Algorithms for Graph Visualization

Summer Semester 2016
Lecture #10

The Crossing Lemma and its Applications

(based on the slides of Alexander Wolff and Philipp Kindermann)

DOI 10.1007/978-3-642-00856-6
Topological Graphs (figures from Proofs from THE BOOK)

Graph with a topological drawing, i.e.,
Topological Graphs  

Graph with a *topological drawing*, i.e.,

⇒ no edge is self-intersecting,
Topological Graphs

Graph with a *topological drawing*, i.e.,

- no edge is self-intersecting,
- Edges with common endpoints do not intersect,
Topological Graphs

Graph with a *topological drawing*, i.e.,

- no edge is self-intersecting,
- Edges with common endpoints do not intersect,
- two edges intersect at most once.

(figures from *Proofs from THE BOOK*)
Topological Graphs (figures from Proofs from THE BOOK)

Graph with a *topological drawing*, i.e.,

- no edge is self-intersecting,
- Edges with common endpoints do not intersect,
- two edges intersect at most once.

For a (topolog.) graph $G$ the *crossing number* of $G$ is:

$$\text{cr}(G) = \text{minimum number of crossings over all (topological) drawings of } G.$$
Topological Graphs

Graph with a *topological drawing*, i.e.,

- no edge is self-intersecting,
- Edges with common endpoints do not intersect,
- two edges intersect at most once.

For a (topolog.) graph $G$ the *crossing number* of $G$ is:

$$\text{cr}(G) = \text{minimum number of crossings over all (topological) drawings of } G.$$
Topological Graphs (figures from *Proofs from THE BOOK*)

Graph with a *topological drawing*, i.e.,

- no edge is self-intersecting,
- Edges with common endpoints do not intersect,
- two edges intersect at most once.

For a (topolog.) graph $G$ the *crossing number* of $G$ is:

$$\text{cr}(G) = \text{minimum number of crossings over all (topological) drawings of } G.$$  

*E.g.* $\text{cr}(K_{3,3}) = 1$
Topological Graphs (figures from Proofs from THE BOOK)

Graph with a topological drawing, i.e.,

- no edge is self-intersecting,
- Edges with common endpoints do not intersect,
- two edges intersect at most once.

For a (topolog.) graph $G$ the crossing number of $G$ is:

$\text{cr}(G) = \text{minimum number of crossings over all (topological) drawings of } G.$

E.g. $\text{cr}(K_{3,3}) = 1$

Remark Drawings realizing the crossing number are topological drawings.
Topological Graphs (figures from Proofs from THE BOOK)

Graph with a topological drawing, i.e.,

- no edge is self-intersecting,
- Edges with common endpoints do not intersect,
- two edges intersect at most once.

For a (topolog.) graph $G$ the crossing number of $G$ is:

$$\text{cr}(G) = \text{minimum number of crossings over all (topological) drawings of } G.$$  

E.g. $\text{cr}(K_{3,3}) = 1$

Remark: Drawings realizing the crossing number are topological drawings.

Remark: Wlog, in a topological drawing, at most two edges intersect at the same point.
Crossings

Computing $\text{cr}(G)$ is NP-hard. [Garey, Johnson, 1983]
Crossings

Computing $cr(G)$ is NP-hard. [Garey, Johnson, 1983]

In practice force-based methods, multidimensional scaling, and heuristics are used.
Crossings

- Computing $\text{cr}(G)$ is NP-hard. [Garey, Johnson, 1983]
- In practice force-based methods, multidimensional scaling, and heuristics are used.
- Planarization: Replace each crossing by a dummy vertex.
Crossings

▷ Computing $\text{cr}(G)$ is NP-hard. [Garey, Johnson, 1983]

▷ In practice force-based methods, multidimensional scaling, and heuristics are used.

▷ Planarization: Replace each crossing by a dummy vertex.

▷ Other variants are also NP-hard, e.g.:
  – One-sided crossing minimization between two layers
Crossings

- Computing $\text{cr}(G)$ is NP-hard. [Garey, Johnson, 1983]
- In practice force-based methods, multidimensional scaling, and heuristics are used.
- Planarization: Replace each crossing by a dummy vertex.
- Other variants are also NP-hard, e.g.:
  - One-sided crossing minimization between two layers
  - Fixed Linear Crossing Number [Masuda et al., 1990]

```
1  2  3  4  5  6
```
Crossings

- Computing \( \text{cr}(G) \) is NP-hard. [Garey, Johnson, 1983]
- In practice force-based methods, multidimensional scaling, and heuristics are used.
- Planarization: Replace each crossing by a dummy vertex.
- Other variants are also NP-hard, e.g.:
  - One-sided crossing minimization between two layers
  - Fixed Linear Crossing Number [Masuda et al., 1990]
Crossings

- Computing $\text{cr}(G)$ is NP-hard. [Garey, Johnson, 1983]
- In practice force-based methods, multidimensional scaling, and heuristics are used.
- Planarization: Replace each crossing by a dummy vertex.
- Other variants are also NP-hard, e.g.:
  - One-sided crossing minimization between two layers
  - Fixed Linear Crossing Number [Masuda et al., 1990]
Crossings

- Computing $\text{cr}(G)$ is NP-hard. [Garey, Johnson, 1983]
- In practice force-based methods, multidimensional scaling, and heuristics are used.
- Planarization: Replace each crossing by a dummy vertex.
- Other variants are also NP-hard, e.g.:
  - One-sided crossing minimization between two layers
  - Fixed Linear Crossing Number [Masuda et al., 1990]

![Diagram of a graph with crossings and vertices labeled 1 to 6. Only two options per edge: up/down.]

only two options per edge: up/down
Rectilinear (straight-line) Crossing Number

For a graph $G$ the *rectilinear crossing number* of $G$ is:

$$\overline{cr}(G) = \text{the minimum number of crossings in a straight-line drawing of } G.$$
Rectilinear (straight-line) Crossing Number

For a graph $G$ the *rectilinear crossing number* of $G$ is:

$$\text{cr}(G) = \text{the minimum number of crossings in a straight-line drawing of } G.$$  

Separation: $\text{cr}(K_8) = 18$
Rectilinear (straight-line) Crossing Number

For a graph $G$ the *rectilinear crossing number* of $G$ is:

$$\overline{cr}(G) = \text{the minimum number of crossings in a straight-line drawing of } G.$$  

Separation: $cr(K_8) = 18$, but $\overline{cr}(K_8) = 19.$
Rectilinear (straight-line) Crossing Number

For a graph $G$ the *rectilinear crossing number* of $G$ is:

$$\overline{cr}(G) = \text{the minimum number of crossings in a straight-line drawing of } G.$$  

Separation: $\cr(K_8) = 18$, but $\overline{cr}(K_8) = 19$.

**Obs.** For each $k \geq 4$ there is a graph $G_k$ with $\cr(G_k) = 4$ and $\overline{cr}(G_k) \geq k$. 
Rectilinear (straight-line) Crossing Number

For a graph $G$ the *rectilinear crossing number* of $G$ is:

$$\text{cr}(G) = \text{the minimum number of crossings in a straight-line drawing of } G.$$  

Separation: $\text{cr}(K_8) = 18$, but $\overline{\text{cr}}(K_8) = 19$.

**Obs.** For each $k \geq 4$ there is a graph $G_k$ with $\text{cr}(G_k) = 4$ and $\overline{\text{cr}}(G_k) \geq k$. 

![Graph $G_1$ with crossings highlighted](image-url)
Rectilinear (straight-line) Crossing Number

For a graph $G$ the *rectilinear crossing number* of $G$ is:

$$\overline{cr}(G) = \text{the minimum number of crossings in a straight-line drawing of } G.$$

Separation: $cr(K_8) = 18$, but $\overline{cr}(K_8) = 19$.

**Obs.** For each $k \geq 4$ there is a graph $G_k$ with $cr(G_k) = 4$ and $\overline{cr}(G_k) \geq k$.

Each straight-line drawing of $G_1$ at least one crossing of the following types: 

- or

$G_1$
Rectilinear (straight-line) Crossing Number

For a graph $G$ the rectilinear crossing number of $G$ is:

\[ \overline{cr}(G) = \text{the minimum number of crossings in a straight-line drawing of } G. \]

Separation: $cr(K_8) = 18$, but $\overline{cr}(K_8) = 19$.

**Obs.** For each $k \geq 4$ there is a graph $G_k$ with $cr(G_k) = 4$ and $\overline{cr}(G_k) \geq k$.

Each straight-line drawing of $G_1$ at least one crossing of the following types: 

$G_1 \rightarrow G_k$:
Rectilinear (straight-line) Crossing Number

For a graph $G$ the *rectilinear crossing number* of $G$ is:

$$\text{cr}(G) = \text{the minimum number of crossings in a straight-line drawing of } G.$$ 

Separation: $\text{cr}(K_8) = 18$, but $\overline{\text{cr}}(K_8) = 19$.

**Obs.** For each $k \geq 4$ there is a graph $G_k$ with $\text{cr}(G_k) = 4$ and $\overline{\text{cr}}(G_k) \geq k$.

Each straight-line drawing of $G_1$ at least one crossing of the following types:

$G_1 \rightarrow G_k$: $\rightarrow$ or $\rightarrow$
A first lower bound

\textbf{Obs}_1 \quad A \text{ drawing of a graph } G \text{ with } n \text{ vertices and } m \text{ edges has at least } m - 3n + 6 \text{ crossings.}
A first lower bound

\textbf{Obs}_1 \quad \text{A drawing of a graph } G \text{ with } n \text{ vertices and } m \text{ edges has at least } m - 3n + 6 \text{ crossings.}

\textbf{Proof.} (blackboard)
A first lower bound

**Obs**₁ A drawing of a graph $G$ with $n$ vertices and $m$ edges has at least $m - 3n + 6$ crossings.

**Proof.** (blackboard)
A first lower bound

\textbf{Obs}_1 \quad \text{A drawing of a graph } G \text{ with } n \text{ vertices and } m \text{ edges has at least } m - 3n + 6 \text{ crossings.}

\textbf{Proof.} \quad \text{(blackboard)
Tighter Bounds

Conj. [Erdős & Guy ’73]
\[ \text{cr}(G) \in \Omega\left(\frac{m^3}{n^2}\right). \]
Tighter Bounds

**Conj.** [Erdős & Guy ’73]
\[ cr(G) \in \Omega(m^3/n^2). \]

**Thm** \(_1\) [Ajtai, Chvátal, Newborn, Szemerédi ’82, Leighton ’84]
\[ m \geq 4n \Rightarrow cr(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}. \][Chazelle, Sharir, Welzl ...]

“BOOK” proof (blackboard)
Tighter Bounds

**Conj.** [Erdős & Guy ’73]
\[ \text{cr}(G) \in \Omega(m^3/n^2). \]

**Thm** \(_1\) [Ajtai, Chvátal, Newborn, Szemerédi ’82, Leighton ’84]
\[ m \geq 4n \Rightarrow \text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}. \]

**Remark** Bounds asymptotically sharp!

[Chazelle, Sharir, Welzl ...] “BOOK” proof (blackboard)
Tighter Bounds

**Conj.** [Erdős & Guy ’73]
\[ \text{cr}(G) \in \Omega(m^3/n^2). \]

**Thm**₁ [Ajtai, Chvátal, Newborn, Szemerédi ’82, Leighton ’84]
\[ m \geq 4n \Rightarrow \text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}. \] [Chazelle, Sharir, Welzl ...] “BOOK” proof (blackboard)

**Remark** Bounds asymptotically sharp!
Consider geom. graph with vertices \( v_0, \ldots, v_{n-1} \) in convex position and \( E = \{v_i v_j \mid i < j \leq i + k \text{ mod } n\} \) for \( 0 < k < n/2 \).

[exercise!]
Tighter Bounds

**Conj.** [Erdős & Guy ’73]
\[ \text{cr}(G) \in \Omega(m^3/n^2). \]

**Thm\textsubscript{1}** [Ajtai, Chvátal, Newborn, Szemerédi ’82, Leighton ’84]
\[ m \geq 4n \implies \text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}. \]

**Remark** Bounds asymptotically sharp!
Consider geom. graph with vertices \( v_0, \ldots, v_{n-1} \) in convex position and \( E = \{v_i v_j \mid i < j \leq i + k \text{ mod } n\} \) for \( 0 < k < n/2 \).

**Thm\textsubscript{2}** Improving the constants [Pach & Tóth ’97]
\[ m \geq 6n \implies \text{cr}(G) \geq \frac{1}{36} \cdot \frac{m^3}{n^2}. \]
Application 1: Point-Line Incidences

For points $P \subset \mathbb{R}^2$ and lines $\mathcal{L}$

$I(P, \mathcal{L}) =$ number of point-line incidences in $(P, \mathcal{L})$. 
Application 1: Point-Line Incidences

For points $P \subset \mathbb{R}^2$ and lines $\mathcal{L}$

$I(P, \mathcal{L}) =$ number of point-line incidences in $(P, \mathcal{L})$. 
Application 1: Point-Line Incidences

For points $P \subset \mathbb{R}^2$ and lines $\mathcal{L}$

$I(P, \mathcal{L}) =$ number of point-line incidences in $(P, \mathcal{L})$. 
Application 1: Point-Line Incidences

For points $P \subset \mathbb{R}^2$ and lines $\mathcal{L}$

$I(P, \mathcal{L}) =$ number of point-line incidences in $(P, \mathcal{L})$.

\[ \Rightarrow I(P, \mathcal{L}) = \]
Application 1: Point-Line Incidences

For points $P \subset \mathbb{R}^2$ and lines $\mathcal{L}$

$I(P, \mathcal{L}) = \text{number of point-line incidences in } (P, \mathcal{L})$. 

$\Rightarrow I(P, \mathcal{L}) =$
Application 1: Point-Line Incidences

For points $P \subset \mathbb{R}^2$ and lines $\mathcal{L}$

$I(P, \mathcal{L}) = \text{number of point-line incidences in } (P, \mathcal{L})$.

$\Rightarrow I(P, \mathcal{L}) = 10$
Application 1: Point-Line Incidences

For points \( P \subset \mathbb{R}^2 \) and lines \( \mathcal{L} \)

\[ I(P, \mathcal{L}) = \text{number of point-line incidences in } (P, \mathcal{L}). \]

\[ \Rightarrow I(P, \mathcal{L}) = 10 \]

Def. \( I(n, k) = \max_{|P|=n, |\mathcal{L}|=k} I(P, \mathcal{L}). \)
Application 1: Point-Line Incidences

For points $P \subset \mathbb{R}^2$ and lines $\mathcal{L}$

$I(P, \mathcal{L}) =$ number of point-line incidences in $(P, \mathcal{L})$.

Def. $I(n, k) = \max_{|P|=n, |\mathcal{L}|=k} I(P, \mathcal{L})$.

For example: $I(4, 4) =$
Application 1: Point-Line Incidences

For points \( P \subset \mathbb{R}^2 \) and lines \( \mathcal{L} \)
\[ I(P, \mathcal{L}) = \text{number of point-line incidences in } (P, \mathcal{L}). \]

For example:
\[ I(4, 4) = 9 \]

\[ \Rightarrow I(P, \mathcal{L}) = 10 \]
Application 1: Point-Line Incidences

For points $P \subset \mathbb{R}^2$ and lines $\mathcal{L}$

$I(P, \mathcal{L}) =$ number of point-line incidences in $(P, \mathcal{L})$.

\[
\Rightarrow I(P, \mathcal{L}) = 10
\]

Def. $I(n, k) = \max_{|P|=n, |\mathcal{L}|=k} I(P, \mathcal{L})$.

For example: $I(4, 4) = 9$

**Thm**$_3$ [Szemerédi & Trotter '83, Székely '97]

$I(n, k) \leq 2.7 n^{2/3} k^{2/3} + 6n + 2k$. 
Application 2: Unit Distances

For points $P \subset \mathbb{R}^2$ define

$U(P) =$ number of pairs in $P$ at unit distance

$U(n) = \max_{|P|=n} U(P)$. 
Application 2: Unit Distances

For points $P \subset \mathbb{R}^2$ define
$U(P) =$ number of pairs in $P$ at unit distance
$U(n) = \max_{|P|=n} U(P)$.

**Thm** [Spencer, Szemerédi, Trotter ’84, Székely ’97]
$U(n) < 6.7n^{4/3}$