Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Computational Geometry

Winter term 2016/17

Linear Programming
or
Profit Maximization

Lecture #4

Joachim Spoerhase

Chair for Computer Science I

Maximizing Profit

You are the boss of a small company that produces two products, P_1 and P_2. If you produce x_1 units of P_1 and x_2 units of P_2, your profit in € is

$$G(x_1, x_2) = 300x_1 + 500x_2$$

Your production runs on three machines M_A, M_B, and M_C with the following capacities:

$$\begin{aligned} M_A: &\quad 4x_1 + 11x_2 \le 880 \\ M_B: &\quad x_1 + x_2 \le 150 \\ M_C: &\quad x_2 \le 60 \end{aligned}$$

Which choice of (x_1, x_2) maximizes your profit?

The Answer

linear constraints:

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M_A: &\quad 4x_1 + 11x_2 \le 880\\
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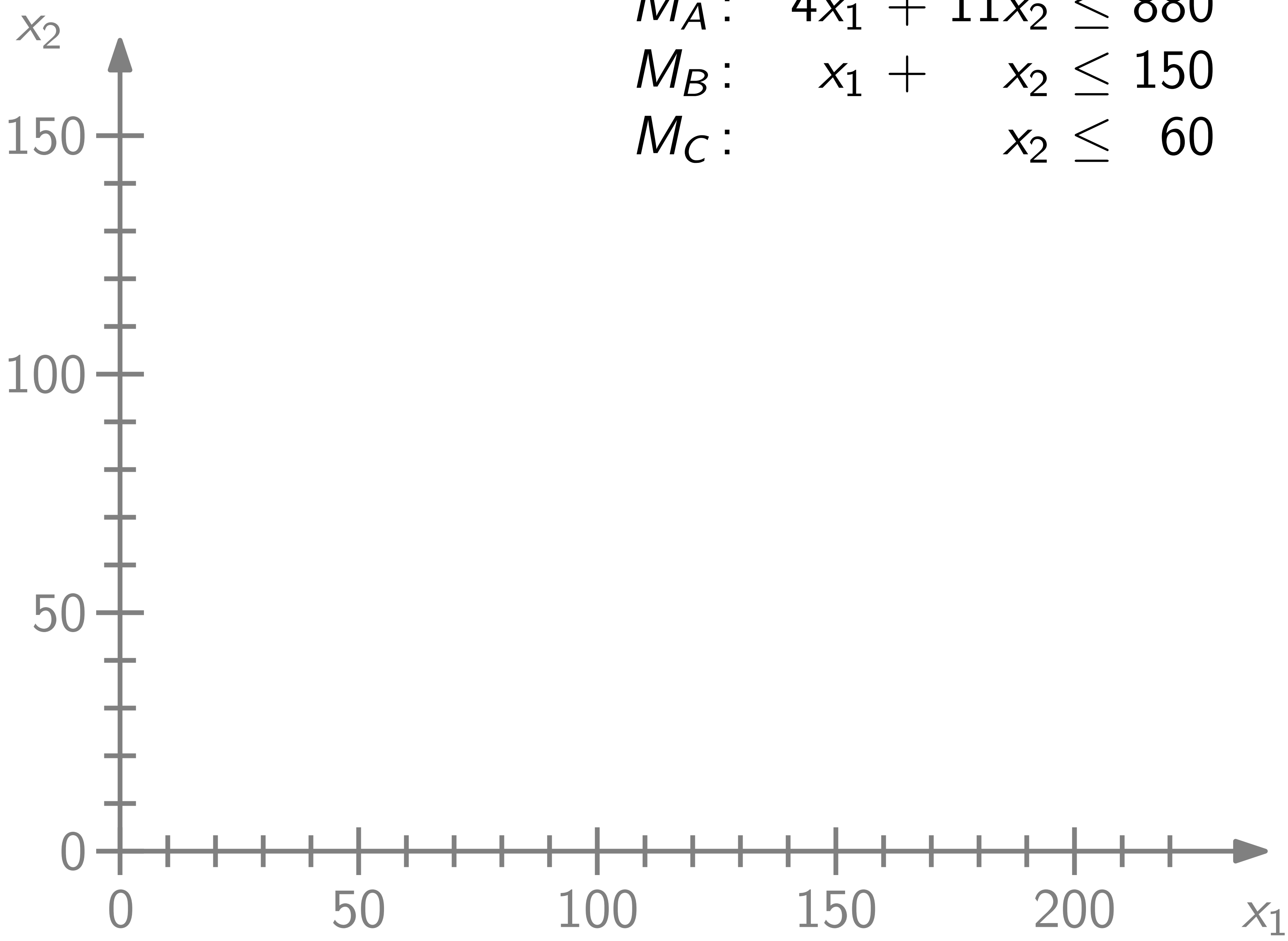

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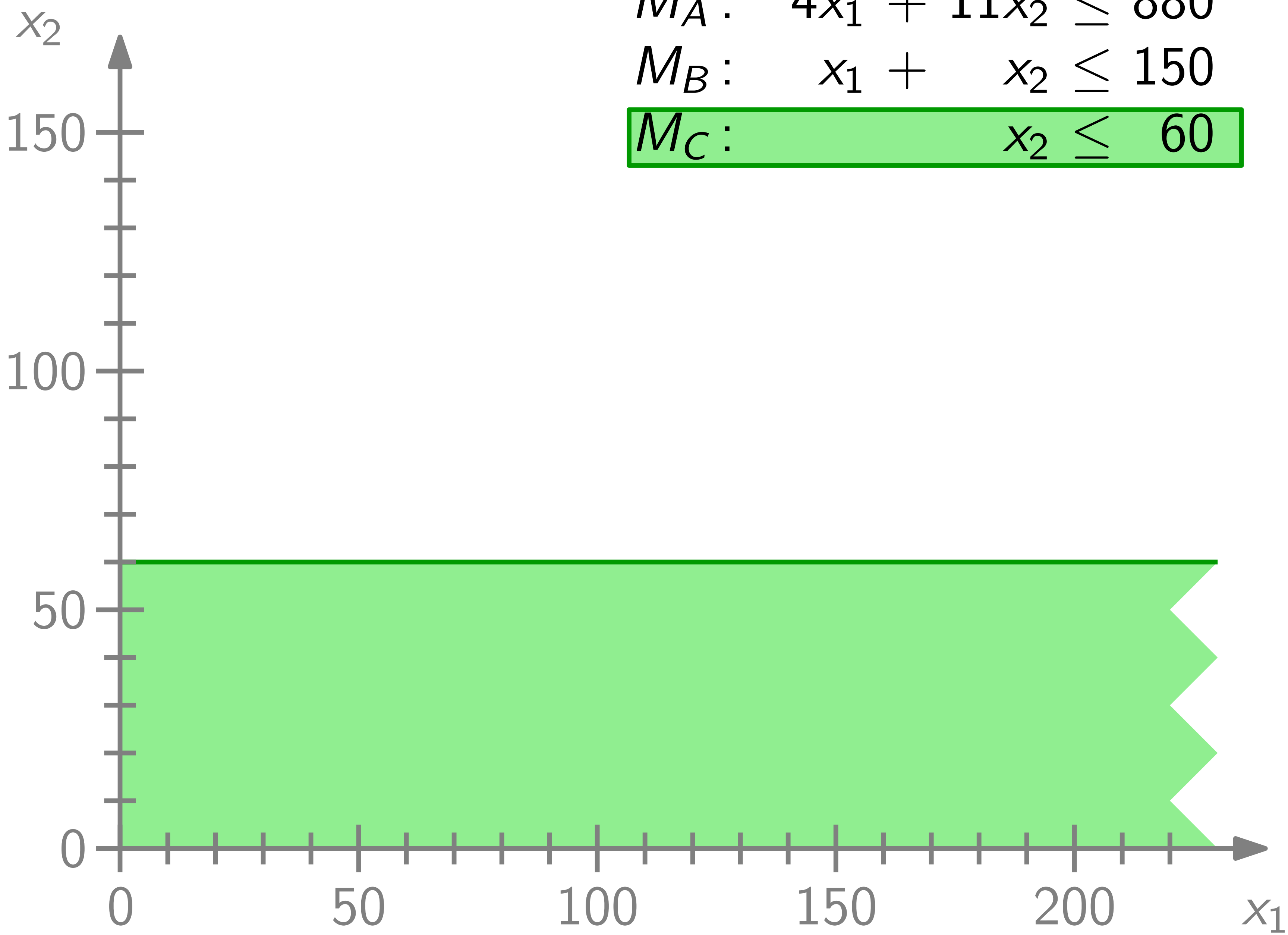

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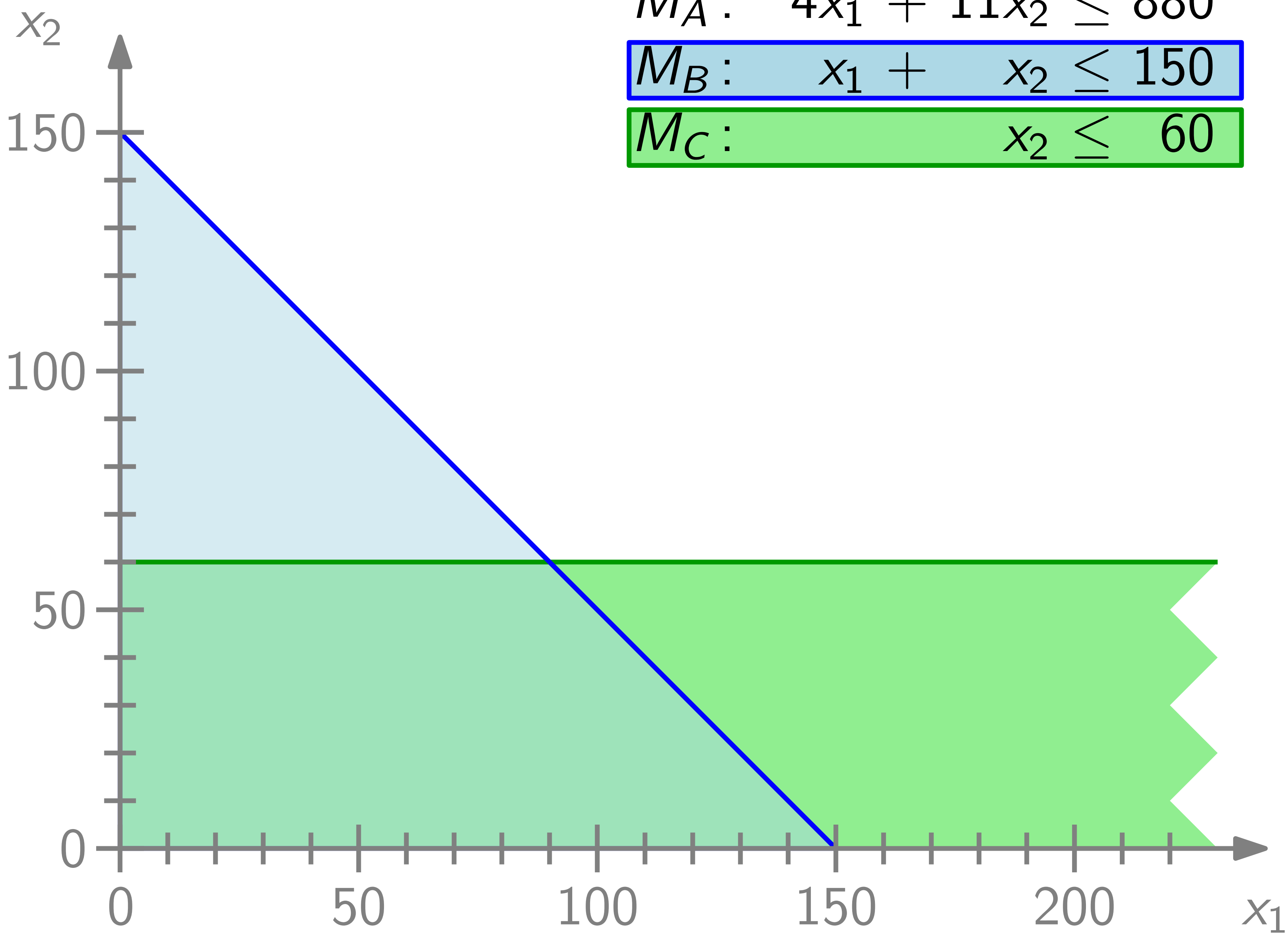

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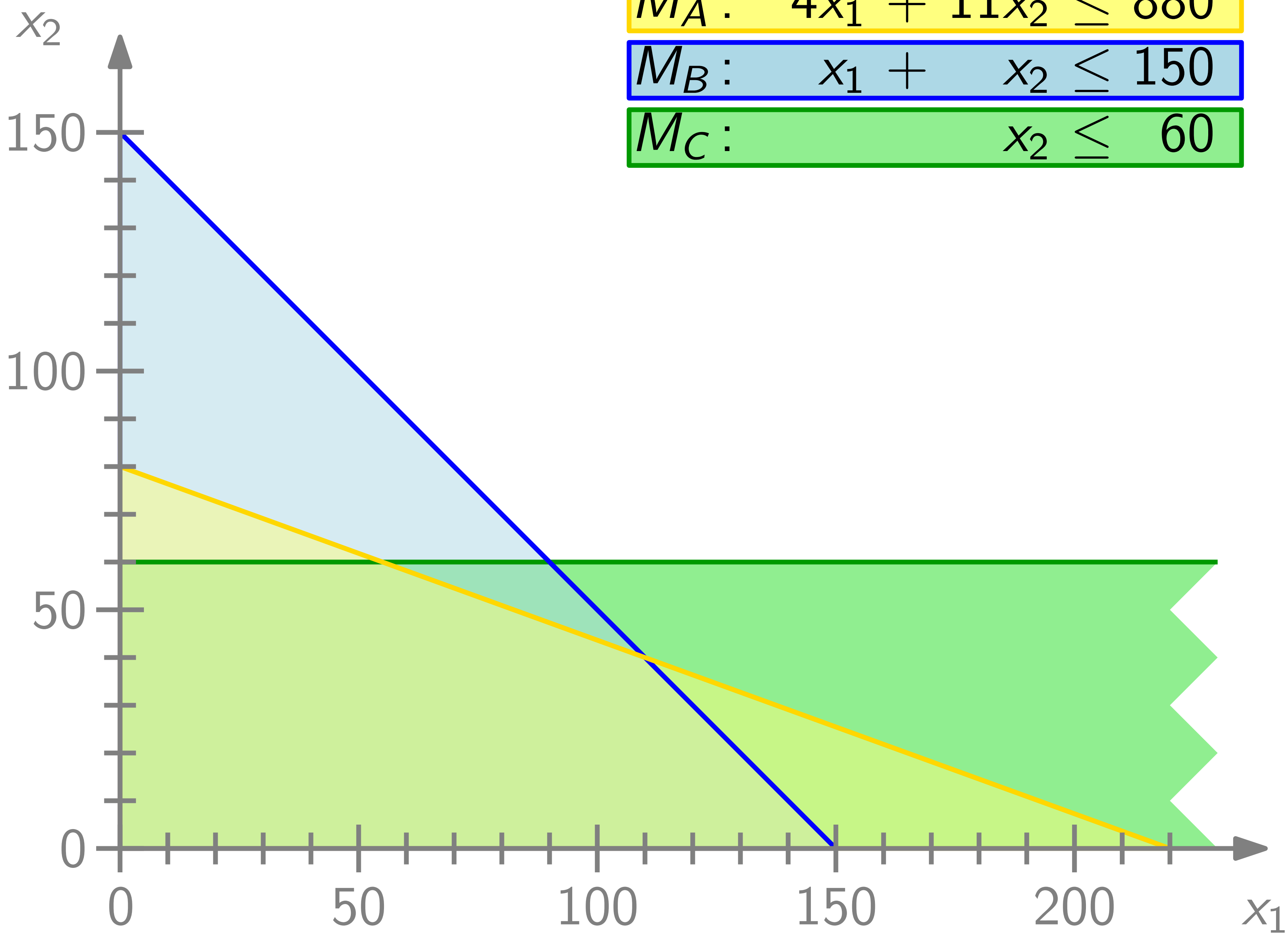

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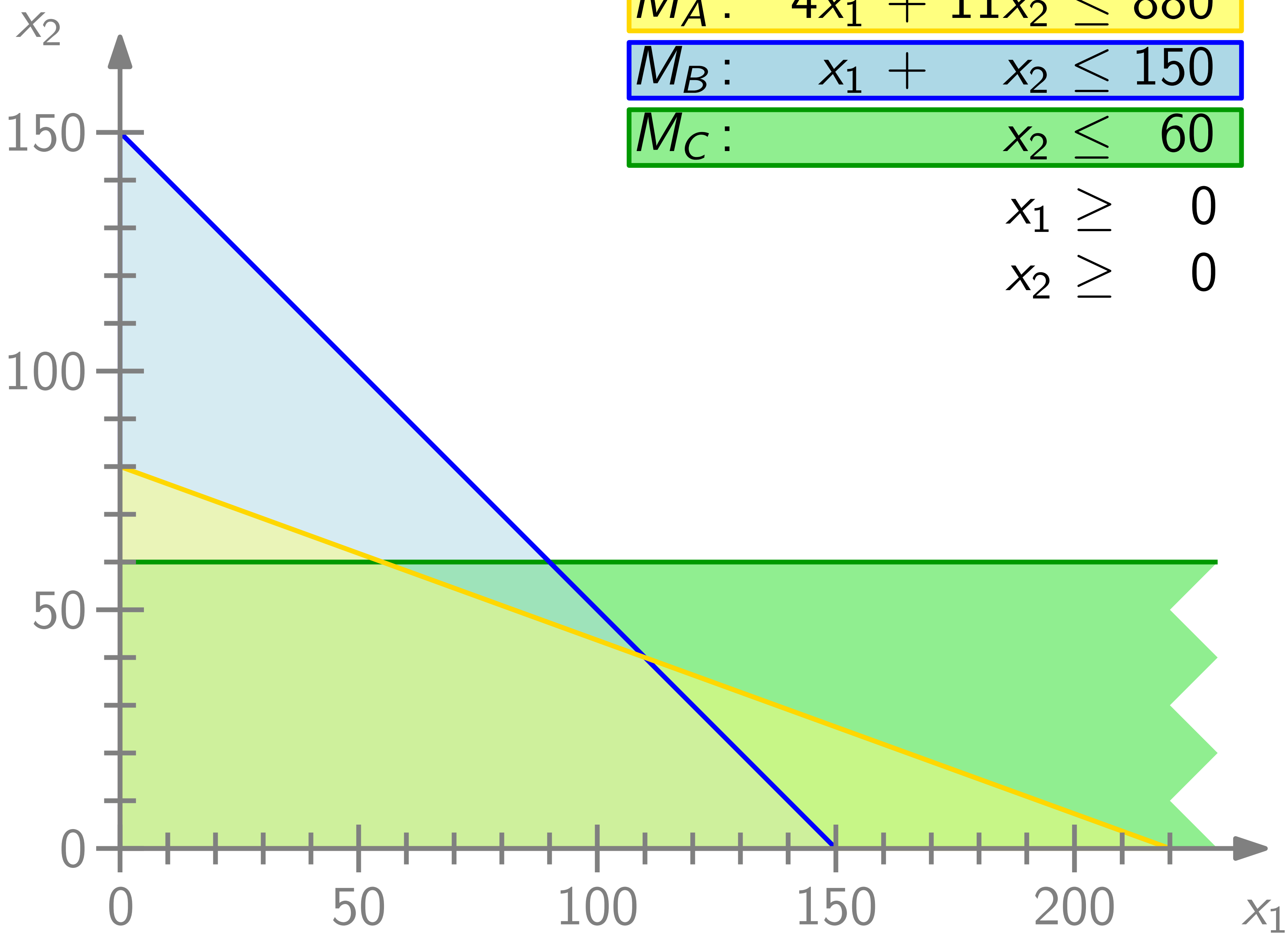

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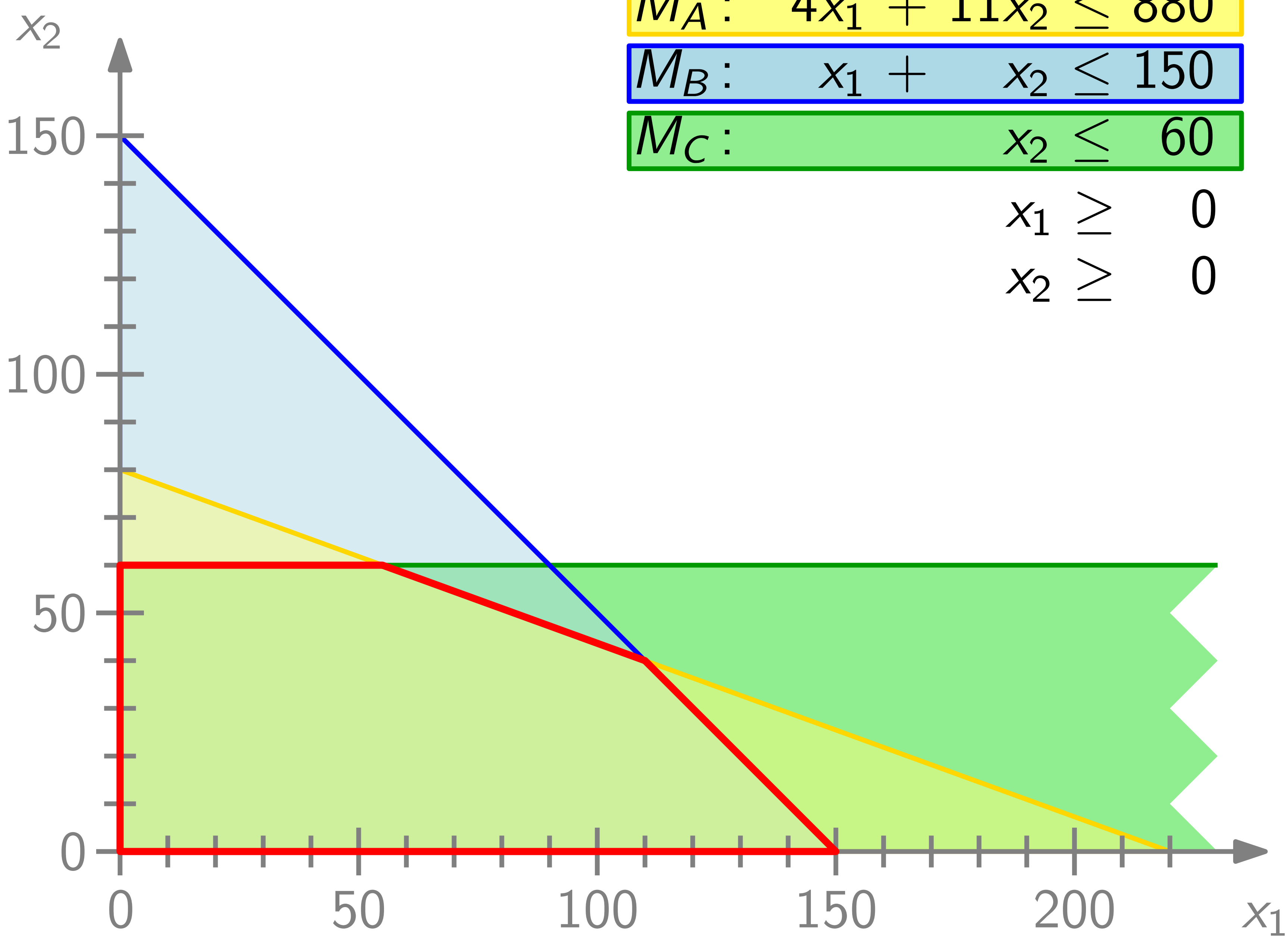

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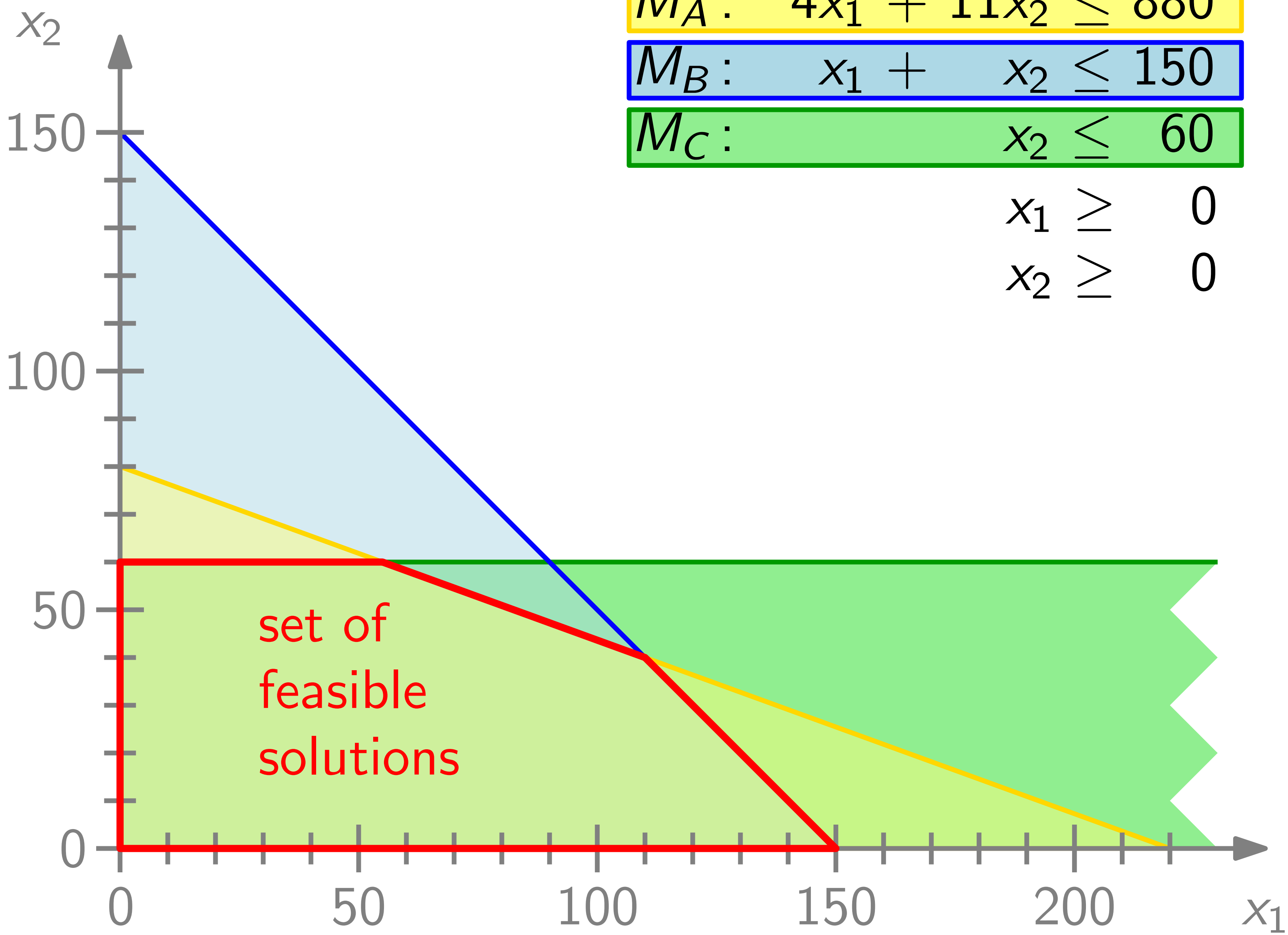

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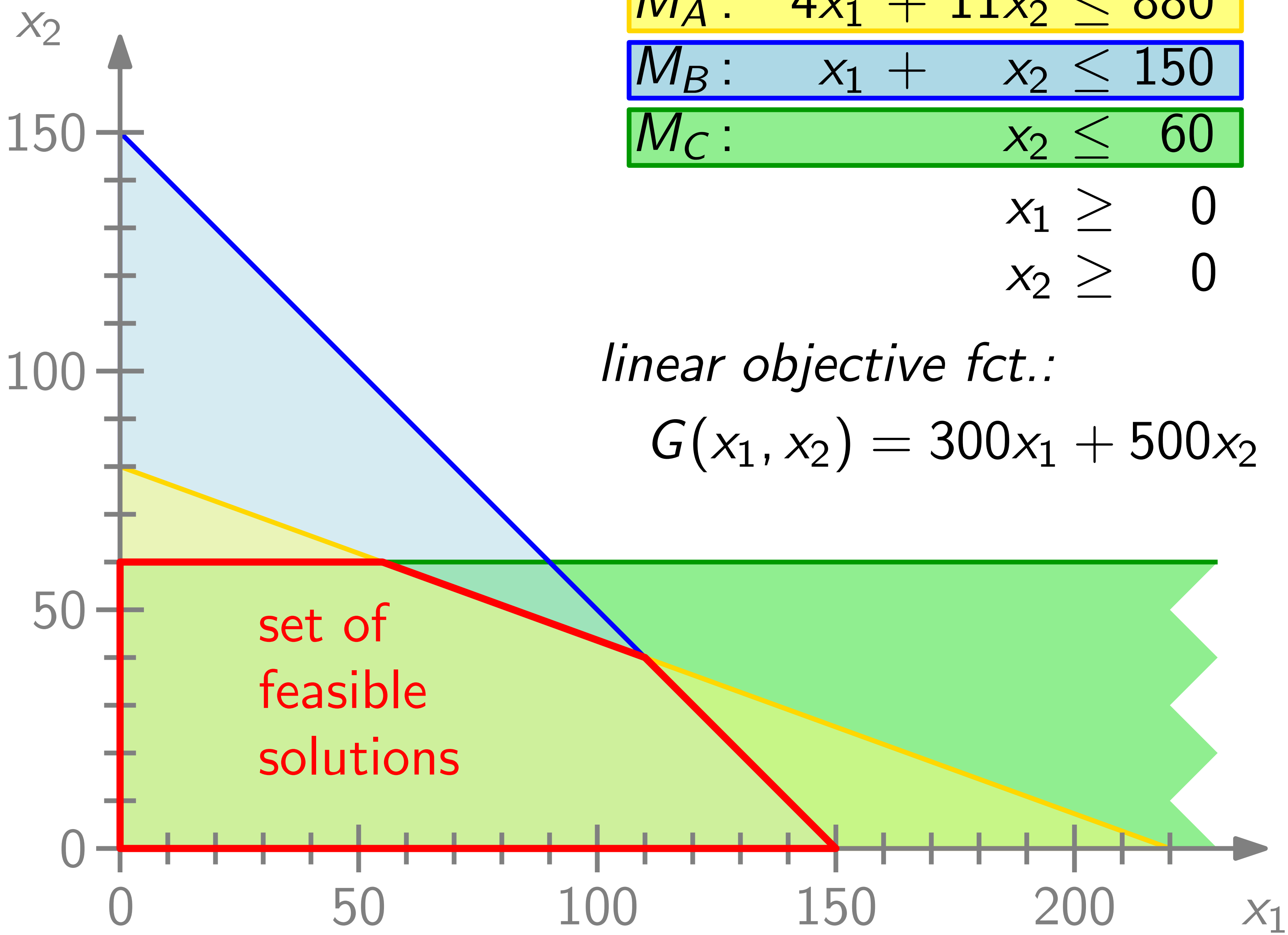

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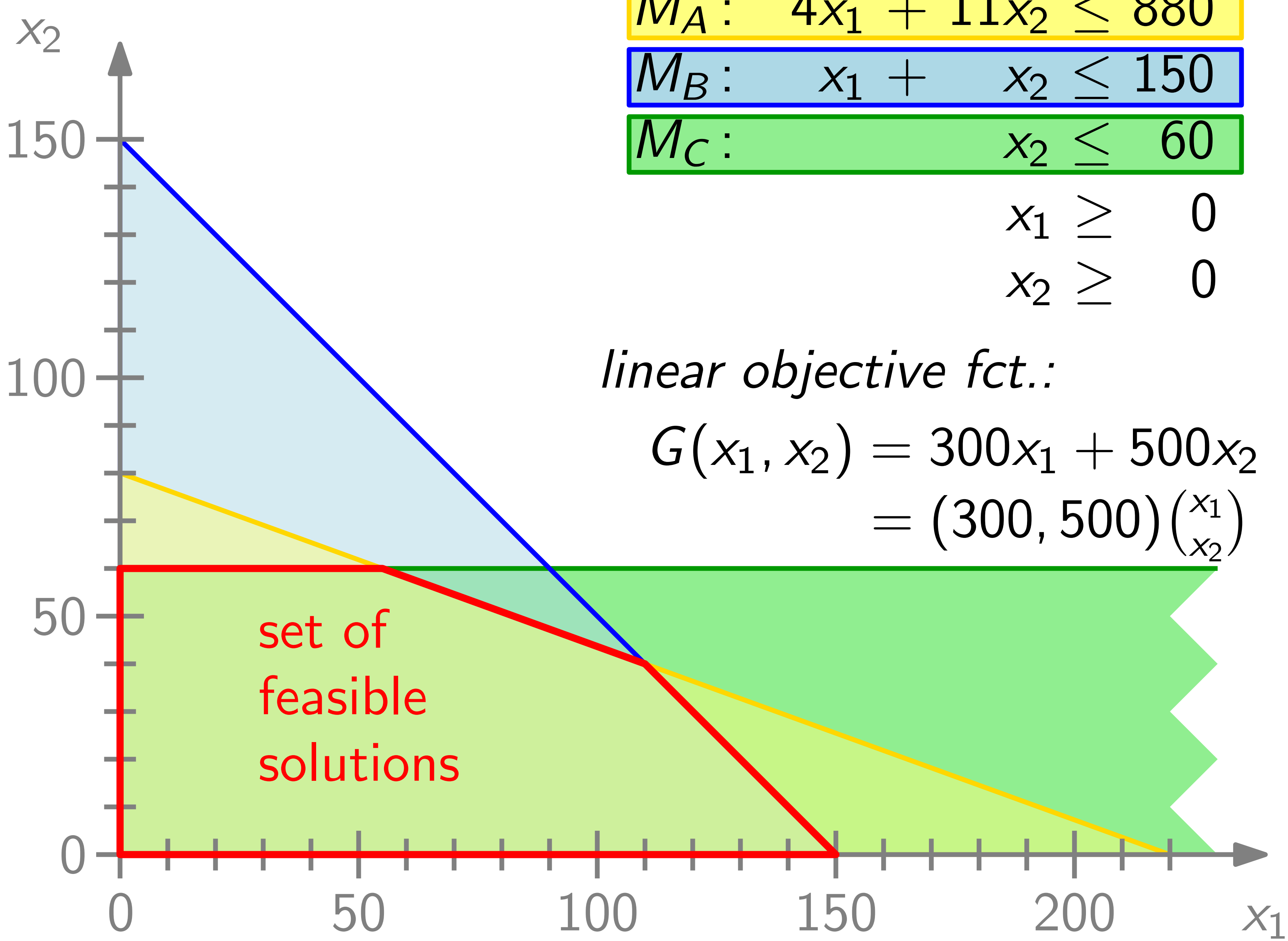

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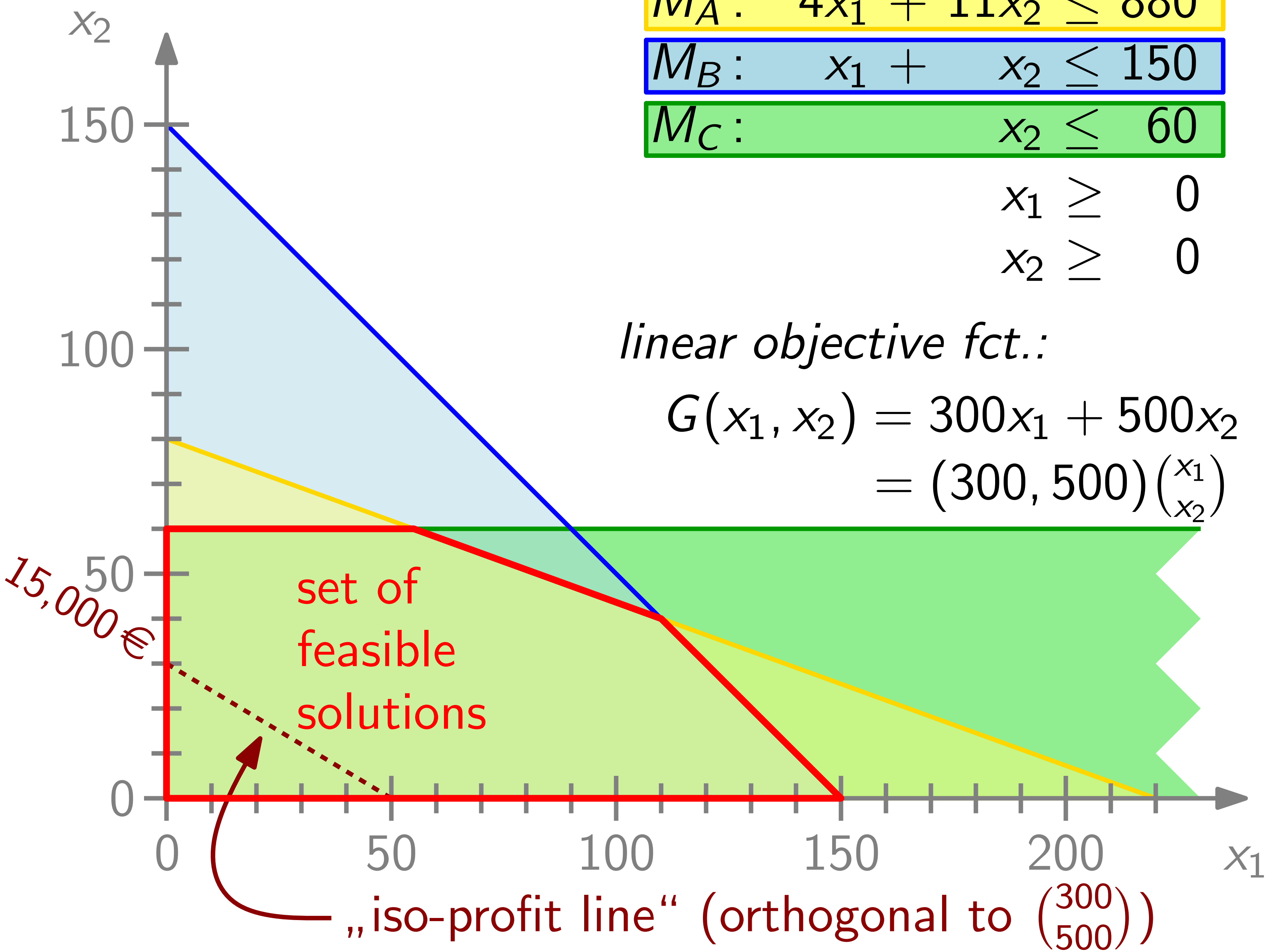

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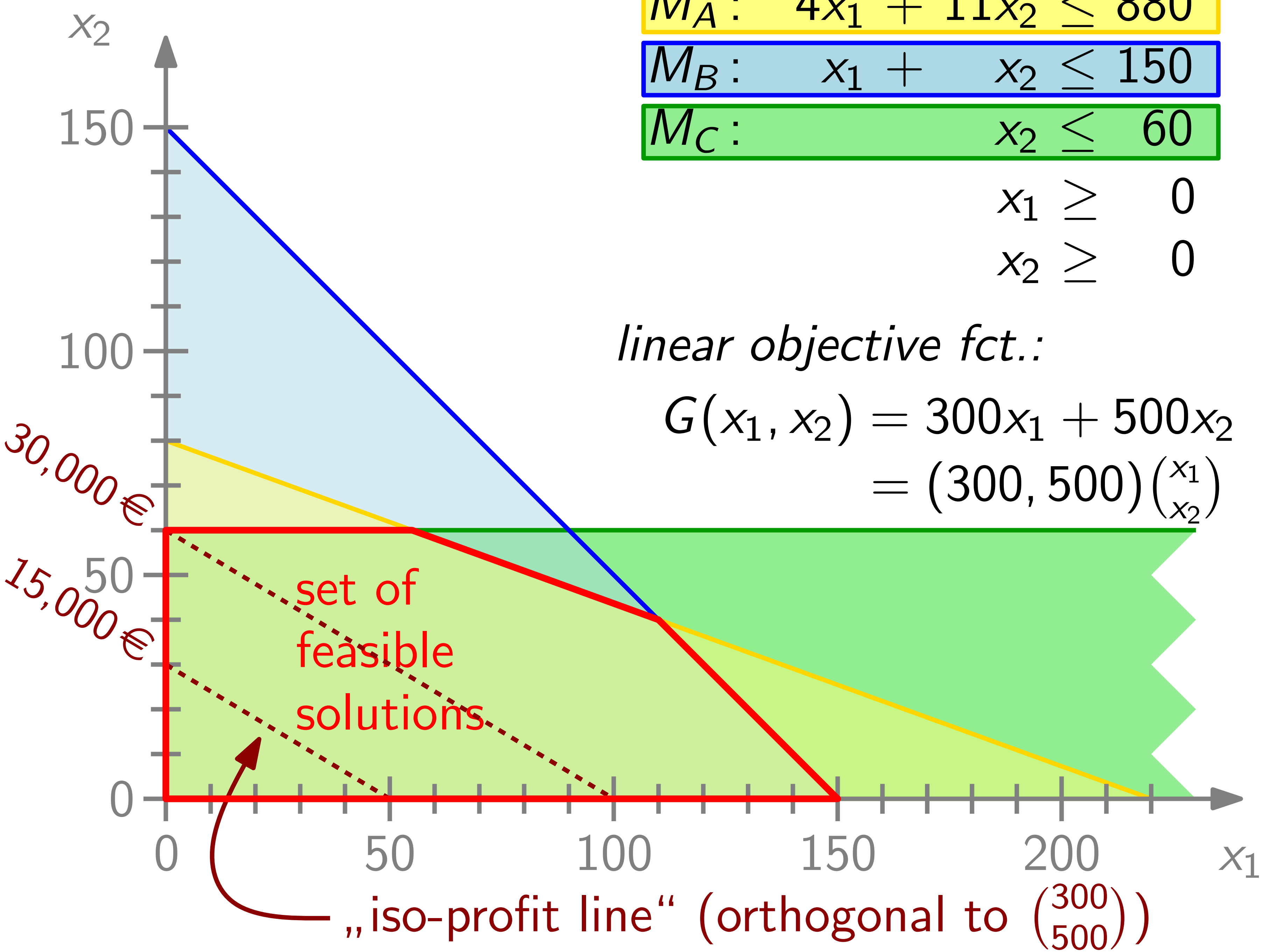

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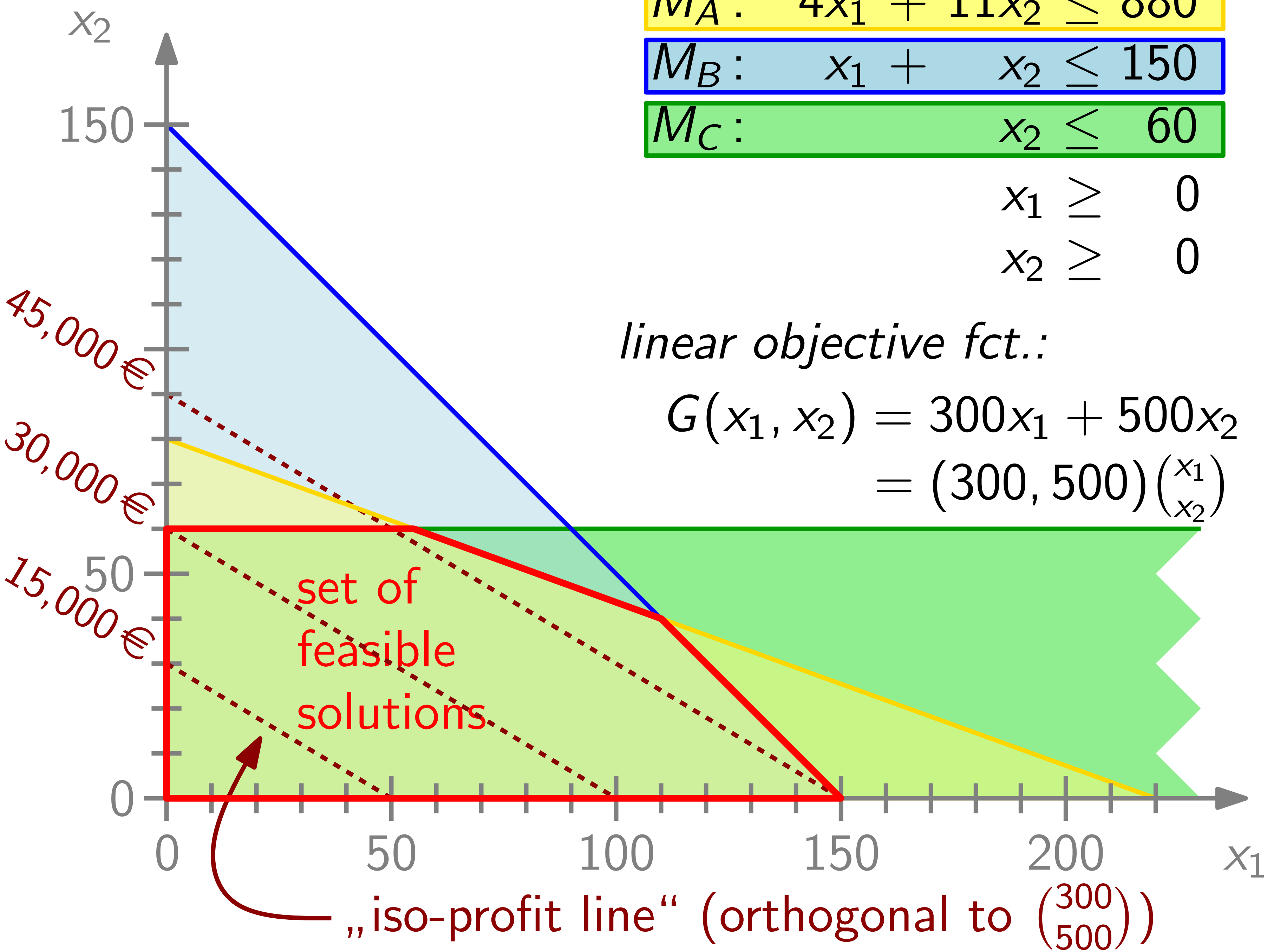

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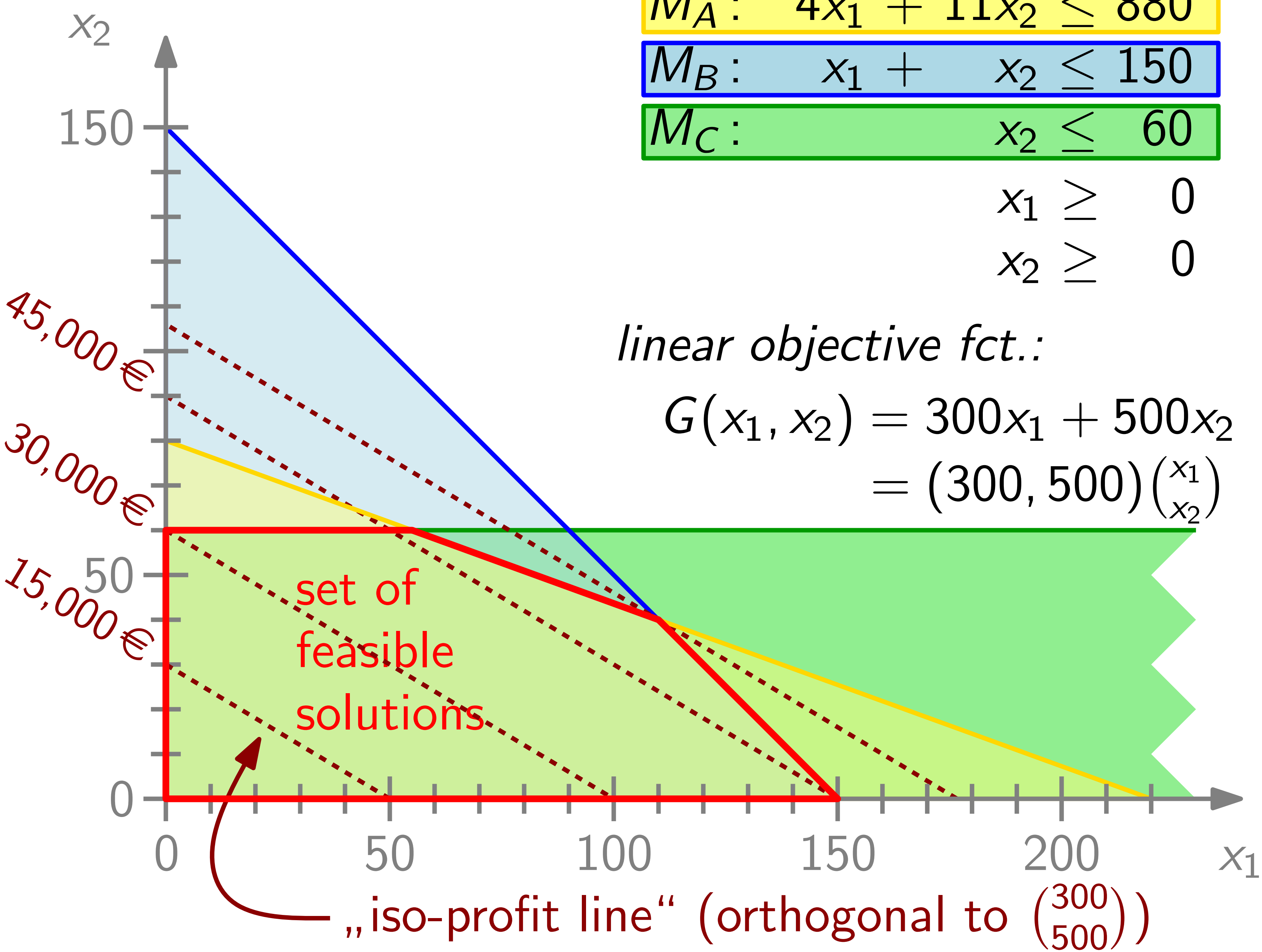

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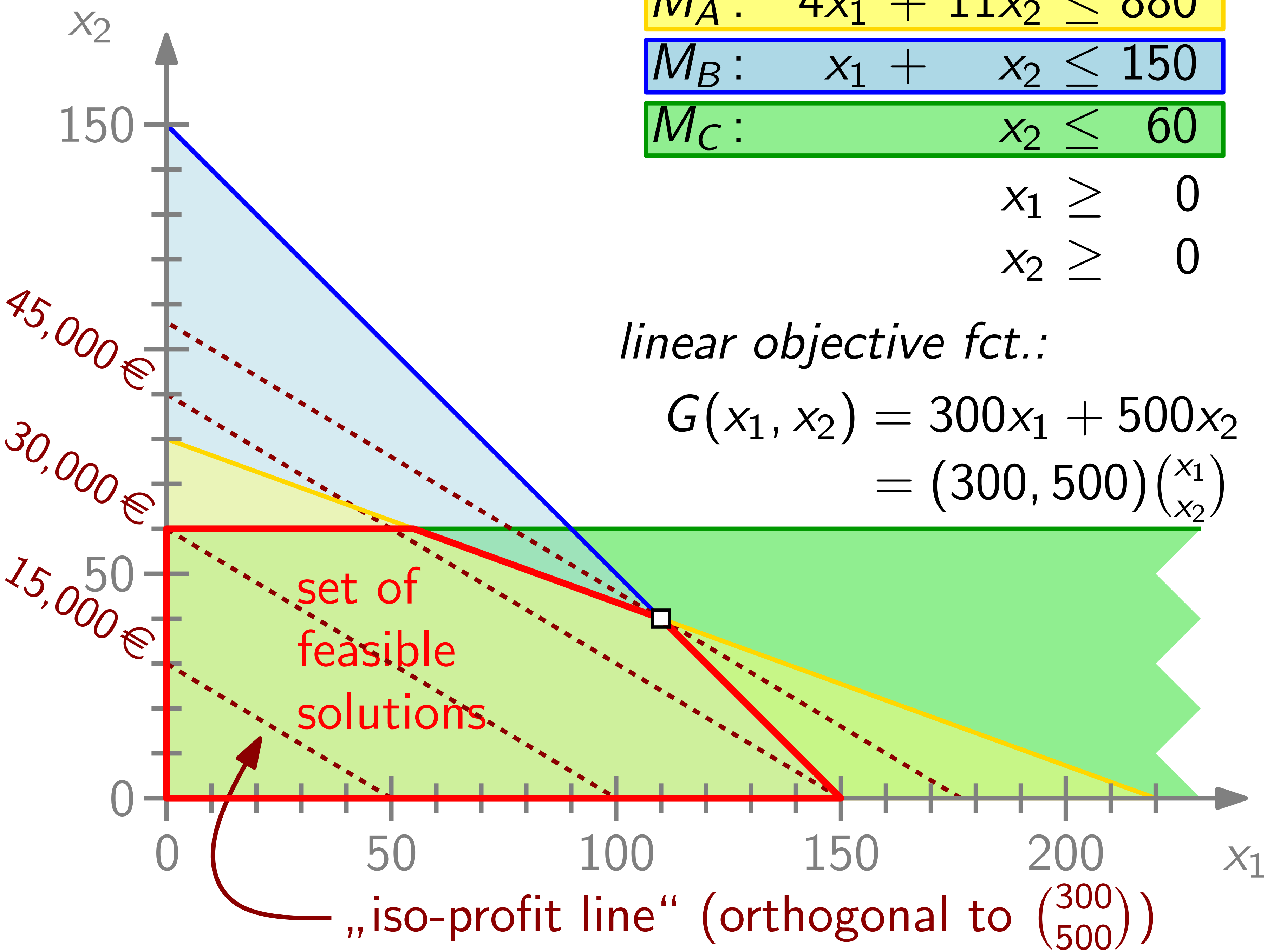

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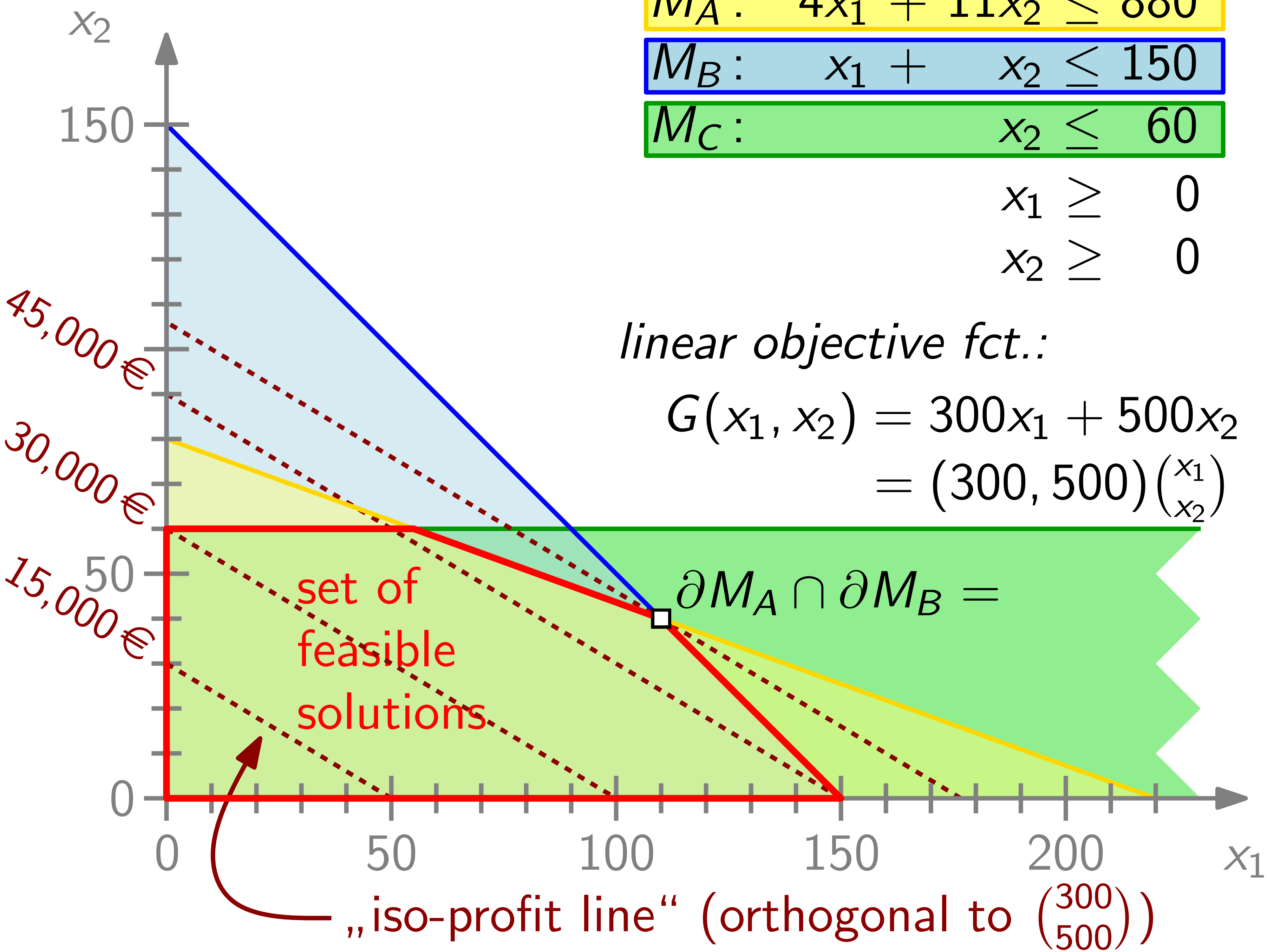

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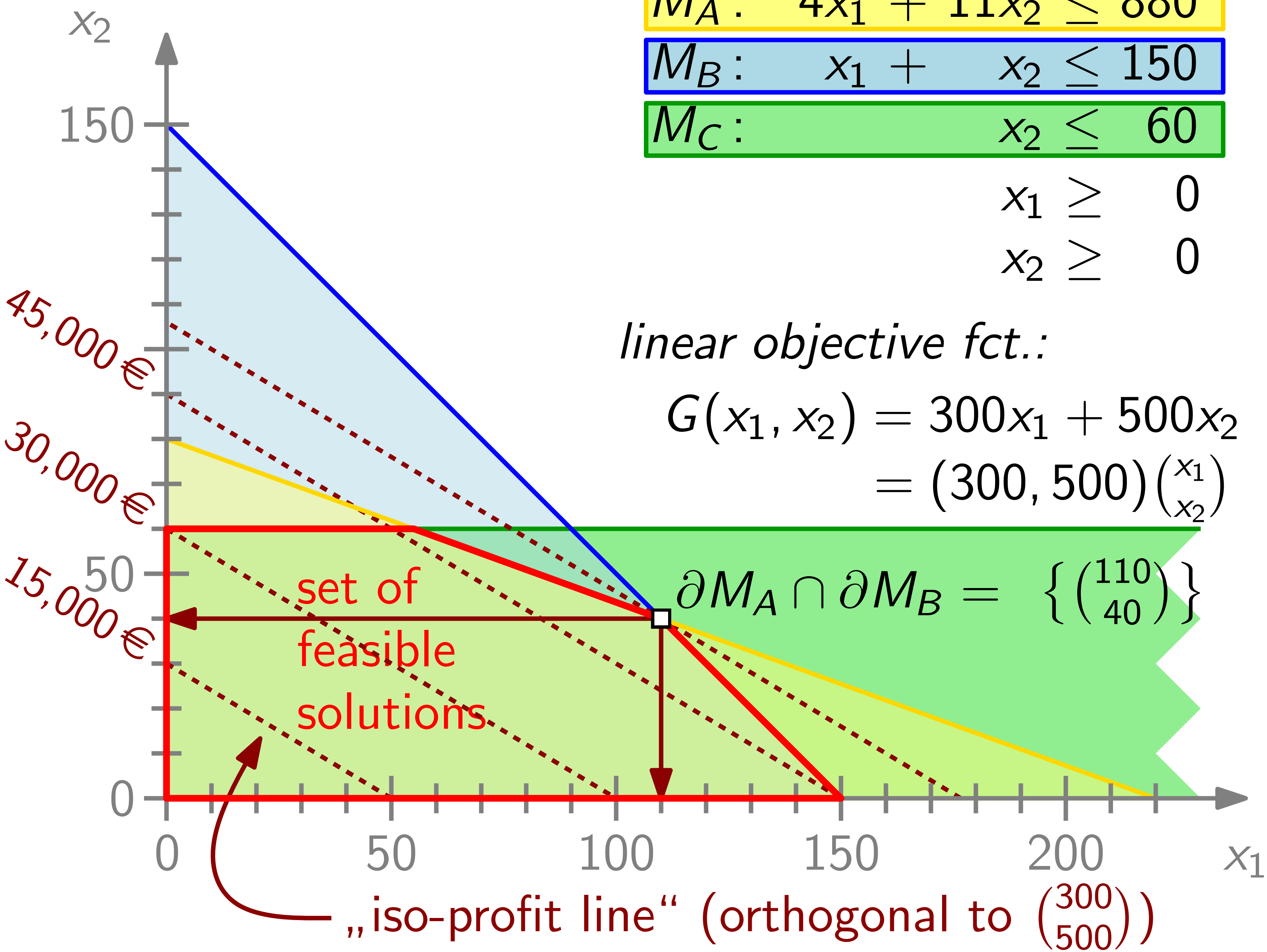
$$\partial M_A \cap \partial M_B = \left\{ \binom{110}{40} \right\}$$



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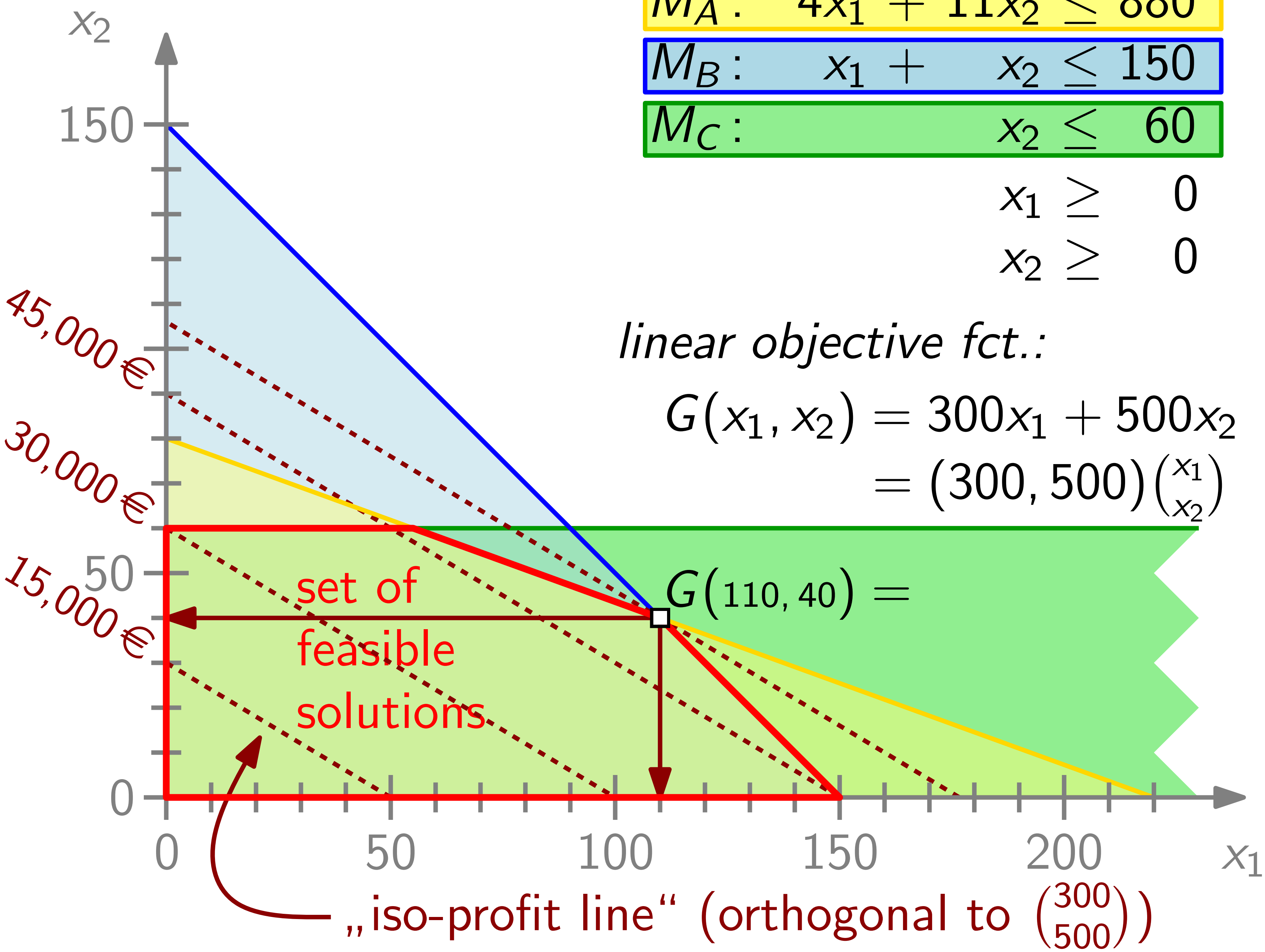

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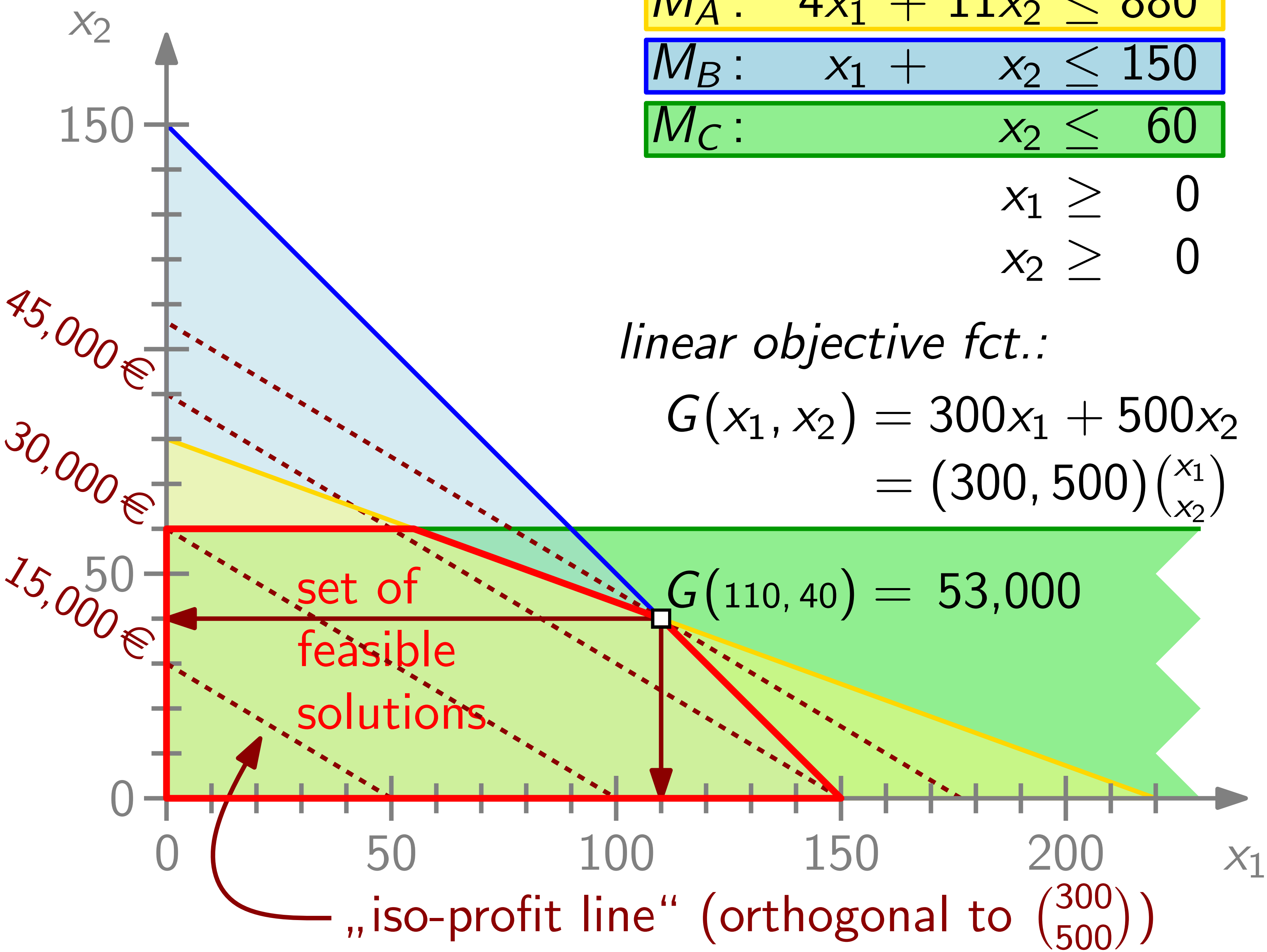

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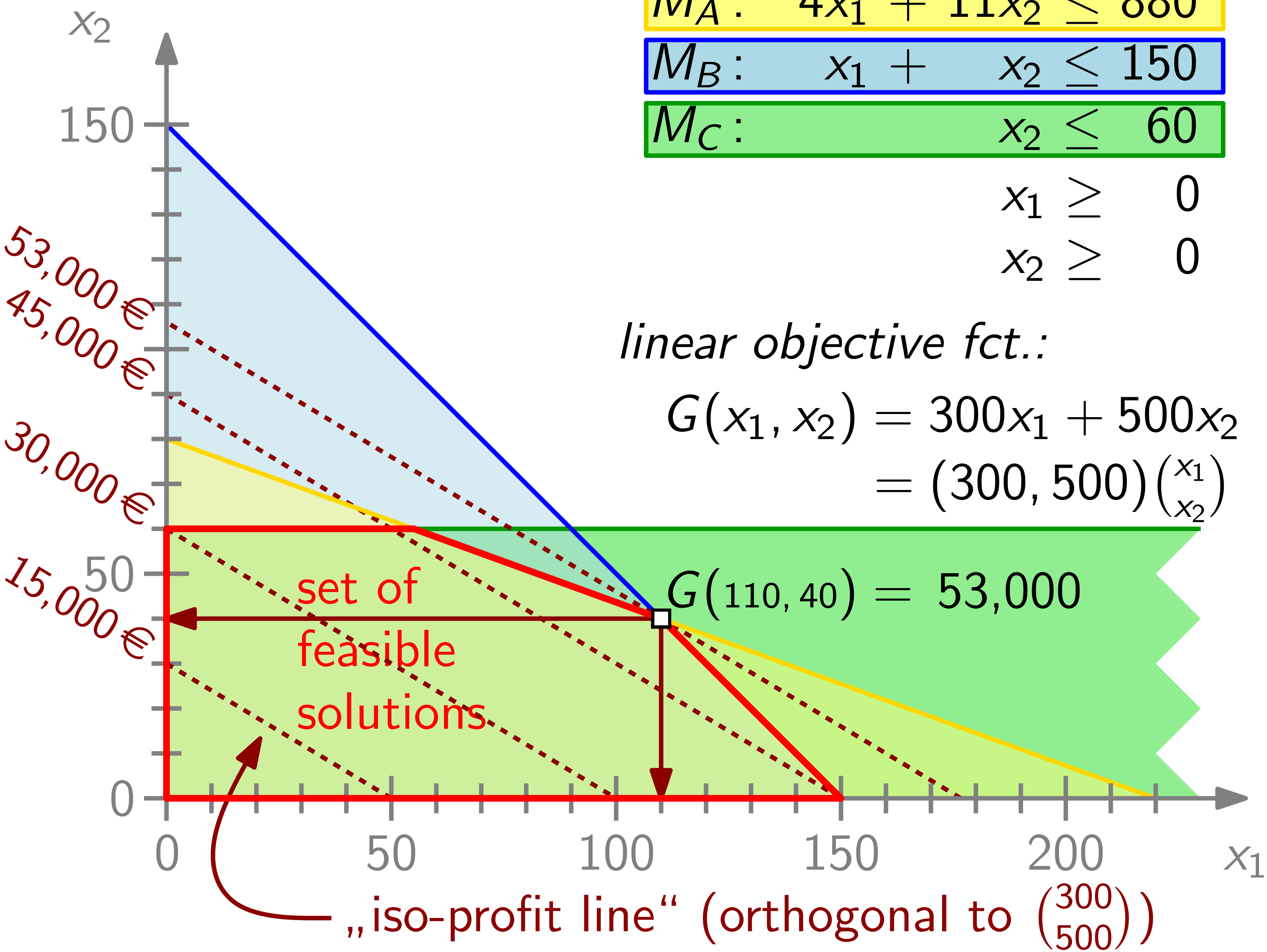

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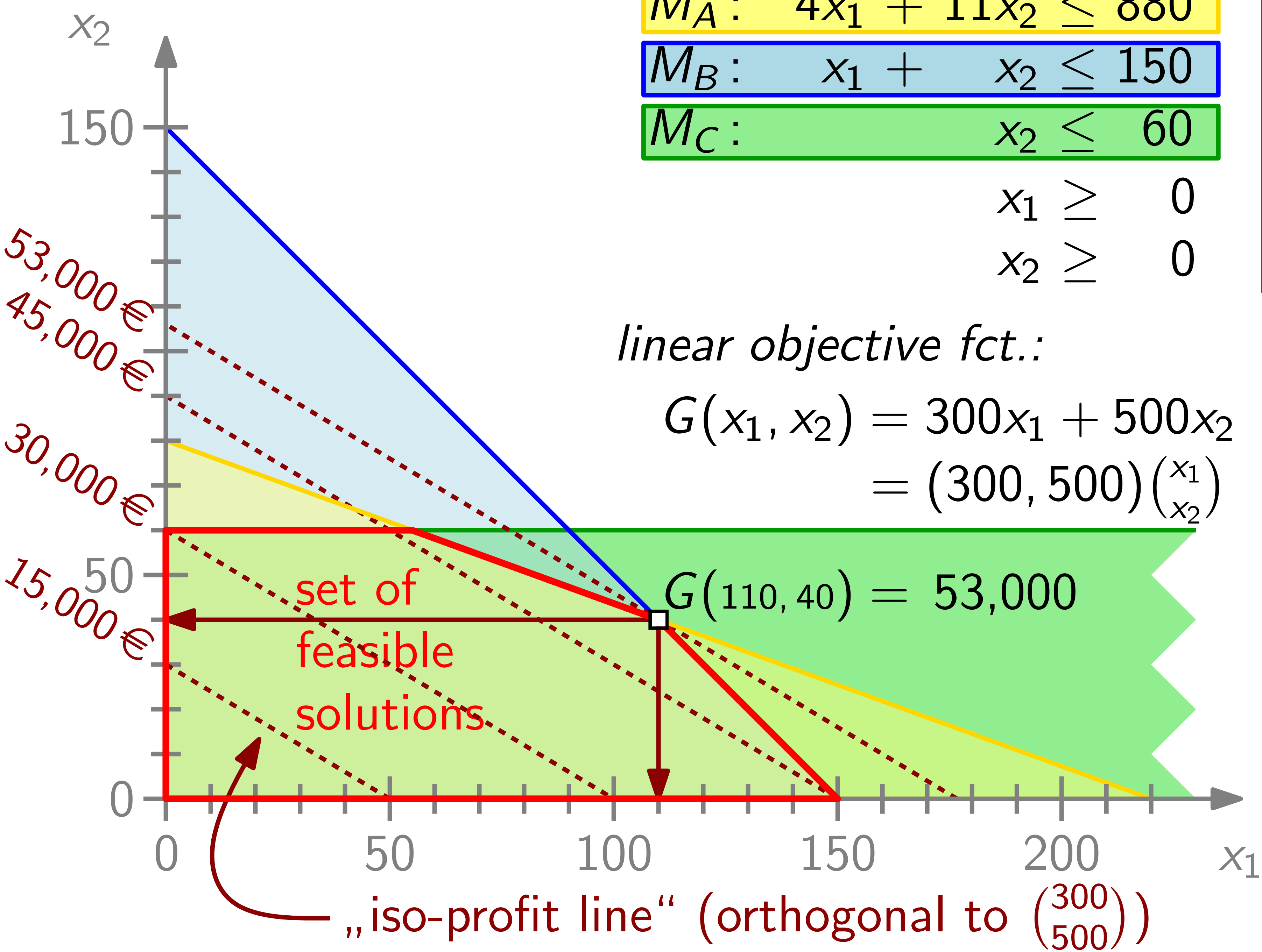

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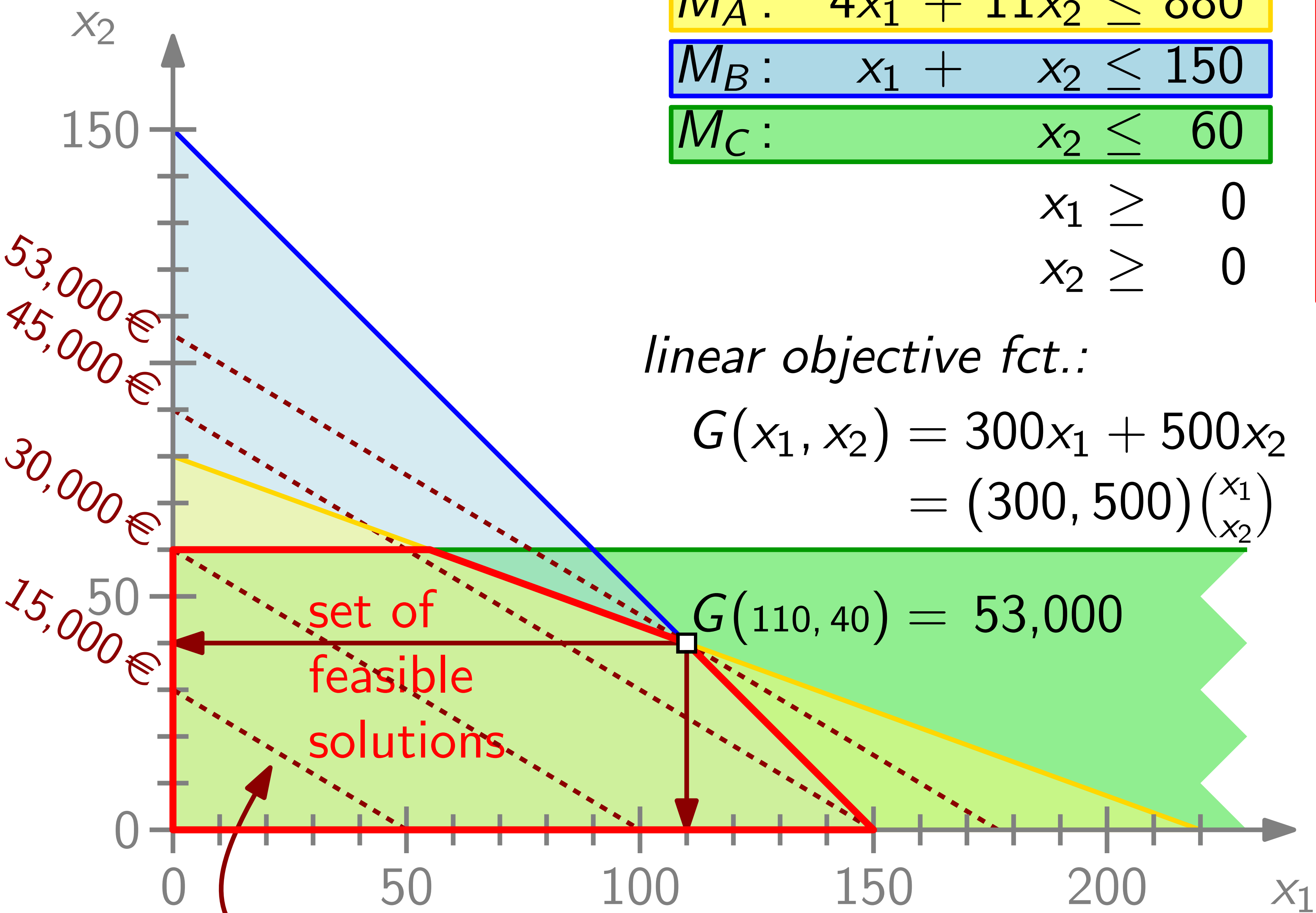

„iso-profit line" (orthogonal to $\binom{300}{500}$)

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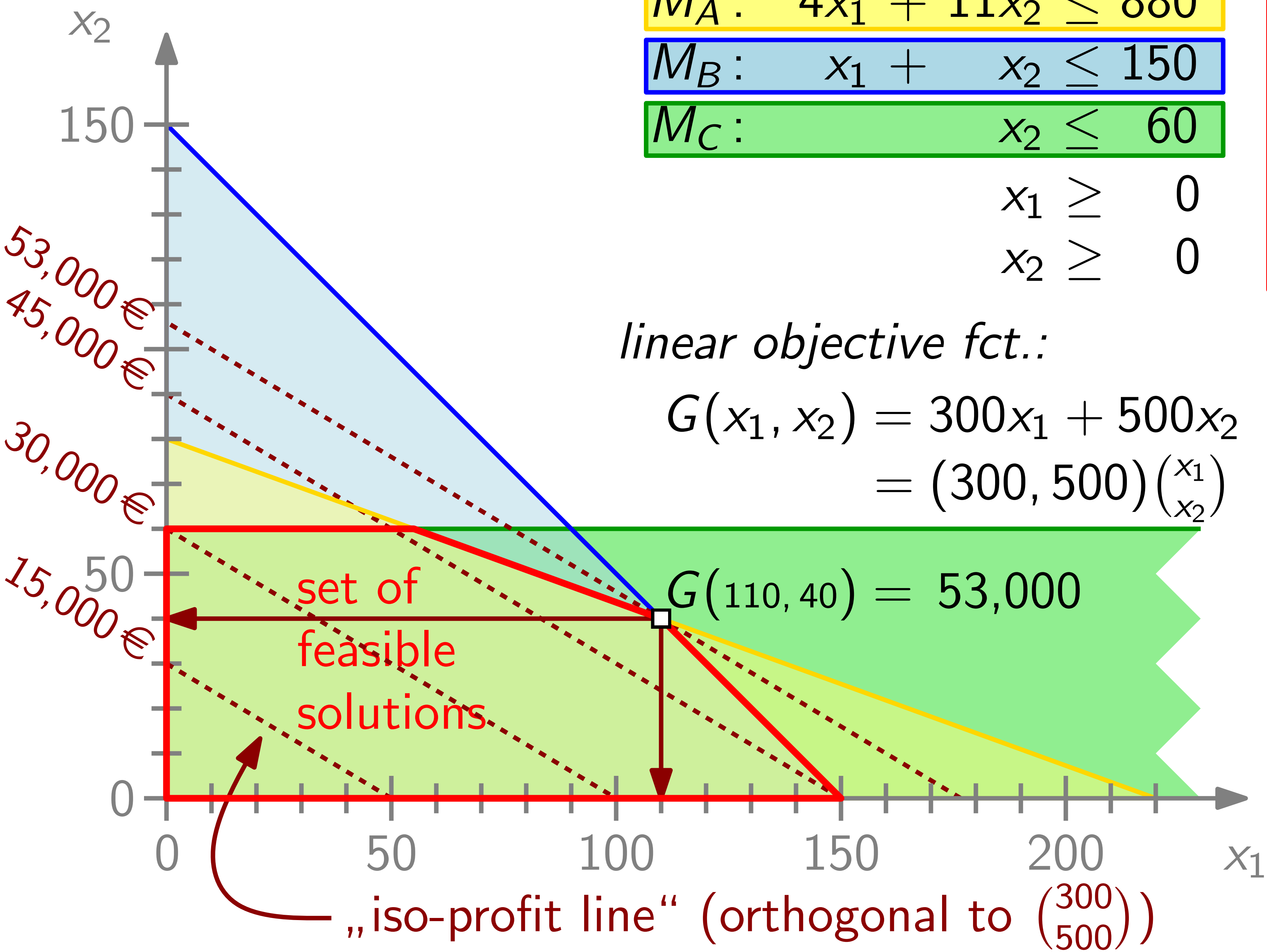

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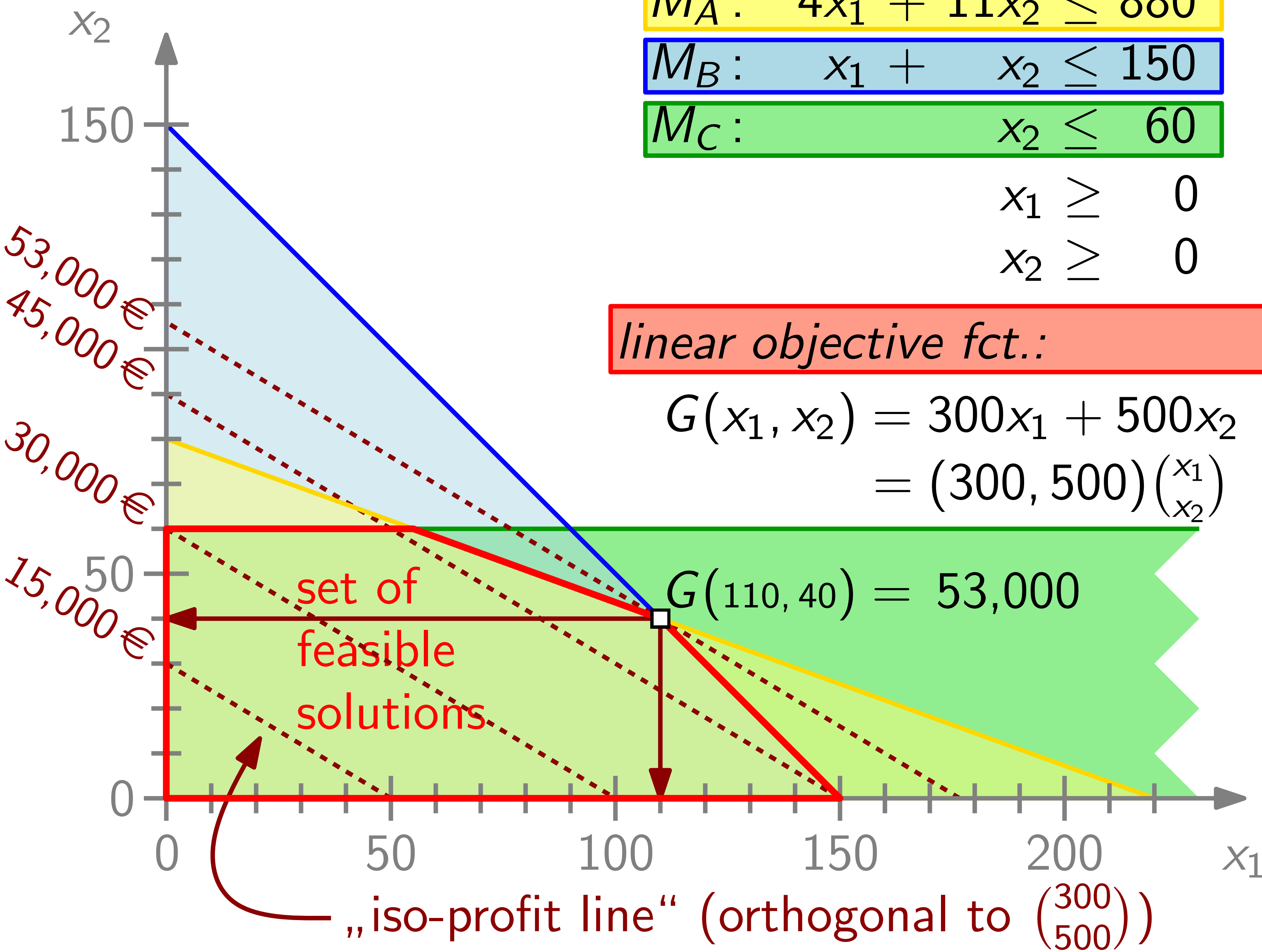

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linear constraints: $Ax \leq b$, $x \geq 0$

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linear objective fct.: maximize $c^T x$

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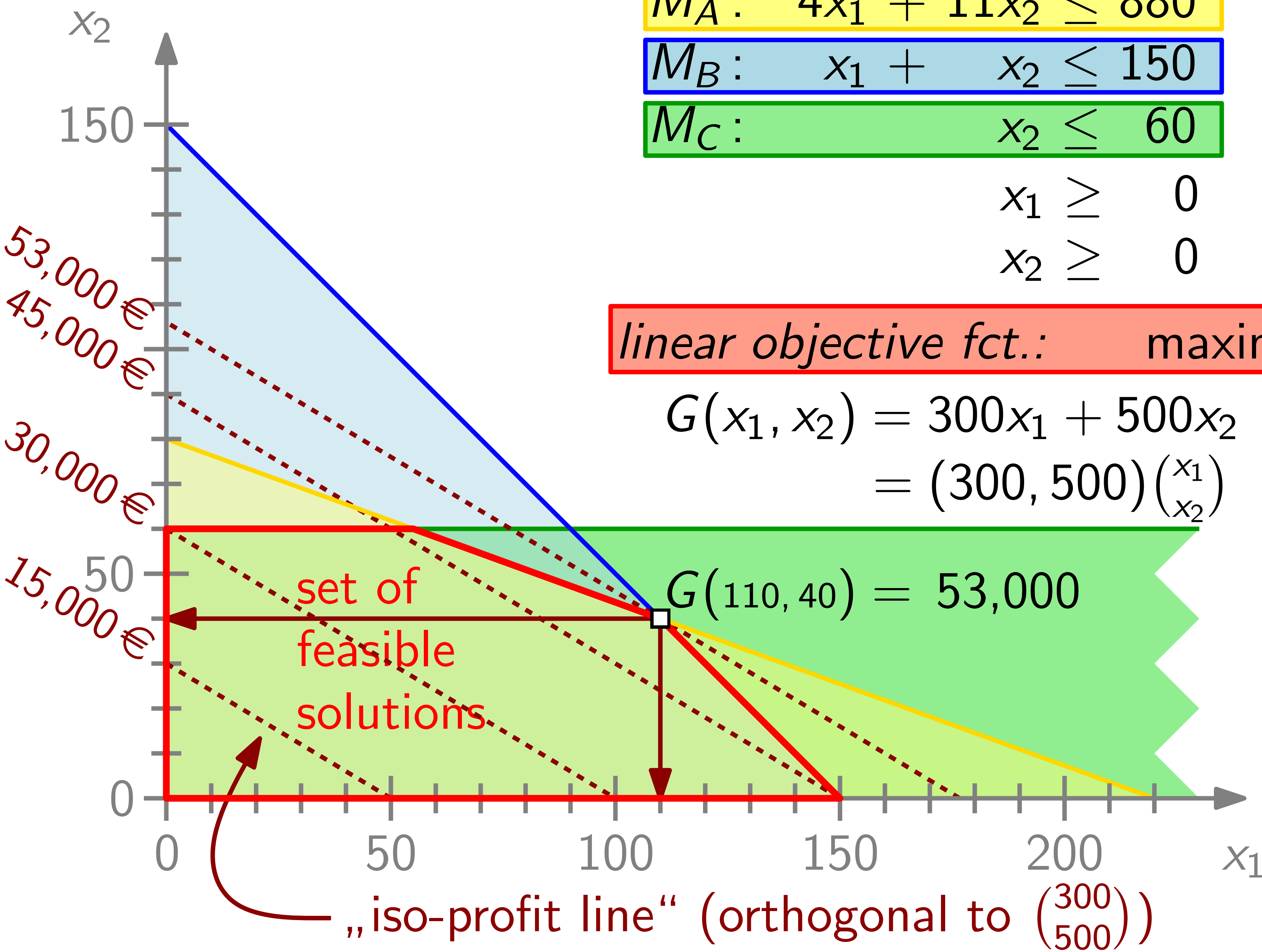

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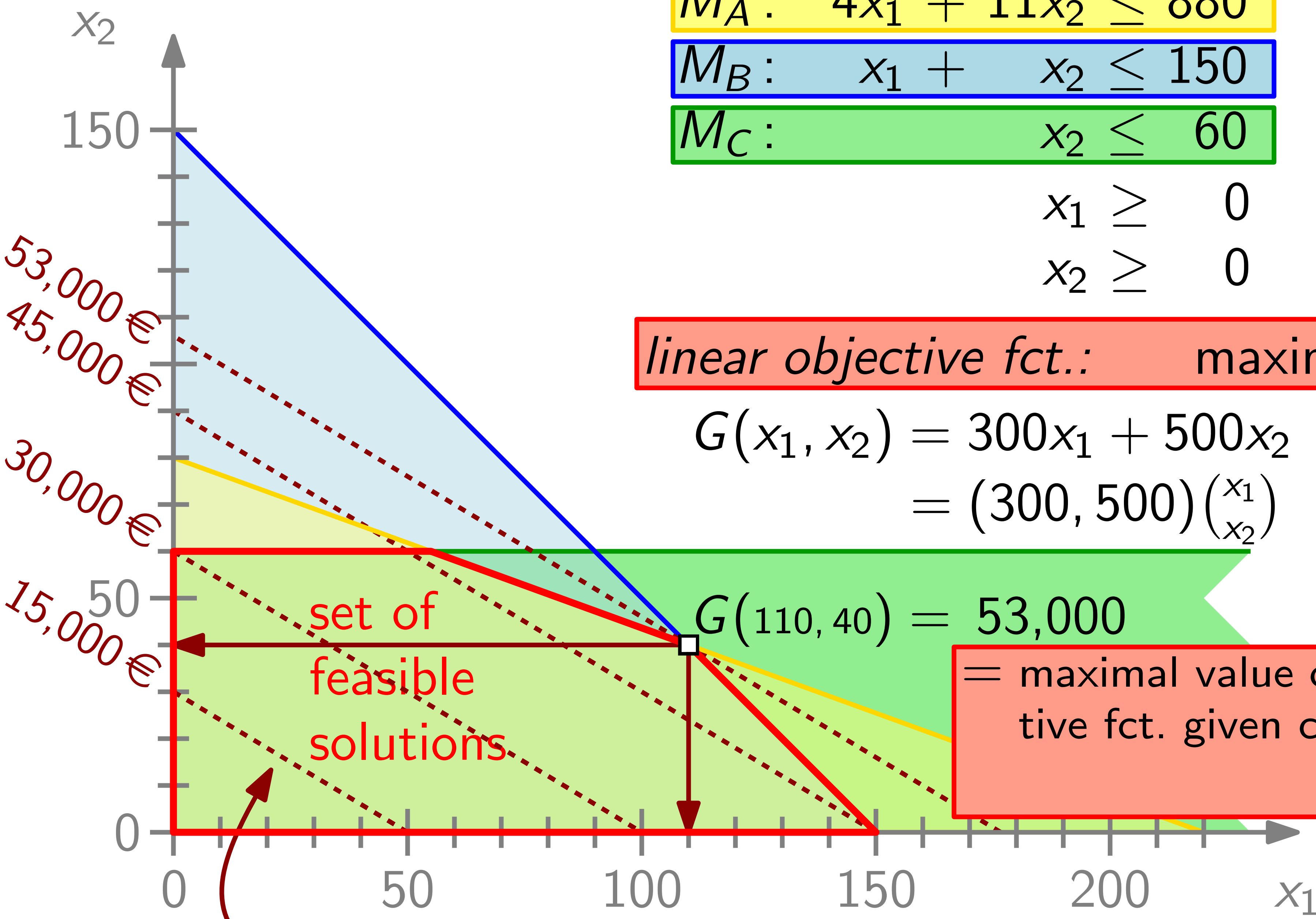

$$G(110, 40) = 53{,}000$$

= maximal value of objective fct. given constraints

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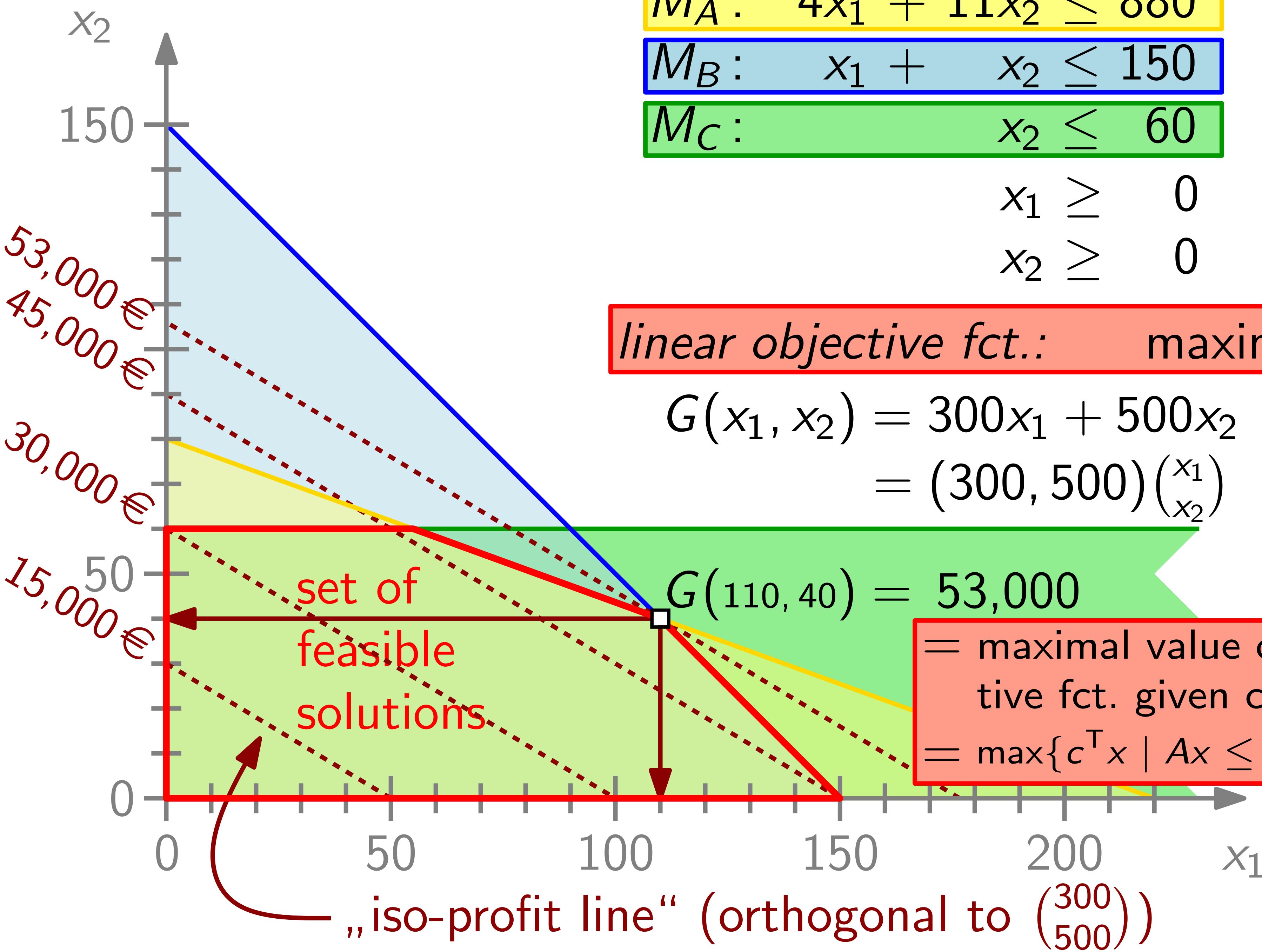

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$$= \max\{c^{\mathsf{T}}x \mid Ax \le b, x \ge 0\}$$

Definition and Known Algorithms

Given a set H of n halfspaces in $\mathbb{R}^d$ and a direction c,
find a point $x \in \bigcap H$ such that cx is maximum (or minimum).

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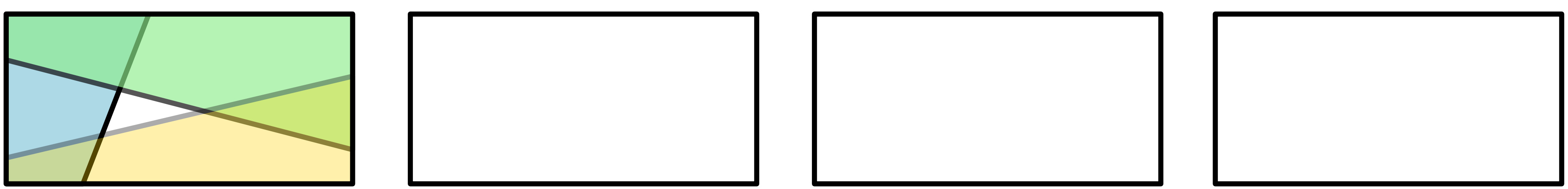

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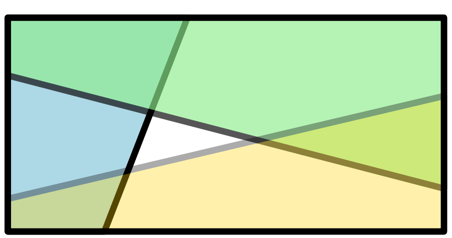

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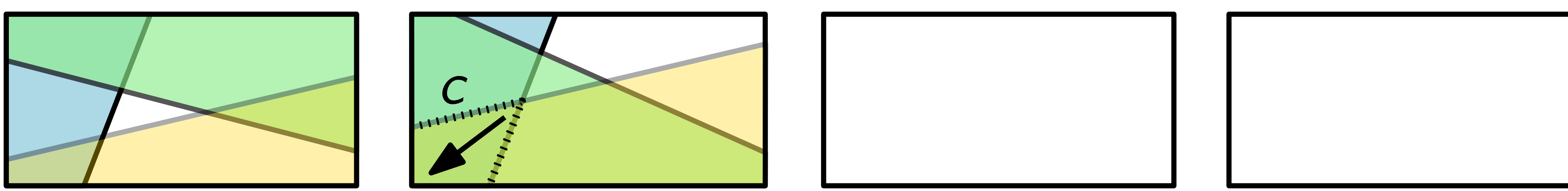

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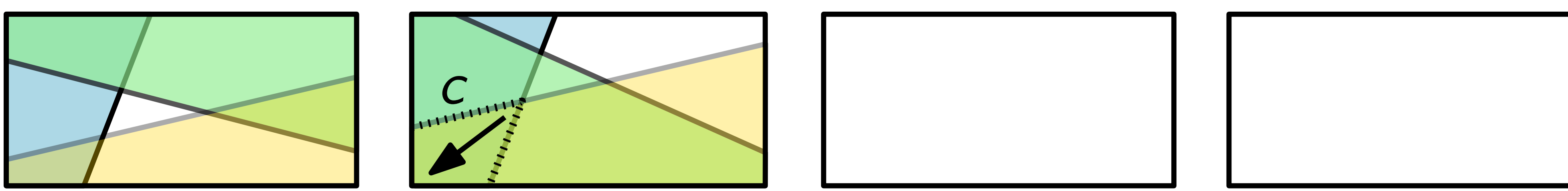

$\bigcap H = \emptyset$ $\bigcap H$ unbnd. in dir. c

Definition and Known Algorithms

Given a set H of n halfspaces in $\mathbb{R}^d$ and a direction c,
find a point $x \in \bigcap H$ such that cx is maximum (or minimum).

Many algorithms known, e.g.:

- Simplex [Dantzig '47]
- Ellipsoid method [Khatchiyan '79]
- Inner-point method [Karmakar' 84]

Good for instances where n *and* d are large.

We consider $d = 2$.

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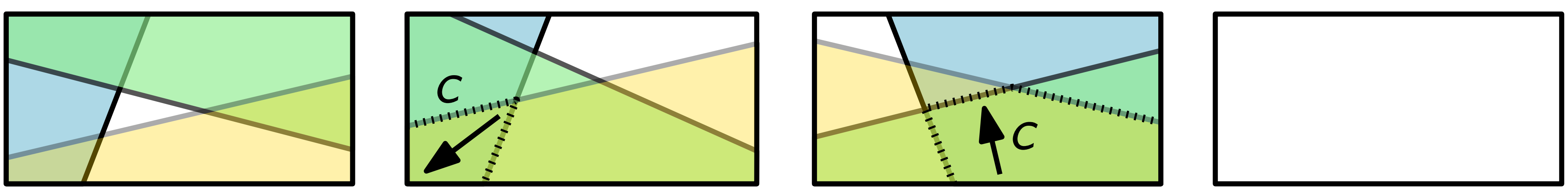

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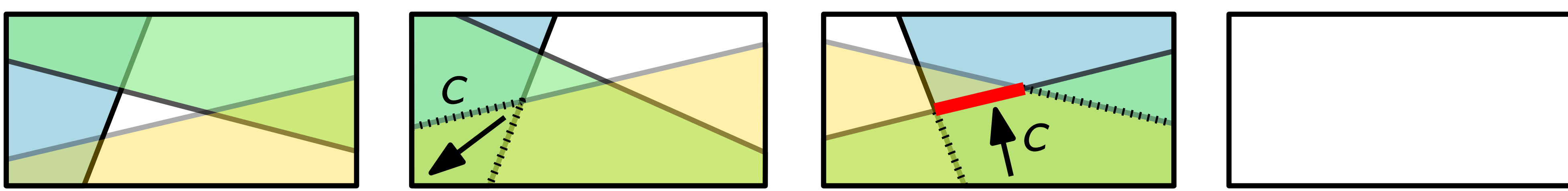

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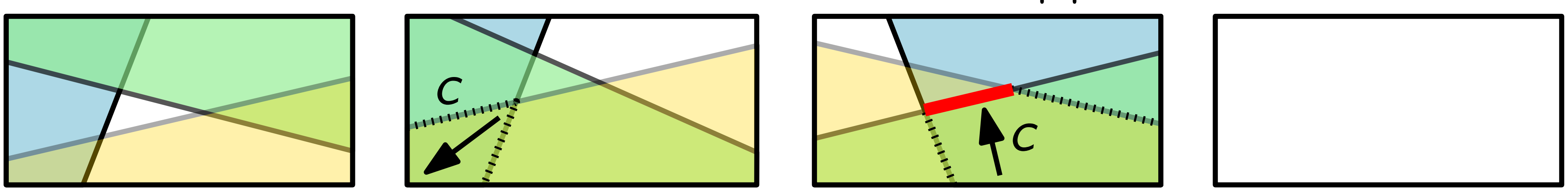

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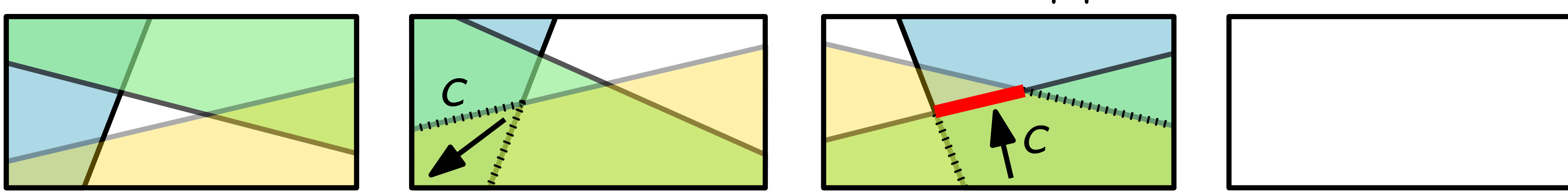

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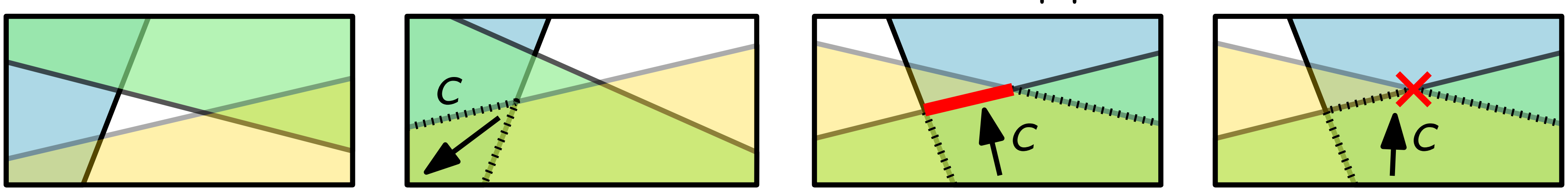

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First Approach

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- compute $\bigcap H$ explicitly
- walk along $\partial\left(\bigcap H\right)$ to find a vertex x with cx maximum

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if $|H| = 1$ **then**
  $C \leftarrow h$, where $\{h\} = H$
else

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Running time: $T_{\mathrm{IH}}(n) = 2T_{\mathrm{IH}}(n/2) + T_{\mathrm{ICR}}(n)$

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  |   C ← IntersectConvexRegions(C₁, C₂)
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How??

Running time: $T_{\text{IH}}(n) = 2T_{\text{IH}}(n/2) + T_{\text{ICR}}(n)$

Intersecting Convex Regions

Any ideas?

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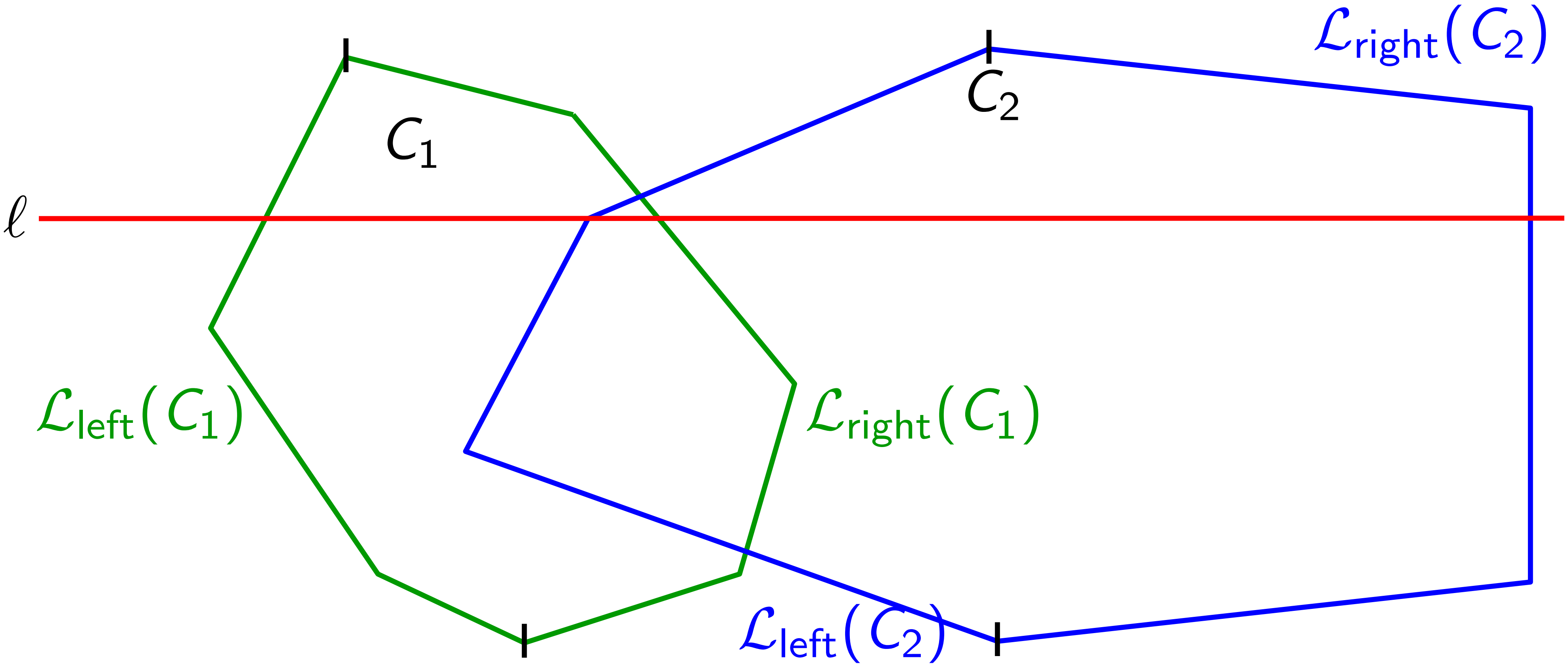
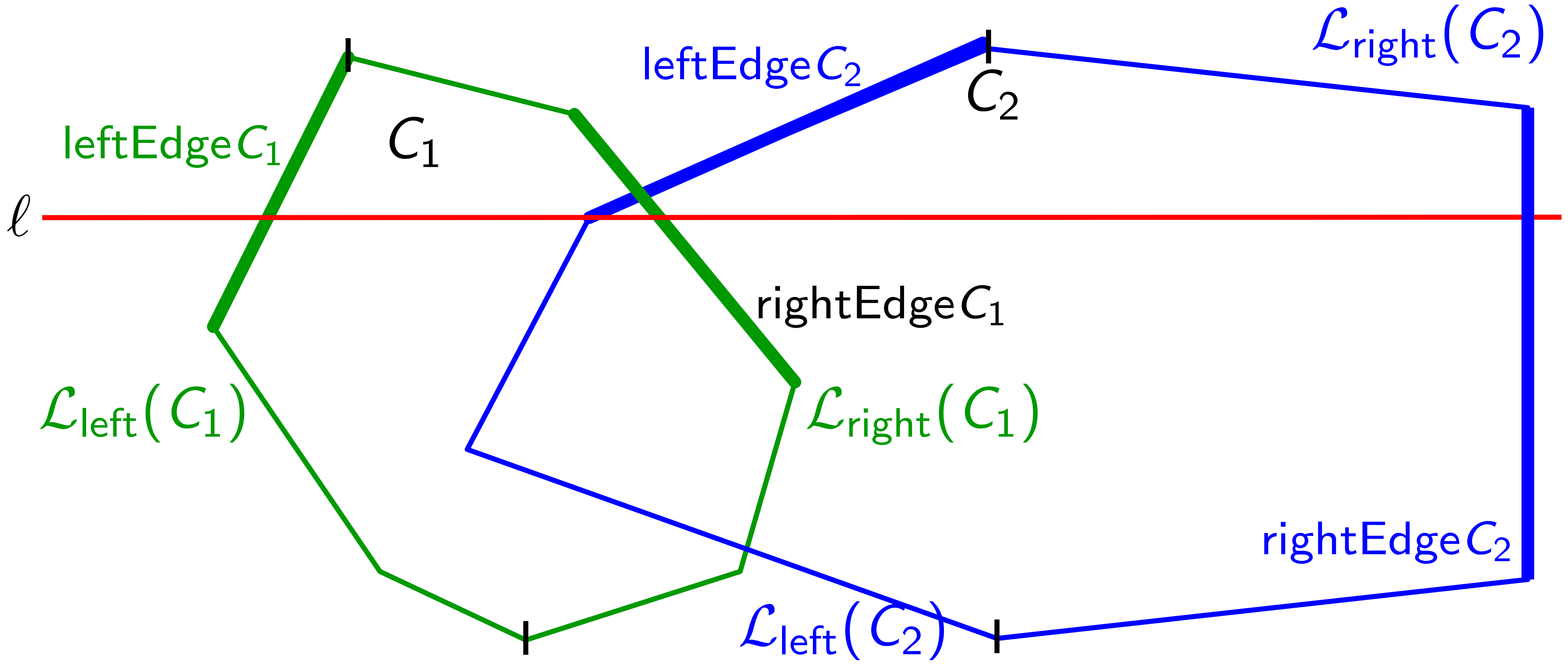
Better ideas?

Use specialized algorithm for intersecting *convex* regions/polyg.

Intersecting Convex Regions Faster



Intersecting Convex Regions Faster



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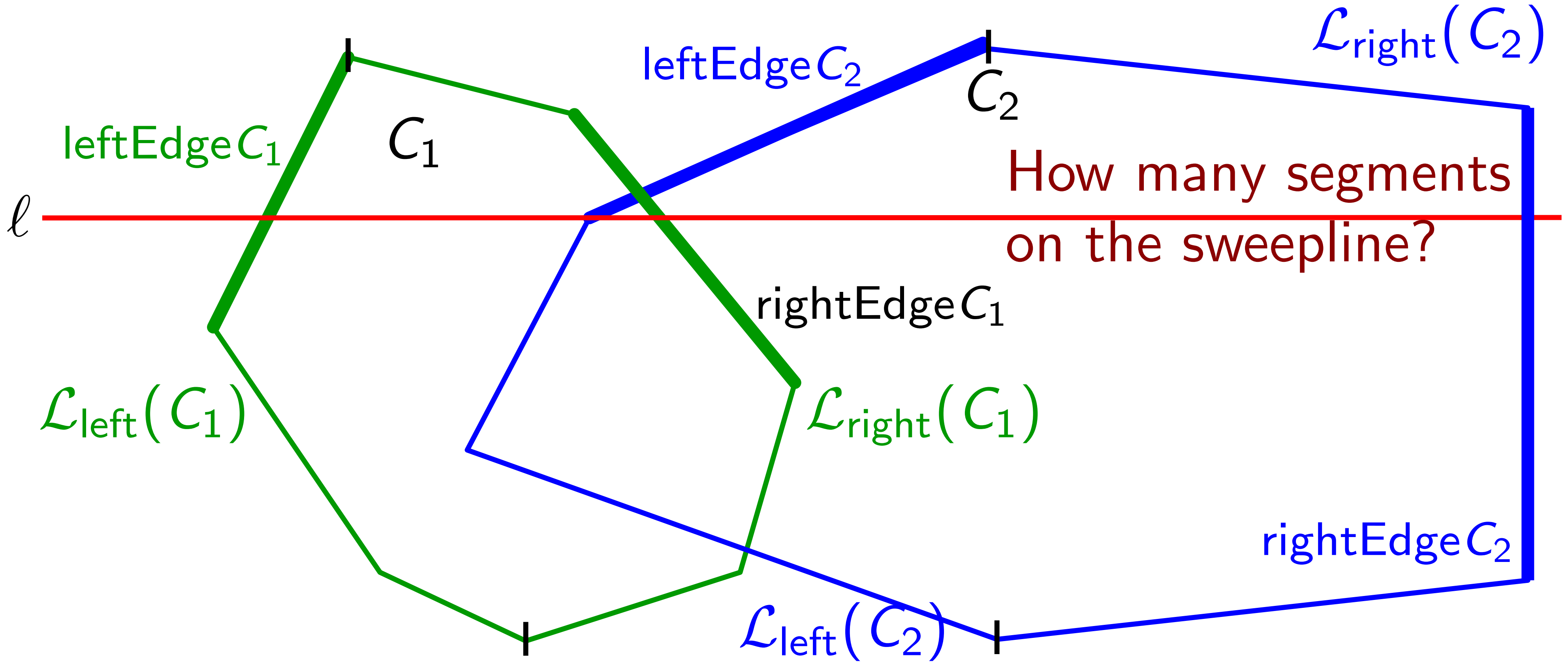

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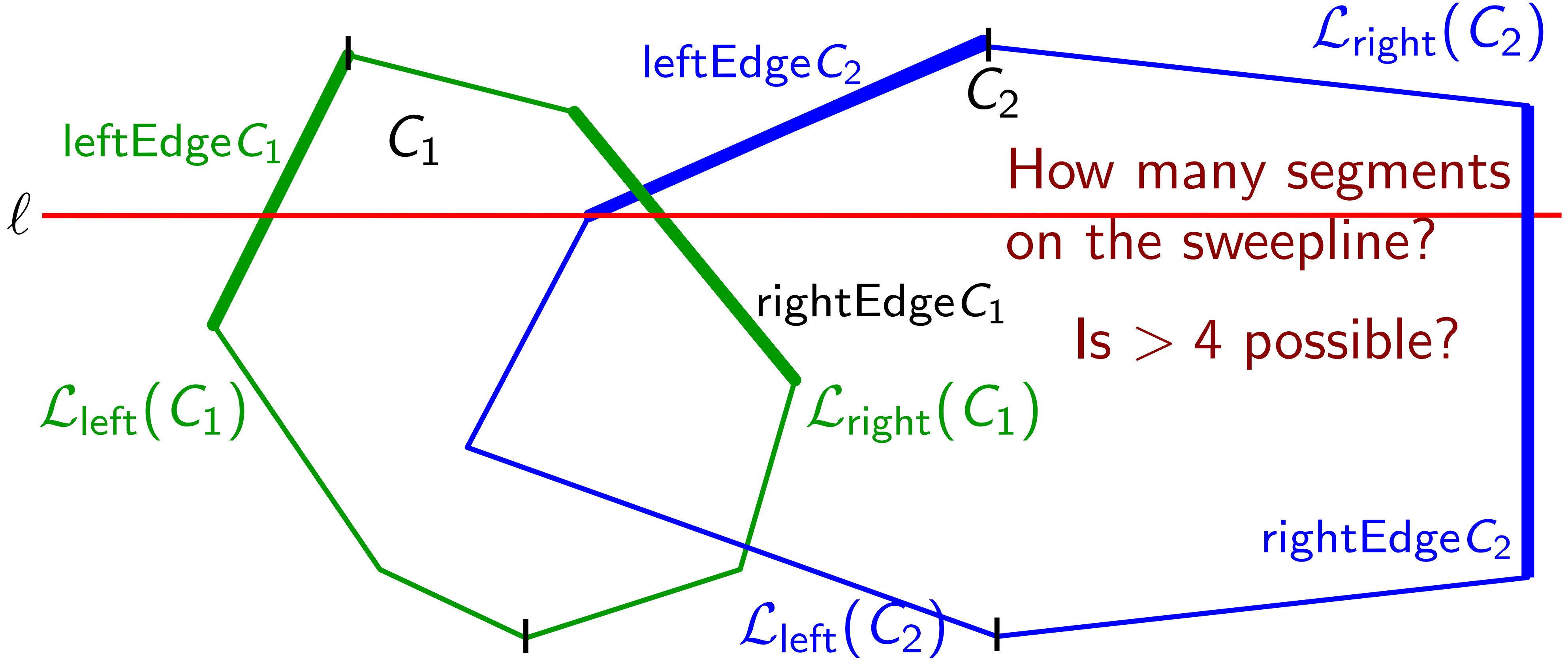

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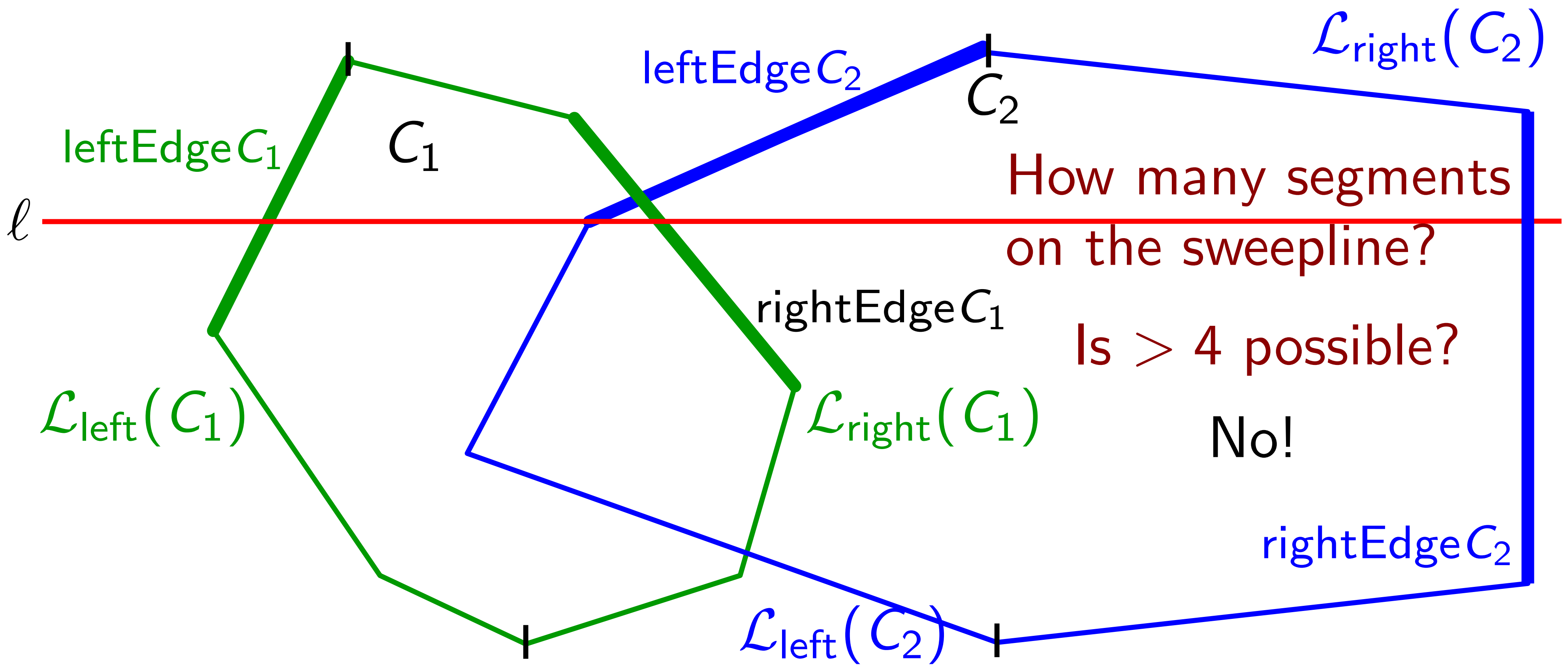
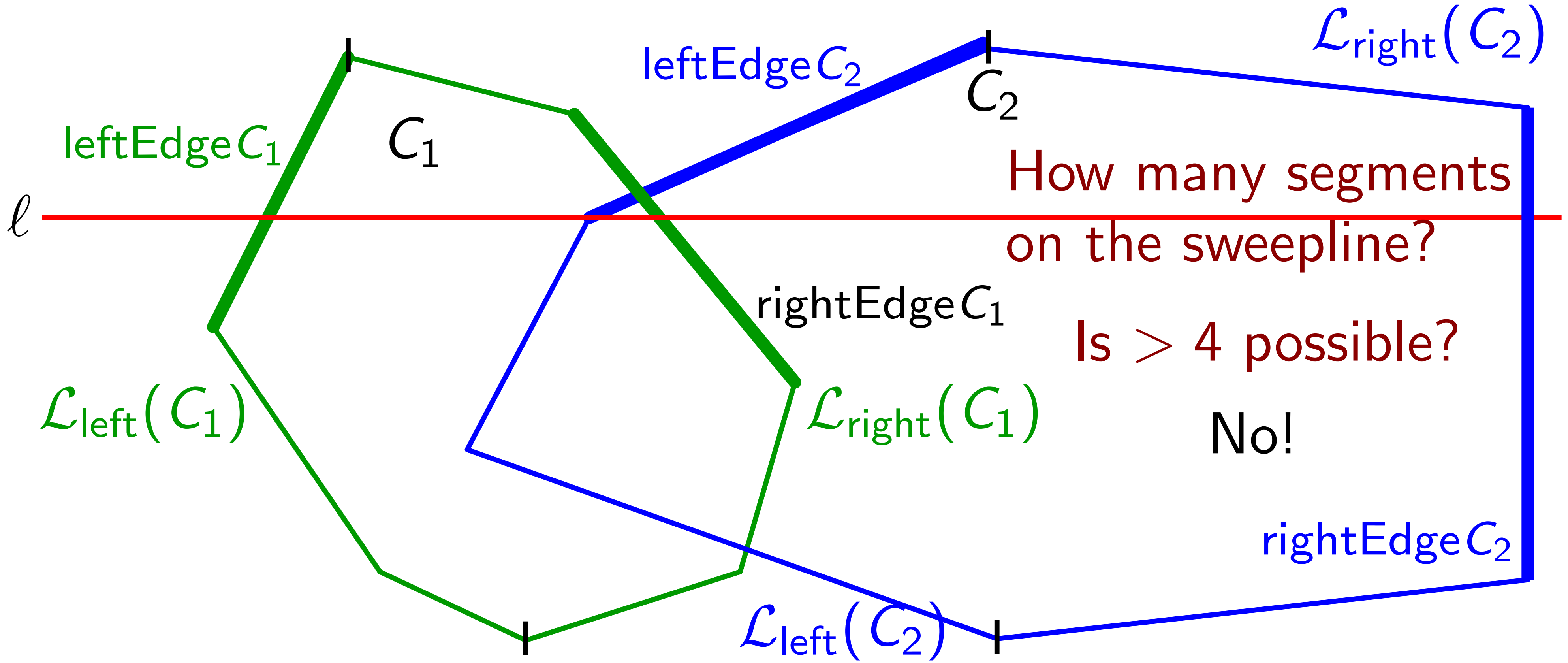


How many segments on the sweepline?

Is > 4 possible?

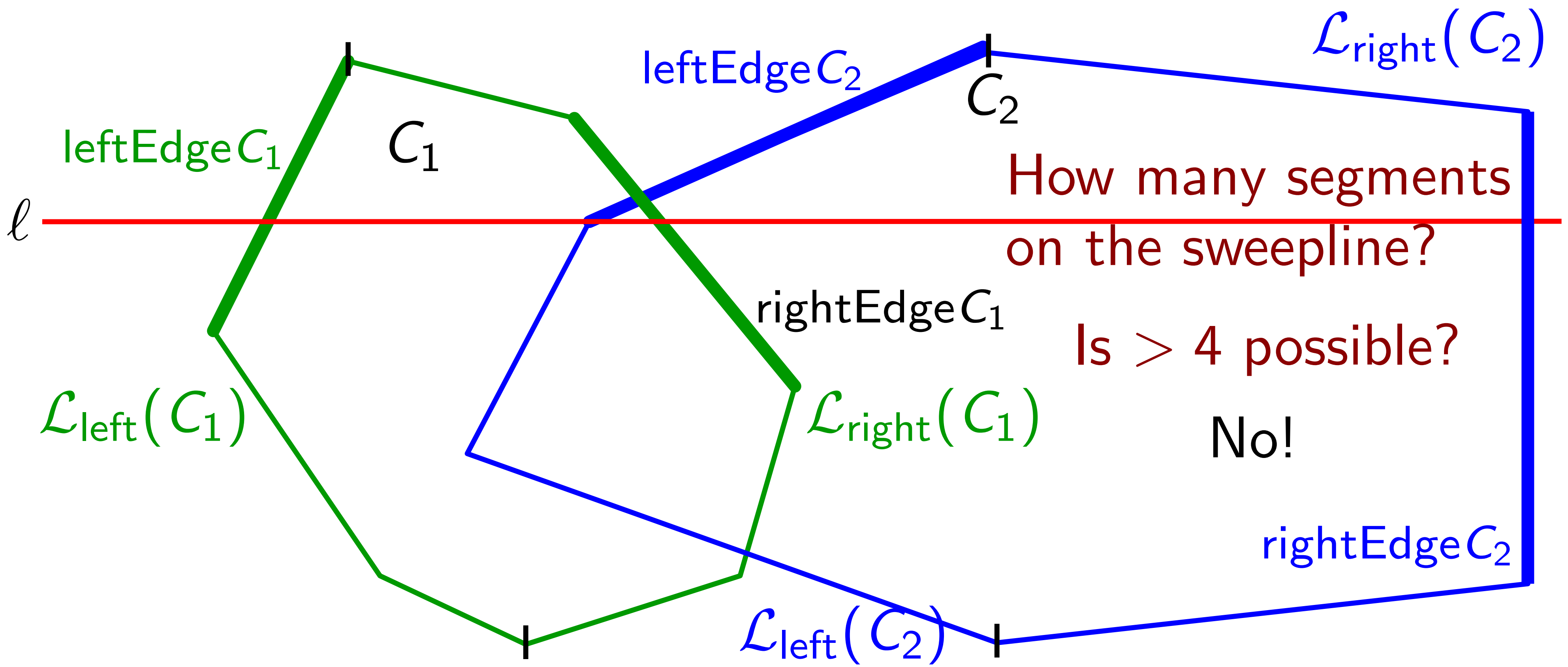
No!

Intersecting Convex Regions Faster



Theorem. The intersection of two convex polygonal regions can be computed in linear time.

Intersecting Convex Regions Faster



Theorem. The intersection of two convex polygonal regions can be computed in linear time.

Corollary. The intersection of n half planes can be computed in $O(n \log n)$ time.

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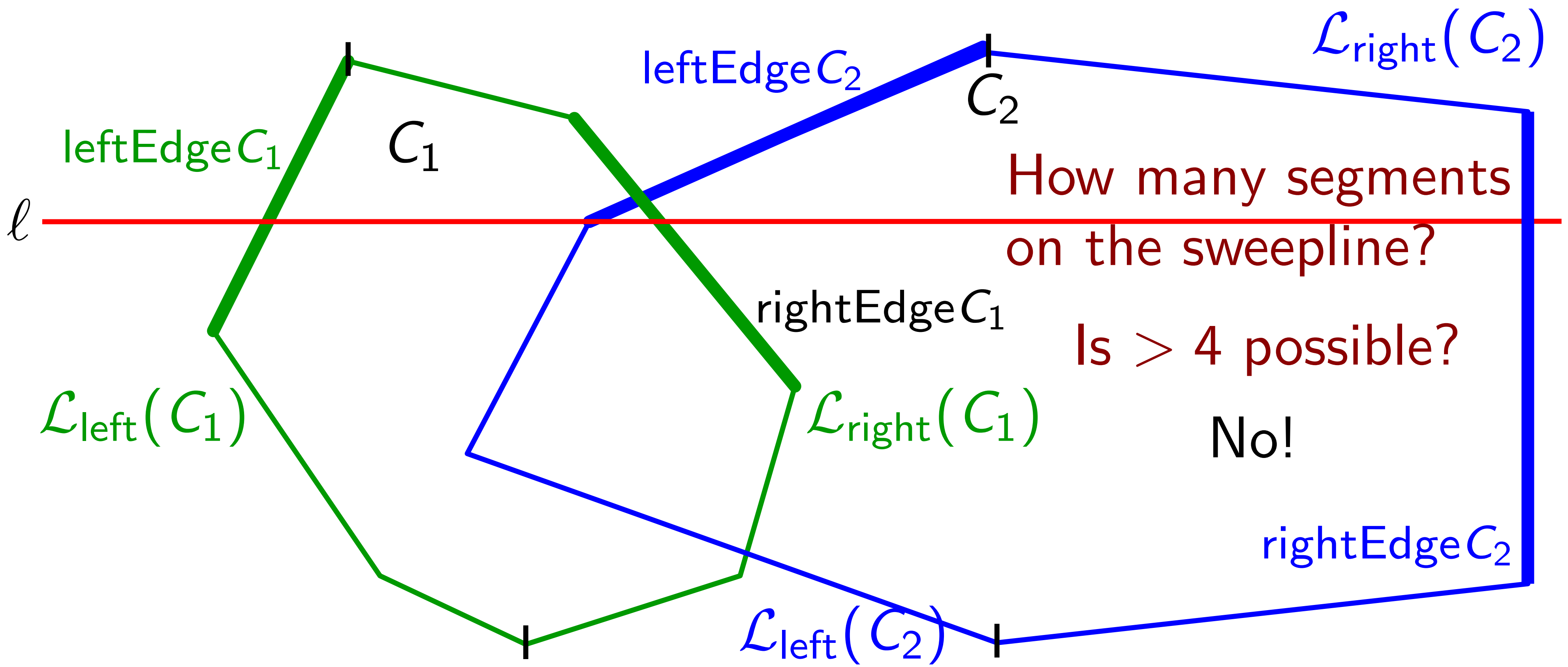

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Can we do better?

A Small Trick: Make Solution Unique

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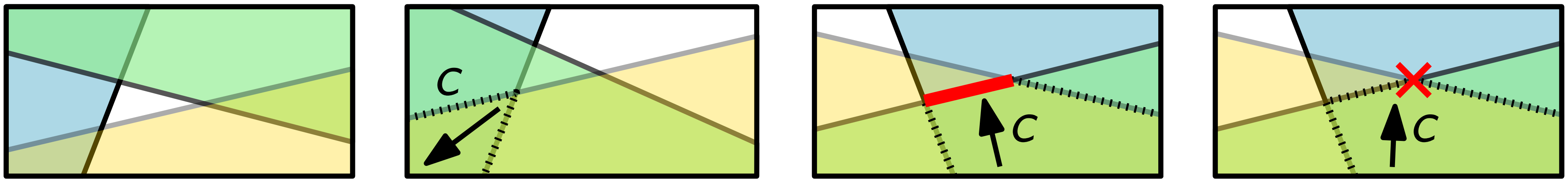

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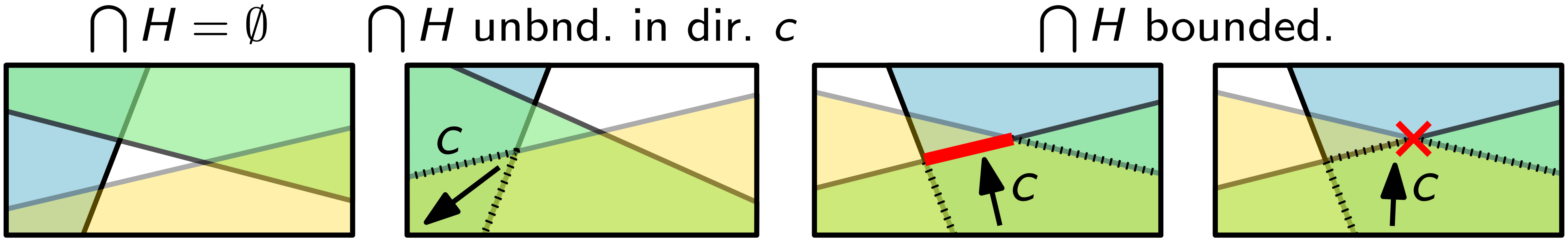

- Add two bounding halfplanes m_1 and m_2

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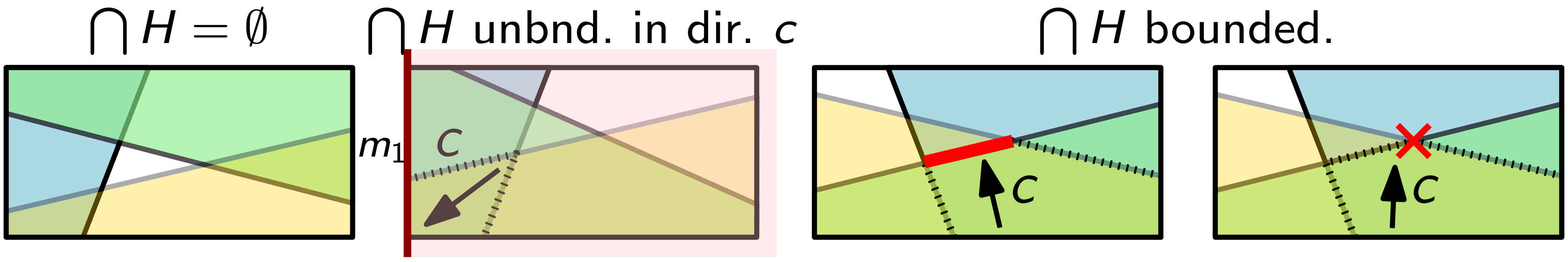

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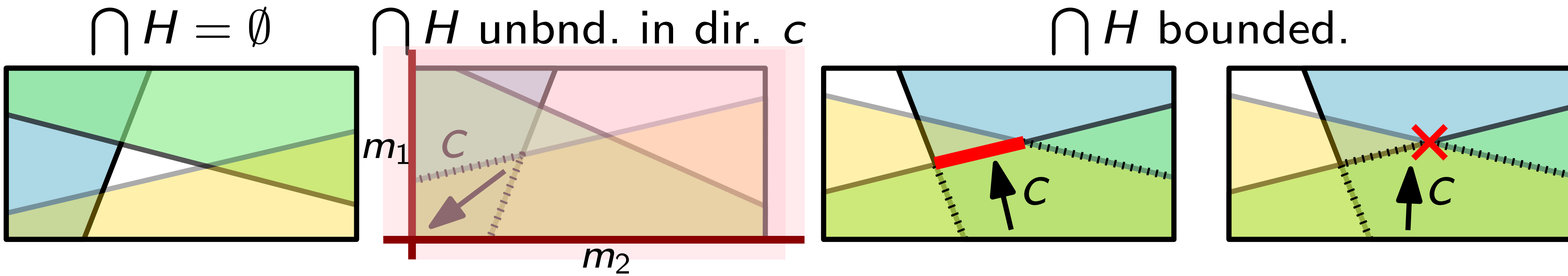

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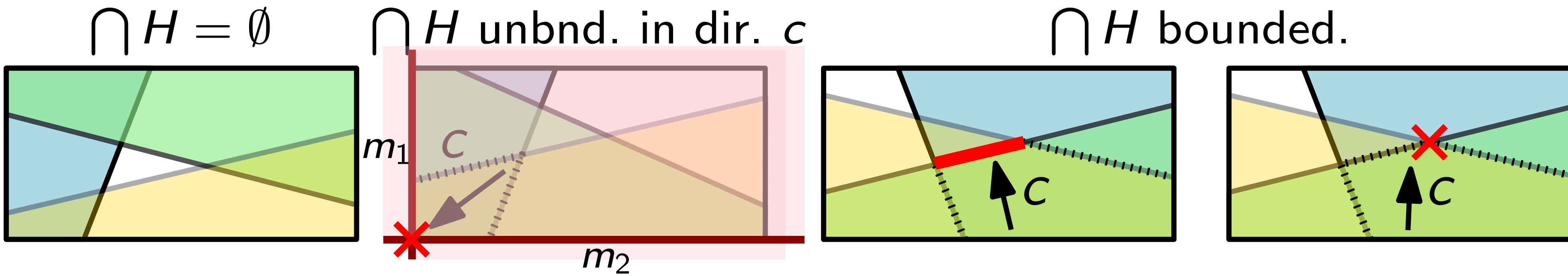

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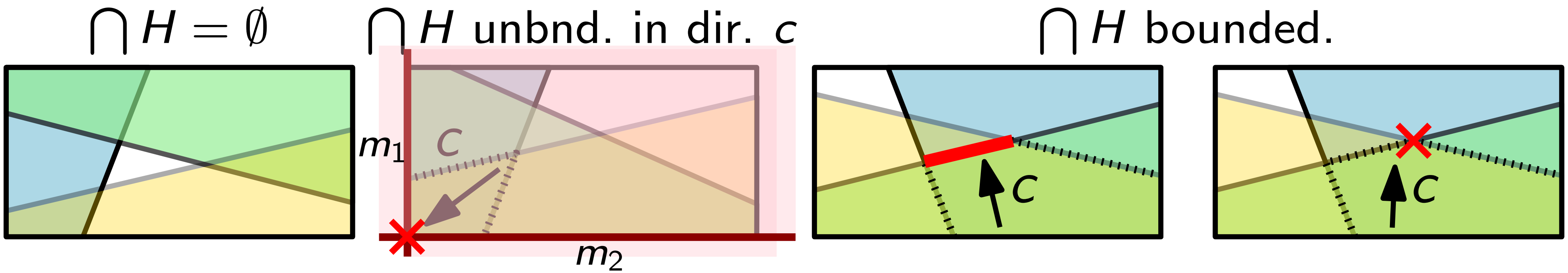

- Add two bounding halfplanes m_1 and m_2

$$m_1 = \begin{cases} x \le M & \text{if } c_x > 0, \\ x \ge M & \text{otherwise,} \end{cases} \quad \text{for some sufficiently large } M$$

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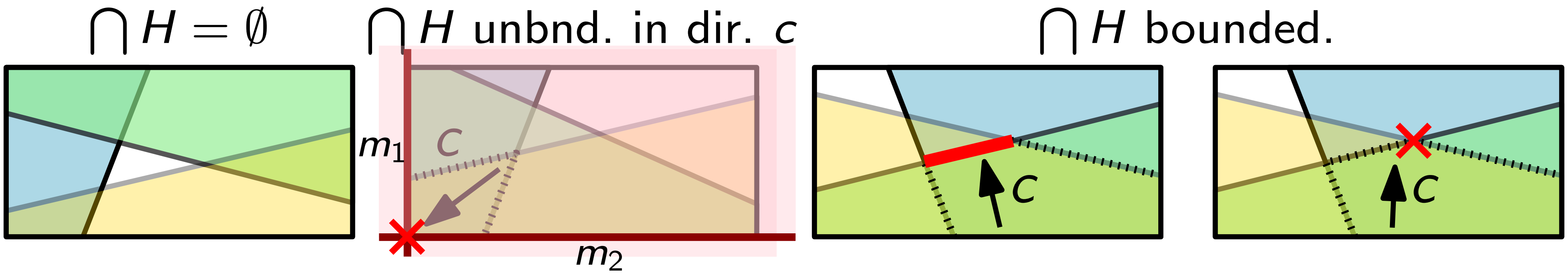

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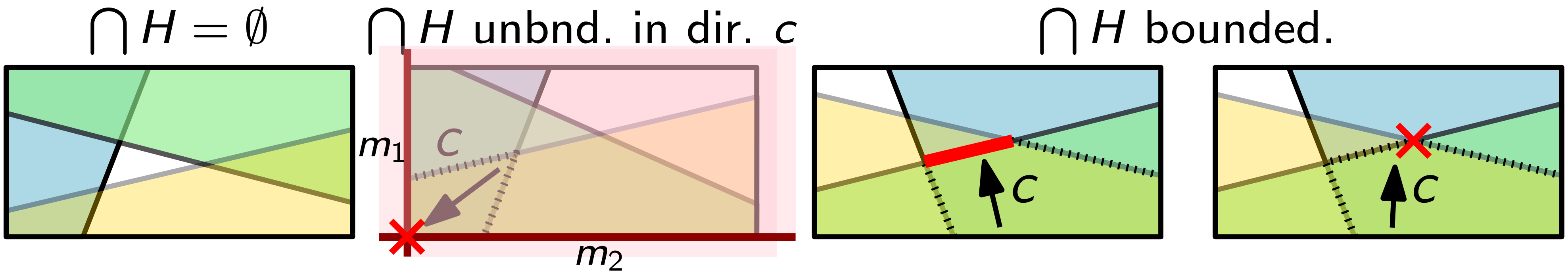

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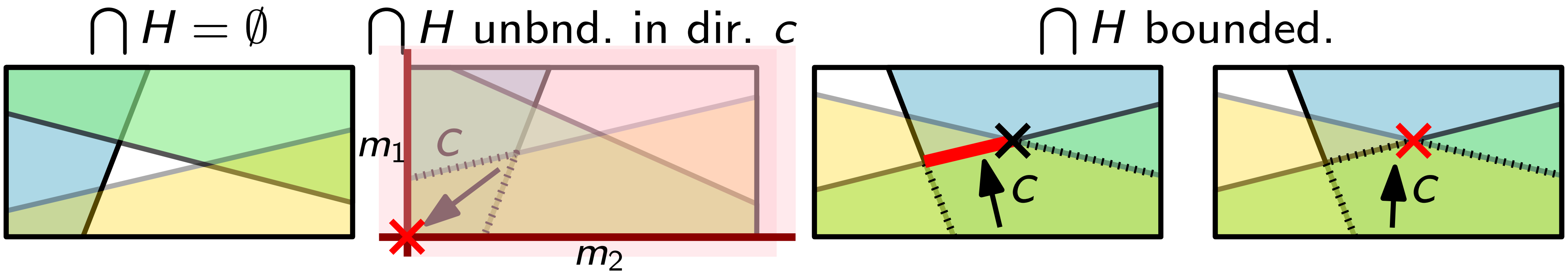

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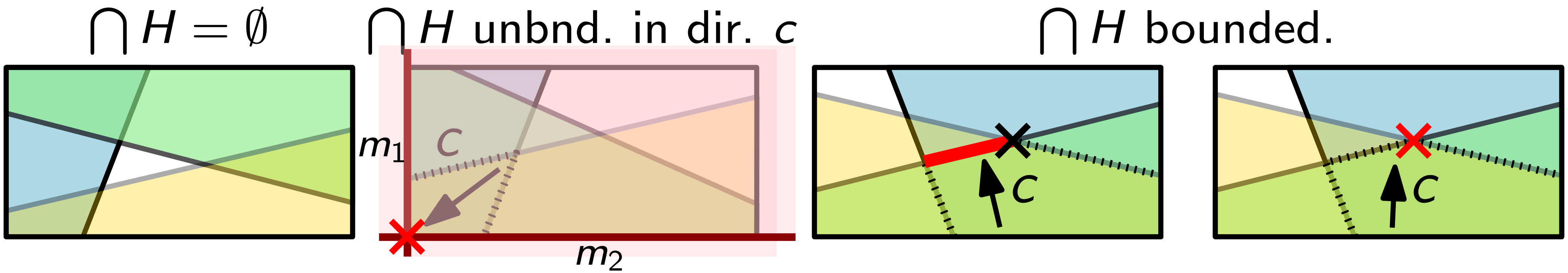

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- Take the lexicographically largest solution.

$\Rightarrow$ Set of solutions is either empty or a uniquely defined point.

Incremental Approach

Idea: Don't compute $\bigcap H$, but just *one* (optimal) point!

C_i = convex hull of H_i

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2dBoundedLP(H, c, m_1, m_2)

$H_0 = \{m_1, m_2\}$; $C_0 \leftarrow m_1 \cap m_2$
$v_0 \leftarrow$ unique optimal vertex of C_0 wrt obj.
for $i \leftarrow 1$ **to** n **do**
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What is this?

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w-c running time:

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  v_0 ← unique optimal vertex of C_0 wrt obj.
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      H_i = H_{i-1} ∪ {h_i}; C_i = C_{i-1} ∩ h_i
      if v_{i-1} ∈ h_i then
          v_i ← v_{i-1}
      else
          v_i ← 1dBoundedLP(π_{∂h_i}(H_{i-1}), π_{∂h_i}(c))
          if v_i = nil then
              return nil
  return v_n
```

w-c running time:

Incremental Approach

Idea: Don't compute $\bigcap H$, but just *one* (optimal) point!

C_i = convex hull of H_i

2dBoundedLP(H, c, m_1, m_2)

$H_0 = \{m_1, m_2\}$; $C_0 \leftarrow m_1 \cap m_2$
$v_0 \leftarrow$ unique optimal vertex of C_0 wrt obj.
for $i \leftarrow 1$ **to** n **do**
  $H_i = H_{i-1} \cup \{h_i\}$; $C_i = C_{i-1} \cap h_i$
  if $v_{i-1} \in h_i$ **then**
    $v_i \leftarrow v_{i-1}$ — $O(1)$
  else
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Randomized

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Incremental Approach

Idea: Don't compute $\bigcap H$, but just *one* (optimal) point!

Randomized

C_i = convex hull of H_i

2dBoundedLP(H, c, m_1, m_2)
    compute random permutation of H
    $H_0 = \{m_1, m_2\}$; $C_0 \leftarrow m_1 \cap m_2$
    $v_0 \leftarrow$ unique optimal vertex of C_0 wrt obj.
    for $i \leftarrow 1$ **to** n **do**
        $H_i = H_{i-1} \cup \{h_i\}$; $C_i = C_{i-1} \cap h_i$
        if $v_{i-1} \in h_i$ **then**
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Result

Theorem. The 2d bounded LP problem can be solved in $O(n)$ expected time.

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Proof. Let $X_i = \begin{cases} 1 & \text{if } v_{i-1} \notin h_i, \\ 0 & \text{else.} \end{cases}$ (indicator random variable).

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Proof. Let $X_i = \left\{ \begin{matrix} 1 & \text{if } v_{i-1} \notin h_i, \\ 0 & \text{else.} \end{matrix} \right\}$ (indicator random variable).

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$$\mathbf{E}[T_{2d}(n)] = \mathbf{E}\left[\sum_{i=1}^{n} (1 - X_i) \cdot O(1) + X_i \cdot O(i)\right]$$
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i.e., when $p_i \in \partial h_i$ and $p_i \in \partial h_j$ for exactly one $j < i$.

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Proof technique: *Backward analysis!*

$=$ probability that the optimal solution changes when h_i is removed from C_i.

$\leq 2/i$. This is independent of the choice of H_i. □