Algorithms for Graph Visualization

Summer Semester 2016
Lecture #5

Upward Planar Drawings

(based on slides from Martin Nöllenburg and Robert Görke, KIT)
The Problem

**Definition.**

A directed Graph $D = (V, A)$ is *upward planar*, when it has a drawing such that:

- all edges are upward $y$-monotone curves, and
- no two edges cross.

**Obvious requirements?**

- Planar & acyclic.

*not sufficient!*
Problem: Upward Planarity Testing

Given a directed acyclic graph $D = (V, A)$. Determine if $D$ is upward planar. If so, construct a corresponding drawing.

NP-hard! [Garg & Tamassia ’95]

Problem’: Embedded Upward Planarity Testing

Given an acyclic graph $D = (V, A)$ with an embedding $\mathcal{F}, f_0$. Determine if $D$ is upward planar with respect to $\mathcal{F}, f_0$. If so, construct a corresponding drawing.

Can be tested efficiently! [this lecture]
The Big Picture: a characterization

**Theorem** [Kelly ’87, Di Battista & Tamassia ’88]

For a directed graph $D = (V, A)$, the following are equivalent.

1. $D$ is upward planar.
2. $D$ has a *straight-line* upward planar drawing.
3. $D$ is a spanning subgraph of a planar *st-graph*.

Additionally:

- embedded so that $s$ and $t$ are on the outer-face $f_0$.
- acyclic directed graph with a single source $s$ and single sink $t$.

\[\begin{align*}
\text{without crossings}
\end{align*}\]
The Big Picture: a characterization

**Theorem** [Kelly ’87, Di Battista & Tamassia ’88]

For a directed graph $D = (V, A)$, the following are equivalent.

1. $D$ is upward planar.
2. $D$ has a *straight-line* upward planar drawing.
3. $D$ is a spanning subgraph of a planar st-graph.

*Proof in textbook [DETT, Sec. 6.1]*

*can be drawn upward planar, see textbook [DETT, Sec. 6.1]*
Bimodality

**Lemma**
An embedded directed graph is upward planar only if it is bimodal.

**Definition**
An embedded directed graph is *bimodal* ⇔ all vertices are bimodal.
Angle Sizes of Sources and Sinks

For a face $f$ of a straight-line drawing, consider angles of
– local sinks (vertices with 2 incoming edges on $\partial f$)
– local sources (vertices with 2 outgoing edges on $\partial f$)

\[ L(f) := \text{number of large angles} \quad (\text{Intuition: in drawing} > \pi) \]
\[ S(f) := \text{number of small angles} \]
\[ A(f) := \text{number of local sources} \quad (= \text{number of local sinks}) \]

Thus:
\[ L(f) + S(f) = 2A(f) \]

By induction:
\[ L(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases} \Rightarrow \]
\[ A(f) = \begin{cases} A(f) - 1, & f \neq f_0 \\ A(f) + 1, & f = f_0 \end{cases} \]

Proof: \( L(f) - S(f) = -2 \) for \( f \neq f_0 \)

\[ \Rightarrow L(f) = 0 \quad \Rightarrow S(f) = 2 \]

\[ \Rightarrow L(f) \geq 1 \]

Separate \( f \) by.

5. \( v \) neither source nor sink:

\[
L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1) = -2
\]

induction hypothesis
Observations

Consider the angle between two incoming/outgoing edges.

**Lemma**

Let $D$ be a directed graph.

In every upward planar drawing of $D$:

1. For each vertex $v \in V$: \( L(v) = \begin{cases} 0 & v \text{ inner vertex}, \\ 1 & v \text{ source/sink}. \end{cases} \)

2. For each face $f \in \mathcal{F}$: \( L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases} \)

\[ \Phi: S \cup T \to \mathcal{F} \quad v \mapsto \text{incid. face} \]

\[ |\Phi^{-1}(f)| = \begin{cases} A(f) - 1 & f \neq f_0 \\ A(f) + 1 & f = f_0 \end{cases} \]

called *consistent* global sources and sinks
Example: Face Assignment

Assignment \( \phi : S \cup T \rightarrow \mathcal{F} \)

- Global sources and sinks

\[
\begin{align*}
A(f_1) &= 3 \\
L(f_1) &= 2 \\
A(f_3) &= 1 \\
L(f_3) &= 0 \\
A(f_4) &= 2 \\
L(f_4) &= 1 \\
A(f_5) &= 2 \\
L(f_5) &= 1 \\
A(f_6) &= 1 \\
L(f_6) &= 0 \\
A(f_7) &= 2 \\
L(f_7) &= 1 \\
A(f_8) &= 1 \\
L(f_8) &= 0 \\
A(f_9) &= 1 \\
L(f_9) &= 0 \\
A(f_0) &= 3 \\
L(f_0) &= 4 \\
A(f_2) &= 1 \\
L(f_2) &= 0 \\
\end{align*}
\]
Main Result

Theorem

If $D = (V, A)$ is a dir. acyclic graph with embedding $\mathcal{F}, f_0$. Then:

$D$ upward planar (resp. $\mathcal{F}, f_0$) $\iff$ bimodal and $\exists$ consistent $\Phi$.

$\Rightarrow$: as constructed before

$\Leftarrow$: ideas

$–$ construct st-Graph $\supseteq D$

$–$ apply equivalence from the beginning of the lecture

First: $D, \mathcal{F}, f_0 \xrightarrow{?} \Phi$ consistent assignment
Flow Network to Construct $\Phi$

**Definition Flow Network** $N_{\mathcal{F}, f_0}(D) = ((W, A_N); l; u; d)$

- $W = \{ v \in V \mid v \text{ is source or sink} \} \cup \mathcal{F}$
- $A_N = \{ (v, f) \mid v \text{ incident to } f \}$
- $l(a) = 0 \quad \forall a \in A_N$
- $u(a) = 1 \quad \forall a \in A_N$
- $d(q) = \begin{cases} 1 & \forall q \in W \cap V \\ -(A(q) - 1) & \forall q \in \mathcal{F} \setminus \{ f_0 \} \\ -(A(q) + 1) & q = f_0 \end{cases}$

idea: flow $(v, f) = 1$ iff $v$ is a global source/sink whose large angle is assigned to $f$
Example Network

- normal vertex
- source / sink
- face vertex
Algorithm: $\Phi, \mathcal{F}, f_0 \rightarrow \text{st-Graph} \supseteq D$

Let $f$ be a face. Consider the clockwise angle sequence $\sigma_f$ of L/S on local sources and sinks of $f$

Goal: Add edges to break large angles (sources and sinks).

$f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle L, S, S \rangle$ at vertices $x, y, z$

- $x$ source $\Rightarrow$ insert edge $(z, x)$
- $x$ sink $\Rightarrow$ insert edge $(x, z)$

Refine the outerface $f_0$

Refine all $f \in \mathcal{F} \Rightarrow D$ contains a planar st-Graph
Example Refinement
Example Refinement
Summary

Given: embedded, directed, acyclic graph \( G = (V, E) \).

- Test for bimodality
- Test for a consistent assignment \( \Phi \) (via flow network).
- If both bimodal and \( \Phi \) exists, draw \( G \) as upward planar.
  - Refine \( G \) to planar \( st \)-graph \( G' \)
  - Draw \( G' \) via \( st \)-graph methods
  - Delete the edges added by refinement.

15 gives up. planar drawing, see textbook [DETT, Sec. 6.1] – but the area usage can be exponential!
Finding the angles via the flow network

\[
\begin{align*}
W & := V \cup \mathcal{F} \\
A & := \{(v, f) \in V \times \mathcal{F} : v \text{ incident (∼) to } f\} \\
\ell(a) & = 0 \quad \forall a \in A \\
u(a) & = 2\pi \quad \forall a \in A \\
d(v) & = 2\pi \quad \forall v \in V \\
d(f) & = \begin{cases} 
-(\deg(f) - 2)\pi & \text{if } f \neq f_0, \\
-(\deg(f) + 2)\pi & \text{otherwise}
\end{cases}
\end{align*}
\]

Flow provides an assignment \(x(\cdot, \cdot)\) of angles where:

1. vertices : \(\forall v \in V: \sum_{f \sim v} x(v, f) = 2\pi\)
2. faces : \(\forall f \in \mathcal{F}: \sum_{v \sim f} x(v, f) = (\deg(f) \pm 2)\pi\)

1. and 2. mean: assignment \(\text{locally consistent}\).

Obs. using edge costs we can maximize \(\text{angular resolution}\).
Locally Consistent $\nRightarrow$ Globally Consistent

not isoceles!
Characterizing Inner Triangulations

**Theorem** [Di Battista & Vismara ’93]

Given planar inner triangulation* with embedding $\mathcal{F}$, $f_0$ and angle assignment $x$, then:

There is a straight-line drawing with $\mathbb{R}^2 \setminus f_0$ convex

\[ \sum \text{vertex angles} = 2\pi \]
\[ \sum \text{face angles} = \pi \]
\[ \text{for every } v \sim f_0, \text{ via radius } R_v: \prod_{i=1}^{\deg v} \frac{\sin \alpha_i}{\sin \beta_i} = 1 \]
\[ \text{for every } v \sim f_0, \sum_{v \sim f \neq f_0} x(v, f) \leq \pi \]

\[ \Leftrightarrow \]

**Problem:** it’s not a linear condition :-(

*) Every face $f \neq f_0$ is a triangle ($C_3$).