Algorithms for Graph Visualization

Summer Semester 2016
Lecture #2

Bend-minimization in Orthogonal Drawings

(based on slides from Martin Nöllenburg and Robert Görke, KIT)
Examples of graph drawing problems for planar graphs.

- orthogonal drawings
- upward drawings of acyclic graphs
- angular resolution in straight line drawings

Model these problems via flow networks.
# Planarity

**Definition**

A graph $G = (V, E)$ is called **planar**, when there is a drawing $\Gamma$ of $G$ in the plane such that no pair of edges of in the drawing cross.

Such a drawing $\Gamma$ is called a **plane embedding**.

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**Definition (combinatorial) embedding**

![Planar Graph Example](image.png)
Vertices, Edges, and Faces

Each drawing \( \Gamma \) of a planar graph divides the plane into faces.

The faces are the connected regions of \( \mathbb{R}^2 \setminus \Gamma \).

\( F := \) the faces of a planar embedding of \( G = (V, E) \).

Is there a relationship between \( |V| = n, |E| = m, |F| =: f \)?

**Euler’s Formula:**

\[
n - m + f = k + 1,
\]

where \( k = \) the number of connected components of \( G \).

\textit{Proof?}
s-t Flow

Definition

Given: flow network \((G = (V, E); s, t; c)\) with

\[
\begin{align*}
\text{directed Graph } & G = (V, E) \\
\text{edge capacities } & c : E \to \mathbb{R}_{>0} \\
\text{source } & s \in V, \text{ sink } t \in V
\end{align*}
\]

An assignment \(f : E \to \mathbb{R}_{\geq 0}\) is an s-t-flow, when:

- for every \(e \in E\): \(f(e) \leq c(e)\)
- for every \(v \in V \setminus \{s, t\}\):
  \[
  \sum_{vw \in E} f(vw) - \sum_{uv \in E} f(uv) = 0
  \]

\[
\text{deficit}_f(v) := \text{outflow}_f(v) - \text{inflow}_f(v)
\]
General Flow Networks

Definition

Given: flow network \((G = (V, E); l; u; b)\) with

- directed graph \(G = (V, E)\)
- edge lowerbounds \(l: E \rightarrow \mathbb{R}_{\geq 0}\)
- edge upperbounds \(u: E \rightarrow \mathbb{R}_{> 0}\)
- vertex deficits \(d: V \rightarrow \mathbb{R}\) with \(\sum_{v \in V} d(v) = 0\)

An assignment \(X: E \rightarrow \mathbb{R}_{\geq 0}\) is a valid flow, when:

- for every \(e \in E\): \(l(e) \leq X(e) \leq u(e)\)
- for every \(v \in V\): \(\sum_{vw \in E} X(vw) - \sum_{uv \in E} X(uv) = d(v)\)

\[\text{deficit}_X(v)\]
Questions on flow networks

Valid flows

Find a valid flow $X : E \rightarrow \mathbb{R}_{\geq 0}$, i.e., an edge assignment where:

$\gg$ the bounds are satisfied, also $l \leq X \leq u$, and

$\gg$ the deficits/demands $d$ are met.

Weighted version, given cost: $E \rightarrow \mathbb{R}_{\geq 0}$

Def. $\text{cost}(X) := \sum_{e \in E} \text{cost}(e) \cdot X(e)$

Minimum cost flow

Find a valid flow $X : E \rightarrow \mathbb{R}_{\geq 0}$, such that

$\gg$ $\text{cost}(X)$ is minimized (among all valid flows).

Runtime (general) $O(n^2 m^3 \log n)$

Planar, edge costs $\leq c$, face size $\leq s$ $O(c\sqrt{s} \cdot n^{3/2})$

[Cornelsen & Karrenbauer, JGAA'12]
(Planar) Orthogonal Drawing

3-Step Approach:  Topology – Shape – Metrics

\[ V = \{v_1, v_2, v_3, v_4\} \]
\[ E = \{\{v_1, v_2\},\{v_1, v_3\},\{v_1, v_4\},\{v_2, v_3\},\{v_2, v_4\}\} \]
The Problem

Problem 2: Bend minimization for fixed embeddings.

Given a graph $G = (V, E)$ with max-degree $\Delta \leq 4$, a combinatorial embedding $\mathcal{F}$, and an outerface $f_0$, find an orthogonal grid-drawing of $(\mathcal{F}, f_0)$ minimizing bends.
The Problem

Problem 2’: Orthogonal Description

Given
Graph \( G = (V, E) \) with max-degree \( \Delta \leq 4 \),
combinatorial embedding \( \mathcal{F} \) and outerface \( f_0 \),
find
orthogonal description \( H(G) \)
of \( (\mathcal{F}, f_0) \) minimizing bends.
Orthogonal Description

**In:** \( G = (V, E) \) planar, \( F, f_0 \)

**Out:** orthogonal description \( H(G) = \{H(f) \mid f \in F\} \)

**Face description** \( H(f) \): edges in clockwise order.
Each \( e \) comes with \( \delta \), and \( \alpha \). (def. as follows):

\[ \delta \text{ is 0-1-sequence (0 = right turn, 1 = left turn)} \]
\[ \alpha \text{ is the angle } \in \left\{ \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \right\} \text{ between } e \text{ and the next edge } e' \text{ around } f. \]

\[ f_0 \text{ ordering!} \]

**Sample** \( G \):

\[
\begin{align*}
H(f_1) &= ((e_1, 00, \frac{3\pi}{2}), (e_2, \emptyset, \frac{\pi}{2}), (e_6, 00, \pi)) \\
H(f_2) &= ((e_5, 000, \frac{\pi}{2}), (e_6, 11, \frac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \frac{\pi}{2})) \\
H(f_0) &= ((e_1, 11, \frac{\pi}{2}), (e_5, 111, \frac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \frac{\pi}{2}))
\end{align*}
\]
Example: Orthogonal Drawing

Task:
Find an orthogonal drawing of $G$ according to the orthogonal description $H(G)$!

Solution:

$H(G)$:

$H(f_1) = ((e_1, 00, \frac{3\pi}{2}), (e_2, \emptyset, \frac{\pi}{2}), (e_6, 00, \pi))$

$H(f_2) = ((e_5, 000, \frac{\pi}{2}), (e_6, 11, \frac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \frac{\pi}{2}))$

$H(f_0) = ((e_1, 11, \frac{\pi}{2}), (e_5, 111, \frac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \frac{\pi}{2}))$
Valid orthogonal descriptions

(H1) $H(G)$ corresponds to $(\mathcal{F}, f_0)$.

(H2) For an edge $\{u, v\}$ between faces $f$ and $g$ with $(uv, \delta_1, \alpha_1) \in H(f)$ and $(vu, \delta_2, \alpha_2) \in H(g)$, need: $\delta_1$ is inverted and reversed $\delta_2$.

(H3) Let $|\delta|_0$ and $|\delta|_1$ be the number of zeros and ones in $\delta$. Let $r = (e, \delta, \alpha)$. For $C(r) := |\delta|_0 - |\delta|_1 + 2 - \alpha / \pi^2$:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{o.w.} \end{cases}$$
Valid orthogonal descriptions

(H1) $H(G)$ corresponds to $(\mathcal{F}, f_0)$.

(H2) For an edge $\{u, v\}$ between faces $f$ and $g$ with $(uv, \delta_1, \alpha_1) \in H(f)$ and $(vu, \delta_2, \alpha_2) \in H(g)$, need: $\delta_1$ is inverted and reversed $\delta_2$.

(H3) Let $|\delta|_0$ and $|\delta|_1$ be the number of zeros and ones in $\delta$. Let $r = (e, \delta, \alpha)$. For $C(r) := |\delta|_0 - |\delta|_1 + 2 - \alpha/\pi$:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{o.w.} \end{cases}$$

(H4) For each node $v$ the sum of adjacent angles is $2\pi$. 

\[12-4\]
Recall

Problem 2’: Orthogonal Description

Given Graph $G = (V, E)$ with max-degree $\Delta \leq 4$, combinatorial embedding $\mathcal{F}$ and outerface $f_0$, find orthogonal description $H(G)$ of $(\mathcal{F}, f_0)$ minimizing bends.

Idea: Build flow network!

$\implies$ value $= \angle \frac{\pi}{2}$

$\implies$ Vertices $\rightarrow$ faces ($\# \angle \frac{\pi}{2}$ per face)

$\implies$ Faces $\rightarrow$ neighbouring faces ($\#$ bends toward the neighbour)
The flow network $N(G)$

Definition flow network $N(G) = ((V \cup \mathcal{F}, A); l; u; b; \text{cost})$

$$A = \left\{ (v, f)_{ee'} \in V \times \mathcal{F} \mid v \text{ between edges } e, e' \text{ of } \partial f \right\} 
\cup \left\{ (f, g)_{e} \in \mathcal{F} \times \mathcal{F} \mid f, g \text{ have common edge } e \right\}$$

*directed Multigraph!*
The flow network $N(G)$

**Definition** flow network $N(G) = ((V \cup \mathcal{F}, A); l; u; b; \text{cost})$

\[ A = \{ (v, f)_{ee'} \in V \times \mathcal{F} \mid v \text{ between edges } e, e' \text{ of } \partial f \} \]
\[ \cup \{ (f, g)_e \in \mathcal{F} \times \mathcal{F} \mid f, g \text{ have common edge } e \} \]

\[ d(v) = 4 \text{ for every } v \in V \]

\[ d(f) = -2 \deg_G(f) + \begin{cases} -4 & \text{if } f = f_0, \\ +4 & \text{otherwise} \end{cases} \Rightarrow \sum d = 0 \quad \text{(Euler)} \]

\[ \text{for every } vf \in A: \quad l(vf) := 1 \leq X(vf) \leq 4 =: u(vf) \]

\[ \text{for every } fg \in A: \quad l(fg) := 0 \leq X(fg) \leq \infty =: u(fg) \]

\[ \text{cost}(vf) = \quad \text{cost}(fg) = \]
Example flow network

cost = 1
one bend! (outward)

Legend
- flow
- 4 = \( d \)-value
- \( \ell/u/cost \)
- 1/4/0
- 0/\( \infty \)/1

\( V \)
Overview + Correctness

**Theorem**

For an embedded graph \((G, \mathcal{F}, f_0)\), there is an orthogonal description \(H(G)\) with \(k\) bends

\[
\iff \text{the flow network } N(G) \text{ has a valid flow } X \text{ of cost } k.
\]

\[
\iff:\text{In.: Flow network } N(G), \text{ flow } X \text{ with cost } k
\]

\[
\text{Out.: orthogonal description } H(G)
\]

(H1) \(H(G)\) matches \(\mathcal{F}, f_0\)

(H2) bend order inverted and reversed on opposite sides

(H3) angle sum of \(f\): \(d(f)\) without flat angles = 4

(H4) Total angle at each node = \(2\pi\)

[Tamassia’87]

**exercise**
Overview + Correctness

Theorem [Tamassia’87]

For an embedded graph \((G, F, f_0)\), there is an orthogonal description \(H(G)\) with \(k\) bends

\[ \iff \text{ the flow network } N(G) \text{ has a valid flow } X \text{ of cost } k. \]

⇒: In.: orthogonal description \(H(G)\)
Out: Flow network \(N(G)\), flow \(X\) with cost \(k\)

\[(N1) \ X(vf) = 1/2/3/4 \checkmark \]
\[(N2) \ X(fg) := |\delta_{fg}|_0, (e, \delta_{fg}, x) \text{ describes } e^* = fg \text{ from } f \checkmark \]
\[(N3) \text{ capacities } \checkmark, \text{ deficit/demand coverage } \checkmark \]
\[(N4) \text{ cost } = k \checkmark \]

Runtime:
Using values \((l, u, c)\), and amount of edges, ...
\[O(n^2 \log n) \quad [\text{Tamassia '87}]\]
\[O(n^{7/4} \sqrt{\log n}) \quad [\text{Garg & Tamassia '96}]\]
\[O(n^{3/2}) \quad [\text{Cornelsen & Karrenbauer '12}]\]
Compactifying

Problem: orthogonal drawing from an orthogonal description

Given planar graph $G = (V, E)$ with $\Delta \leq 4$ and an orthogonal description $H(G)$.
Find an orthogonal drawing of $G$, realizing $H(G)$.

Special Case: all faces are rectangles
⇒ guarantees:  
  - minimum total edge length
  - minimum area

  bends are on the outer face
  opposite sides have equal length ⇒ Layout ok
Flow network length assignment

**Definition** flow network $N_{\text{hor}} = ((V_{\text{hor}}, A_{\text{hor}}); l; u; b; \text{cost})$

- $V_{\text{hor}} = F$
- $A_{\text{hor}} = \{(f, g) \mid f, g \text{ have a common horizontal line segment and } f \text{ is below } g\}$
- $l \equiv 1$
- $u \equiv \infty$
- $\text{cost} \equiv 1$
- $d \equiv 0$
Flow network length assignment

Definition flow network $N_{\text{ver}} = ((V_{\text{ver}}, A_{\text{ver}}); l; u; b; \text{cost})$

- $V_{\text{ver}} = \mathcal{F}$
- $A_{\text{ver}} = \{(f, g) \mid f, g \text{ have a common vertical line segment and } f \text{ is left of } g\}$
- $l \equiv 1$
- $u \equiv \infty$
- $\text{cost} \equiv 1$
- $d \equiv 0$
Optimal with Rectangles

Theorem

Min-Cost-flow for $N_{\text{hor}}$ and $N_{\text{ver}}$ implies:

1. $X$ valid flow $\iff$ corr. edge lengths induce orth. drawing
2. $|X(N_{\text{hor}})| = \text{width}$, $|X(N_{\text{ver}})| = \text{height}$
3. $\text{cost}(X(N_{\text{hor}})) + \text{cost}(X(N_{\text{ver}})) = \text{total edge length}$
Refinement of \((G,H)\) – inner face
Refinement of \((G, H)\) – outerface

Area minimized?  
No!

But …. do we get some bounds on the area?  
Yes! \(O((n + b)^2)\)
Summary Bend-minimization

- bend-minimization for fixed embeddings via flow networks.
- optimal compactifying for rectangular faces
- extension to the general faces is not optimal :(
- compactification is NP-hard !!

Possible Extensions:
- degree $\geq 4$
- non-planar graphs

exercise
Compactifying is NP-hard [Patrignani ’01]

Reduction via SAT

- $n$ variables $x_1, \ldots, x_n$
- $m$ clauses $C_1, \ldots, C_m$
- each clause: Disjunction of literals $x_i/\overline{x_i}$
  e.g.: $C_1 = x_1 \lor \overline{x_2} \lor x_3$
- Is $\Phi = C_1 \land C_2 \land \ldots \land C_m$ satisfiable, i.e., is there an assignment to the variables satisfying every clause?

Find an appropriate value $K$ such that $(G, H)$ can be drawn in $K$ area $\iff \Phi$ is satisfiable.
Compactifying is NP-hard [Patrignani ’01]

» High level structure of \((G, H)\)
  » boundary.
  » belts, and pistons
  » clause gadgets.
  » variable gadgets.
Boundary, belt, and “piston” gadget
Clause Gadgets

Example:

\[ C_1 = x_2 \lor \overline{x_4} \]
\[ C_2 = x_1 \lor x_2 \lor \overline{x_3} \]
\[ C_3 = x_5 \]
\[ C_4 = x_4 \lor \overline{x_5} \]

insert \((2n - 1)\)-chain through each clause
Pick
\[ K = (9n + 2) \cdot (9m + 7) \]

Then:
\[(G, H) \text{ has an area } K \text{ drawing } \iff \Phi \text{ satisfiable} \]