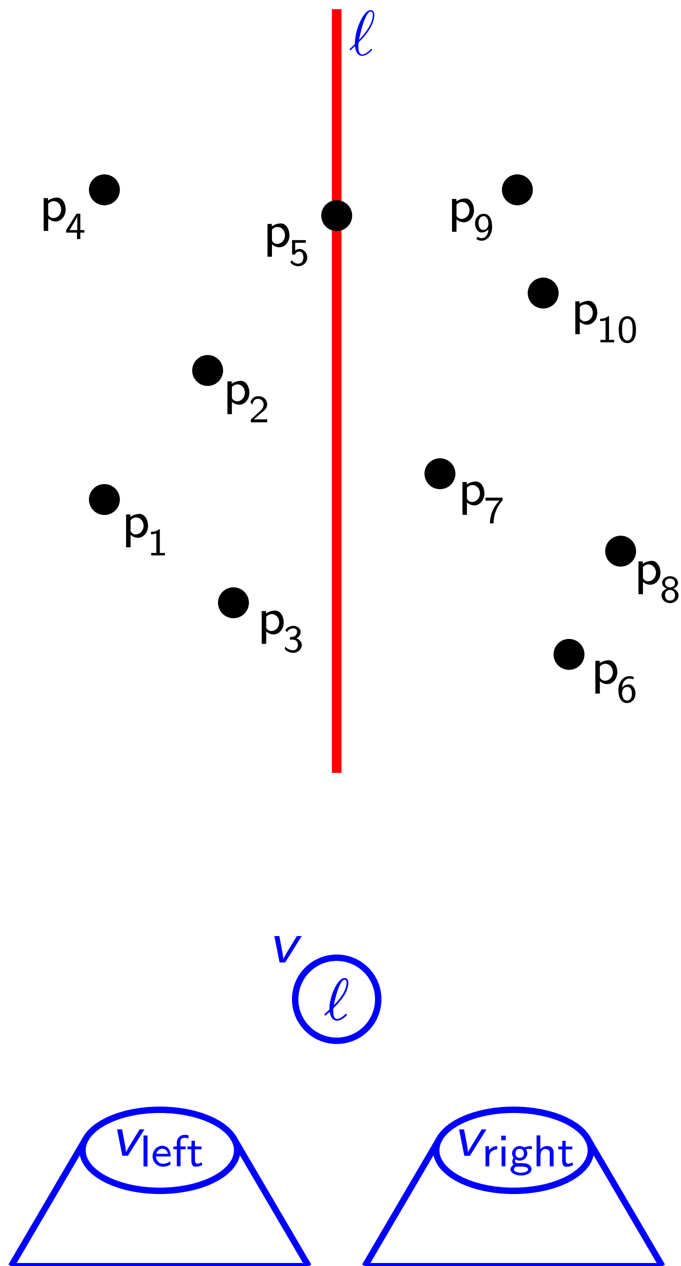

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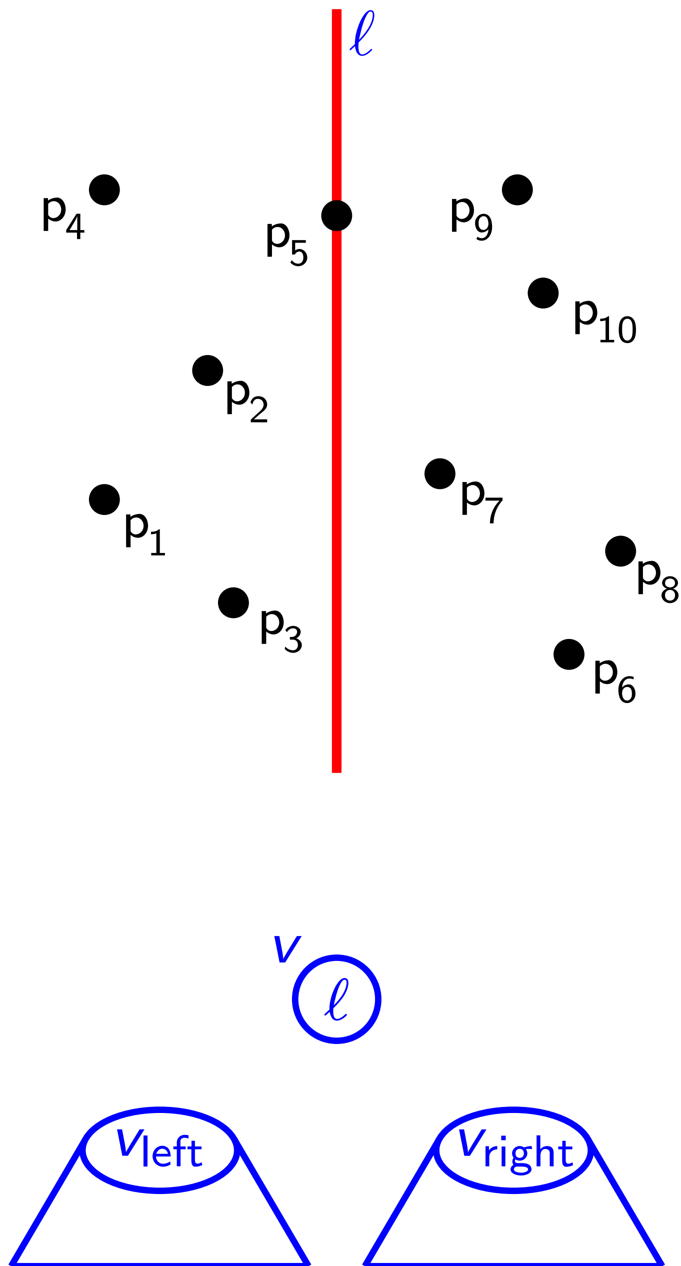

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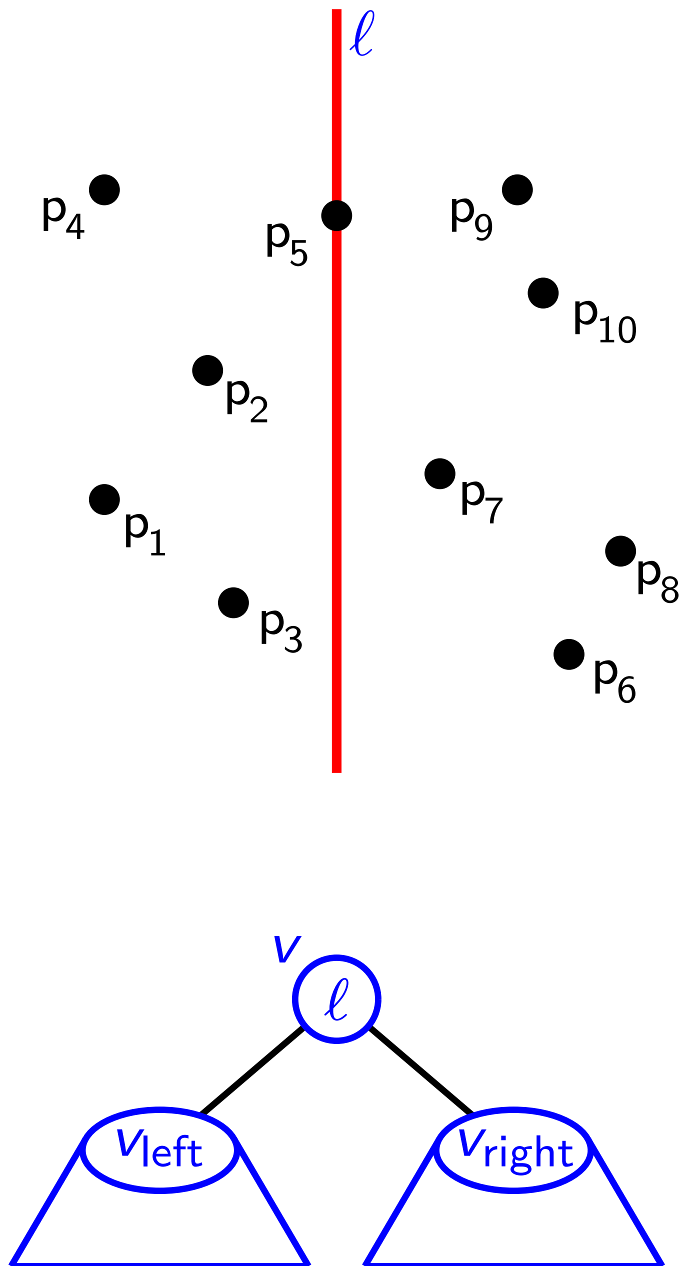

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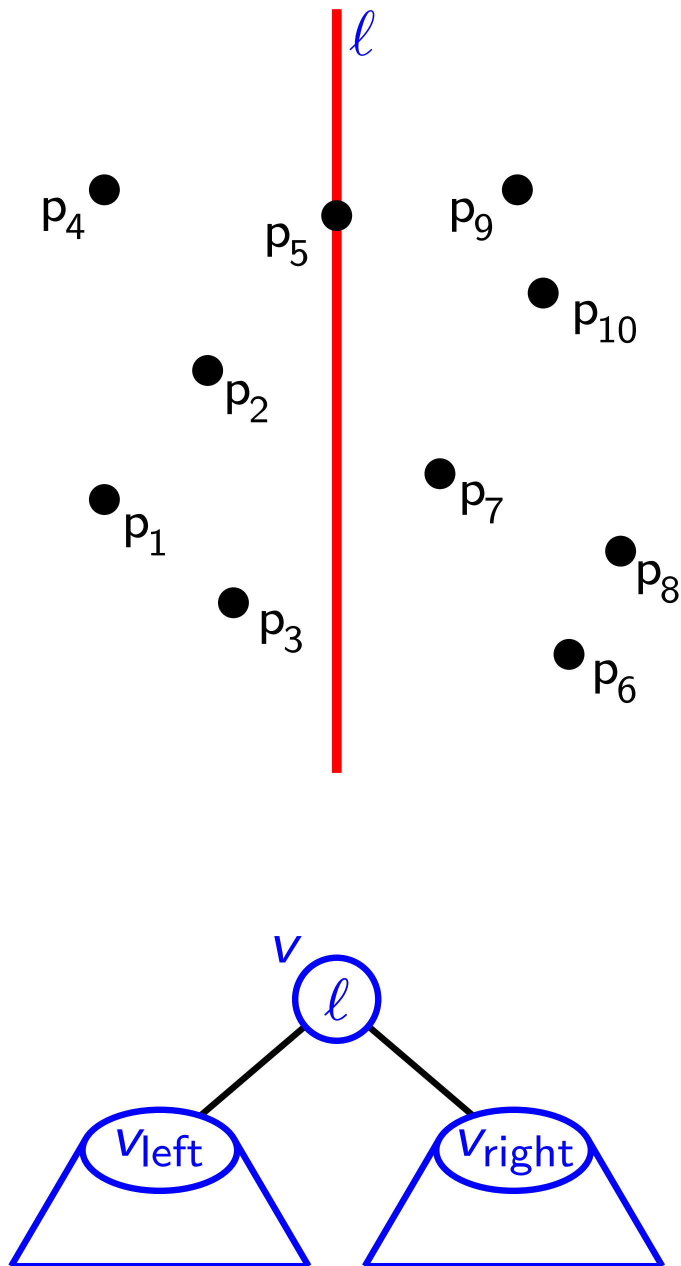

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  - make v_{left} and v_{right} the children of v
  - **return** (v)

Kd-Trees: Analysis

Construction time?

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$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ O(n) + 2T(\lceil n/2 \rceil) & \text{else.} \end{cases}$$

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Construction time?

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see Mergesort!

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Lemma: A kd-tree for a set of n pts in the plane takes $O(n \log n)$ time to construct and uses $O(n)$ storage.

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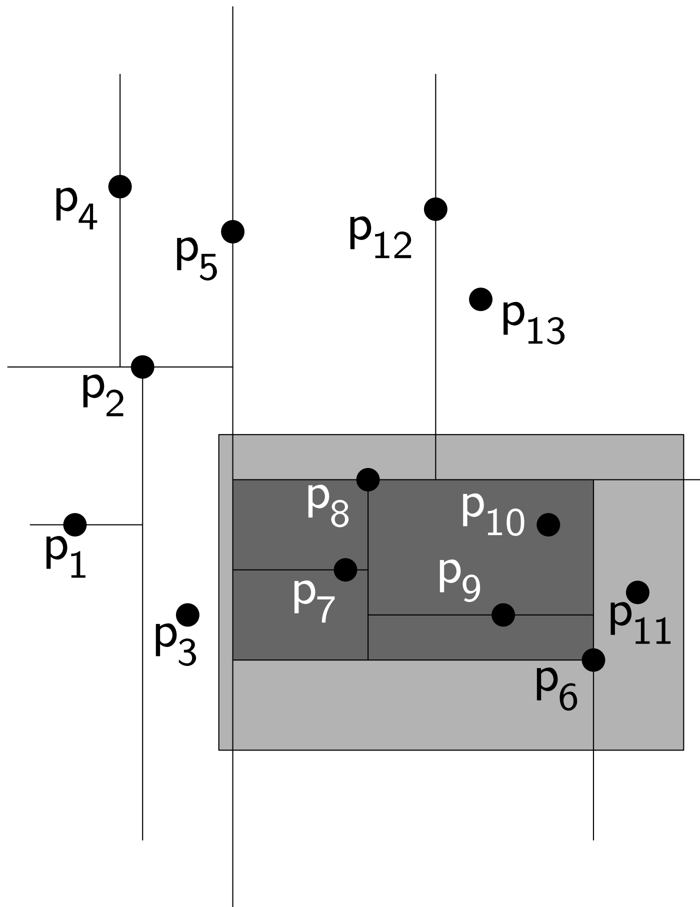
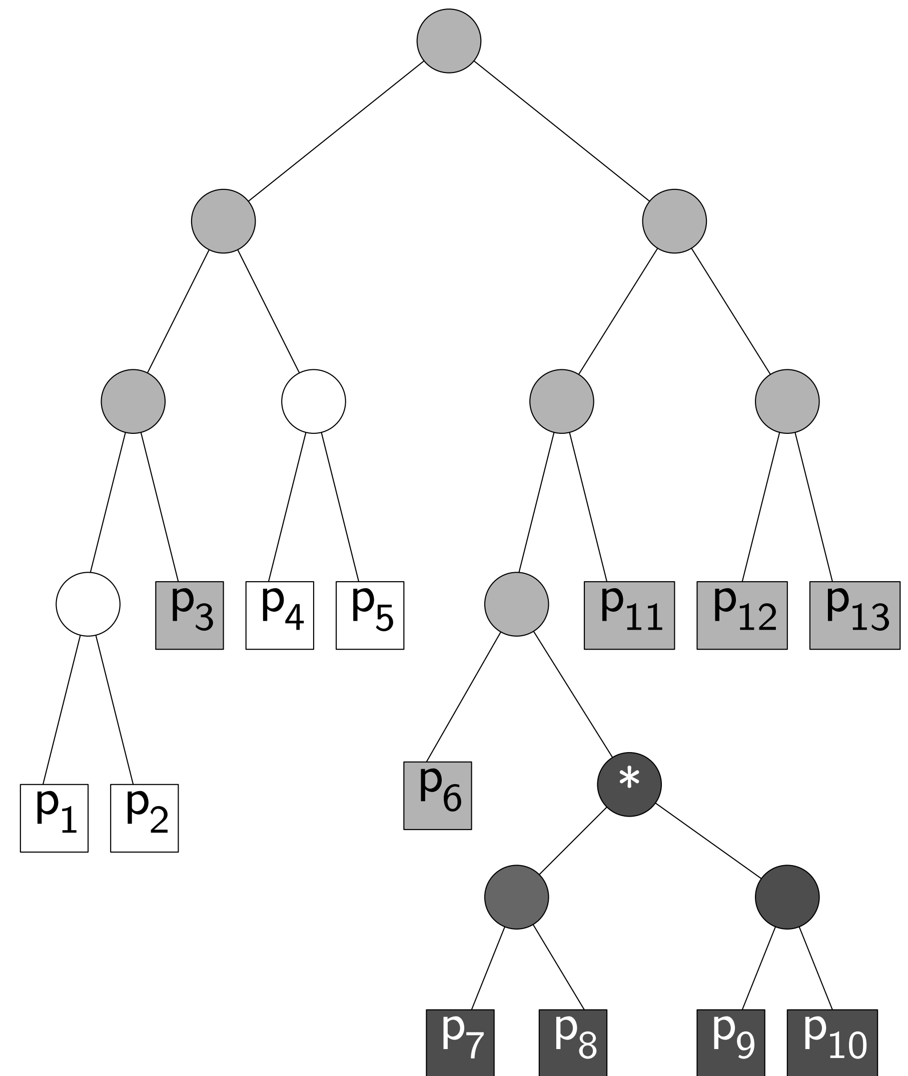
Kd-Trees: Querying





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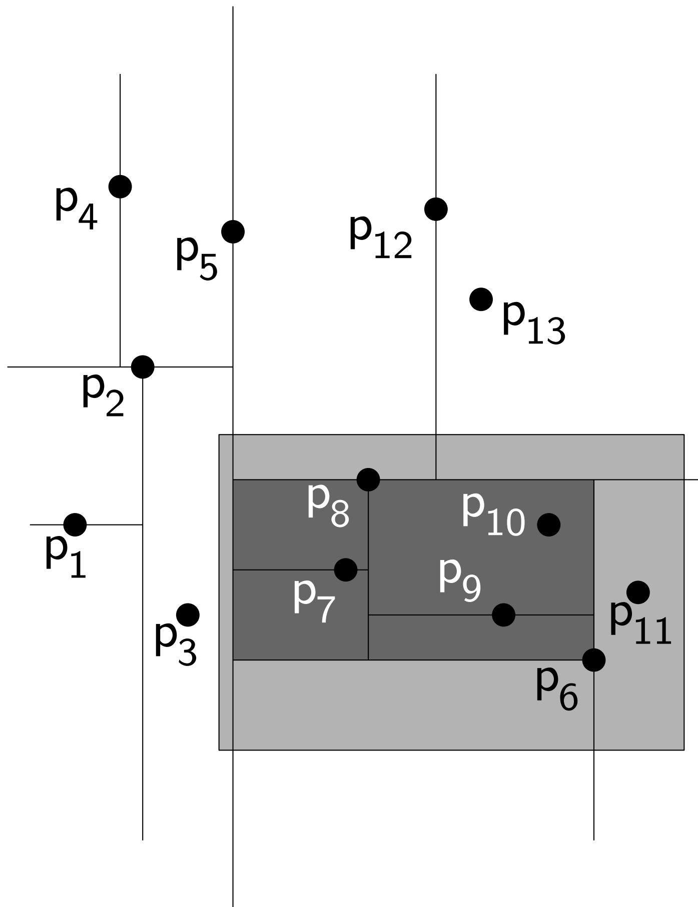

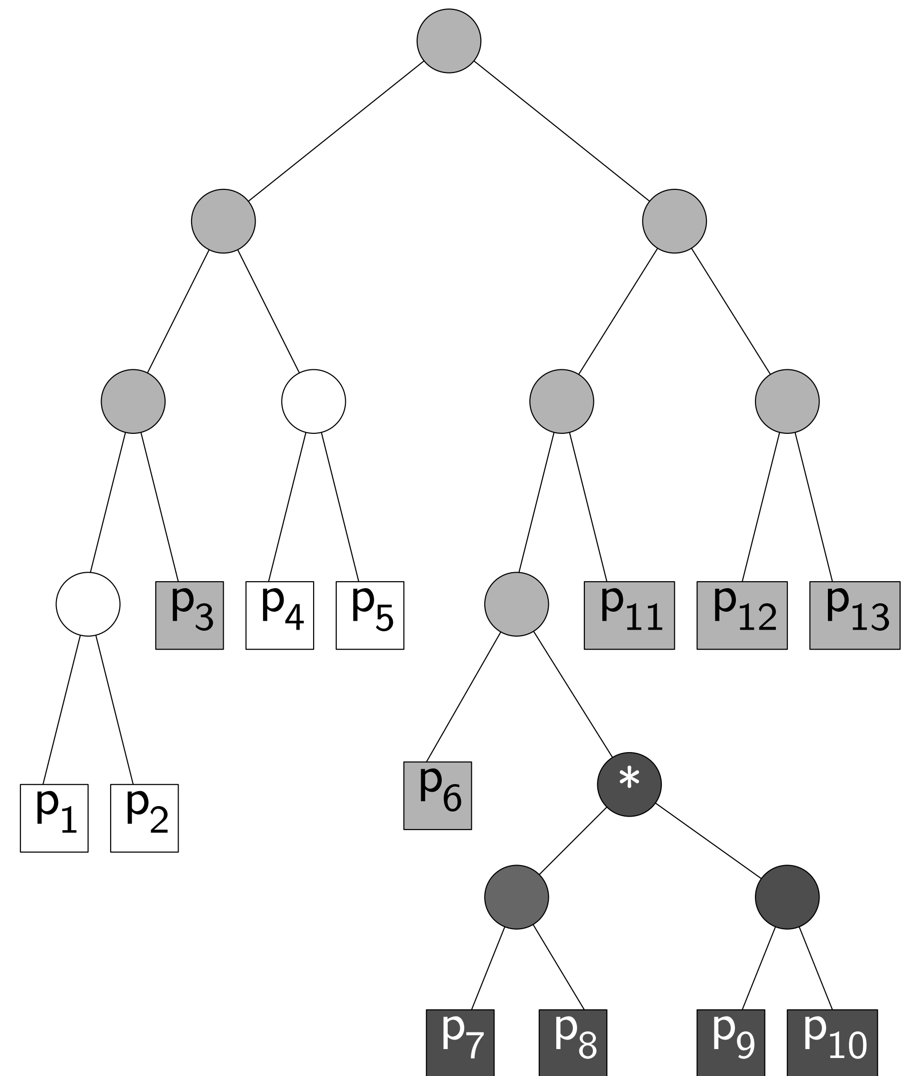

Lemma. Querying a kd-tree for n pts in the plane with an axis-parallel rectangle R takes $O(k + \sqrt{n}\,)$ time, where $k = |\text{output}|$.

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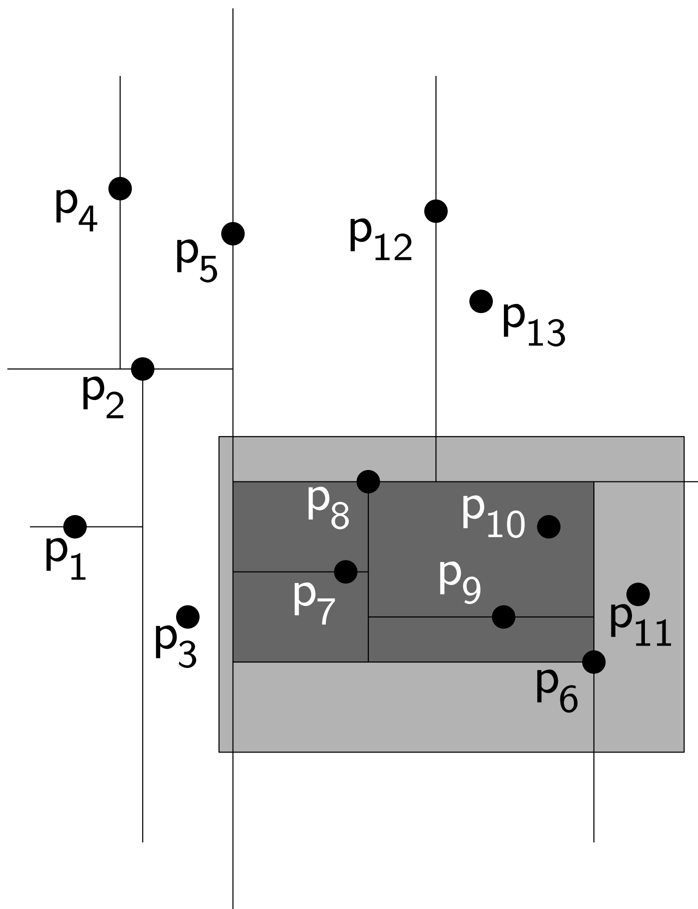

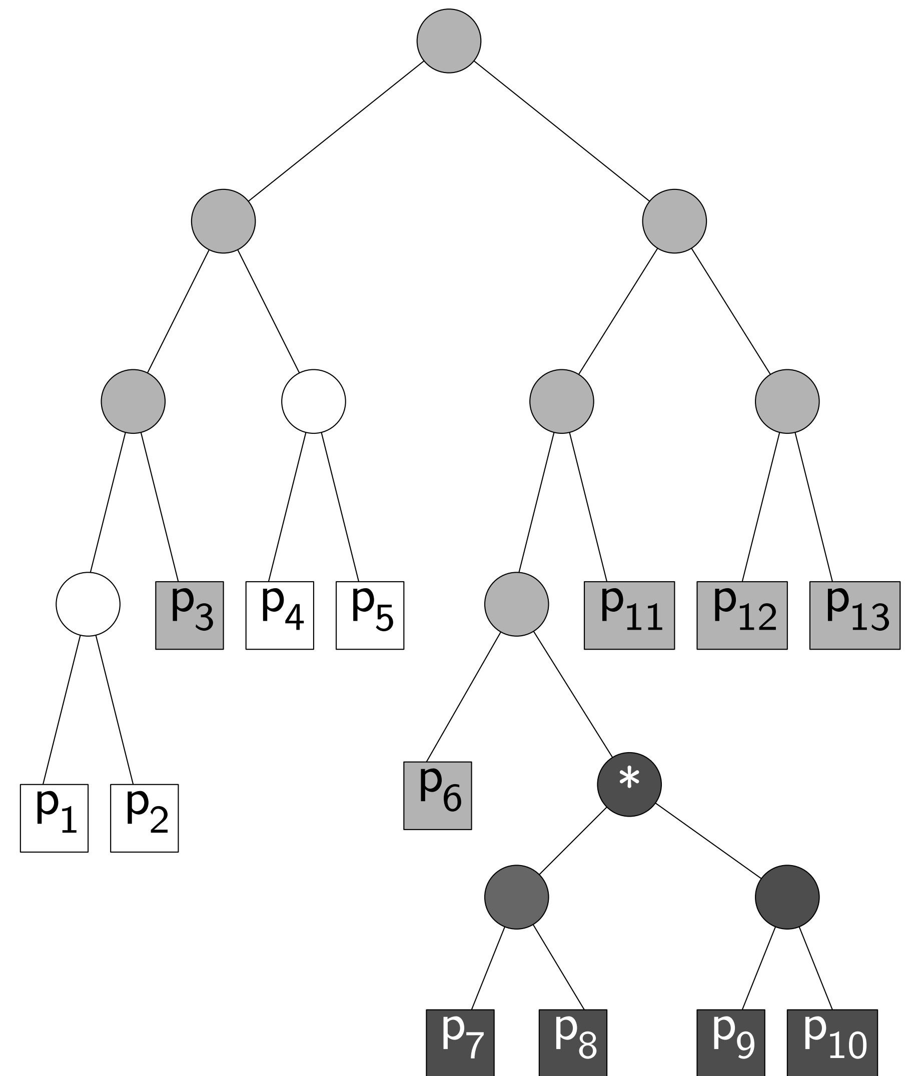

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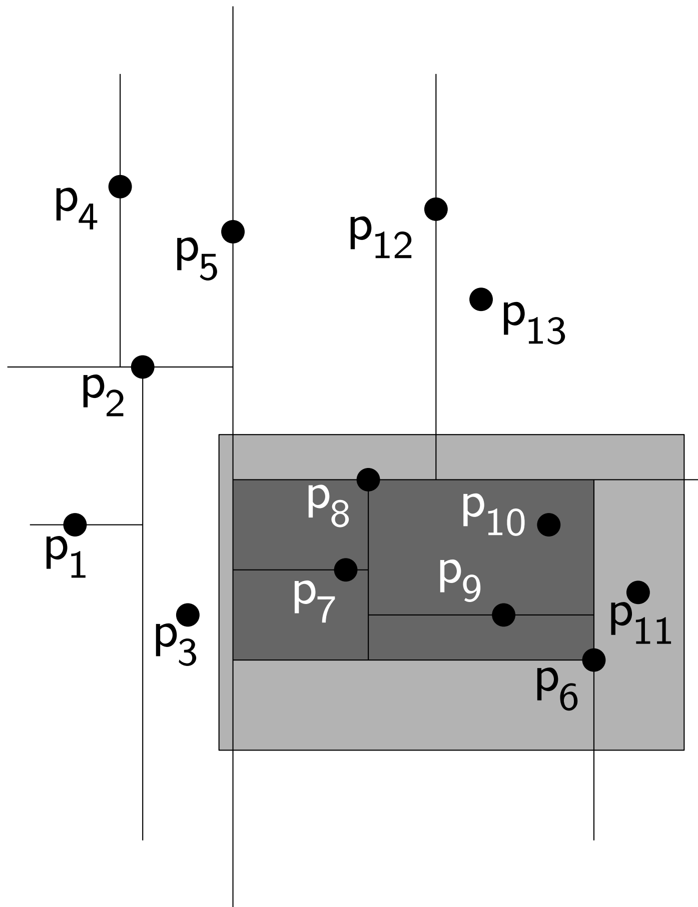

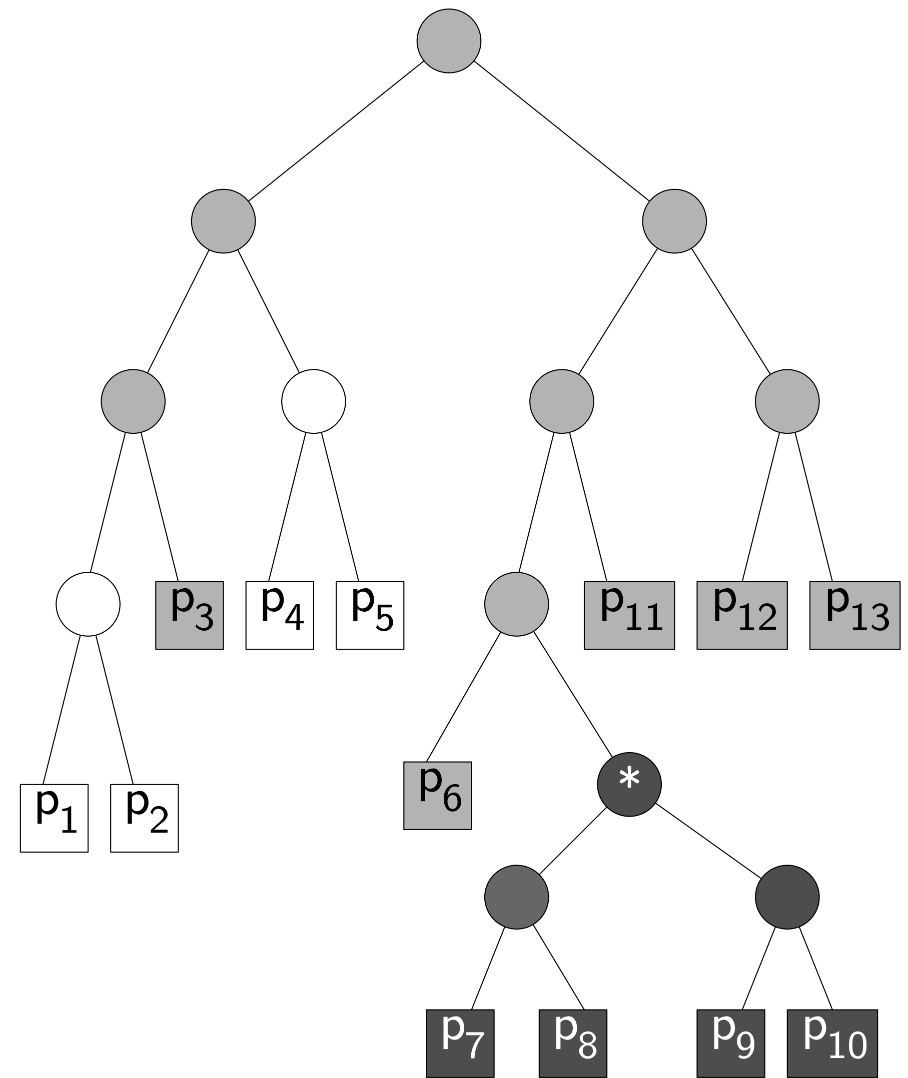

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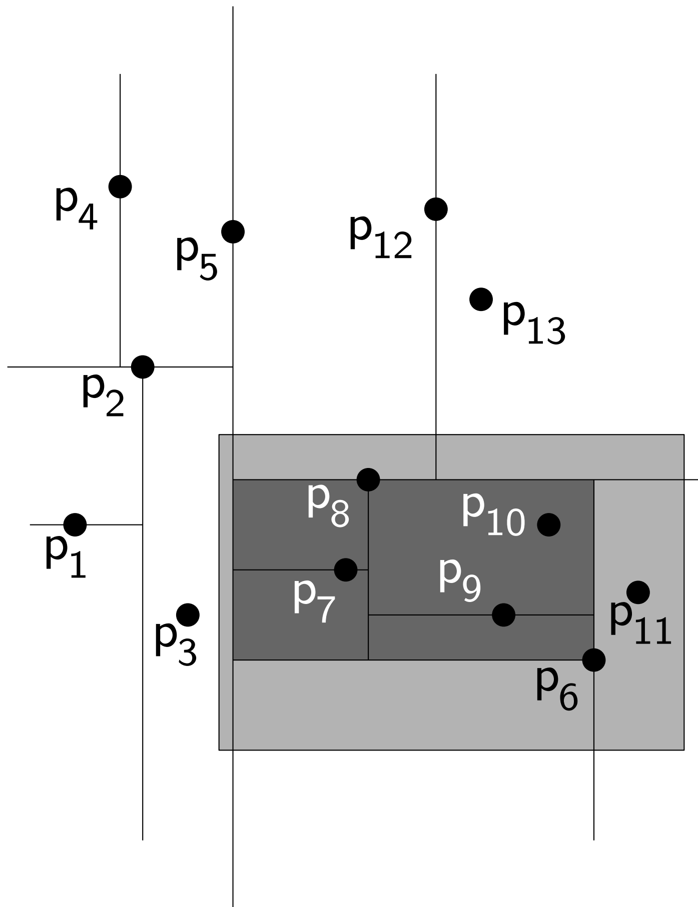

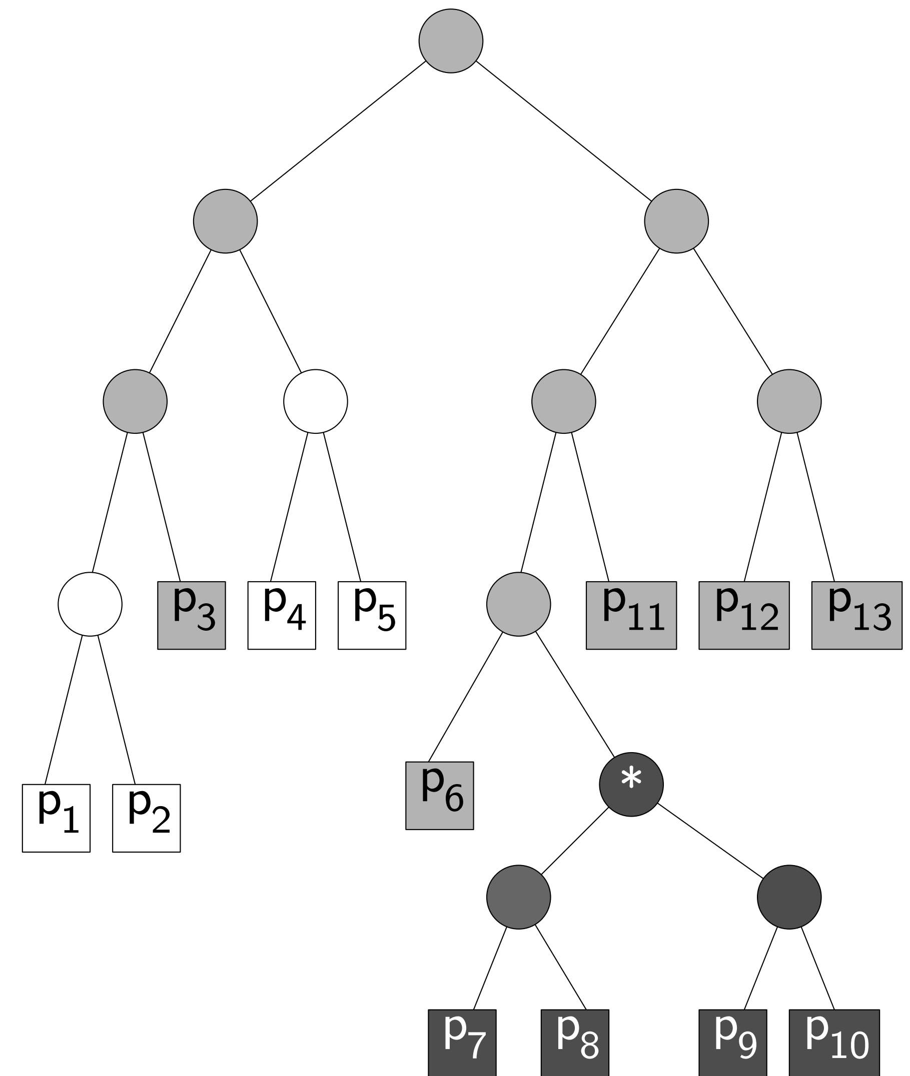

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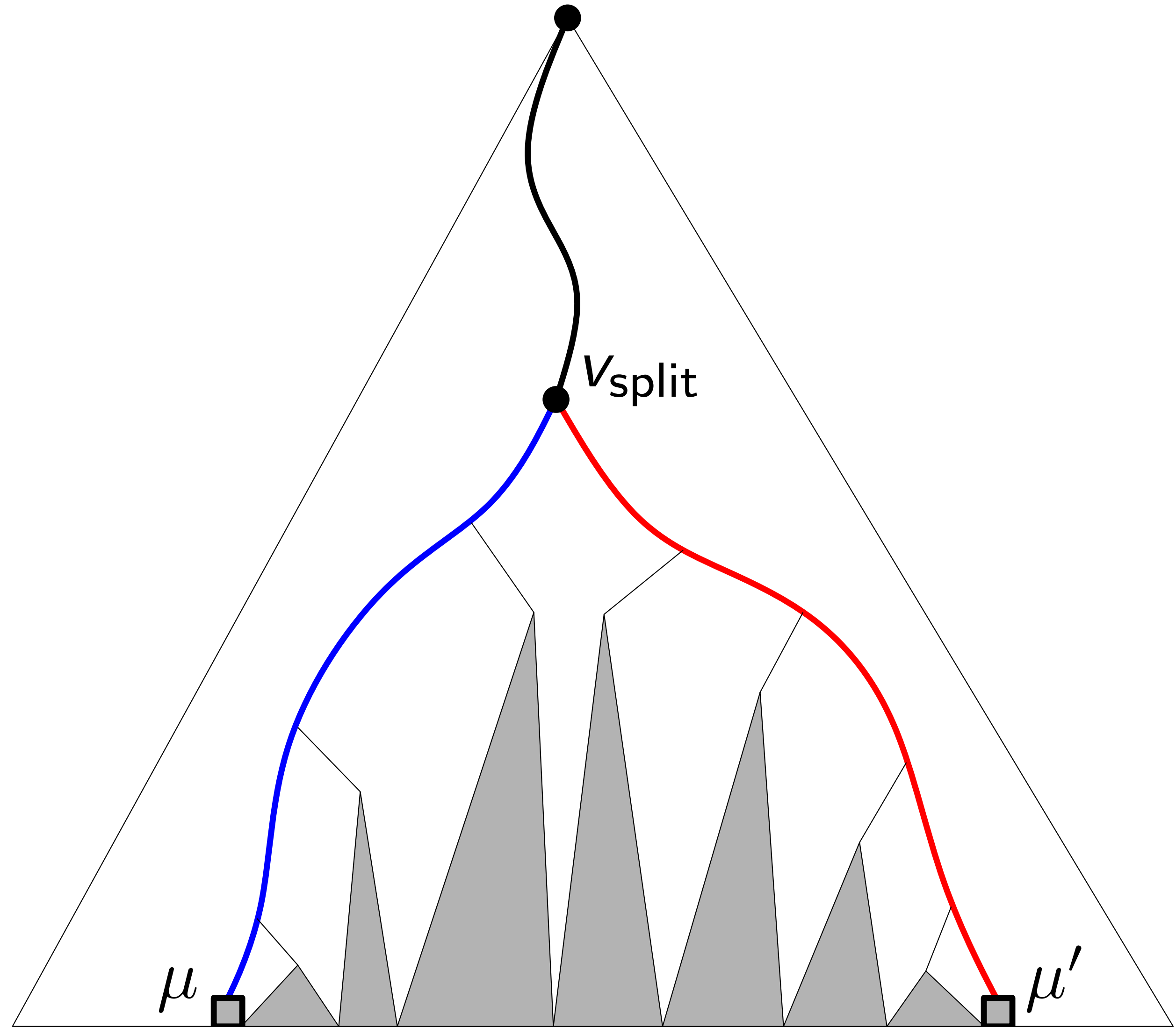

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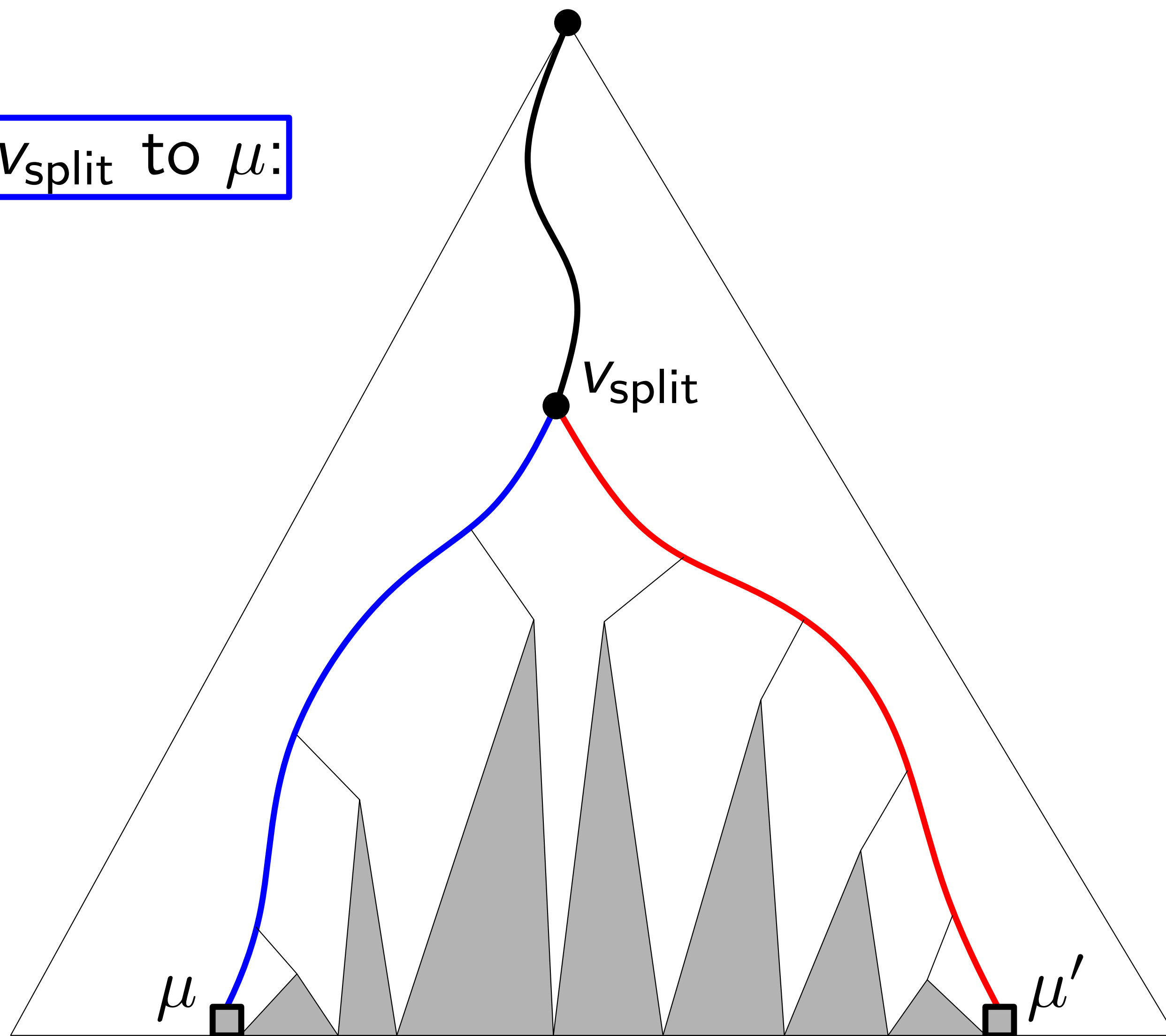

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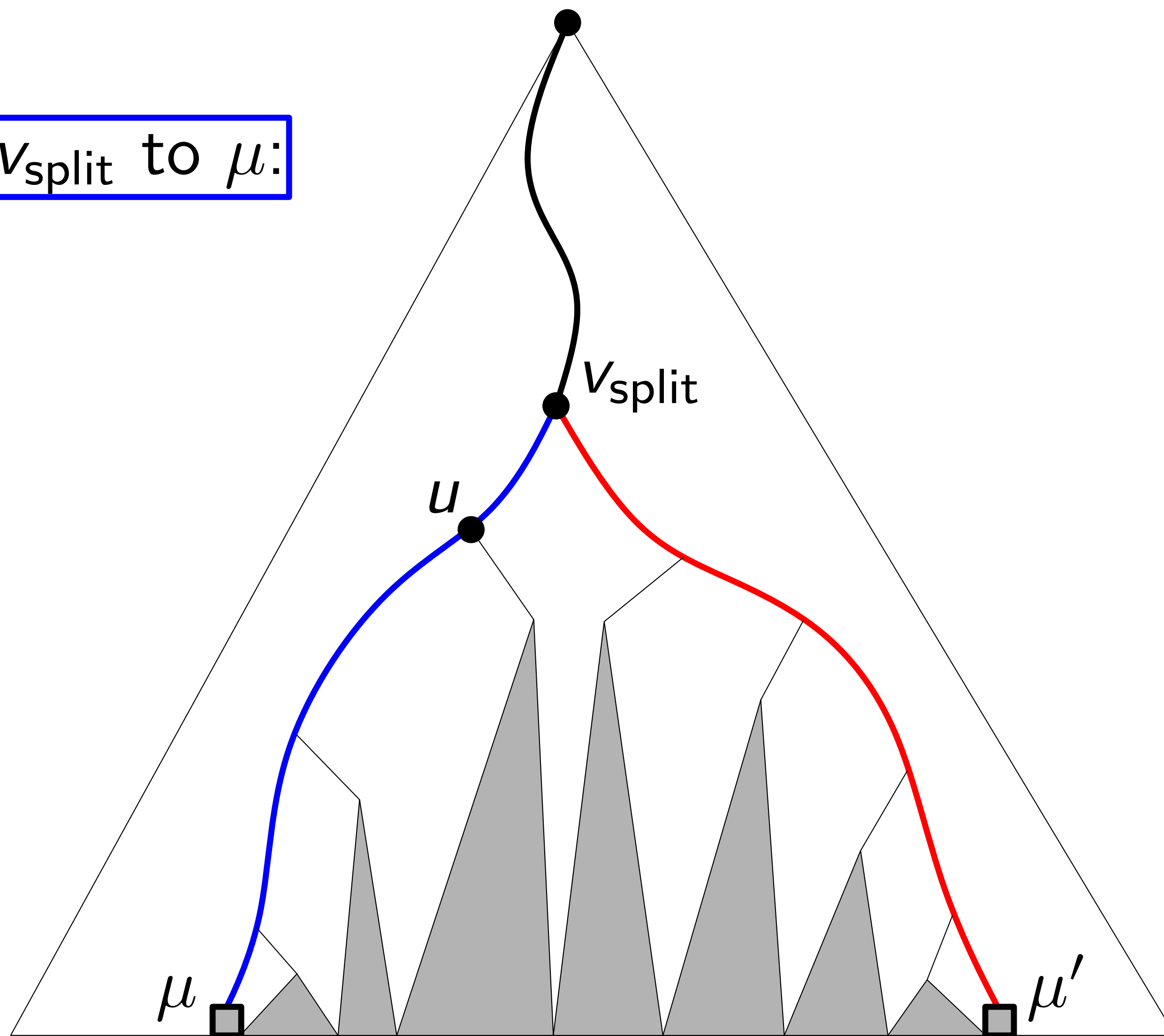

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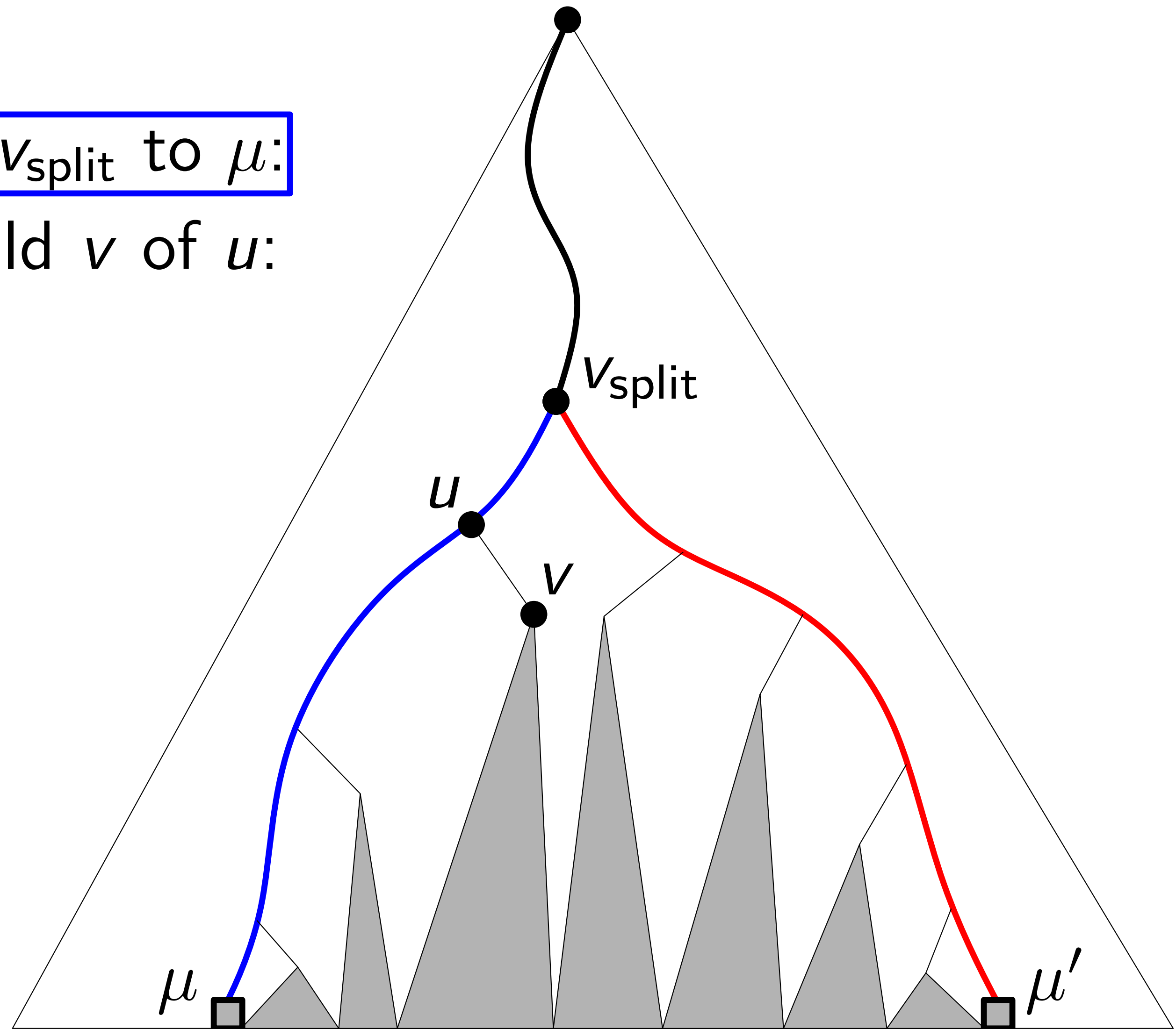

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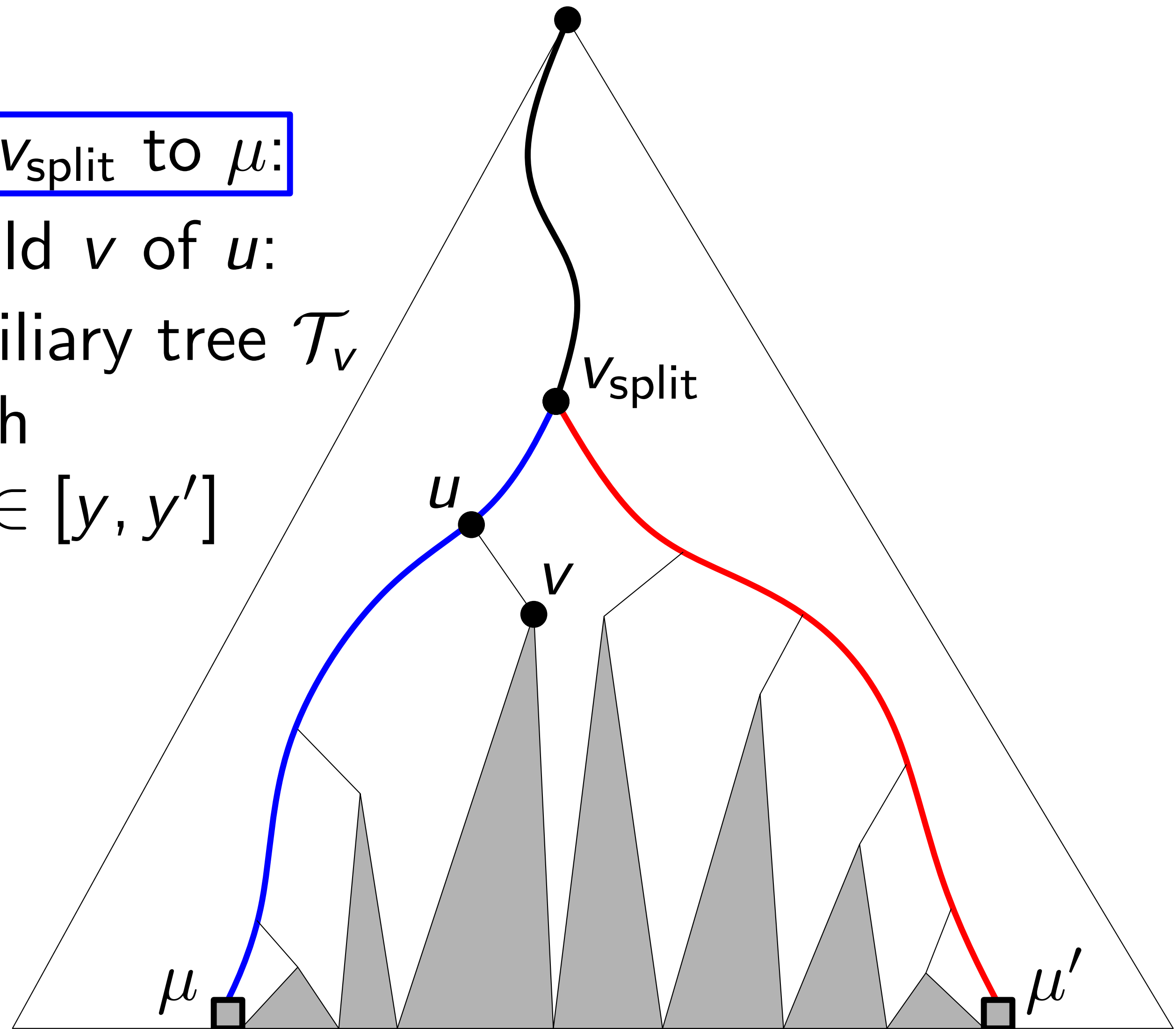

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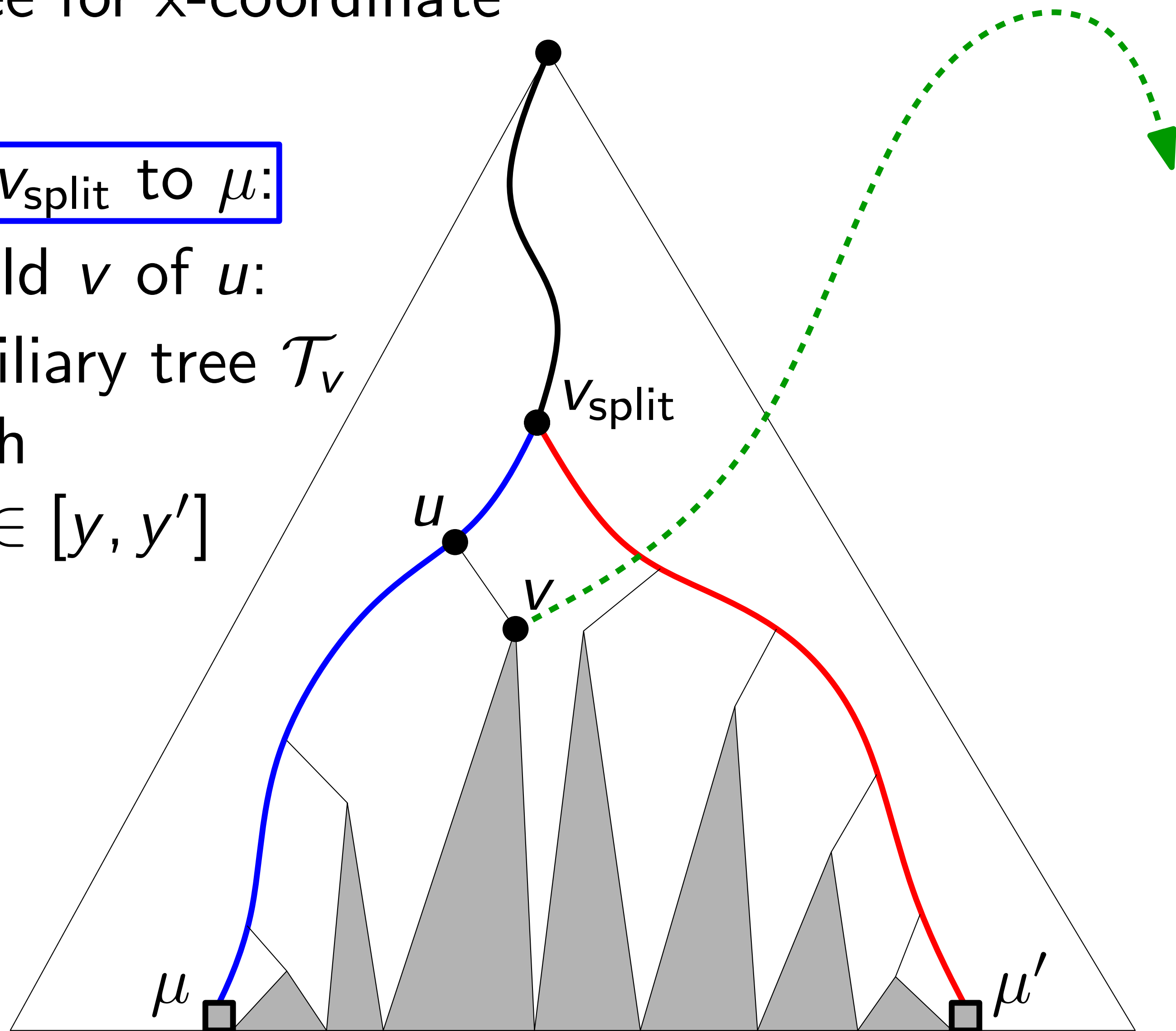

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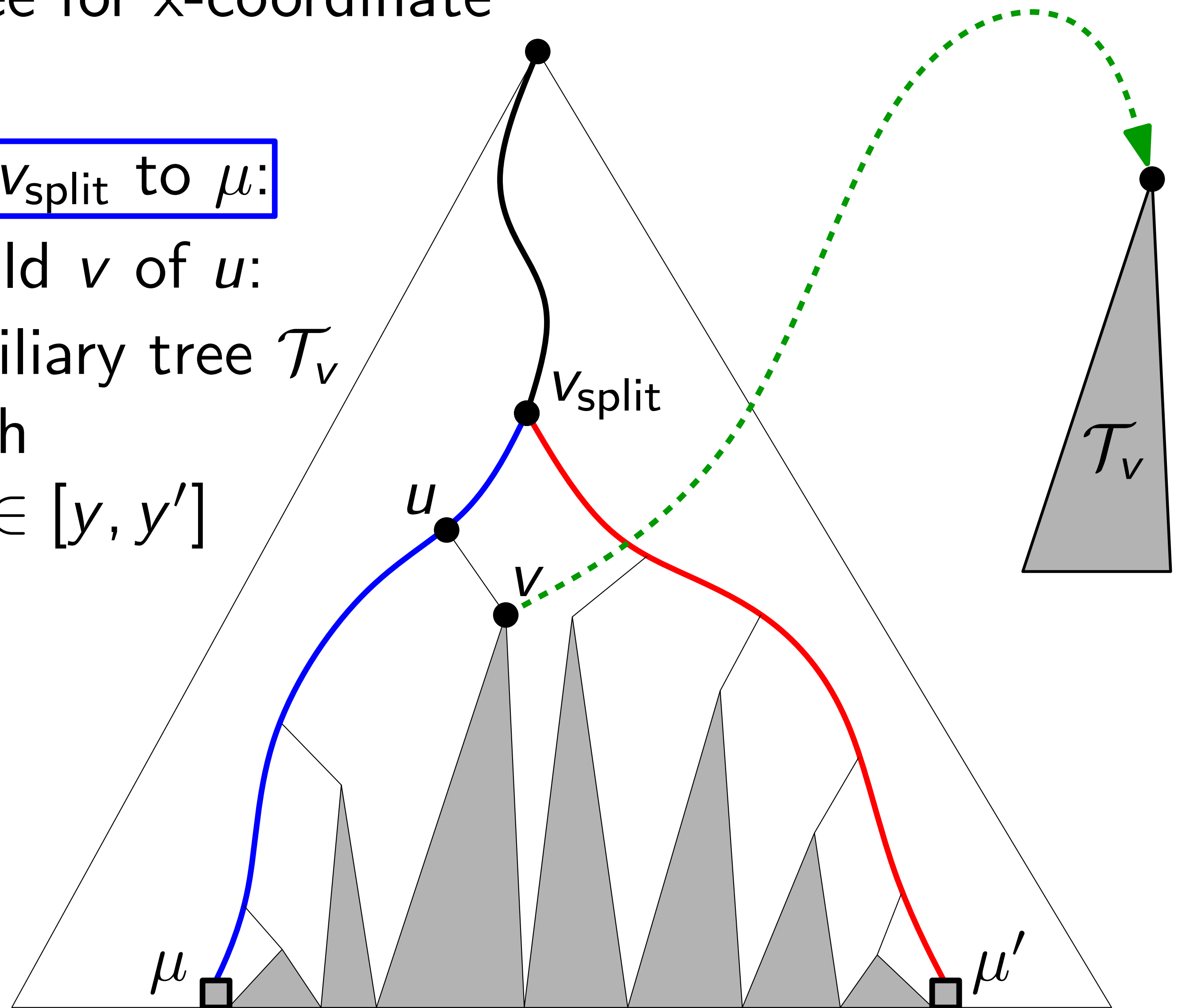

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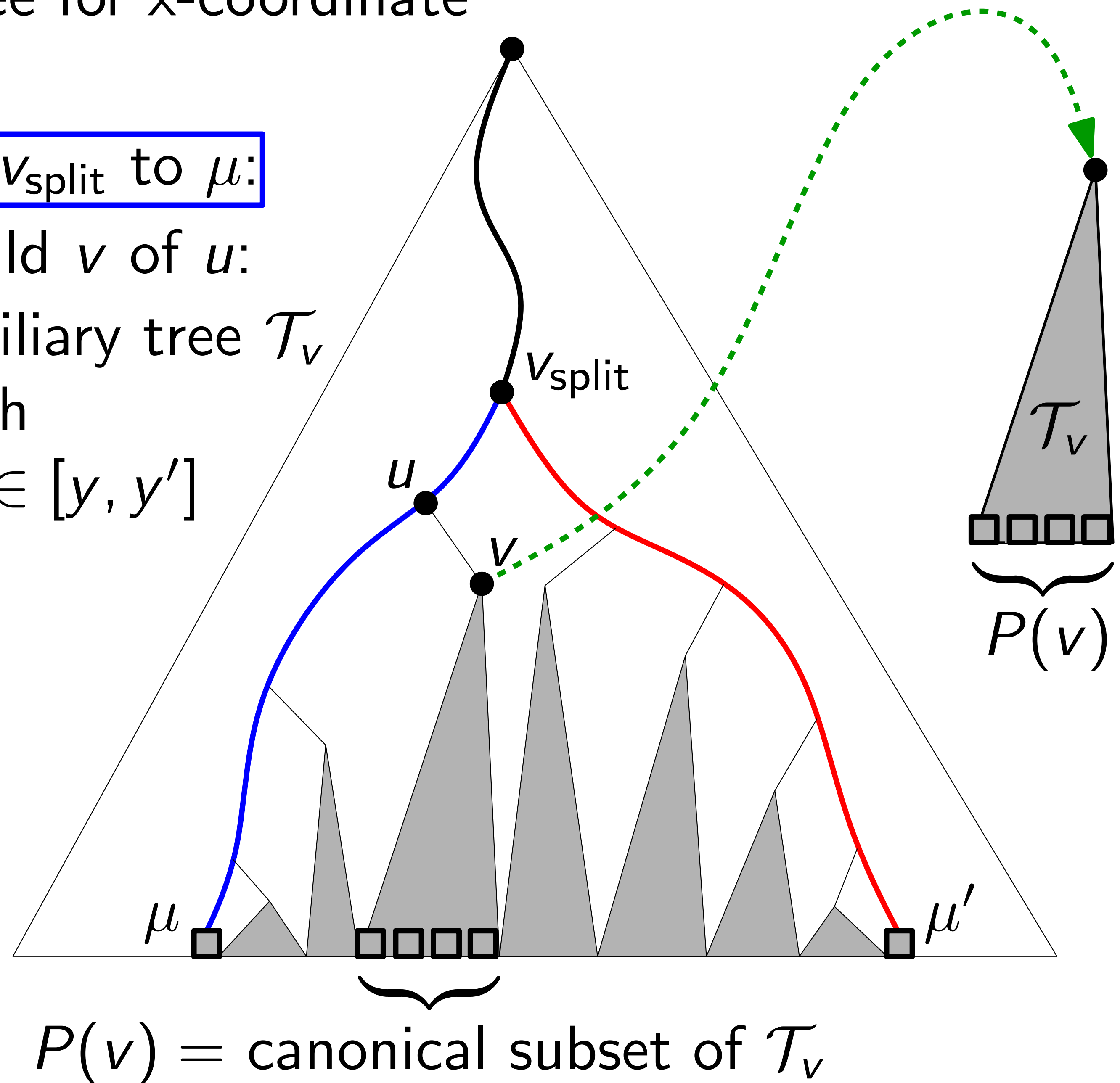

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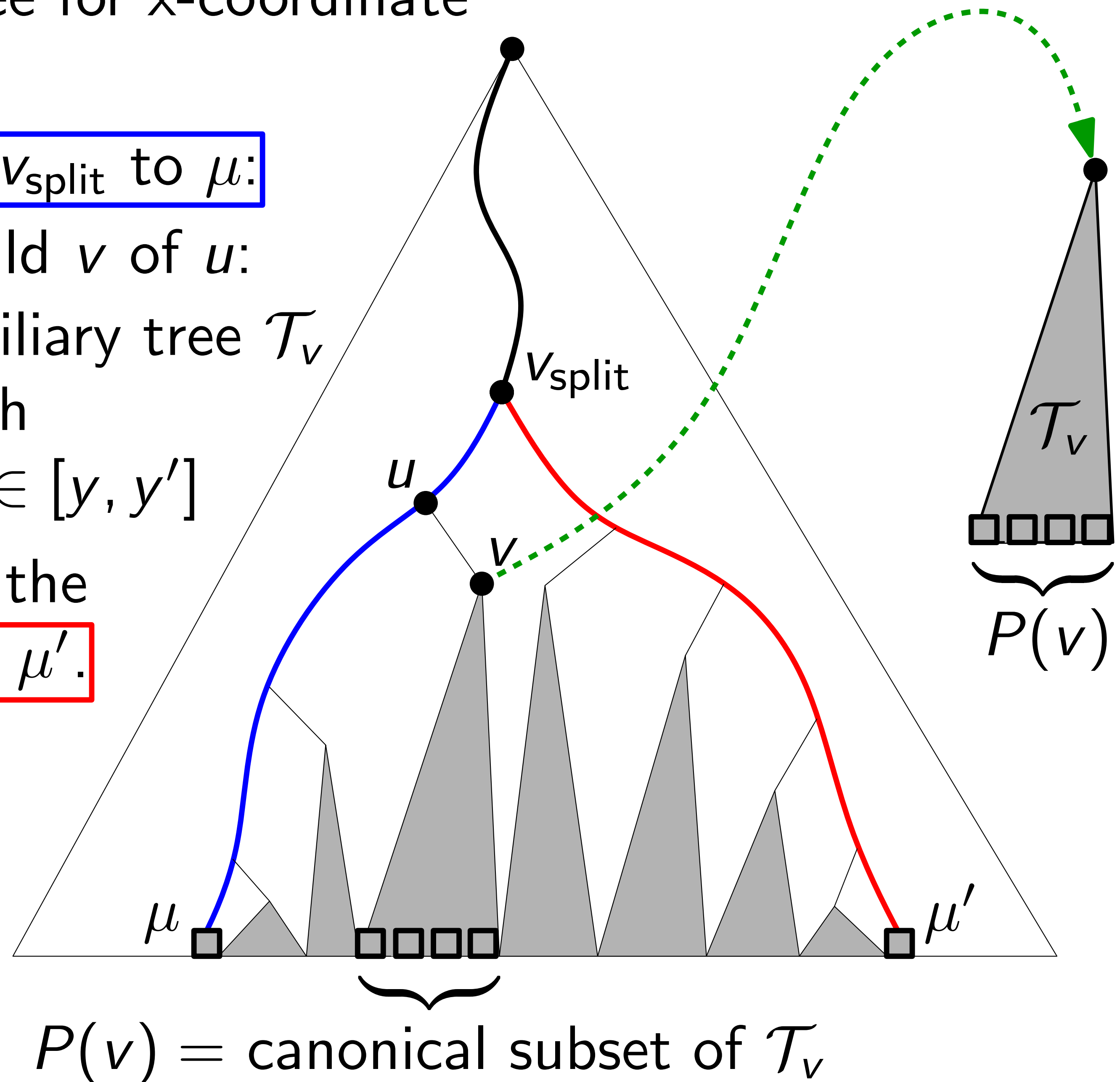

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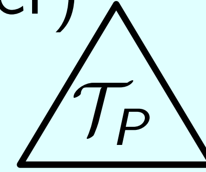

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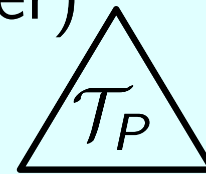

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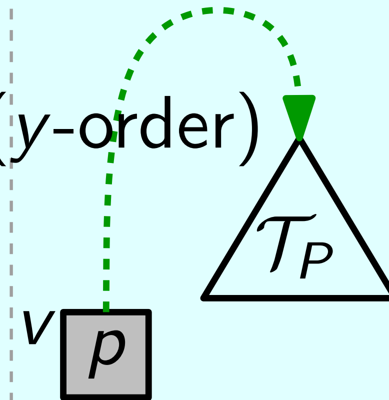

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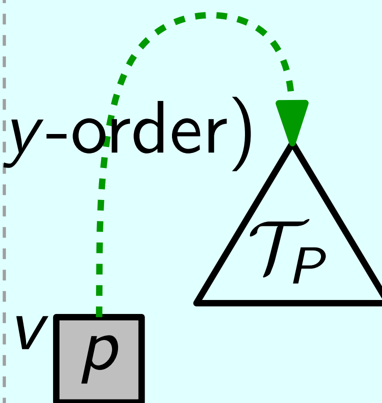

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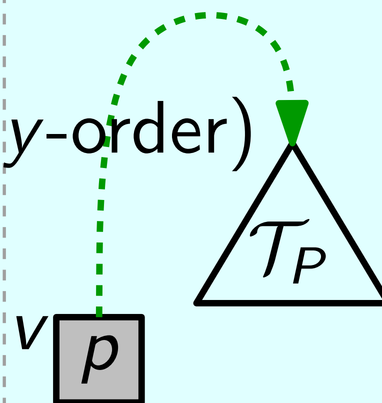

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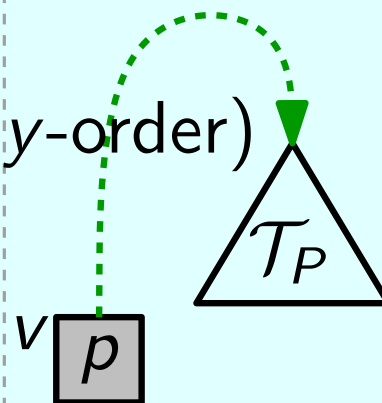

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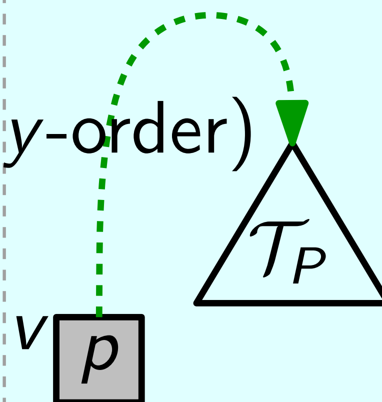

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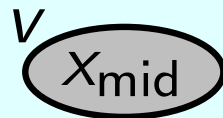

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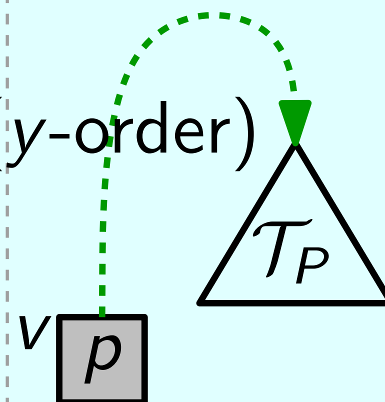

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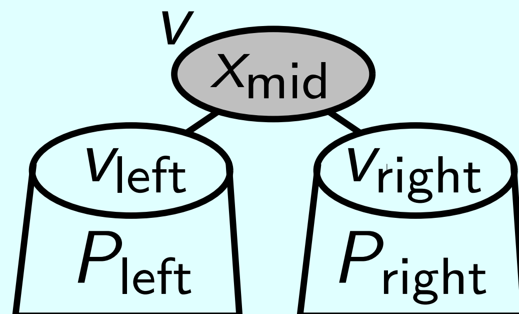

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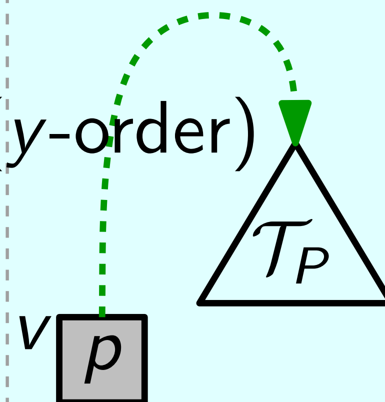

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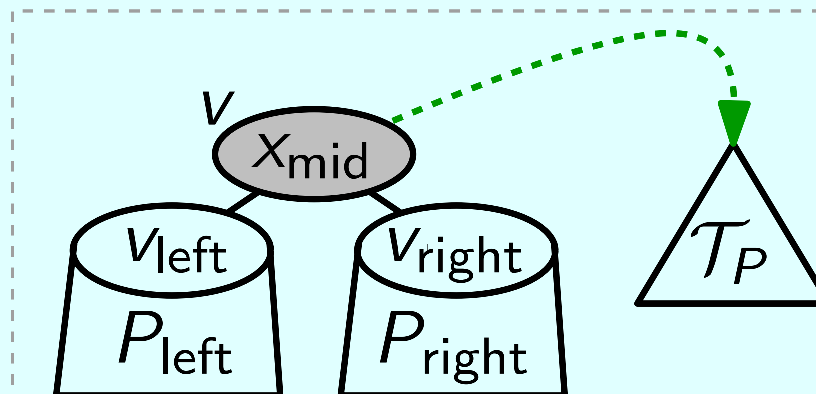

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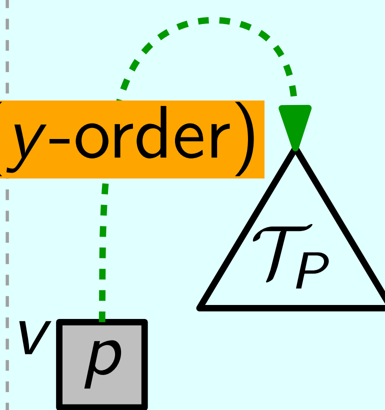

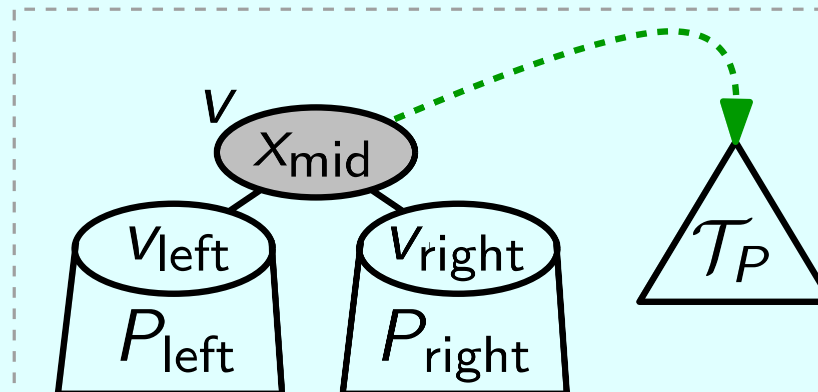

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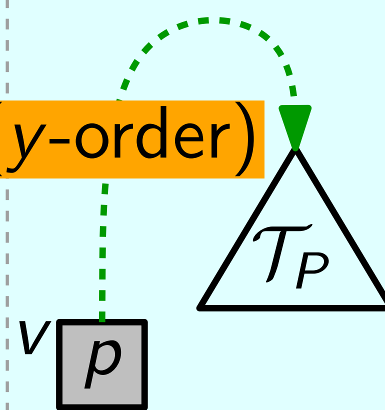

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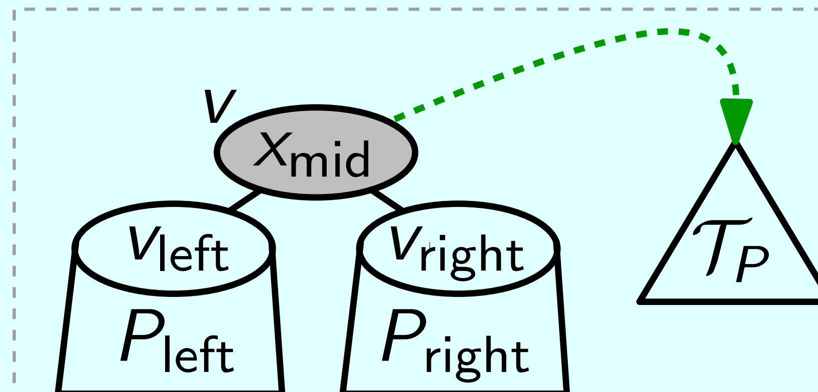

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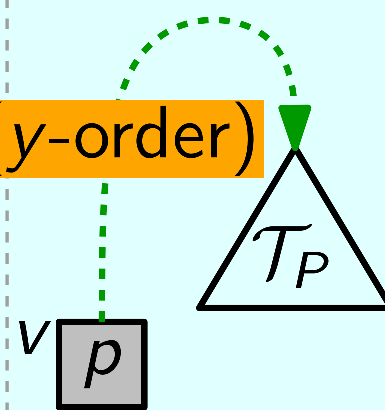

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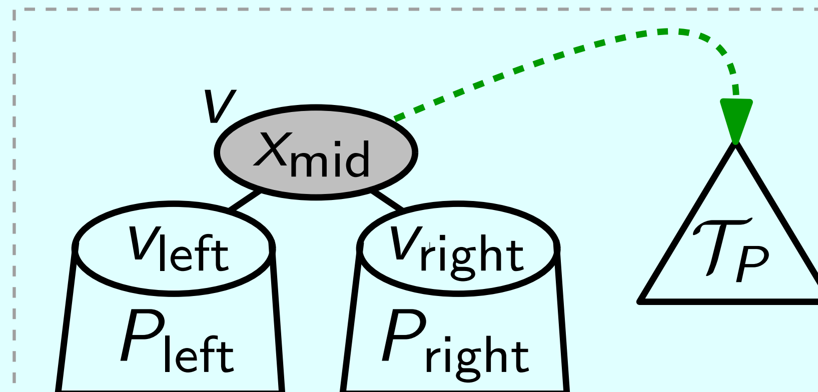

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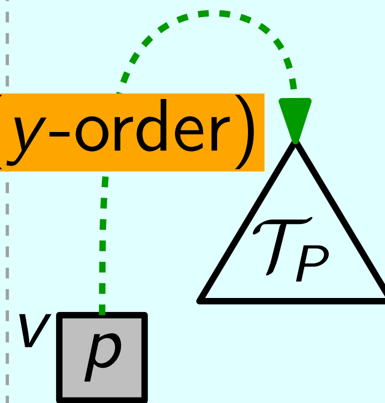

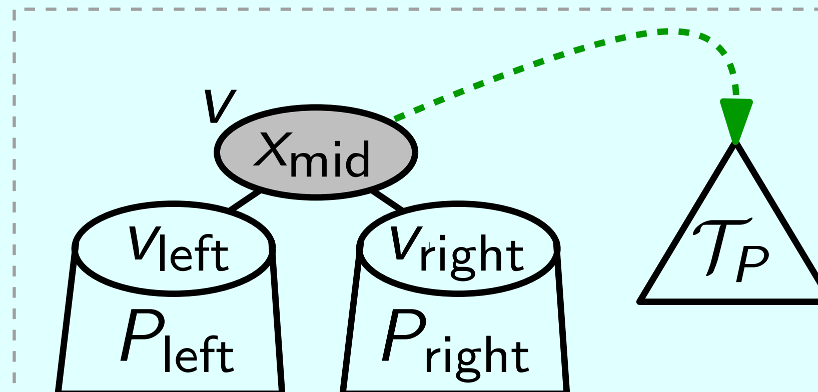

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$\Downarrow$
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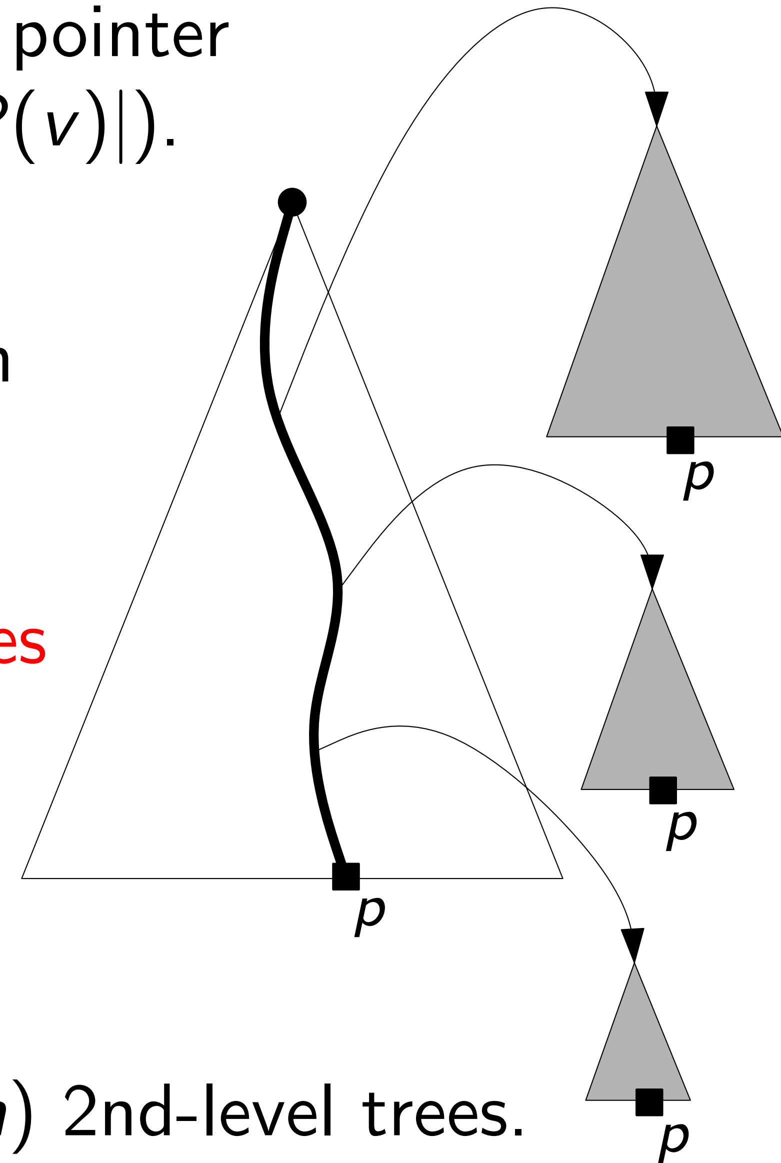

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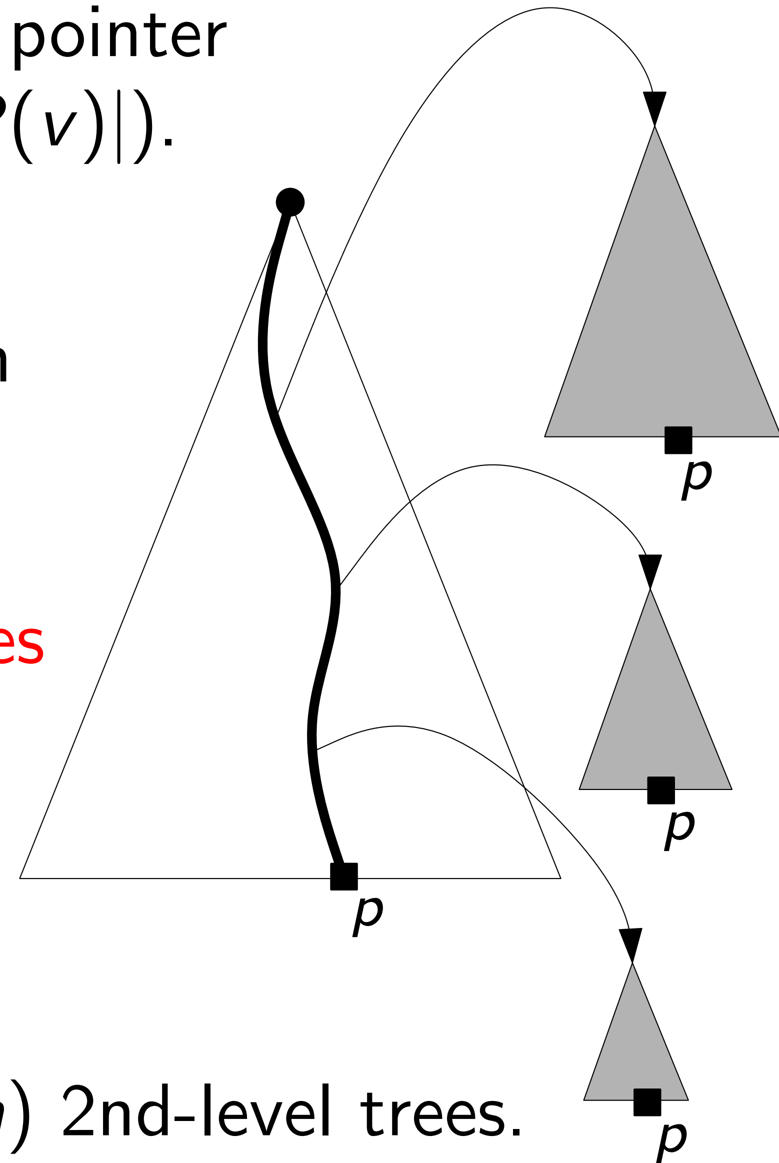

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$\Rightarrow \Theta(n \log n)$ space

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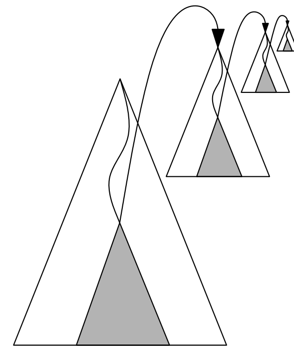

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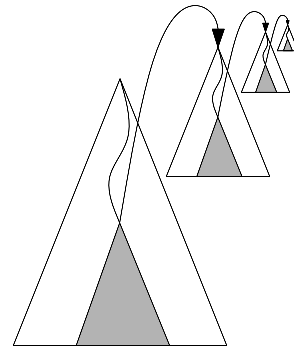

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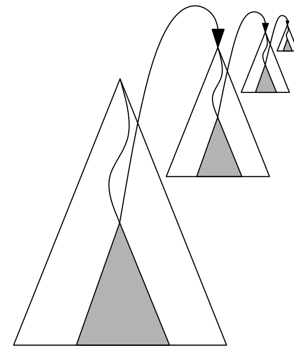

See Chapter 5.4 in the book
Computational Geometry: Algorithms and Applications
by de Berg et al., Springer-Verlag, 3rd edition 2008.

Comparison

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Note: *trade-off* between space and query time

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We can use kd-trees and range trees for *any* set of points; no matter how many points have the same x- or y-coordinates.

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Task 1: Given sets $B \subset A \subset \mathbb{N}$ stored in sorted order in arrays $A[1..n]$ and $B[1..m]$, support 1d range queries in the multiset $A \cup B$ in $k + 1 \cdot \log n$ time!

A

3	10	19	23	30	37	59	62	70	80	100	105

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[dBCvKO'08]

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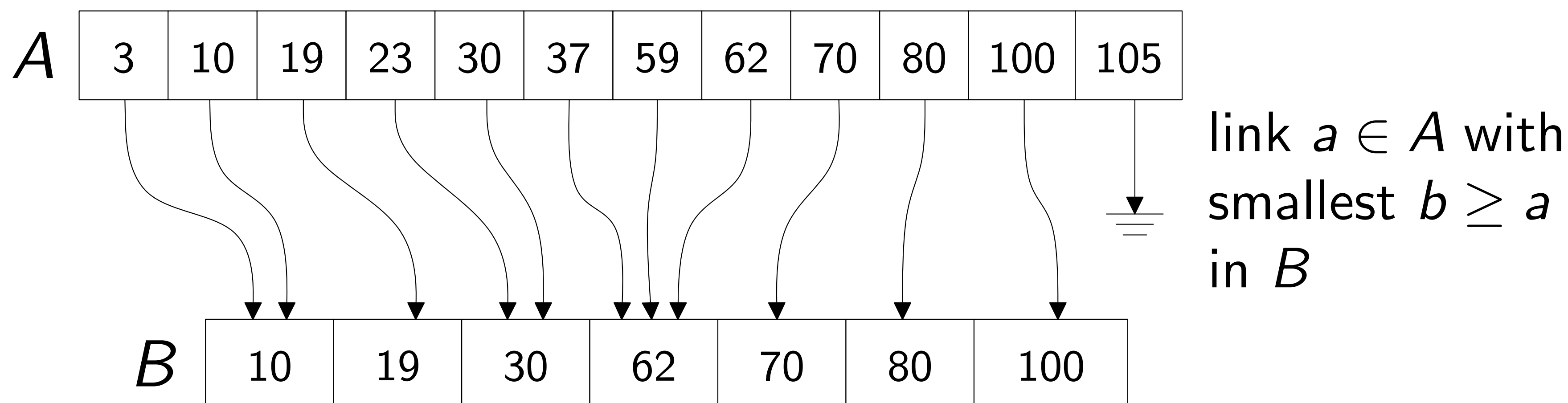

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query with range $[20, 65]$

link $a \in A$ with smallest $b \geq a$ in B

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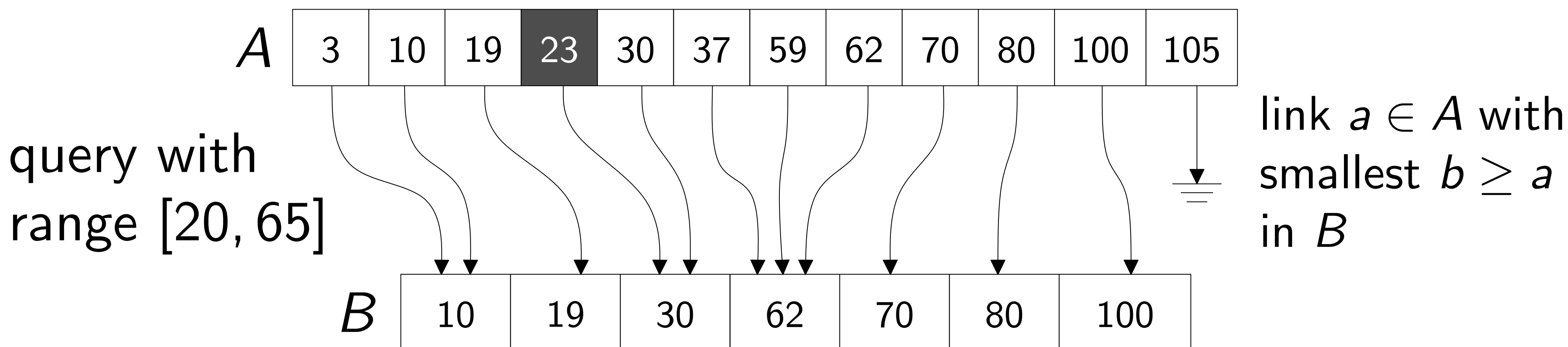

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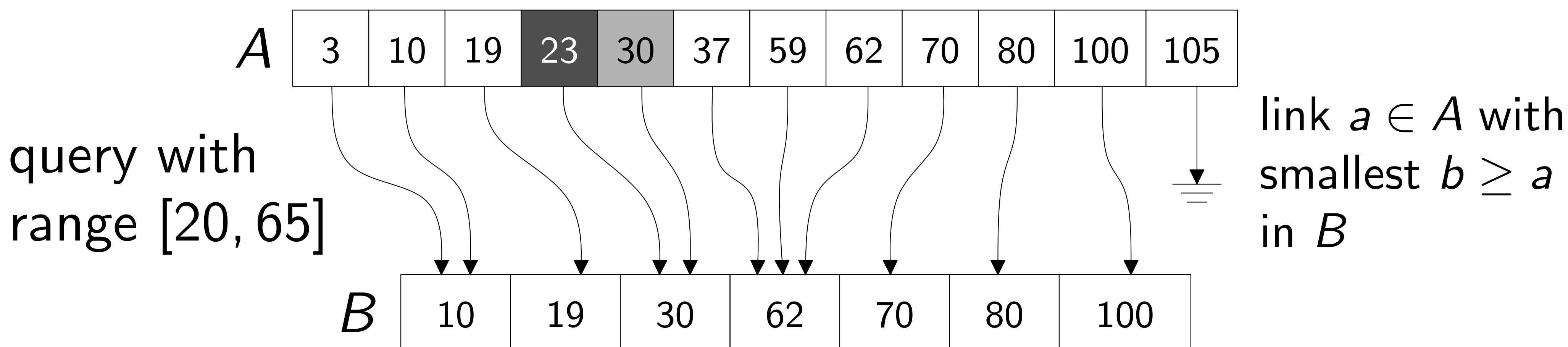

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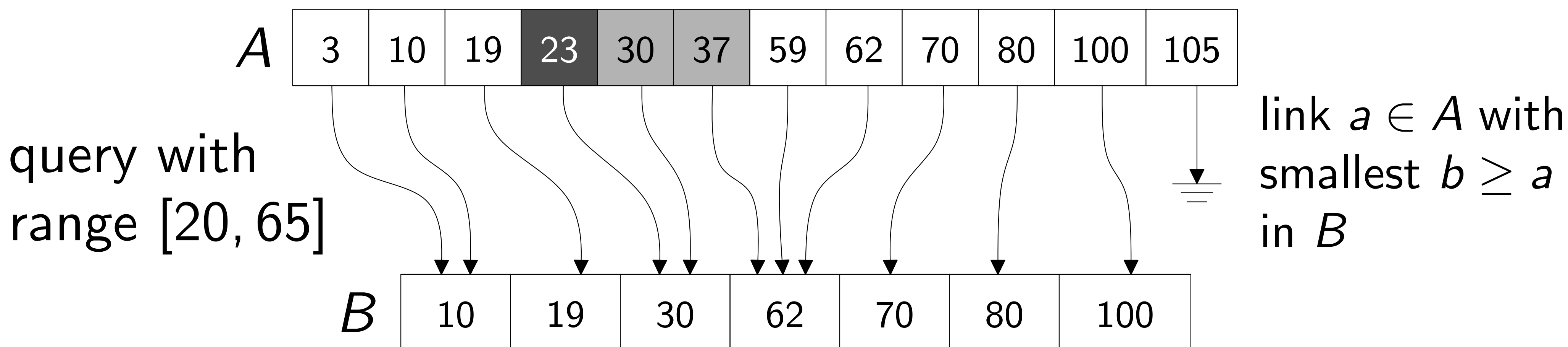

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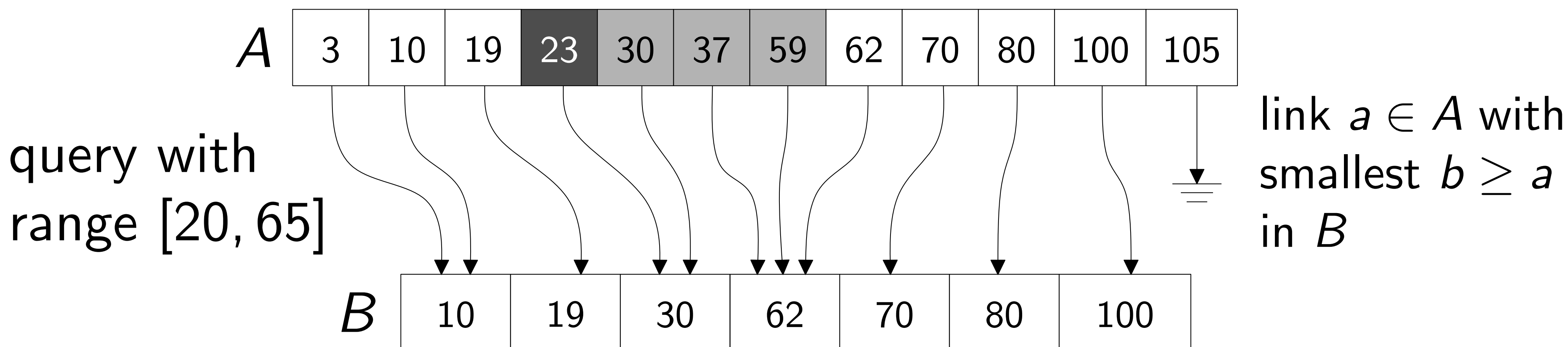

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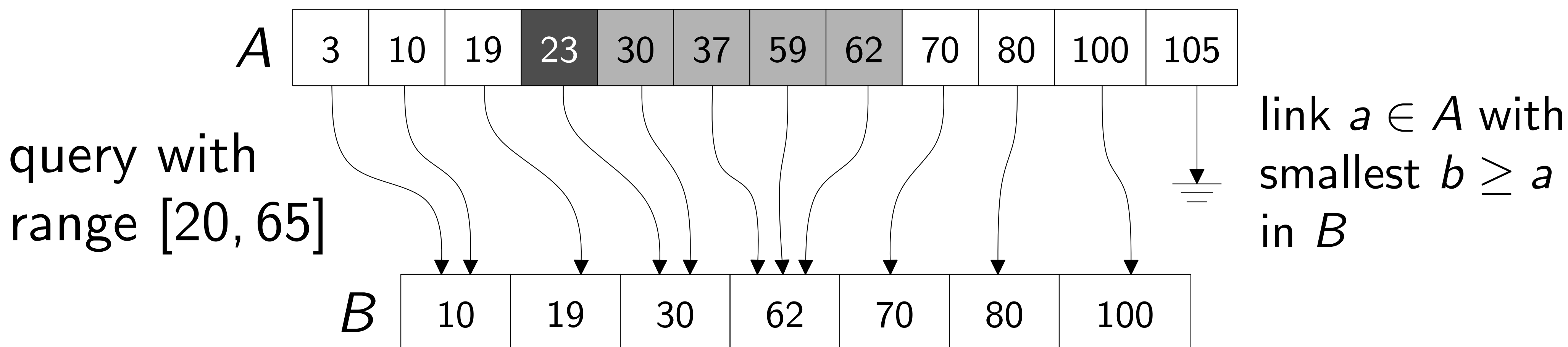

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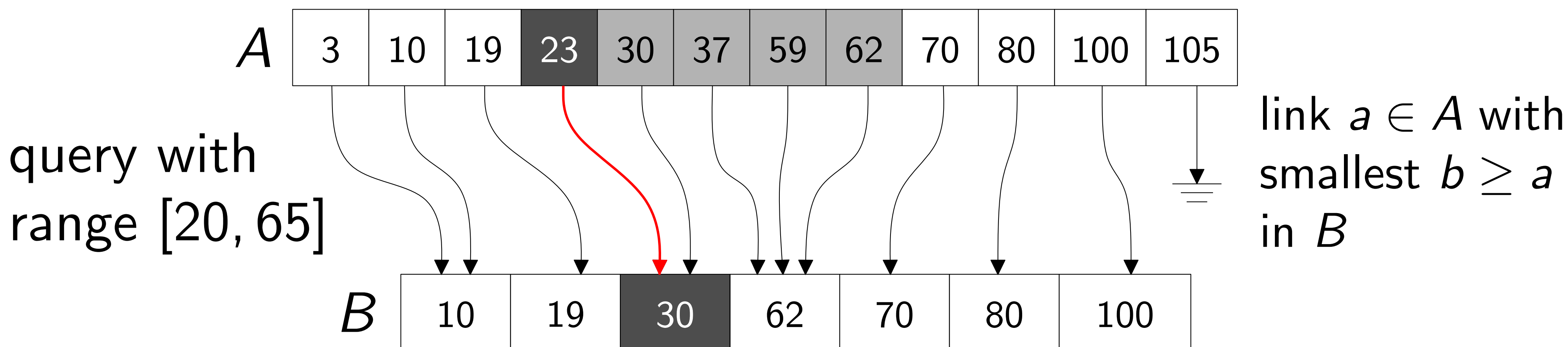

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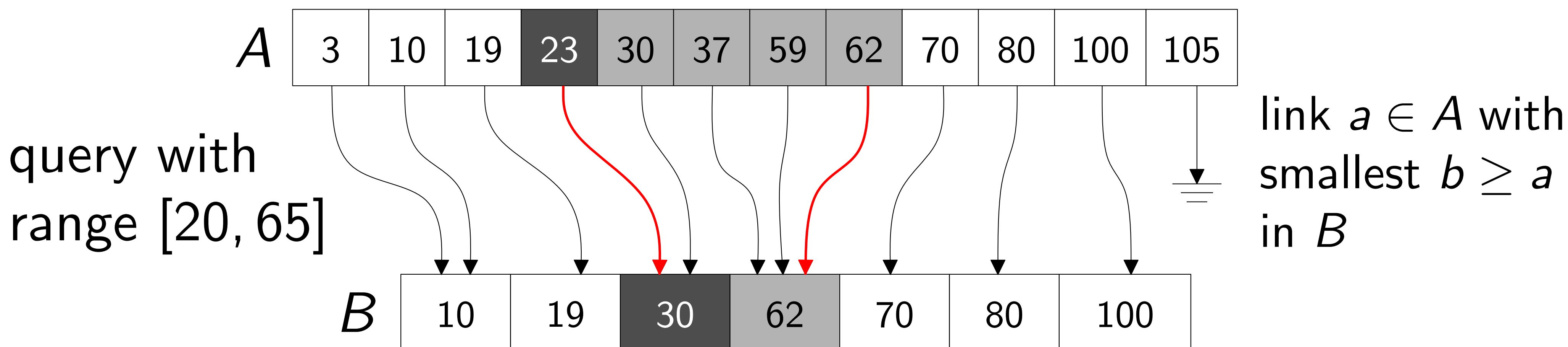
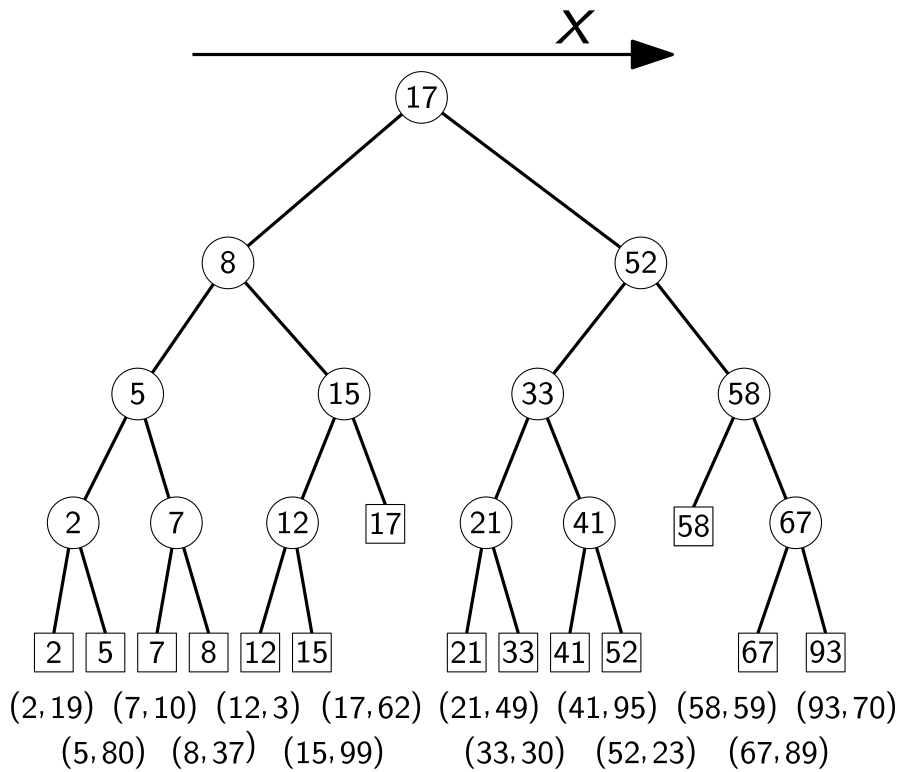


[dBCvKO'08]

Layered Range Trees



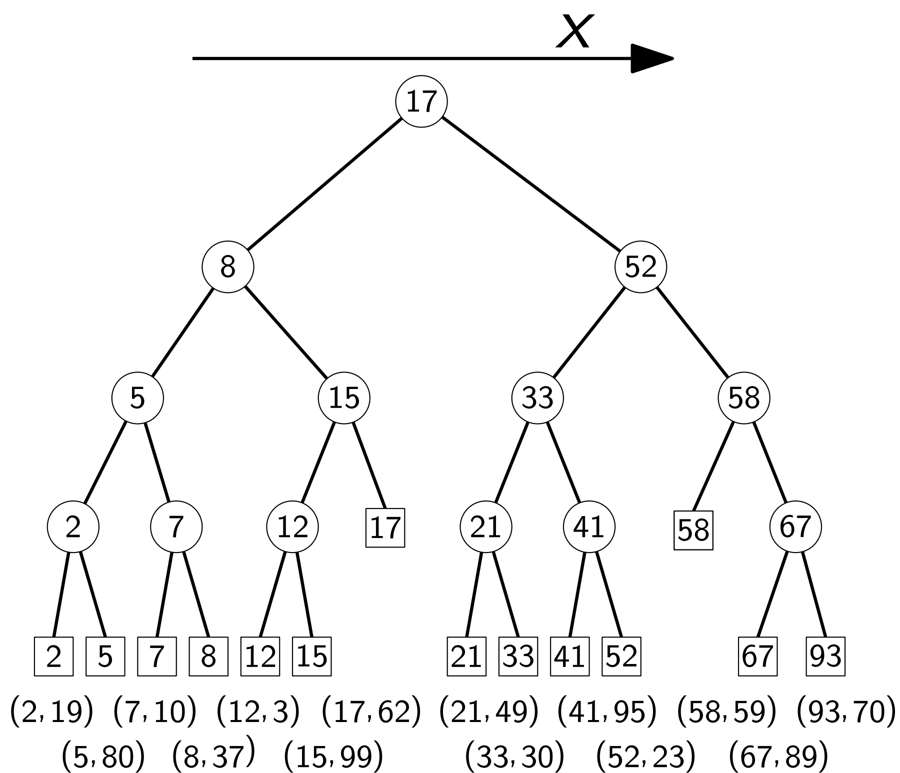
Layered Range Trees

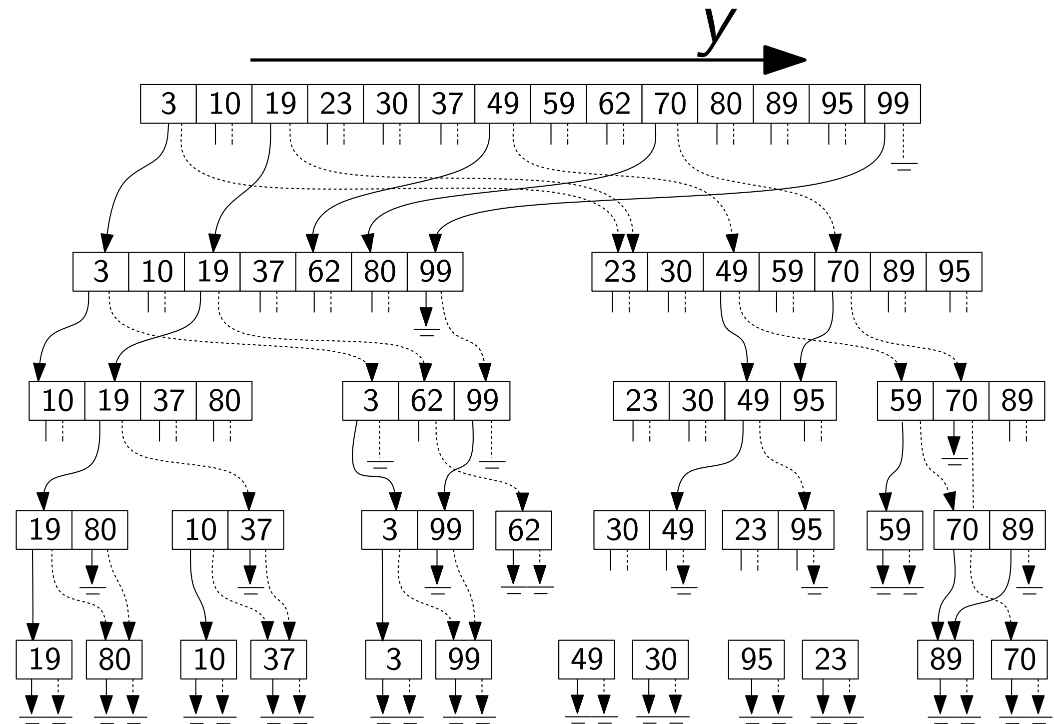
x

y

(2, 19) (7, 10) (12, 3) (17, 62) (21, 49) (41, 95) (58, 59) (93, 70)
(5, 80) (8, 37) (15, 99) (33, 30) (52, 23) (67, 89)

Layered Range Trees





[dBCvKO'08]

Theorem: Let $d \geq 2$ and let P be a set of n pts in $\mathbb{R}^d$. Given $O(n \log^{d-1} n)$ preprocessing time & storage, d-dim range queries on P can be answered in $O(k + \log^{d-1} n)$ time.

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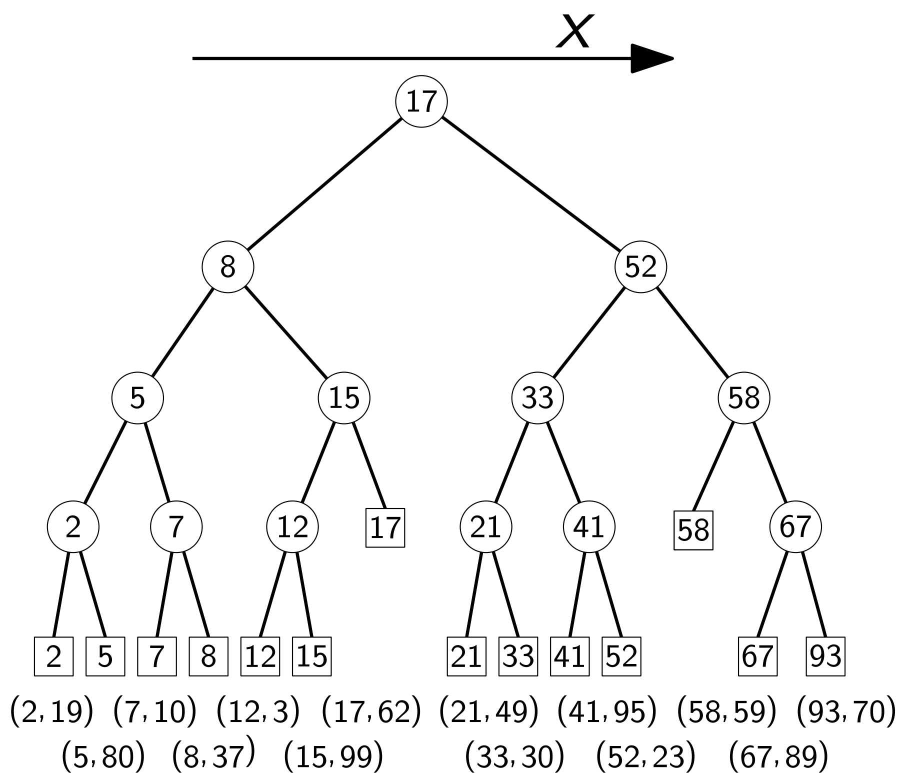

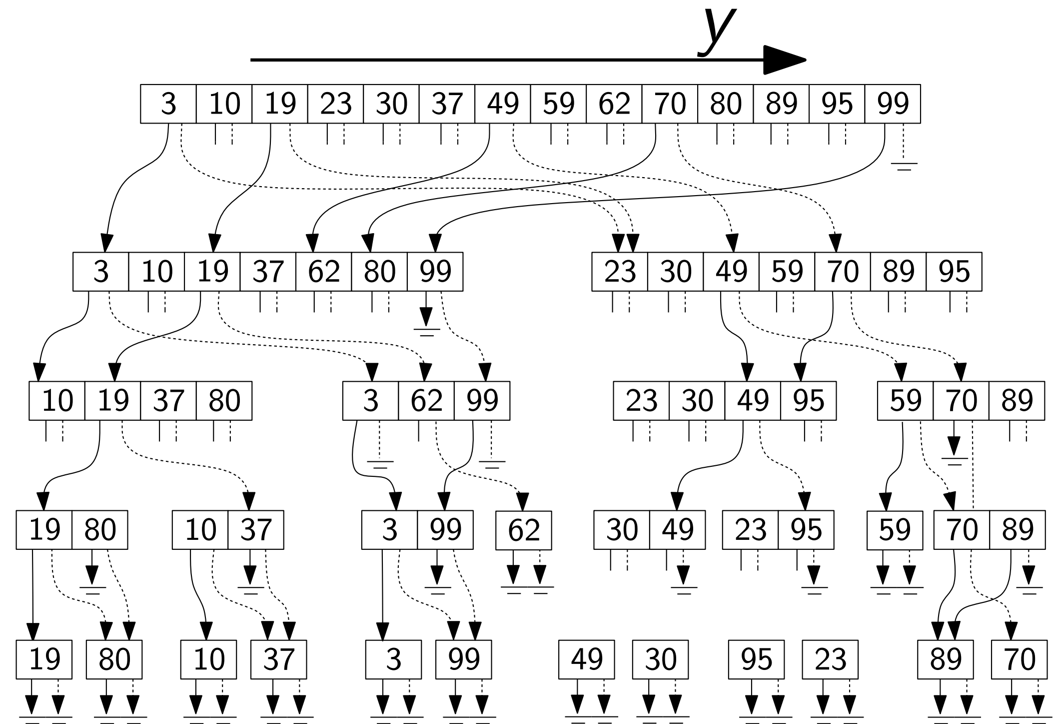

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