Algorithms for Graph Visualization

Summer Semester 2016
Lecture #5

Upward Planar Drawings

(based on slides from Martin Nöllenburg and Robert Görke, KIT)
The Problem

Definition.

A directed Graph $D = (V, A)$ is *upward planar*, when it has a drawing such that:

- all edges are upward y-monotone curves, and
- no two edges cross.
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Obvious requirements?

- Planar & acyclic.

—not sufficient!
Upward Planarity

Problem: Upward Planarity Testing

Given a directed acyclic graph $D = (V, A)$. Determine if $D$ is upward planar. If so, construct a corresponding drawing.
Upward Planarity

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Given a directed acyclic graph $D = (V, A)$. Determine if $D$ is upward planar. If so, construct a corresponding drawing.

Problem’: Embedded Upward Planarity Testing

Given an acyclic graph $D = (V, A)$ with an embedding $\mathcal{F}, f_0$. Determine if $D$ is upward planar with respect to $\mathcal{F}, f_0$. If so, construct a corresponding drawing.
## Upward Planarity

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⇒ NP-hard! [Garg & Tamassia ’95]

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NP-hard! \cite{Garg & Tamassia '95}

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Given an acyclic graph $D = (V, A)$ with an embedding $\mathcal{F}, f_0$. Determine if $D$ is upward planar with respect to $\mathcal{F}, f_0$. If so, construct a corresponding drawing.

Can be tested efficiently! \[this\ lecture\]
The Big Picture: a characterization

**Theorem** [Kelly ’87, Di Battista & Tamassia ’88]

For a directed graph $D = (V, A)$, the following are equivalent.

1. $D$ is upward planar.
2. $D$ has a *straight-line* upward planar drawing.
3. $D$ is a spanning subgraph of a planar st-graph.
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An acyclic directed graph with a single source $s$ and single sink $t$.
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Additionally:
- embedded so that $s$ and $t$ are on the outer-face $f_0$.
- without crossings
- acyclic directed graph with a single source $s$ and single sink $t$.  

4-6
The Big Picture: a characterization

Theorem [Kelly '87, Di Battista & Tamassia '88]
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---

![Diagram of a directed graph](image-url)
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*Proof in textbook [DETT, Sec. 6.1]*
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\( \text{Proof in textbook [DETT, Sec. 6.1]} \)

\[ \text{can be drawn upward planar, see textbook [DETT, Sec. 6.1]} \]
Bimodality

bimodal vertex
Bimodality

bimodal vertex

not bimodal
Bimodality

bimodal vertex

\[\text{not bimodal}\]

Definition

An embedded directed graph is \textit{bimodal} \iff all vertices are bimodal.
Lemma

An embedded directed graph is upward planar only if it is bimodal.

Definition

An embedded directed graph is \textit{bimodal} ⇔ all vertices are bimodal.
Angle Sizes of Sources and Sinks

For a face $f$ of a straight-line drawing, consider angles of
- **local sinks** (vertices with 2 incoming edges on $\partial f$)
- **local sources** (vertices with 2 outgoing edges on $\partial f$)
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$\Rightarrow L(f) := \text{number of large angles}$ (Intuition: in drawing $> \pi$)
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Thus:
$L(f) + S(f) =$

```latex
gf
```
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By induction:
\[ L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases} \]
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By induction:

$L(f) = \begin{cases} 
-2, & f \neq f_0 \\
+2, & f = f_0 
\end{cases}$

$L(f) = \begin{cases} 
A(f) - 1, & f \neq f_0 \\
A(f) + 1, & f = f_0 
\end{cases}$
Proof: $L(f) - S(f) = -2$ for $f \neq f_0$

$\Rightarrow L(f) = 0$
Proof: \( L(f) - S(f) = -2 \) for \( f \neq f_0 \)

\[ \Rightarrow L(f) = 0 \quad \Rightarrow S(f) = 2 \]
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\[ L(f) \geq 1 \]
Proof: $L(f) - S(f) = -2$ for $f \neq f_0$

$\implies L(f) = 0$  $\implies S(f) = 2$

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Separate $f$ by.
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Separate \( f \) by.

1. \( v \) sink with a small angle:
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1. \( v \) sink with a small angle:

\[
L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1) = -2
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\[\text{induction hypothesis}\]
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Separate $f$ by.

2. $v$ sink with a big angle:

$L(f) - S(f) = L(f_1) + L(f_2) + 1$

$\quad \quad - (S(f_1) + S(f_2) - 1)$

$\quad \quad = -2$

induction hypothesis
Proof: $L(f) - S(f) = -2$ for $f \neq f_0$

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Separate $f$ by.

3. $v$ source with big angle:

\[ L(f) - S(f) = L(f_1) + L(f_2) + 2 - (S(f_1) + S(f_2)) = -2 \]

induction hypothesis
Proof: $L(f) - S(f) = -2$ for $f \neq f_0$

$\implies L(f) = 0 \quad \implies S(f) = 2$

$\implies L(f) \geq 1$

Separate $f$ by.

4. $v$ source with small angle:
Proof: \( L(f) - S(f) = -2 \) for \( f \neq f_0 \)

\[ L(f) = 0 \quad \Rightarrow \quad S(f) = 2 \]

\[ L(f) \geq 1 \]

Separate \( f \) by.

5. \( v \) neither source nor sink:

\[
L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1) = -2
\]

induction hypothesis
Observations

Consider the angle between two incoming/outgoing edges.
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Consider the angle between two incoming/outgoing edges.

**Lemma**

Let $D$ be a directed graph. In every upward planar drawing of $D$:

1. For each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex}, \\ 1 & v \text{ source/sink.} \end{cases}$

2. For each face $f \in F$: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$
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$\Phi: S \cup T \to F$

$\nu \mapsto \text{incid. face}$

$|\Phi^{-1}(f)| = \begin{cases} A(f) - 1 & f \neq f_0 \\ A(f) + 1 & f = f_0 \end{cases}$

$\text{global sources and sinks}$
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$\Phi: S \cup T \to F$ called consistent
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Consider the angle between two incoming/outgoing edges.

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(2) for each face $f \in \mathcal{F}$: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$

\[ \Phi: S \cup T \rightarrow \mathcal{F} \] 
\[ v \mapsto \text{incid. face} \]
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\[ |\Phi^{-1}(f)| = \begin{cases} A(f) - 1 & f \neq f_0 \\ A(f) + 1 & f = f_0 \end{cases} \]
Example: Face Assignment
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- $v_1$, $v_2$, $v_3$, $v_4$, $v_5$, $v_6$, $v_7$, $v_8$, $v_9$
- $f_0$, $f_1$, $f_2$, $f_3$, $f_4$, $f_5$, $f_6$, $f_7$, $f_8$, $f_9$

- Global sources and sinks
Example: Face Assignment

\[ A(f_1) = 3 \]

\[ A(f_2) = 1 \]

\[ A(f_3) = 1 \]

\[ A(f_4) = 2 \]

\[ A(f_5) = 2 \]

\[ A(f_6) = 1 \]

\[ A(f_7) = 2 \]

\[ A(f_8) = 1 \]

\[ A(f_9) = 1 \]

\[ v_1 \]

\[ v_2 \]

\[ v_3 \]

\[ v_4 \]

\[ v_5 \]

\[ v_6 \]

\[ v_7 \]

\[ v_8 \]

\[ v_9 \]

\[ f_0 \]

\[ f_1 \]

\[ f_2 \]

\[ f_3 \]

\[ f_4 \]

\[ f_5 \]

\[ f_6 \]

\[ f_7 \]

\[ f_8 \]

\[ f_9 \]

---

Global sources and sinks
Example: Face Assignment

- $A(f_3) = 1$, $L(f_3) = 0$
- $A(f_4) = 2$, $L(f_4) = 1$
- $A(f_5) = 2$, $L(f_5) = 1$
- $A(f_6) = 1$, $L(f_6) = 0$
- $A(f_7) = 2$, $L(f_7) = 1$
- $A(f_8) = 1$, $L(f_8) = 0$
- $A(f_9) = 1$, $L(f_9) = 0$

- Global sources and sinks
Example: Face Assignment

Assignment $\phi : S \cup T \rightarrow F$

- $A(f_3) = 1, L(f_3) = 0$
- $A(f_1) = 3, L(f_1) = 2$
- $A(f_2) = 1, L(f_2) = 0$
- $A(f_0) = 3, L(f_0) = 4$
- $A(f_4) = 2, L(f_4) = 1$
- $A(f_5) = 2, L(f_5) = 1$
- $A(f_6) = 1, L(f_6) = 0$
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- $A(f_8) = 1, L(f_8) = 0$
- $A(f_9) = 1, L(f_9) = 0$
Main Result

**Theorem**

If $D = (V, A)$ is a dir. acyclic graph with embedding $\mathcal{F}, f_0$. Then:

$D$ upward planar (resp. $\mathcal{F}, f_0$) $\iff$ bimodal and $\exists$ consistent $\Phi$. 
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– construct st-Graph $\supseteq D$
Main Result

**Theorem**

If \( D = (V, A) \) is a dir. acyclic graph with embedding \( \mathcal{F}, f_0 \).
Then:
\( D \) upward planar (resp. \( \mathcal{F}, f_0 \)) \iff \text{bimodal and } \exists \text{ consistent } \Phi.

\( \Rightarrow \): as constructed before

\( \Leftarrow \): ideas
  - construct st-Graph \( \supseteq D \)
  - apply equivalence from the beginning of the lecture
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If $D = (V, A)$ is a dir. acyclic graph with embedding $\mathcal{F}, f_0$. Then:

$D$ upward planar (resp. $\mathcal{F}, f_0 \iff$ bimodal and $\exists$ consistent $\Phi$.

$\Rightarrow$: as constructed before

$\Leftarrow$: ideas

- construct st-Graph $\supseteq D$
- apply equivalence from the beginning of the lecture

First: $D, \mathcal{F}, f_0 \rightarrow \Phi$ consistent assignment
Flow Network to Construct $\Phi$

Definition Flow Network $N_{\mathcal{F},f_0}(D) = ((W, A_N); l; u; d)$

- $W = \{v \in V \mid v$ is source or sink$\} \cup \mathcal{F}$
- $A_N = \{(v, f) \mid v$ incident to $f\}$
- $l(a) = 0 \quad \forall a \in A_N$
- $u(a) = 1 \quad \forall a \in A_N$
- $d(q) = \begin{cases} 
1 & \forall q \in W \cap V \\
-(A(q) - 1) & \forall q \in \mathcal{F} \setminus \{f_0\} \\
-(A(q) + 1) & q = f_0
\end{cases}$

idea: flow $(v, f) = 1$ iff $v$ is a global source/sink whose large angle is assigned to $f$
Example Network

- normal vertex
- source / sink
Example Network

- normal vertex
- source / sink
- face vertex
Example Network

- normal vertex
- source / sink
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Algorithm: $\Phi, \mathcal{F}, f_0 \rightarrow \text{st-Graph} \supseteq D$
Algorithm: $\Phi, \mathcal{F}, f_0 \rightarrow \text{st-Graph} \subseteq D$

Let $f$ be a face. Consider the clockwise angle sequence $\sigma_f$ of L/S on local sources and sinks of $f$. 
Algorithm: $\Phi, \mathcal{F}, f_0 \rightarrow \text{st-Graph} \supseteq D$

Let $f$ be a face. Consider the clockwise angle sequence $\sigma_f$ of L/S on local sources and sinks of $f$

Goal: Add edges to break large angles (sources and sinks).
Algorithm: $\Phi, F, f_0 \rightarrow \text{st-Graph} \supseteq D$

Let $f$ be a face. Consider the clockwise angle sequence $\sigma_f$ of $L/S$ on local sources and sinks of $f$.

Goal: Add edges to break large angles (sources and sinks).

$f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle L, S, S \rangle$ at vertices $x, y, z$. 

![Diagram of st-Graph](image)
Algorithm: $\Phi, \mathcal{F}, f_0 \rightarrow \text{st-Graph} \supseteq D$

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Algorithm: \( \Phi, \mathcal{F}, f_0 \rightarrow \text{st-Graph} \supseteq D \)

Let \( f \) be a face. Consider the clockwise angle sequence \( \sigma_f \) of L/S on local sources and sinks of \( f \)

\[ \Rightarrow \text{Goal: Add edges to break large angles (sources and sinks).} \]

\[ \Rightarrow f \neq f_0 \text{ with } |\sigma_f| \geq 2 \text{ containing } \langle L, S, S \rangle \text{ at vertices } x, y, z \]

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Refine all $f \in \mathcal{F} \Rightarrow \mathcal{D}$ contains a planar st-Graph
Example Refinement
Example Refinement
Example Refinement
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Finding the angles via the flow network

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Flow provides an assignment \( x(\cdot, \cdot) \) of angles where:

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1. and 2. mean: assignment \textit{locally consistent}. 
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1. and 2. mean: assignment locally consistent.

Obs. using edge costs we can maximize angular resolution.
Locally Consistent $\not\Rightarrow$ Globally Consistent
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![Diagram](image)

- Not isoceles!
Characterizing Inner Triangulations

**Theorem** [Di Battista & Vismara ’93]

Given planar inner triangulation* with embedding $\mathcal{F}, f_0$ and angle assignment $x$, then:

There is a straight-line drawing with $\mathbb{R}^2 \setminus f_0$ convex

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\begin{align*}
1. & \quad \sum \text{ vertex angles} = 2\pi \\
2. & \quad \sum \text{ face angles} = \pi \\
3. & \quad \text{for every } v \sim f_0, \text{ via radius } R_v: \prod_{i=1}^{\deg v} \frac{\sin \alpha_i}{\sin \beta_i} = 1 \\
4. & \quad \text{for every } v \sim f_0, \sum_{v \sim f \neq f_0} x(v, f) \leq \pi
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*) Every face $f \neq f_0$ is a triangle ($C_3$).
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\[\iff\]

*Problem:* it’s not a linear condition :-(

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