Computational Geometry
Winter term 2016/17

Triangulating Polygons
or
Guarding Art Galleries

Lecture #3

Joachim Spoerhase
Guarding an Art Gallery

Given a simple polygon $P$ (i.e., no holes, no self-intersection)...

![Diagram of a simple polygon](attachment:diagram.png)
Guarding an Art Gallery

Given a *simple* polygon $P$ (i.e., no holes, no self-intersection)…
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**Observation.** Camera $c$ “sees” a star-shaped region
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**Theorem.** 1. Every simple polygon can be triangulated.
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**Theorem.**
1. Every simple polygon can be triangulated.
2. Any triangulation of a simple polygon with $n$ vertices consists of $n - 2$ triangles.
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**Theorem.** How can we prove these?

1. Every simple polygon can be triangulated.
2. Any triangulation of a simple polygon with $n$ vertices consists of $n - 2$ triangles.
The Art Gallery Theorem

Theorem. For surveilling a simple polygon with \( n \) vertices, \( \lfloor n/3 \rfloor \) cameras are sometimes necessary and always sufficient.
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**Exercise.** Find, for arbitrarily large \( n \), a polygon with \( n \) vertices, where \( \approx n/3 \) cameras are necessary.
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[2 minutes]

[Chvátal ’75]

[dBCvKO’08]
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**To do:** Find algorithm for triangulating a simple polygon!
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**Brute force:**
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**To do:** Find algorithm for triangulating a simple polygon!

**Brute force:** follow existence proof, using recursion
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running time:
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running time: $O(n^2)$
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Faster triangulation in two steps:
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**Faster triangulation in two steps:**

$n$-vtx polygon
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n-vtx polygon  \rightarrow \text{“nice” pieces, } n’ \text{ vtc} \rightarrow \text{ } n'' \text{ triangles}
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Faster triangulation in two steps:

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$O(n \log n)$
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O(n \log n)  \rightarrow  O(n')
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**Definition.** A polygon $P$ is $y$-monotone if, for any horizontal line $\ell$, $\ell \cap P$ is connected.
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Faster triangulation in two steps:

<table>
<thead>
<tr>
<th>n-vtx polygon</th>
<th>“nice” pieces, ( n' ) vtc</th>
<th>( n'' ) triangles</th>
</tr>
</thead>
<tbody>
<tr>
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<td></td>
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Definition. A polygon \( P \) is \( y \)-monotone if, for any horizontal line \( \ell \), \( \ell \cap P \) is connected.
Partitioning a Polygon into Monotone Pieces

Idea: Classify vertices of given simple polygon $P$
Partitioning a Polygon into Monotone Pieces

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– *turn* vertices:

– *regular* vertices
Partitioning a Polygon into Monotone Pieces

**Idea:** Classify vertices of given simple polygon $P$

- **turn** vertices:
  
  vertical component of walking direction changes

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  if $\alpha < 180^\circ$

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Partitioning a Polygon into Monotone Pieces

Idea: Classify vertices of given simple polygon $P$

– *turn* vertices:
  vertical component of walking direction changes
  - start vertex
  - split vertex
    - if $\alpha < 180^\circ$
    - if $\beta > 180^\circ$

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**Idea:** Classify vertices of given simple polygon $P$

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  - *end* vertex

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  - **split vertex**
  
  - **end vertex**
  
  - **merge vertex**

- **regular vertices**
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- **turn vertices:**
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  - **start vertex**
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- **regular vertices**

Lemma: Let $P$ be a simple polygon. Then $P$ is $y$-monotone $\iff P$ has neither split vertices nor merge vertices.
Towards an Algorithm

Idea: Add diagonals to “destroy” split and merge vertices.
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1) Treating split vertices
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1) Treating split vertices

Connect $v$ to vertex $w^*$ having minimum $y$-coordinate among all vertices $w$ above $v$ and with $\text{left}(w) = \text{left}(v)$. 
Towards an Algorithm

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Think of a sweep-line algorithm:
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Think of a sweep-line algorithm:
Connect \( v \) to \( \text{helper}(\text{left}(v)) \).
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An Algorithm

2) Treating merge vertices
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\textbf{makeMonotone}(polygon P)

\( D \leftarrow \text{DCEL}(V(P), E(P)) \)

\( Q \leftarrow \text{priority queue on } V(P) \)

\( T \leftarrow \text{empty bin. search tree} \)
An Algorithm

2) Treating merge vertices

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\{doubly-connected edge list: data structure for planar subdivisions\}
An Algorithm

2) Treating merge vertices

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\begin{equation}
\{ \text{doubly-connected edge list: data structure for planar subdivisions} \}
\end{equation}
\[ (x, y) \prec (x', y') :\Leftrightarrow \]
\[ y > y' \lor (y = y' \land x < x') \]
An Algorithm

2) Treating merge vertices

makeMonotone \(\text{(polygon } P)\)

\(D \leftarrow \text{DCEL}(V(P), E(P))\)

\(Q \leftarrow \text{priority queue on } V(P)\)

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\[\begin{align*}
\text{while } Q \neq \emptyset \hspace{1em} \text{do} \\
\quad & v \leftarrow Q.\text{extractMax}() \\
\quad & \text{type} \leftarrow \text{type of vertex } v \\
\quad & \text{handleTypeVertex}(v) \\
\text{return } DCEL \ D
\end{align*}\]
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\text{makeMonotone}(\text{polygon } P) & \quad \mathcal{D} \leftarrow \text{DCEL}(V(P), E(P)) \\
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& \quad \text{type } \leftarrow \text{type of vertex } \mathcal{v} \\
& \quad \text{handleTypeVertex}(\mathcal{v}) \\
\text{return } \text{DCEL } \mathcal{D}
\end{align*}

\begin{align*}
\text{handleMergeVertex}(\text{vertex } \mathcal{v}) & \quad \mathcal{e} \leftarrow \text{edge following } \mathcal{v} \text{ cw} \\
& \quad \text{if helper}(\mathcal{e}) \text{ merge vtx then} \\
& \quad \quad \mathcal{D} . \text{insert}(\text{diag}(\mathcal{v}, \text{helper}(\mathcal{e}))) \\
& \quad \mathcal{T} . \text{delete}(\mathcal{e}) \\
& \quad \mathcal{e}' \leftarrow \mathcal{T} . \text{edgeLeftOf}(\mathcal{v}) \\
& \quad \text{if helper}(\mathcal{e}') \text{ merge vtx then} \\
& \quad \quad \mathcal{D} . \text{insert}(\text{diag}(\mathcal{v}, \text{helper}(\mathcal{e}'))) \\
& \quad \text{helper}(\mathcal{e}') \leftarrow \mathcal{v}
\end{align*}
An Algorithm

2) Treating merge vertices

makeMonotone(polygon P)
\[ D \leftarrow \text{DCEL}(V(P), E(P)) \]
\[ Q \leftarrow \text{priority queue on } V(P) \]
\[ T \leftarrow \text{empty bin. search tree} \]
while \( Q \neq \emptyset \) do
\[ v \leftarrow Q.\text{extractMax}() \]
\[ \text{type} \leftarrow \text{type of vertex } v \]
handleTypeVertex(v)
return DCEL \( D \)

handleMergeVertex(vertex v)
\[ e \leftarrow \text{edge following } v \text{ cw} \]
if helper(e) merge vtx then
\[ D.\text{insert}((v, \text{helper(e)})) \]
\[ T.\text{delete}(e) \]
\[ e' \leftarrow T.\text{edgeLeftOf}(v) \]
if helper(e') merge vtx then
\[ D.\text{insert}((v, \text{helper(e')}) \]
\[ \text{helper(e')} \leftarrow v \]
An Algorithm

2) Treating merge vertices

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\text{makeMonotone}(\text{polygon } P) \\
D \leftarrow \text{DCEL}(V(P), E(P)) \\
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\quad v \leftarrow Q.\text{extractMax}() \\
\quad \text{type} \leftarrow \text{type of vertex } v \\
\quad \text{handleTypeVertex}(v) \\
\text{return DCEL } D
\]

\[
\text{handleMergeVertex}(\text{vertex } v) \\
e \leftarrow \text{edge following } v \text{ cw} \\
\text{if } \text{helper}(e) \text{ merge vtx then} \\
\quad D.\text{insert(diag}(v, \text{helper}(e)))) \\
T.\text{delete}(e) \\
e' \leftarrow T.\text{edgeLeftOf}(v) \\
\text{if } \text{helper}(e') \text{ merge vtx then} \\
\quad D.\text{insert(diag}(v, \text{helper}(e')))) \\
\text{helper}(e') \leftarrow v\]
An Algorithm

2) Treating merge vertices

```
makeMonotone(polygon P)
D ← DCEL(V(P), E(P))
Q ← priority queue on V(P)
T ← empty bin. search tree
while Q ≠ ∅ do
    v ← Q.extractMax()
    type ← type of vertex v
    handleTypeVertex(v)
handleMergeVertex(vertex v)
e ← edge following v cw
if helper(e) merge vtx then
    D.insert(diag(v, helper(e)))
T.delete(e)
e' ← T.edgeLeftOf(v)
if helper(e') merge vtx then
    D.insert(diag(v, helper(e')))
helper(e') ← v
return DCEL D
```
An Algorithm

2) Treating merge vertices

makeMonotone(polygon P)
\[ \mathcal{D} \leftarrow \text{DCEL}(V(P), E(P)) \]
\[ Q \leftarrow \text{priority queue on } V(P) \]
\[ T \leftarrow \text{empty bin. search tree} \]
while \( Q \neq \emptyset \) do
\[ v \leftarrow Q.\text{extractMax}() \]
\[ \text{type} \leftarrow \text{type of vertex } v \]
handleTypeVertex(v)
return \text{DCEL} \( \mathcal{D} \)

handleMergeVertex(vertex v)
\[ e \leftarrow \text{edge following } v \text{ cw} \]
if helper(e) merge vtx then
\[ D.\text{insert}(\text{diag}(v, \text{helper}(e))) \]
\[ T.\text{delete}(e) \]
e' \leftarrow T.\text{edgeLeftOf}(v)
if helper(e') merge vtx then
\[ D.\text{insert}(\text{diag}(v, \text{helper}(e'))) \]
helper(e') \leftarrow v
An Algorithm

2) Treating merge vertices

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helper(e') ← v
```
An Algorithm

2) Treating merge vertices

\texttt{makeMonotone(polygon } P \texttt{)}
\begin{align*}
\mathcal{D} & \leftarrow \text{DCEL}(V(P), E(P)) \\
\mathcal{Q} & \leftarrow \text{priority queue on } V(P) \\
\mathcal{T} & \leftarrow \text{empty bin. search tree}
\end{align*}
\textbf{while } \mathcal{Q} \neq \emptyset \textbf{ do}
\begin{align*}
v & \leftarrow \mathcal{Q}.\text{extractMax}() \\
\text{type} & \leftarrow \text{type of vertex } v \\
\text{handleTypeVertex}(v)
\end{align*}
\textbf{return } \text{DCEL } \mathcal{D}

\texttt{handleMergeVertex(vertex } v \texttt{)}
\begin{align*}
e & \leftarrow \text{edge following } v \text{ cw} \\
\text{if } & \text{helper}(e) \text{ merge vtx } \text{ then} \\
& \mathcal{D}.\text{insert(diag}(v, \text{helper}(e))) \\
& \mathcal{T}.\text{delete}(e) \\
e' & \leftarrow \mathcal{T}.\text{edgeLeftOf}(v) \\
\text{if } & \text{helper}(e') \text{ merge vtx } \text{ then} \\
& \mathcal{D}.\text{insert(diag}(v, \text{helper}(e'))) \\
\text{helper}(e') & \leftarrow v
An Algorithm

2) Treating merge vertices

\textbf{makeMonotone(polygon } \ P) \quad D \leftarrow \text{DCEL(} V(P), \ E(P) \text{)}

Q \leftarrow \text{priority queue on } V(P)

\mathcal{T} \leftarrow \text{empty bin. search tree}

\textbf{while } Q \neq \emptyset \textbf{ do}

\quad v \leftarrow Q.\text{extractMax}()

\quad \text{type} \leftarrow \text{type of vertex } v

\quad \text{handleTypeVertex}(v)

\textbf{return } \text{DCEL } D

\textbf{handleMergeVertex(vertex } \ v) \quad e \leftarrow \text{edge following } v \text{ cw}

\quad \textbf{if } \text{helper}(e) \text{ merge vtx then}

\quad \quad \mathcal{D}.\text{insert(diag}(v, \text{helper}(e)))

\quad \mathcal{T}.\text{delete}(e)

\quad e' \leftarrow \mathcal{T}.\text{edgeLeftOf}(v)

\quad \textbf{if } \text{helper}(e') \text{ merge vtx then}

\quad \quad \mathcal{D}.\text{insert(diag}(v, \text{helper}(e')))

\quad \text{helper}(e') \leftarrow v
makeMonotone(polygon \( P \))
\[
\mathcal{D} \leftarrow \text{DCEL}(V(P), E(P))
\]
\( Q \leftarrow \text{priority queue on } V(P) \)
\( \mathcal{T} \leftarrow \text{empty bin. search tree} \)

while \( Q \neq \emptyset \) do
\[
\begin{align*}
\text{\( v \leftarrow Q.\text{extractMax}() \)} \\
\text{type} \leftarrow \text{type of vertex } v \\
\text{handleTypeVertex}(v)
\end{align*}
\]

handleMergeVertex(vertex \( v \))
\[
\begin{align*}
e \leftarrow \text{edge following } v \text{ cw} \\
\text{if } \text{helper}(e) \text{ merge vtx then} \\
\quad \mathcal{D}.\text{insert}(\text{diag}(v, \text{helper}(e))) \\
\quad \mathcal{T}.\text{delete}(e) \\
\text{if } \text{helper}(e') \text{ merge vtx then} \\
\quad \mathcal{D}.\text{insert}(\text{diag}(v, \text{helper}(e')))
\end{align*}
\]
\[
\begin{align*}
\text{helper}(e') & \leftarrow v
\end{align*}
\]

return DCEL \( \mathcal{D} \)
An Algorithm

2) Treating merge vertices

```
makeMonotone(polygon P)
D ← DCEL(V(P), E(P))
Q ← priority queue on V(P)
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```
An Algorithm

2) Treating merge vertices

\[
\text{makeMonotone}(\text{polygon } P) \\
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\text{return } \text{DCEL } D
\]

\[
\text{handleMergeVertex}(\text{vertex } v) \\
e \leftarrow \text{edge following } v \text{ cw} \\
\text{if } \text{helper}(e) \text{ merge vtx then} \\
\quad D.\text{insert}((\text{diag}(v, \text{helper}(e)))) \\
\quad T.\text{delete}(e) \\
\quad e' \leftarrow T.\text{edgeLeftOf}(v) \\
\text{if } \text{helper}(e') \text{ merge vtx then} \\
\quad D.\text{insert}((\text{diag}(v, \text{helper}(e')))) \\
\quad \text{helper}(e') \leftarrow v
\]
Analysis

Lemma. makeMonotone() adds a set of non-intersecting diagonals to $P$ such that $P$ is partitioned into $y$-monotone subpolygons.
Analysis

**Lemma.** makeMonotone() adds a set of non-intersecting diagonals to $P$ such that $P$ is partitioned into $y$-monotone subpolygons.

**Lemma.** A simple polygon with $n$ vertices can be subdivided into $y$-monotone polygons in $O(n \log n)$ time.
Triangulating a $y$-Monotone Polygon $P$

**Approach:** greedy, going from top to bottom
Triangulating a \( y\)-Monotone Polygon \( P \)

**Approach:** greedy, going from top to bottom
Triangulating a $y$-Monotone Polygon $P$

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Invariant?
Triangulating a $y$-Monotone Polygon $P$

**Approach:** greedy, going from top to bottom

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Invariant?
Triangulating a $y$-Monotone Polygon $P$

**Approach:** greedy, going from top to bottom

**Invariant?**

The part of $P$ that we have seen but not yet triangulated is a *funnel*. 
Triangulating a $y$-Monotone Polygon $P$

**Approach:** greedy, going from top to bottom

**Invariant?**

The part of $P$ that we have seen but not yet triangulated is a *funnel*.

chains of reflex vtc
Triangulating a \( y \)-Monotone Polygon \( P \)

**Approach:** greedy, going from top to bottom

**Invariant?**

The part of \( P \) that we have seen but not yet triangulated is a *funnel*.

*angle in \( P \) > 180°*

*reflex vtc*
Triangulating a y-Monotone Polygon \( P \)

**Approach:** greedy, going from top to bottom

**Invariant?**

The part of \( P \) that we have seen but not yet triangulated is a *funnel*.

- angle in \( P \) > 180°
- reflex vtc
- convex vtc.
- chains of reflex vtc
Triangulating a $y$-Monotone Polygon $P$

**Approach:** greedy, going from top to bottom

**Invariant?**

The part of $P$ that we have seen but not yet triangulated is a *funnel*.

Our funnels are special:

- angle in $P > 180^\circ$
- reflex vtc
- convex vtc.
Triangulating a $y$-Monotone Polygon $P$

**Approach:** greedy, going from top to bottom

The part of $P$ that we have seen but not yet triangulated is a *funnel*.

Our funnels are special: just 1 chain!

angle in $P > 180^\circ$

reflex vtc

convex vtc.
Triangulating a $y$-Monotone Polygon $P$

**Approach:** greedy, going from top to bottom

*Invariant?*

The part of $P$ that we have seen but not yet triangulated is a *funnel*.

Our funnels are special:

- **Convex vtc.**
- **Reflex vtc**

*Easy!*

*angle in $P > 180^\circ$*
Algorithm

TriangulateMonotonePolygon(Polygon $P$ as circular vertex list)
merge left and right chain $\rightarrow$ sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$
Stack $S$; $S$.push($u_1$); $S$.push($u_2$)
for $i \leftarrow 3$ to $n - 1$ do
Algorithm

**TriangulateMonotonePolygon** (Polygon $P$ as circular vertex list)

merge left and right chain $\rightarrow$ sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$

Stack $S$; $S$.push($u_1$); $S$.push($u_2$)

for $j \leftarrow 3$ to $n - 1$ do

\begin{enumerate}
\item if $u_j$ and $S$.top() lie on different chains then
\begin{enumerate}
\item while not $S$.empty() do
\end{enumerate}
\item else
\end{enumerate}
**Algorithm**

**TriangulateMonotonePolygon** (Polygon $P$ as circular vertex list)

merge left and right chain $\rightarrow$ sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$

Stack $S$; $S$.push($u_1$); $S$.push($u_2$)

**for** $j \leftarrow 3$ **to** $n - 1$ **do**

**if** $u_j$ and $S$.top() lie on different chains **then**

**while** not $S$.empty() **do**

$v \leftarrow S$.pop()

**if** not $S$.empty() **then** draw diag. $(u_j, v)$

**else**

$S$.push($u_j - 1$);

$S$.push($u_j$)

**draw diagonals from** $u_n$ **to** all vtc on $S$ **except** first and last one

$S$.top()
Algorithm

**TriangulateMonotonePolygon** (Polygon \( P \) as circular vertex list)

merge left and right chain → sequence \( u_1, \ldots, u_n \) with \( y_1 \geq \cdots \geq y_n \)

Stack \( S \); \( S \).push(\( u_1 \)); \( S \).push(\( u_2 \))

for \( j \leftarrow 3 \) to \( n - 1 \) do

- if \( u_j \) and \( S \).top() lie on different chains then
  - while not \( S \).empty() do
    - \( v \leftarrow S \).pop()
    - if not \( S \).empty() then draw diag. \( (u_j, v) \)

- else

- draw diagonals from \( u_n \) to all vtc on \( S \) except first and last one
Algorithm

**TriangulateMonotonePolygon**(Polygon $P$ as circular vertex list)
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draw diagonals from $u_n$ to all vtc on $S$ except first and last one
Algorithm

TriangulateMonotonePolygon(Polygon \( P \) as circular vertex list)
merge left and right chain \( \rightarrow \) sequence \( u_1, \ldots, u_n \) with \( y_1 \geq \cdots \geq y_n \)
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**TriangulateMonotonePolygon** (Polygon $P$ as circular vertex list)
merge left and right chain $\rightarrow$ sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$
Stack $S$; $S$.push($u_1$); $S$.push($u_2$)
for $j \leftarrow 3$ to $n - 1$ do
  if $u_j$ and $S$.top() lie on different chains then
    while not $S$.empty() do
      $v \leftarrow S$.pop()
      if not $S$.empty() then draw diag. ($u_j$, $v$)
  else
    $v \leftarrow S$.pop()
    while not $S$.empty() and $u_j$ sees $S$.top() do
      $v \leftarrow S$.pop()
      draw diagonal ($u_j$, $v$)
    $S$.push($v$);
    $S$.push($u_j$)

draw diagonals from $u_n$ to all vtc on $S$ except first and last one
Algorithm

TriangulateMonotonePolygon(Polygon \( P \) as circular vertex list)

merge left and right chain \( \rightarrow \) sequence \( u_1, \ldots, u_n \) with \( y_1 \geq \cdots \geq y_n \)

Stack \( S; S.push(u_1); S.push(u_2) \)

for \( j \leftarrow 3 \) to \( n - 1 \) do

    if \( u_j \) and \( S.top() \) lie on different chains then

        while not \( S.empty() \) do

            \( v \leftarrow S.pop() \)

            if not \( S.empty() \) then draw diag. \( (u_j, v) \)

        else

            \( v \leftarrow S.pop() \)

            while not \( S.empty() \) and \( u_j \) sees \( S.top() \) do

                \( v \leftarrow S.pop() \)

                draw diagonal \( (u_j, v) \)

            \( S.push(v) \);

            \( S.push(u_j) \)

else


draw diagonals from \( u_n \) to all vtc on \( S \) except first and last one
Algorithm

TriangulateMonotonePolygon(Polygon P as circular vertex list)
merge left and right chain → sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$
Stack $S$; $S$.push($u_1$); $S$.push($u_2$)
for $j \leftarrow 3$ to $n - 1$ do
  if $u_j$ and $S$.top() lie on different chains then
    while not $S$.empty() do
      $v \leftarrow S$.pop()
      if not $S$.empty() then draw diag. $(u_j, v)$

else
  draw diagonals from $u_n$ to all vtc on $S$ except first and last one
Algorithm

\textbf{TriangulateMonotonePolygon}(Polygon \( P \) as circular vertex list)
merge left and right chain \( \rightarrow \) sequence \( u_1, \ldots, u_n \) with \( y_1 \geq \cdots \geq y_n \)
Stack \( S; S.push(u_1); S.push(u_2) \)
for \( j \leftarrow 3 \) to \( n - 1 \) do
  \hspace{1em} if \( u_j \) and \( S.\text{top()} \) lie on different chains then
  \hspace{2em} while not \( S.\text{empty()} \) do
  \hspace{3em} \hspace{1em} \( v \leftarrow S.\text{pop()} \)
  \hspace{3em} \hspace{1em} if not \( S.\text{empty()} \) then draw diag. \((u_j, v)\)
  else
  \hspace{1em} .
  \hspace{1em} .
  \hspace{1em} .

draw diagonals from \( u_n \) to all vtc on \( S \) except first and last one
Algorithm

**TriangulateMonotonePolygon** (Polygon \( P \) as circular vertex list)

merge left and right chain \( \rightarrow \) sequence \( u_1, \ldots, u_n \) with \( y_1 \geq \cdots \geq y_n \)

Stack \( S \); \( S \).push\((u_1)\); \( S \).push\((u_2)\)

\[
\text{for } j \leftarrow 3 \text{ to } n - 1 \text{ do} \\
\hspace{1em} \text{if } u_j \text{ and } S\text{.top()} \text{ lie on different chains then} \\
\hspace{2em} \text{while not } S\text{.empty()} \text{ do} \\
\hspace{3em} v \leftarrow S\text{.pop()} \\
\hspace{3em} \text{if not } S\text{.empty()} \text{ then draw diag. } (u_j, v) \\
\hspace{2em} S\text{.push}(u_{j-1}); S\text{.push}(u_j) \\
\hspace{1em} \text{else} \\
\hspace{2em} \]

draw diagonals from \( u_n \) to all vtc on \( S \) except first and last one
Algorithm

**TriangulateMonotonePolygon** (Polygon \( P \) as circular vertex list)

merge left and right chain \( \rightarrow \) sequence \( u_1, \ldots, u_n \) with \( y_1 \geq \cdots \geq y_n \)

Stack \( S; \) \( S\.push(u_1); \) \( S\.push(u_2) \)

**for** \( j \leftarrow 3 \) **to** \( n - 1 \) **do**

\[ \text{if } u_j \text{ and } S\.top() \text{ lie on different chains then} \]
\[ \text{while not } S\.empty() \text{ do} \]
\[ v \leftarrow S\.pop() \]
\[ \text{if not } S\.empty() \text{ then draw diag. } (u_j, v) \]
\[ S\.push(u_{j-1}); S\.push(u_j) \]

**else**

\[ \text{draw diagonals from } u_n \text{ to all vtc on } S \text{ except first and last one} \]
Algorithm

**TriangulateMonotonePolygon** (Polygon $P$ as circular vertex list)
merge left and right chain $\rightarrow$ sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$
Stack $S$; $S$.push($u_1$); $S$.push($u_2$)

for $j \leftarrow 3$ to $n - 1$ do
  if $u_j$ and $S$.top() lie on different chains then
    while not $S$.empty() do
      $v \leftarrow S$.pop()
      if not $S$.empty() then draw diag. ($u_j$, $v$)
    $S$.push($u_{j-1}$); $S$.push($u_j$)
  else
    for each $u_j$ see $v$ on $S$ do
      draw diagonal ($u_j$, $v$)
    $S$.push($v$)

draw diagonals from $u_n$ to all vtc on $S$ except first and last one.
Algorithm

TriangulateMonotonePolygon(Polygon $P$ as circular vertex list)

merge left and right chain $\rightarrow$ sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$

Stack $S$; $S$.push($u_1$); $S$.push($u_2$)

for $j \leftarrow 3$ to $n - 1$ do
  if $u_j$ and $S$.top() lie on different chains then
    while not $S$.empty() do
      $v \leftarrow S$.pop()
      if not $S$.empty() then draw diag. $(u_j, v)$
    $S$.push($u_{j-1}$); $S$.push($u_j$)
  else
    $v \leftarrow S$.pop()

draw diagonals from $u_n$ to all vtc on $S$ except first and last one
**Algorithm**

**TriangulateMonotonePolygon** (Polygon $P$ as circular vertex list)

merge left and right chain → sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$

Stack $S$; $S$.push($u_1$); $S$.push($u_2$)

for $j \leftarrow 3$ to $n - 1$ do

  if $u_j$ and $S$.top() lie on different chains then

    while not $S$.empty() do

      $v \leftarrow S$.pop()

      if not $S$.empty() then draw diag. ($u_j, v$)

    $S$.push($u_{j-1}$); $S$.push($u_j$)

  else

    $v \leftarrow S$.pop()

    while not $S$.empty() and $u_j$ sees $S$.top() do

      $v \leftarrow S$.pop()

      draw diagonal ($u_j, v$)

  draw diagonals from $u_n$ to all vtc on $S$ except first and last one
Algorithm

TriangulateMonotonePolygon(Polygon \( P \) as circular vertex list)

merge left and right chain \( \rightarrow \) sequence \( u_1, \ldots, u_n \) with \( y_1 \geq \cdots \geq y_n \)
Stack \( S \); \( S.push(u_1) \); \( S.push(u_2) \)

for \( j \leftarrow 3 \) to \( n - 1 \) do

if \( u_j \) and \( S.top() \) lie on different chains then

\[ \begin{align*}
&\text{while not } S.empty() \text{ do} \\
&\quad \nu \leftarrow S.pop() \\
&\quad \text{if not } S.empty() \text{ then draw diag. } (u_j, \nu) \\
&\quad S.push(u_{j-1}); S.push(u_j)
\end{align*} \]

else

\[ \begin{align*}
&\nu \leftarrow S.pop() \\
&\text{while not } S.empty() \text{ and } u_j \text{ sees } S.top() \text{ do} \\
&\quad \nu \leftarrow S.pop() \\
&\quad \text{draw diagonal } (u_j, \nu)
\end{align*} \]
Algorithm

TriangulateMonotonePolygon(Polygon $P$ as circular vertex list)
merge left and right chain $\rightarrow$ sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$
Stack $S$; $S.push(u_1)$; $S.push(u_2)$
for $j \leftarrow 3$ to $n - 1$ do
  if $u_j$ and $S$.top() lie on different chains then
    while not $S$.empty() do
      $v \leftarrow S$.pop()
      if not $S$.empty() then draw diag. $(u_j, v)$
    $S$.push($u_{j-1}$); $S$.push($u_j$)
  else
    $v \leftarrow S$.pop()
    while not $S$.empty() and $u_j$ sees $S$.top() do
      $v \leftarrow S$.pop()
      draw diagonal $(u_j, v)$
    draw diagonals from $u_n$ to all vtc on $S$ except first and last one
Algorithm

**TriangulateMonotonePolygon** (Polygon $P$ as circular vertex list)

merge left and right chain $\rightarrow$ sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$

Stack $S$; $S$.push($u_1$); $S$.push($u_2$)

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      if not $S$.empty() then draw diag. $(u_j, v)$

  $S$.push($u_{j-1}$); $S$.push($u_j$)

else

  $v \leftarrow S$.pop()

  while not $S$.empty() and $u_j$ sees $S$.top() do

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    draw diagonal $(u_j, v)$

draw diagonals from $u_n$ to all vtc on $S$ except first and last one
Algorithm

TriangulateMonotonePolygon(Polygon $P$ as circular vertex list)
merge left and right chain $\rightarrow$ sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$
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Algorithm

`TriangulateMonotonePolygon(Polygon P as circular vertex list)`
merge left and right chain → sequence \( u_1, \ldots, u_n \) with \( y_1 \geq \cdots \geq y_n \)
Stack \( S; S.push(u_1); S.push(u_2) \)

for \( j \leftarrow 3 \) to \( n - 1 \) do
  if \( u_j \) and \( S.top() \) lie on different chains then
    while not \( S.empty() \) do
      \( v \leftarrow S.pop() \)
      if not \( S.empty() \) then draw diag. \( (u_j, v) \)
    \( S.push(u_{j-1}); S.push(u_j) \)
  else
    \( v \leftarrow S.pop() \)
    while not \( S.empty() \) and \( u_j \) sees \( S.top() \) do
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      draw diagonal \( (u_j, v) \)

draw diagonals from \( u_n \) to all vtc on \( S \) except first and last one
Algorithm

TriangulateMonotonePolygon(Polyigon \( P \) as circular vertex list)

merge left and right chain \( \rightarrow \) sequence \( u_1, \ldots, u_n \) with \( y_1 \geq \cdots \geq y_n \)

Stack \( S \); \( S.push(u_1) \); \( S.push(u_2) \)

for \( j \leftarrow 3 \) to \( n-1 \) do

\[ \text{if } u_j \text{ and } S.top() \text{ lie on different chains then} \]

\[ \text{while not } S.empty() \text{ do} \]

\[ v \leftarrow S.pop() \]

\[ \text{if not } S.empty() \text{ then draw diag. } (u_j, v) \]

\[ S.push(u_{j-1}); S.push(u_j) \]

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\[ \text{draw diagonal } (u_j, v) \]

draw diagonals from \( u_n \) to all vtc on \( S \) except first and last one
Algorithm

**TriangulateMonotonePolygon** (Polygon $P$ as circular vertex list)

merge left and right chain $\rightarrow$ sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$

Stack $S$; $S$.push($u_1$); $S$.push($u_2$)

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      if not $S$.empty() then draw diag. $(u_j, v)$

    $S$.push($u_{j-1}$); $S$.push($u_j$)

  else

    $v \leftarrow S$.pop()

    while not $S$.empty() and $u_j$ sees $S$.top() do

      $v \leftarrow S$.pop()

      draw diagonal $(u_j, v)$

  end if

end for

draw diagonals from $u_n$ to all vtc on $S$ except first and last one
Algorithm

TriangulateMonotonePolygon(Polygon $P$ as circular vertex list)

merge left and right chain $\rightarrow$ sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$

Stack $S$; $S$.push($u_1$); $S$.push($u_2$)

\begin{enumerate}
\item for $j \leftarrow 3$ to $n-1$ do
  \begin{enumerate}
  \item if $u_j$ and $S$.top() lie on different chains then
    \begin{enumerate}
    \item while not $S$.empty() do
      \begin{enumerate}
      \item $v \leftarrow S$.pop()
      \item if not $S$.empty() then draw diag. $(u_j, v)$
      \end{enumerate}
    \end{enumerate}
  \item $S$.push($u_{j-1}$); $S$.push($u_j$)
  \end{enumerate}
  \item else
    \begin{enumerate}
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      \end{enumerate}
    \end{enumerate}
    \item $S$.push($v$); $S$.push($u_j$)
  \end{enumerate}
\end{enumerate}
Algorithm

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draw diagonals from $u_n$ to all vtc on $S$ except first and last one
**Algorithm**

**TriangulateMonotonePolygon** *(Polygon $P$ as circular vertex list)*

merge left and right chain → sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$

Stack $S$; $S$.push($u_1$); $S$.push($u_2$)

for $j \leftarrow 3$ to $n - 1$ do

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  draw diagonals from $u_n$ to all vtc on $S$ except first and last one
Algorithm

TriangulateMonotonePolygon(Polygon \( P \) as circular vertex list)
merge left and right chain \( \rightarrow \) sequence \( u_1, \ldots, u_n \) with \( y_1 \geq \cdots \geq y_n \)
Stack \( S \); \( S.push(u_1); S.push(u_2) \)
for \( j \leftarrow 3 \) to \( n - 1 \) do
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    \( S.push(u_{j - 1}); S.push(u_j) \)
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    \( S.push(v); S.push(u_j) \)

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Algorithm

TriangulateMonotonePolygon(Polygon \( P \) as circular vertex list)

merge left and right chain \( \rightarrow \) sequence \( u_1, \ldots, u_n \) with \( y_1 \geq \cdots \geq y_n \)

Stack \( S \); \( S.push(u_1); S.push(u_2) \)

\( \text{for } j \leftarrow 3 \text{ to } n - 1 \text{ do} \)

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\( S.push(u_{j-1}); S.push(u_j) \)

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\( v \leftarrow S.pop() \)

\( \text{draw diagonal } (u_j, v) \)

\( S.push(v); S.push(u_j) \)

draw diagonals from \( u_n \) to all vtc on \( S \) except first and last one
Summary

- n-vtx polygon
- “nice” pieces
- $n'$ vtc
- $n''$ triangles

$O(n \log n)$

$O(n')$
A simple polygon with $n$ vertices can be subdivided into $y$-monotone polygons in $O(n \log n)$ time.
Lemma. A simple polygon with $n$ vertices can be subdivided into $y$-monotone polygons in $O(n \log n)$ time.

Lemma. A $y$-monotone polygon with $n$ vertices can be triangulated in $O(n)$ time.
A simple polygon with $n$ vertices can be subdivided into $y$-monotone polygons in $O(n \log n)$ time.

A $y$-monotone polygon with $n$ vertices can be triangulated in $O(n)$ time.

Subdividing a simple polygon with $n$ vertices by drawing $d$ (pairwise non-crossing) diagonals yields $d + 1$ simple polygons of total complexity $O(n)$. 

**Summary**

- **Lemma (old)**: A simple polygon with $n$ vertices can be subdivided into $y$-monotone polygons in $O(n \log n)$ time.
- **Lemma (new)**: A $y$-monotone polygon with $n$ vertices can be triangulated in $O(n)$ time.
- **Lemma (homework)**: Subdividing a simple polygon with $n$ vertices by drawing $d$ (pairwise non-crossing) diagonals yields $d + 1$ simple polygons of total complexity $O(n)$. 

**Diagram**

- $n$-vtx polygon $\rightarrow$ “nice” pieces $\rightarrow$ $O(n \log n)$
- $n'$ vtc $\rightarrow$ $n''$ triangles $\rightarrow$ $O(n')$
Summary

**Lemma.** A simple polygon with $n$ vertices can be subdivided into $y$-monotone polygons in $O(n \log n)$ time.

**Lemma.** A $y$-monotone polygon with $n$ vertices can be triangulated in $O(n)$ time.

**Lemma.** Subdividing a simple polygon with $n$ vertices by drawing $d$ (pairwise non-crossing) diagonals yields $d + 1$ simple polygons of total complexity $O(n)$.

**Theorem.** A simple polygon with $n$ vertices can be triangulated in $O(n \log n)$ time.
Summary

Lemma. A simple polygon with $n$ vertices can be subdivided into $y$-monotone polygons in $O(n \log n)$ time.

Lemma. A $y$-monotone polygon with $n$ vertices can be triangulated in $O(n)$ time.

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Is this it?
**Summary**

**Lemma.** A simple polygon with $n$ vertices can be subdivided into $y$-monotone polygons in $O(n \log n)$ time.

**Lemma.** A $y$-monotone polygon with $n$ vertices can be triangulated in $O(n)$ time.

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**Theorem.** A simple polygon with $n$ vertices can be triangulated in $O(n \log n)$ time.

**Is this it?** Tarjan & van Wyk [1988]:

```
n-vtx polygon ➔ “nice” pieces ➔ n’ vtc ➔ n’’ triangles
                O(n log n)                O(n’)
```
Summary

**Lemma.** A simple polygon with $n$ vertices can be subdivided into $y$-monotone polygons in $O(n \log n)$ time.

**Lemma.** A $y$-monotone polygon with $n$ vertices can be triangulated in $O(n)$ time.

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**Is this it?** Tarjan & van Wyk [1988]: $O(n \log \log n)$
**Summary**

**Lemma.** A simple polygon with $n$ vertices can be subdivided into $y$-monotone polygons in $O(n \log n)$ time.

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**Is this it?**

Tarjan & van Wyk [1988]: $O(n \log n)$

Clarkson, Tarjan, van Wyk [1989]: $O(n \log \log n)$
Summary

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Is this it? Tarjan & van Wyk [1988]: $O(n \log \log n)$
Clarkson, Tarjan, van Wyk [1989]: $O(n \log^* n)$
Summary

**Lemma.** A simple polygon with $n$ vertices can be subdivided into $y$-monotone polygons in $O(n \log n)$ time.

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**Is this it?** Tarjan & van Wyk [1988]: $O(n \log \log n)$
Clarkson, Tarjan, van Wyk [1989]: $O(n \log^* n)$
Chazelle [1991]: $O(n \log^* n)$
Summary

**Lemma.** A simple polygon with $n$ vertices can be subdivided into $y$-monotone polygons in $O(n \log n)$ time.

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**Is this it?**

Tarjan & van Wyk [1988]: $O(n \log \log n)$

Clarkson, Tarjan, van Wyk [1989]: $O(n \log^* n)$

Chazelle [1991]: $O(n)$
Summary

Lemma. A simple polygon with $n$ vertices can be subdivided into $y$-monotone polygons in $O(n \log n)$ time.

Lemma. A $y$-monotone polygon with $n$ vertices can be triangulated in $O(n)$ time.

Lemma. Subdividing a simple polygon with $n$ vertices by drawing $d$ (pairwise non-crossing) diagonals yields $d + 1$ simple polygons of total complexity $O(n)$.

Theorem. A simple polygon with $n$ vertices can be triangulated in $O(n \log n)$ time.

Is this it? Tarjan & van Wyk [1988]: $O(n \log \log n)$; Clarkson, Tarjan, van Wyk [1989]: $O(n \log^* n)$; Chazelle [1991]: $O(n)$; Kirkpatrick, Klawe, Tarjan [1992]: $O(n)$.
Summary

**Lemma.** A simple polygon with $n$ vertices can be subdivided into $y$-monotone polygons in $O(n \log n)$ time.

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**Is this it?**

- Tarjan & van Wyk [1988]: $O(n \log \log n)$
- Clarkson, Tarjan, van Wyk [1989]: $O(n \log^* n)$
- Chazelle [1991]: $O(n)$
- Kirkpatrick, Klawe, Tarjan [1992]: $O(n \log \log n)$
- Seidel [1991]: randomized