Algorithms for Graph Visualization

Summer Semester 2016
Lecture #9

Planar Orientations
Tessellations and Visibility Representations
Topological Numbering

Let $G = (V, E)$ be a directed graph.

-topological numbering of $G$: mapping $\mu: V \rightarrow \mathbb{N}$ where $\mu(u) < \mu(v)$ for every edge $(u, v)$
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  *optimal* when: $\max_{v \in V} \mu(v) - \min_{v \in V} \mu(v)$ is minimized

- Can be calculated in $O(n + m)$ time.

Exercise!
$st$-graphs

$st$-graph: a directed *acyclic* graph $G = (V, E)$ with exactly one source and exactly one sink.
**st-graphs**

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$\implies$ $G$ numbered topologically: each path traverses nodes in increasing order.
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- For each vertex $v$, there is a directed $(s, t)$-path containing $v$. 
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**Planar st-graph**: an st-graph with a planar embedding such that $s$ and $t$ are on the outerface.
Planar $st$-graphs

$\Rightarrow$ Normally drawn upwards planar.
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- Normally drawn upwards planar.
- Two outerfaces $s^*/t^*$ left/right.
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- For each $e = (u, v) \in E$: 

![Graph Diagram]
Planar $st$-graphs

- Normally drawn upwards planar.
- Two outerfaces $s^*/t^*$ left/right.
- For each $e = (u, v) \in E$: $\text{orig}(e) = u$ and $\text{dest}(e) = v$
Planar $st$-graphs

- Normally drawn upwards planar.
- Two outerfaces $s^* / t^*$ left/right.
- For each $e = (u, v) \in E$: orig($e$) = $u$ and dest($e$) = $v$
  - left($e$), right($e$) $\in F$:
    - face left of $e$, right of $e$
Planar $st$-graphs

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- For each $e = (u, v) \in E$: $\text{orig}(e) = u$ and $\text{dest}(e) = v$,
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$$G^* = (V^* = F, E^*)$$
\[ e \in E \Rightarrow (\text{left}(e), \text{right}(e)) \in E^* \]
Planar $st$-graphs

- Normally drawn upwards planar.
- Two outerfaces $s^*/t^*$ left/right.
- For each $e = (u, v) \in E$:
  \[ \text{orig}(e) = u \text{ and } \text{dest}(e) = v \]
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  face left of $e$. right of $e$}
- $G^* = (V^* = F, E^*)$:
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Planar $st$-graphs

▶ Normally drawn upwards planar.

▶ Two outerfaces $s^*/t^*$ left/right.

▶ For each $e = (u, v) \in E$: 
  $\text{orig}(e) = u$ and $\text{dest}(e) = v$
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Planar $st$-graphs

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- $G^* = (V^* = F, E^*)$: $e \in E \Rightarrow (\text{left}(e), \text{right}(e)) \in E^*$
- Multigraph
Planar $st$-graphs

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- $G^* = (V^* = F, E^*)$: $e \in E \Rightarrow (\text{left}(e), \text{right}(e)) \in E^*$
- Multigraph
- $s^*t^*$-graph
Properties of Planar $st$-graphs

**Lemma** Every face $f$ of $G$ consists of two paths from its source $\text{orig}(f)$ to its sink $\text{dest}(f)$.
Properties of Planar $st$-graphs

**Lemma** 1  
Every face $f$ of $G$ consists of two paths from its source $\text{orig}(f)$ to its sink $\text{dest}(f)$.

**Proof:**

![Diagram showing a face $f$ with source $\text{orig}(f)$ and sink $\text{dest}(f)$, illustrating two paths within the face.](image)
Properties of Planar $st$-graphs

**Lemma** Every face $f$ of $G$ consists of two paths from its source $\text{orig}(f)$ to its sink $\text{dest}(f)$.

**Proof:** Assume: $\exists f$, violating the statement.
Properties of Planar $st$-graphs

**Lemma** 1. Every face $f$ of $G$ consists of two paths from its source $\text{orig}(f)$ to its sink $\text{dest}(f)$.

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Properties of Planar $st$-graphs

**Lemma**$_1$ Every face $f$ of $G$ consists of two paths from its source $\text{orig}(f)$ to its sink $\text{dest}(f)$.

**Proof:** Assume: $\exists f$, violating the statement.

$\implies G$ acyclic $\Rightarrow f$ has a source $q$
Properties of Planar $st$-graphs

**Lemma** Every face $f$ of $G$ consists of two paths from its source $\text{orig}(f)$ to its sink $\text{dest}(f)$.

**Proof:** Assume: $\exists f$, violating the statement.

$\Rightarrow \ G$ acyclic $\Rightarrow f$ has a source $q$

$\Rightarrow$ Follow the paths $\pi_{li}, \pi_{re}$ from $q$ along $\partial f$ until a reversed edge is found.
Properties of Planar \textit{st}-graphs

\textbf{Lemma}\textsubscript{1} Every face \(f\) of \(G\) consists of two paths from its source \(\text{orig}(f)\) to its sink \(\text{dest}(f)\).

\textbf{Proof:} Assume: \(\exists f\), violating the statement.
\(\Rightarrow\ G\ \text{acyclic} \Rightarrow f\ \text{has a source}\ q\)
\(\Rightarrow\ \text{Follow the paths}\ \pi_{\text{li}}, \pi_{\text{re}}\ \text{from} q\ \text{along}\ \partial f\ \text{until a reversed edge is found.}\)
\(\Rightarrow\ \text{Obs.}: \exists s, w\text{-path}\ \pi_s\ \text{and}\ u, t\text{-path}\ \pi_t\)
Properties of Planar $st$-graphs

**Lemma** Every face $f$ of $G$ consists of two paths from its source $\text{orig}(f)$ to its sink $\text{dest}(f)$.

**Proof:** Assume: $\exists f$, violating the statement.

- $G$ acyclic $\Rightarrow f$ has a source $q$
- Follow the paths $\pi_{li}, \pi_{re}$ from $q$ along $\partial f$ until a reversed edge is found.
- Obs.: $\exists s, w$-path $\pi_s$ and $u, t$-path $\pi_t$
- $G$ planar $\Rightarrow \exists$ vertex $x \in \pi_s \cap \pi_t$
Properties of Planar $st$-graphs

**Lemma** Every face $f$ of $G$ consists of two paths from its source $\text{orig}(f)$ to its sink $\text{dest}(f)$.

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- $G$ acyclic $\Rightarrow f$ has a source $q$
- Follow the paths $\pi_{li}, \pi_{re}$ from $q$ along $\partial f$ until a reversed edge is found.
- Obs.: $\exists s$, $w$-path $\pi_s$ and $u$, $t$-path $\pi_t$
- $G$ planar $\Rightarrow \exists$ vertex $x \in \pi_s \cap \pi_t$
- $\Rightarrow$ directed cycle

Other cases?
Properties of Planar \( st \)-graphs

**Lemma_1** Every face \( f \) of \( G \) consists of two paths from its source \( \text{orig}(f) \) to its sink \( \text{dest}(f) \).

**Lemma_2** At each vertex \( v \in V \) the incoming/outgoing edges each form an interval and these intervals are separated by the faces \( \text{left}(v)/\text{right}(v) \).
Properties of Planar \( st \)-graphs

**Lemma 1** Every face \( f \) of \( G \) consists of two paths from its source \( \text{orig}(f) \) to its sink \( \text{dest}(f) \).

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**Proof:**
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**Proof:**

\(\implies\) Clear for \(s\) and \(t\)
Properties of Planar $st$-graphs

**Lemma 1** Every face $f$ of $G$ consists of two paths from its source $\text{orig}(f)$ to its sink $\text{dest}(f)$.

**Lemma 2** At each vertex $v \in V$ the incoming/outgoing edges each form an interval and these intervals are separated by the faces $\text{left}(v)/\text{right}(v)$.

**Proof:**

$\Rightarrow$ Clear for $s$ and $t$

$\Rightarrow$ Let $v \in V \setminus \{s, t\}$, so that the statement is false.
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\textbf{Lemma}_1 \quad \text{Every face } f \text{ of } G \text{ consists of two paths from its source } \text{orig}(f) \text{ to its sink } \text{dest}(f).

\textbf{Lemma}_2 \quad \text{At each vertex } v \in V \text{ the incoming/outgoing edges each form an interval and these intervals are separated by the faces left}(v)/\text{right}(v).

\textbf{Proof:}

\begin{itemize}
  \item \text{Clear for } s \text{ and } t
  \item \text{Let } v \in V \setminus \{s, t\}, \text{ so that the statement is false.}
\end{itemize}
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**Lemma**$_1$ Every face $f$ of $G$ consists of two paths from its source $\text{orig}(f)$ to its sink $\text{dest}(f)$.

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Properties of Planar $st$-graphs

**Lemma** 1  Every face $f$ of $G$ consists of two paths from its source $\text{orig}(f)$ to its sink $\text{dest}(f)$.

**Lemma** 2  At each vertex $v \in V$ the incoming/outgoing edges each form an interval and these intervals are separated by the faces $\text{left}(v)/\text{right}(v)$.

**Proof:**

⇒ Clear for $s$ and $t$

⇒ Let $v \in V \setminus \{s, t\}$, so that the statement is false.

⇒ directed cycle
Properties of Planar $st$-graphs

**Lemma 1** Every face $f$ of $G$ consists of two paths from its source $\text{orig}(f)$ to its sink $\text{dest}(f)$.

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Properties of Planar \textit{st}-graphs

\textbf{Lemma}_1 \, \textit{Every face} \, f \, \textit{of} \, G \, \textit{consists of two paths from its source} \, \text{orig}(f) \, \textit{to its sink} \, \text{dest}(f). \n
\textbf{Lemma}_2 \, \textit{At each vertex} \, v \, \in \, V \, \textit{the incoming/outgoing edges each form an interval and these intervals are separated by the faces} \, \text{left}(v)/\text{right}(v). \n
\textit{Statements imply the same in the dual.}
Properties of Planar $st$-graphs

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Statements imply the same in the dual.
Properties of Planar $st$-graphs

Lemma 3 For faces $f$ and $g$ exactly one of the following is true:

- There is a path from $\text{dest}(f)$ to $\text{orig}(g)$ in $G$.
- There is a path from $\text{dest}(g)$ to $\text{orig}(f)$ in $G$.
- There is a path from $f$ to $g$ in $G^*$.
- There is a path from $g$ to $f$ in $G^*$.
Properties of Planar $st$-graphs

**Lemma** For faces $f$ and $g$ exactly one of the following is true:

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**Proof:**
Properties of Planar \textit{st}-graphs

\textbf{Lemma}_3 For faces \(f\) and \(g\) exactly one of the following is true:
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  \item There is a path from \(\text{dest}(f)\) to \(\text{orig}(g)\) in \(G\).
  \item There is a path from \(\text{dest}(g)\) to \(\text{orig}(f)\) in \(G\).
  \item There is a path from \(f\) to \(g\) in \(G^*\).
  \item There is a path from \(g\) to \(f\) in \(G^*\).
\end{itemize}

\textbf{Proof:}
\begin{itemize}
  \item Let \(\mu\) be a top. sort. of \(G\)
\end{itemize}
Properties of Planar $st$-graphs

**Lemma** For faces $f$ and $g$ exactly one of the following is true:

- There is a path from $\text{dest}(f)$ to $\text{orig}(g)$ in $G$.
- There is a path from $\text{dest}(g)$ to $\text{orig}(f)$ in $G$.
- There is a path from $f$ to $g$ in $G^*$.
- There is a path from $g$ to $f$ in $G^*$.

**Proof:**

- Let $\mu$ be a top. sort. of $G$
- Suppose: $\mu(\text{dest}(f)) < \mu(\text{orig}(g))$
Properties of Planar $st$-graphs

**Lemma $3$** For faces $f$ and $g$ exactly one of the following is true:

- There is a path from $\text{dest}(f)$ to $\text{orig}(g)$ in $G$.
- There is a path from $\text{dest}(g)$ to $\text{orig}(f)$ in $G$.
- There is a path from $f$ to $g$ in $G^*$.
- There is a path from $g$ to $f$ in $G^*$.

**Proof:**

- Let $\mu$ be a top. sort. of $G$
- Suppose: $\mu(\text{dest}(f)) < \mu(\text{orig}(g))$
- Leftmost path always follows the leftmost edge
  (similarly for rightmost path)
Properties of Planar \textit{st}-graphs

\textbf{Lemma$_3$} For faces \(f\) and \(g\) exactly one of the following is true:
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  \item There is a path from \(\text{dest}(f)\) to \(\text{orig}(g)\) in \(G\).
  \item There is a path from \(\text{dest}(g)\) to \(\text{orig}(f)\) in \(G\).
  \item There is a path from \(f\) to \(g\) in \(G^*\).
  \item There is a path from \(g\) to \(f\) in \(G^*\).
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\textbf{Proof:}
\begin{itemize}
  \item Let \(\mu\) be a top. sort. of \(G\)
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Properties of Planar $st$-graphs

**Lemma** For faces $f$ and $g$ exactly one of the following is true:

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- There is a path from $g$ to $f$ in $G^*$.

**Proof:**

- Let $\mu$ be a top. sort. of $G$
- Suppose: $\mu(\text{dest}(f)) < \mu(\text{orig}(g))$
- Leftmost path always follows the leftmost edge
  (similarly for rightmost path)
Properties of Planar $st$-graphs

**Lemma** Let $f$ and $g$ be faces in a planar $st$-graph $G$.

- There is a path from $\text{dest}(f)$ to $\text{orig}(g)$ in $G$.
- There is a path from $\text{dest}(g)$ to $\text{orig}(f)$ in $G$.
- There is a path from $f$ to $g$ in $G^*$.
- There is a path from $g$ to $f$ in $G^*$.

**Proof:**

- Let $\mu$ be a top. sort. of $G$.
- Suppose: $\mu(\text{dest}(f)) < \mu(\text{orig}(g))$.
- Leftmost path always follows the leftmost edge.
  (similarly for rightmost path.)
Properties of Planar $st$-graphs

**Lemma** For faces $f$ and $g$ exactly one of the following is true:

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- There is a path from $f$ to $g$ in $G^*$.
- There is a path from $g$ to $f$ in $G^*$.

For $v \in V$: let $\text{orig}(v) = \text{dest}(v) = v$;
For $f \in F$: let $\text{left}(f) = \text{right}(f) = f$. 
Properties of Planar $st$-graphs

**Lemma 3** For faces $f$ and $g$ exactly one of the following is true:

- There is a path from $\text{dest}(f)$ to $\text{orig}(g)$ in $G$.
- There is a path from $\text{dest}(g)$ to $\text{orig}(f)$ in $G$.
- There is a path from $f$ to $g$ in $G^*$.
- There is a path from $g$ to $f$ in $G^*$.

For $v \in V$: let $\text{orig}(v) = \text{dest}(v) = v$;
For $f \in F$: let $\text{left}(f) = \text{right}(f) = f$.

**Lemma 4** For objects $o_1, o_2 \in V \cup E \cup F$ exactly one of the following is true:

- There is a path from $\text{dest}(o_1)$ to $\text{orig}(o_2)$ in $G$.
- There is a path from $\text{dest}(o_2)$ to $\text{orig}(o_1)$ in $G$.
- There is a path from $\text{right}(o_1)$ to $\text{left}(o_2)$ in $G^*$.
- There is a path from $\text{right}(o_2)$ to $\text{left}(o_1)$ in $G^*$.

**Proof:** Exercise!
Tessellation / Tiling

Tiles: axis-parallel rectangles
Tessellation / Tiling

- Tiles: axis-parallel rectangles
- Can be unbounded, or degenerate (line segment/point)
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- \( \theta_1, \theta_2 \) horizontally/vertically adjacent \iff\ common vertical/horizontal boundary
Tessellation / Tiling

- Tiles: axis-parallel rectangles
- Can be unbounded, or degenerate (line segment/point)
- $\theta_1, \theta_2$ horizontally/vertically adjacent $\iff$ common vertical/horizontal boundary
- We write $\theta = [x_1(\theta), x_2(\theta)] \times [y_1(\theta), y_2(\theta)]$
Tessellation / Tiling

**Def.** A *tessellation* $\theta$ of a planar $st$-graph $G$ places each object $o \in V \cup E \cup F$ onto a *tile* $\theta(o)$, so that:
Tessellation / Tiling

**Def.** A *tessellation* $\theta$ of a planar $st$-graph $G$ places each object $o \in V \cup E \cup F$ onto a *tile* $\theta(o)$, so that:

$$o_1 \neq o_2 \Rightarrow \text{int}(\theta(o_1)) \cap \text{int}(\theta(o_2)) = \emptyset$$
**Tessellation / Tiling**

**Def.** A *tessellation* $\theta$ of a planar $st$-graph $G$ places each object $o \in V \cup E \cup F$ onto a *tile* $\theta(o)$, so that:

1. $o_1 \neq o_2 \implies \text{int}(\theta(o_1)) \cap \text{int}(\theta(o_2)) = \emptyset$
2. $\bigcup_{o \in V \cup E \cup F} \theta(o)$ is a rectangle.
Tessellation / Tiling

**Def.** A *tessellation* \( \theta \) of a planar *st*-graph \( G \) places each object \( o \in V \cup E \cup F \) onto a *tile* \( \theta(o) \), so that:

\[ o_1 \neq o_2 \Rightarrow \text{int}(\theta(o_1)) \cap \text{int}(\theta(o_2)) = \emptyset \]

\[ \bigcup_{o \in V \cup E \cup F} \theta(o) \text{ is a rectangle.} \]
Tessellation / Tiling

**Def.** A *tessellation* $\theta$ of a planar $st$-graph $G$ places each object $o \in V \cup E \cup F$ onto a *tile* $\theta(o)$, so that:

- $o_1 \neq o_2 \Rightarrow \text{int}(\theta(o_1)) \cap \text{int}(\theta(o_2)) = \emptyset$
- $\bigcup_{o \in V \cup E \cup F} \theta(o)$ is a rectangle.
- $\theta(o_1)$ and $\theta(o_2)$ *horizontally* adjacent $\iff$
  - $o_1 = \text{left}(o_2)$ or $o_1 = \text{right}(o_2)$ or
  - $o_2 = \text{left}(o_1)$ or $o_2 = \text{right}(o_1)$
Tessellation / Tiling

**Def.** A *tessellation* $\theta$ of a planar *st*-graph $G$ places each object $o \in V \cup E \cup F$ onto a *tile* $\theta(o)$, so that:

$\implies o_1 \neq o_2 \Rightarrow \text{int}(\theta(o_1)) \cap \text{int}(\theta(o_2)) = \emptyset$

$\implies \bigcup_{o \in V \cup E \cup F} \theta(o)$ is a rectangle.

$\implies \theta(o_1)$ and $\theta(o_2)$ *horizontally* adjacent $\iff$

$\begin{align*}
o_1 &= \text{left}(o_2) \text{ or } o_1 = \text{right}(o_2) \text{ or } \\
o_2 &= \text{left}(o_1) \text{ or } o_2 = \text{right}(o_1)
\end{align*}$
Tessellation / Tiling

**Def.** A *tessellation* $\theta$ of a planar *st*-graph $G$ places each object $o \in V \cup E \cup F$ onto a *tile* $\theta(o)$, so that:

1. $o_1 \neq o_2 \Rightarrow \text{int}(\theta(o_1)) \cap \text{int}(\theta(o_2)) = \emptyset$

2. $\bigcup_{o \in V \cup E \cup F} \theta(o)$ is a rectangle.

3. $\theta(o_1)$ and $\theta(o_2)$ *horizontally* adjacent $\iff$ $o_1 = \text{left}(o_2)$ or $o_1 = \text{right}(o_2)$ or $o_2 = \text{left}(o_1)$ or $o_2 = \text{right}(o_1)$
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\[ \theta(o_1) \text{ and } \theta(o_2) \text{ horizontally adjacent} \iff \]

\( o_1 = \text{left}(o_2) \text{ or } o_1 = \text{right}(o_2) \text{ or } o_2 = \text{left}(o_1) \text{ or } o_2 = \text{right}(o_1) \)

(neither \( o_1 \) nor \( o_2 \) has distinct neighbours!)
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   - $o_2 = \text{left}(o_1)$ or $o_2 = \text{right}(o_1)$
4. $\theta(o_1)$ and $\theta(o_2)$ *vertically* adjacent $\Leftrightarrow$
   - $o_1 = \text{orig}(o_2)$ or $o_1 = \text{dest}(o_2)$ or
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Tessellation Algorithm (planar $st$-graph $G$)

$\Rightarrow$ Compute the dual $G^*$. 
Tessellation Algorithm (planar $st$-graph $G$)

$\Rightarrow$ Compute the dual $G^\ast$.

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\[\begin{align*}
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& X \text{ of } G^* \text{ and } Y \text{ of } G. \\
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$\implies$ Compute the dual $G^*$.  
$\implies$ Compute topological numbering $X$ of $G^*$ and $Y$ of $G$.  
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$$\theta(o) = [X(\text{left}(o)), X(\text{right}(o))] \times [Y(\text{orig}(o)), Y(\text{dest}(o))].$$

vertices $\equiv$ horizontal segments  
faces $\equiv$ vertical segments  
edges $\equiv$ regions
Tessellation Algorithm (planar \(st\)-graph \(G\))

\[\text{Compute the dual } G^* .\]
\[\text{Compute topological numbering } X \text{ of } G^* \text{ and } Y \text{ of } G .\]
\[\text{For each object } o \in V \cup E \cup F \text{ set: } \theta(o) = [X(\text{left}(o)), X(\text{right}(o))] \times [Y(\text{orig}(o)), Y(\text{dest}(o))].\]

**Correctness:**

\[\text{Lemma}_4 \text{ guarantees disjointness.}\]
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**Correctness:**

- Lemma 4 guarantees disjointness.
- Neighbourhood conditions follow from the coordinate mapping.
Tessellation Algorithm (planar \textit{st}-graph \(G\))

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**Correctness:**

\begin{itemize}
  \item Lemma\(_4\) guarantees disjointness.
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\end{itemize}

**Runtime:**
Tessellation Algorithm (planar \(st\)-graph \(G\))

\[
\begin{align*}
\quad & \text{›› Compute the dual } \(G^*\).
\quad & \text{›› Compute topological numbering}
\quad & \begin{align*}
X \text{ of } G^* \text{ and } Y \text{ of } G.
\quad & \text{›› For each object } o \in V \cup E \cup F \text{ set:}
\quad & \theta(o) = [X(\text{left}(o)), X(\text{right}(o))] \times [Y(\text{orig}(o)), Y(\text{dest}(o))].
\end{align*}
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**Runtime:** \(O(n)\)
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**Correctness:**

- Lemma 4 guarantees disjointness.
- Neighbourhood conditions follow from the coordinate mapping.

**Runtime:** $O(n)$ but ... this is degenerate
Size Conditions

Minimum height/width $h, w : E \rightarrow \mathbb{R}_{\geq 0}$ for each edge tile.
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Minimum height/width $h, w : E \to \mathbb{R}_{\geq 0}$ for each edge tile.

Compute optimal weighted topological numbering $Y$ of $G = (V, E; h)$ and $X$ of $G^* = (F, E^*; w)$. 
Size Conditions

Minimum height/width \( h, w : E \rightarrow \mathbb{R}_{\geq 0} \) for each edge tile.

\( \Rightarrow \) Compute optimal *weighted* topological numbering \( Y \) of \( G = (V, E; h) \) and \( X \) of \( G^* = (F, E^*; w) \).

\( \Rightarrow \) Vertex/Face tiles: modify \( G \) to \( G' \)
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- Now each object of $G$ is an edge in $G'$. 
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- Vertex/Face tiles: modify $G$ to $G'$

- Now each object of $G$ is an edge in $G'$.

**Thm:** A minimum area tessellation of a planar $st$-graph $G$ with minimum height/width $h, w: V \cup E \cup F \rightarrow \mathbb{R}_{\geq 0}$ can be computed in $O(n)$ time.
Visibility Representations

**Def.** A visibility representation $\Gamma$ of a planar $st$-graph $G$ has

$\Rightarrow$ vertex $v$ as a horizontal segment $\Gamma(v)$
Visibility Representations

**Def.** A visibility representation \( \Gamma \) of a planar \( st \)-graph \( G \) has

- vertex \( v \) as a horizontal segment \( \Gamma(v) \)
- and edge \((u, v)\) as a vertical segment \( \Gamma(u, v) \)
Visibility Representations

**Def.** A visibility representation $\Gamma$ of a planar $st$-graph $G$ has

- vertex $v$ as a horizontal segment $\Gamma(v)$
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such that

- vertex segments are pairwise disjoint,
- edge segments are pairwise disjoint, and
Visibility Representations

**Def.** A visibility representation $\Gamma$ of a planar $st$-graph $G$ has

- vertex $v$ as a horizontal segment $\Gamma(v)$
- and edge $(u, v)$ as a vertical segment $\Gamma(u, v)$

such that

- vertex segments are pairwise disjoint,
- edge segments are pairwise disjoint, and

- the edge segment $\Gamma(u, v)$ starts from the top of the vertex segment $\Gamma(u)$, ends on the bottom of vertex segment $\Gamma(v)$, and does not intersect other vertex segments.
Computing a visibility representation
Computing a visibility representation

Use the tessellation; vertices are degenerate (i.e., line segments); faces are not degenerate
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Use the tessellation; vertices are degenerate (i.e., line segments); faces are not degenerate

edge segments
Algorithm Visibility(planar $st$-graph $G$)

$\Rightarrow$ Compute the dual $G^*$. 
Algorithm Visibility(planar \( st \)-graph \( G \))

\[
\begin{align*}
\triangleright & \quad \text{Compute the dual } G^*. \\
\triangleright & \quad \text{Compute an optimal weighted topological numbering } Y \text{ of } G, X \text{ of } G^* \text{ with } \textit{unit weights}
\end{align*}
\]
Algorithm Visibility(planar \textit{st}-graph \textit{G})

\begin{itemize}
\item Compute the dual \(G^*\).
\item Compute an optimal weighted topological numbering \(Y\) of \(G\), \(X\) of \(G^*\) with \textit{unit weights}
\item For each vertex \(v \in V\) set 
\(\Gamma(v) = [X(\text{left}(v)), X(\text{right}(v)) - 1] \times \{Y(v)\}\).
\end{itemize}
Algorithm Visibility(planar *st*-graph *G*)

\[\begin{align*}
\text{Compute the dual } G^\ast. \\
\text{Compute an optimal weighted topological numbering } Y \text{ of } G, \ X \text{ of } G^\ast \text{ with unit weights} \\
\text{For each vertex } v \in V \text{ set} \\
\Gamma(v) = [X(\text{left}(v)), X(\text{right}(v)) - 1] \times \{Y(v)\}. \\
\text{For each edge } e \in E \text{ set} \\
\Gamma(e) = \{X(\text{left}(e))\} \times [Y(\text{orig}(e)), Y(\text{dest}(e))].
\end{align*}\]
Algorithm Visibility(planar $st$-graph $G$)

$\Rightarrow$ Compute the dual $G^*$.  
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$\Rightarrow$ 

$\Rightarrow$
Algorithm Visibility(planar st-graph G)

- Compute the dual $G^*$.
- Compute an optimal weighted topological numbering $Y$ of $G$, $X$ of $G^*$ with *unit weights*
Algorithm Visibility(planar $st$-graph $G$)

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**Thm:** In $O(n)$ time, the Visibility algorithm generates a visibility representation with integer coordinates and at most $O(n^2)$ total area.