Algorithms for Graph Visualization

Summer Semester 2016
Lecture #4

Divide-and-Conquer Algorithms: Trees and Series-Parallel Graphs

(based on slides from Martin Nöllenburg and Robert Görke, KIT)
Uses of Divide & Conquer

Well suited for inductively/recursively defined Graph Classes
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Rooted Binary Trees:
1. draw the left subtree
2. draw the right subtree
3. combine together + draw root
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Terminology

$\triangleright\triangleright$ depth$(v)$: distance from the root
$\triangleright\triangleright$ traversal
	• preorder  
	• inorder  
	• postorder
Overview

- balanced drawings of binary trees \( O(nh) \)
- radial drawings of trees \( O(nh) \)
- compact drawings of trees \( O(n \log n) \)
- upward drawings of series parallel graphs \( \exp \)
## Overview

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<th>Radial Drawings of Trees</th>
<th>Compact Drawings of Trees</th>
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Algorithm of Reingold and Tilford (’81)

Trivial: If $T = \{v\}$, draw $v$ (e.g., as a small disk).
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Conquer: shift the partial drawings
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place the root $r$ one above and centrally btw. children.
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Grid Drawing?
Algorithm of Reingold and Tilford ('81)

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place the root \( r \) one above and centrally btw. children.

Grid Drawing?
Algorithm of Reingold and Tilford ('81)

Implementation in 2 Phases:

1. postorder (bottom-up):
   - contours and x-offsets
   - gather the predecessors

2. preorder (top-down):
   - calculate absolute coordinates
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Implementation in 2 Phases:

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Contour: linked list of vertices (-coordinates)
Algorithm of Reingold and Tilford ('81)

Phase 1:

1. compute $T_\ell(v)$ und $T_r(v)$
2. trace the right contour of $T_\ell(v)$ and left of $T_r(v)$
3. Find $d_v = \text{min. horiz. distance between } v_\ell \text{ und } v_r$
4. $x\text{-offset}(v_\ell) = -\lceil d_v/2 \rceil$, $x\text{-offset}(v_r) = \lceil d_v/2 \rceil$
5. Build left contour of $T_v$ from:
   - $\gg v$,
   - $\gg$ left contour of $T_\ell(v)$,
   - $\gg$ left contour of any low hanging part of $T_r(v)$
6. Symmetrically for right contour.
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Runtime? $\sum_v (\cdots) =$
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Runtime? $\sum_v (1 + \min\{h_\ell(v), h_r(v)\}) = $
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**Runtime?** $\sum_v (1 + \min\{h_\ell(v), h_r(v)\}) = n + \sum_v \min\{\ldots\} \leq n + n$
Algorithm of Reingold und Tilford (’81)

Phase 2:

\[ y(v) = -\text{depth}(v) \text{ for each vertex } v. \]
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Phase 2:

- Set $y$-coordinate $y(v) = -\text{depth}(v)$ for each vertex $v$.
- Set $x(w) := 0$ for the root $w$, then in preorder for $v \in V$:
  
  $x(v_\ell) := x(v) + x\text{-offset}(v_\ell)$ and
  
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Runtime? $O(n)$
## Theorem [Reingold & Tilford ’81]

For a binary tree with \( n \) vertices, in \( O(n) \) time we can produce a drawing \( \Gamma \) such that:
**Summary for Balanced Drawings of Binary Trees**

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<td></td>
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<tr>
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<td></td>
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<tr>
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Easily generalizes to arbitrary trees!
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Min. width (but without the grid):

$$\gg$$

Easily generalizes to arbitrary trees!

example?
## Summary for Balanced Drawings of Binary Trees

**Theorem** \([\text{Reingold \& Tilford '81}]\)

For a binary tree with \(n\) vertices, in \(O(n)\) time we can produce a drawing \(\Gamma\) such that:

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Theorem [Reingold & Tilford '81]

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Min. width and on the grid:
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Min. width (but without the grid): by LP!

Min. width and on the grid: NP-hard!  

\[\text{[Supowit & Reingold ’83]}\]
Example of width variation

Output of the Algorithm:
Example of width variation

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Output of the Algorithm:

Optimal Drawing:
Example of width variation

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2. Radial Drawings of Trees

- balanced drawings of binary trees \( O(nh) \)
- radial drawings of trees \( O(nh) \)
- compact drawings of trees \( O(n \log n) \)
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Example: Radial Tree Layouts
An Algorithm for Radial Layout?
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Restricting to Smaller Sectors
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\cos \tau = \frac{\rho_i}{\rho_{i+1}}
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Restricting to Smaller Sectors

\[ \cos \tau = \frac{\rho_i}{\rho_{i+1}} \]

\[ \Rightarrow \begin{cases} \alpha_{\text{min}} = \alpha_v - \arccos \frac{\rho_i}{\rho_{i+1}} \\ \alpha_{\text{max}} \end{cases} \]
Restricting to Smaller Sectors

\[ \cos \tau \equiv \frac{\rho_i}{\rho_{i+1}} \]

\[ \alpha_{\text{min}} = \alpha_v - \arccos \frac{\rho_i}{\rho_{i+1}} \]

\[ \alpha_{\text{max}} = \alpha_v + \arccos \frac{\rho_i}{\rho_{i+1}} \]
Pseudocode for radial tree layout

RadialTreeLayout(tree $T$, root $r \in T$, radii $\rho_1 < \cdots < \rho_k$)

begin
    $postorder(r)$
    $preorder(r, 0, 0, 2\pi)$
    return $(d_v, \alpha_v)_{v \in V(T)}$
    \{vertex pos./ polar coord.\}
end

postorder(vertex $v$)

$calculate the size of the subtree recursively$
Pseudocode for radial tree layout

RadialTreeLayout(tree $T$, root $r \in T$, radii $\rho_1 < \cdots < \rho_k$)

begin
  
  \hspace{1em} postorder($r$)
  \hspace{1em} preorder($r, 0, 0, 2\pi$)
  \hspace{1em} return $((d_v, \alpha_v))_{v \in V(T)}$
  \hspace{1em} \{vertex pos./ polar coord.\}

postorder(vertex $v$)
  
  $n_v \leftarrow 1$
  \hspace{1em} foreach child $w$ von $v$ do
    \hspace{1em} postorder($w$)
    \hspace{2em} $n_v \leftarrow n_v + n_w$
Pseudocode for radial tree layout

RadialTreeLayout(tree $T$, root $r \in T$, radii $\rho_1 < \cdots < \rho_k$)

\begin{verbatim}
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size of the subtree $T(v)$
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RadialTreeLayout(tree $T$, root $r \in T$, radii $\rho_1 < \cdots < \rho_k$)

begin

  preorder($r$, 0, 0, 2$\pi$)

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  {vertex pos./ polar coord.}

end

preorder(vertex $v$, $t$, $\alpha_{\text{min}}$, $\alpha_{\text{max}}$)

$$d_v \leftarrow \rho_t$$
$$\alpha_v \leftarrow (\alpha_{\text{min}} + \alpha_{\text{max}})/2$$

if $t > 0$ then

  $$\alpha_{\text{min}} \leftarrow \max\{\alpha_{\text{min}}, \alpha_v - \arccos \frac{\rho_t}{\rho_{t+1}}\}$$

  $$\alpha_{\text{max}} \leftarrow \min\{\alpha_{\text{max}}, \alpha_v + \arccos \frac{\rho_t}{\rho_{t+1}}\}$$

left $\leftarrow \alpha_{\text{min}}$

foreach child $w$ von $v$ do

  preorder($w$, $t + 1$, $\alpha_{\text{min}}$, $\alpha_{\text{max}}$)

  $$\alpha_v \leftarrow \arccos \frac{\rho_t}{\rho_{t+1}}$$

  $$\alpha_{\text{max}} \leftarrow \min\{\alpha_{\text{max}}, \alpha_v + \arccos \frac{\rho_t}{\rho_{t+1}}\}$$

  left $\leftarrow$ right

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postorder(vertex $v$)

$$n_v \leftarrow 1$$

foreach child $w$ von $v$ do

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size of the subtree $T(v)$
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size of the subtree $T(v)$

preorder(vertex $v$, $t$, $\alpha_{\text{min}}$, $\alpha_{\text{max}}$)
  $d_v \leftarrow \rho_t$
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    $\alpha_{\text{max}} \leftarrow \min\{\alpha_{\text{max}}, \alpha_v + \arccos \frac{\rho_t}{\rho_{t+1}}\}$
  left $\leftarrow \alpha_{\text{min}}$
  foreach child $w$ von $v$ do
    right $\leftarrow$ left $+ \frac{n_w}{n_v - 1} \cdot (\alpha_{\text{max}} - \alpha_{\text{min}})$
    preorder($w, t + 1, \text{left}, \text{right}$)
    left $\leftarrow$ right
Pseudocode for radial tree layout

RadialTreeLayout(tree \( T \), root \( r \in T \), radii \( \rho_1 < \cdots < \rho_k \))

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    postorder(r)
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postorder(vertex \( v \))

\( n_v \leftarrow 1 \)
foreach child \( w \) von \( v \) do
    postorder(w)
    \( n_v \leftarrow n_v + n_w \)

preorder(vertex \( v \), \( t, \alpha_{\text{min}}, \alpha_{\text{max}} \))

\( d_v \leftarrow \rho_t \)
\( \alpha_v \leftarrow (\alpha_{\text{min}} + \alpha_{\text{max}})/2 \) \{ output \}
if \( t > 0 \) then
    \( \alpha_{\text{min}} \leftarrow \max\{\alpha_{\text{min}}, \alpha_v - \arccos \frac{\rho_t}{\rho_{t+1}}\} \)
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left \( \leftarrow \alpha_{\text{min}} \)
foreach child \( w \) von \( v \) do
    right \( \leftarrow \) left + \( \frac{n_w}{n_v-1} \cdot (\alpha_{\text{max}} - \alpha_{\text{min}}) \)
    preorder(w, \( t + 1 \), left, right)
left \( \leftarrow \) right
Pseudocode for radial tree layout

RadialTreeLayout(tree $T$, root $r \in T$, radii $\rho_1 < \cdots < \rho_k$)

begin
  \text{postorder}(r)
  \text{preorder}(r, 0, 0, 2\pi)
  return $(d_v, \alpha_v)_{v \in V(T)}$
  \{vertex pos./ polar coord.\}

\text{postorder}(vertex $v$)
  $n_v \leftarrow 1$
  \textbf{foreach} child $w$ von $v$ \textbf{do}
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    \text{right} $\leftarrow \text{left} + \frac{n_w}{n_v-1} \cdot (\alpha_{\max} - \alpha_{\min})$
    \text{preorder}(w, $t+1$, left, right)
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size of the subtree $T(v)$

Runtime?
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size of the subtree $T(v)$

Runtime? $O(n)$. 
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size of the subtree $T(v)$

Runtime? $O(n)$.  
Correctness?
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  preorder($w, t+1, \text{left}, \text{right}$)

  left $\leftarrow$ right

size of the subtree $T(v)$

Runtime? $O(n)$.  Correctness? ✓
Overview

- balanced drawings of binary trees \( O(nh) \)
- radial drawings of trees \( O(nh) \)
- compact drawings of trees \( O(n \log n) \)
- upward drawings of series parallel graphs \( \exp. \)
**Definition.**

An *hv-drawing* of a binary tree is a straight line drawing, so that for each vertex $v$: 
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$\Rightarrow$ each child of $v$ is either directly right or directly below $v$. 
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An *hv-drawing* of a binary tree is a straight line drawing, so that for each vertex $v$:

- each child of $v$ is either directly right or directly below $v$.

- the smallest axis-parallel rectangle enclosing the subtrees of the children of $v$ are disjoint.
hv-Drawings

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*horizontal combination*
**hv-Drawings**

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Algorithm \textit{RightHeavyHVTreeDraw}

\begin{itemize}
\item[(⇒)] Recursively construct drawings of the left and right subtrees from the root.
\end{itemize}
Algorithm \textit{RightHeavyHVTreeDraw}

$\gg$ Recursively construct drawings of the left and right subtrees from the root.

$\gg$ Place the larger subtree on the right using the horizontal combination, and the smaller on the left
Algorithm *RightHeavyHVTreeDraw*

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*Size of a subtree :\(=\) number of vertices*
Algorithm RightHeavyHVTreDraw

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Size of a subtree := number of vertices
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Size of a subtree := number of vertices
Algorithm *RightHeavyHVTeeDraw*

- Recursively construct drawings of the left and right subtrees from the root.
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**Obs.** The drawing has width $\leq$
Algorithm RightHeavyHVTreDraw

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Obs. The drawing has width ≤ \( n \), height ≤
Algorithm *RightHeavyHVTreeDraw*

- Recursively construct drawings of the left and right subtrees from the root.
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Algorithm *RightHeavyHVTreeDraw*

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Size of a subtree := number of vertices

\[\text{Observe.} \quad \text{The drawing has width } \leq n, \quad \text{height } \leq \]

\[\text{16}\]
Algorithm *RightHeavyHVTTreeDraw*

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Algorithm $RightHeavyHVTreeDraw$

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Algorithm \textit{RightHeavyHVTreeDraw}

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  \item Recursively construct drawings of the left and right subtrees from the root.
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\end{itemize}

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\textbf{Obs.} \quad The drawing has width $\leq n$, height $\leq \cdot 2$
Algorithm *RightHeavyHVTREEDRAW*

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\[\text{at least } \cdot 2\]

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\[
\text{at least } \cdot 2 \\
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Size of a subtree := number of vertices

\[
\text{at least } 2^{\cdot 2} \quad \text{at least } 2^{\cdot 2} \quad \text{at least } 2^{\cdot 2}
\]

\[16\]

**Obs.** The drawing has width \( \leq n \), height \( \leq \)
Algorithm *RightHeavyHVTreeDraw*

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Size of a subtree $:= \text{number of vertices}$

```
16
```

Obs. The drawing has width $\leq n$, height $\leq \log_2 n$. 

```
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Overview

- balanced drawings of binary trees $O(nh)$
- radial drawings of trees $O(nh)$
- compact drawings of trees $O(n \log n)$
- upward drawings of series parallel graphs $\text{exp.}$
Series Parallel Graphs

simple series parallel graph
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- Induction: combining two series parallel graphs $G_1, G_2 \ldots$
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- simple series parallel graph

- Induction: combining two series parallel graphs $G_1, G_2, \ldots$

\[ t_1 = s_2 \]

\[ s_1 \quad \quad t_1 \quad \quad t_2 \quad \quad s_2 \]

\[ G_1 \quad G_2 \]
Series Parallel Graphs

- simple series parallel graph

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- \ldots series \ldots

- \ldots or parallel.
Decomposition Tree for SP-graphs
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Generalization: SPQR-Tree
SP-Graphs: applications

Flow Charts

PERT-Diagrams
(Program Evaluation and Review Technique)
SP-Graphs: applications

Flow Charts

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PERT-Diagrams

(Program Evaluation and Review Technique)

Provides:

Linear time algorithms for NP-complete problems (e.g., Maximum Independent Set)
Grid Size

**Theorem** [Bertolazzi et al. ’92]

There is a family \((G_n)_{n \in \mathbb{N}}\) of embedded SP-graphs where \(G_n\) has \(2^n\) vertices and every upward planar drawing of \(G_n\) requires \(\Omega(4^n)\) area.
Theorem [Bertolazzi et al. '92]

There is a family $(G_n)_{n \in \mathbb{N}}$ of embedded SP-graphs where $G_n$ has $2^n$ vertices and every upward planar drawing of $G_n$ requires $\Omega(4^n)$ area.

Proof:

21
Theorem [Bertolazzi et al. ’92]

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Proof:

\[ G_0 \]
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\[
\begin{align*}
G_0 & \quad G_n & \quad G_{n+1} \\
t_0 & \quad t_n & \quad t_{n+1} \\
s_0 & \quad s_n & \quad s_{n+1}
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\[
a(G_{n+1}) \geq a(\Pi) + a(\Delta_1) + a(\Delta_2)
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**Proof:**

\[
a(G_{n+1}) \geq a(\Pi) + a(\Delta_1) + a(\Delta_2) \geq 2 \cdot a(\Pi) \geq 4 \cdot a(G_n)
\]