Algorithmic Graph Theory

Sommer Term 2015

Lecture #9 (part II)

Coloring Planar Graphs
Coloring (Planar) Graphs

**Def.** Let $G = (V, E)$ be a graph. A map $f : V \rightarrow \{1, \ldots, k\}$ is a $k$-coloring if, for each edge $uv \in E$, it holds that $f(u) \neq f(v)$.

**Obs.** $G$ bipartite $\iff G$ 2-colorable. $G$ $k$-partite $\iff G$ $k$-colorable.

**Obs.** Every planar graph is 6-colorable.

**Proof.** $G$ has a vertex $v$ of degree $\leq 5$. Color $G - v$ inductively. Take 6th color for $v$.

**Thm.** *Four-Color Theorem*
Every planar graph is 4-colorable...

*[Appel & Haken 1976]*

*[Robertson, Sanders, Seymour, Thomas 1997]*

...and there are planar graphs that need 4 colors.
**The Five-Color Theorem of 1890**

**Thm.** *Five-Color Theorem*  
Every planar graph is 5-colorable.

**Def.** Given a graph $G = (V, E)$ and, for each vertex $v$ of $G$, a list $L_v$ of “colors”, a list coloring of $G$ is a map $\lambda : V \rightarrow \bigcup_v L_v$ s.t.  
• $\lambda(v) \in L_v$ and  
• $\lambda(u) \neq \lambda(v)$ for each $uv \in E$.

**Obs.** A “normal” coloring $c : V \rightarrow \{1, \ldots, k\}$ is a list coloring with $L_v = \{1, \ldots, k\}$ for every $v \in V$.

**Expl.** Does this graph have a coloring with the given lists?  
Yes!
List Colorability

Def. A graph $G = (V, E)$ is \textit{k-list colorable} if $G$ has a list coloring for \textit{any} choice of lists of length $k$.

Obs. $G$ \textit{k-list colorable} $\Rightarrow$ \textit{k-colorable}.

Expl. Every bipartite graph is 2-colorable – but not necessarily 2-list colorable.
List Colorability of Planar Graphs

**Thm.** *Not-Four-Color Theorem*
Not every planar graph is 4-list colorable. 

[Voigt 1993]

**Thm.** Every planar graph is 5-*list* colorable. [Thomassen 1994]
(and hence also 5-colorable!)

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Proof of Thomassen’s Theorem

Wlog., $G$ is nearly triangulated, i.e. all interior faces are triangles and the boundary of the outer face is elementary.

Claim. Let $G$ nearly triang., $K$ boundary of the outer face, and

- (i) two adj. vertices $x, y \in K$ are colored with $\alpha \neq \beta$.
- (ii) $|L_v| \geq 3$ for every $v \in K \setminus \{x, y\}$
- (iii) $|L_v| \geq 5$ for every $v \in V \setminus K$.

then the coloring of $x$ and $y$ can be extended to $G$.

Proof. By induction on $n = |V|$.

$n = 3$: 

Color $z$ with $\gamma \in L_z \setminus \{\alpha, \beta\}$.
Induction Step \((n > 3)\)

Case 1: \(K\) has a chord \(uv\).

\(uv\) splits \(K\) into \(K_1\) and \(K_2\).

Let \(G_1\) be the subgraph of \(G\) inside \(K_1 + uv\) (incl. boundary).

Apply induction hypothesis (IH) to \(G_1\).

Apply IH (with \(u\) and \(v\) already being colored!) to \(G_2\).

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Case 2: \(K\) has no chord.

Let \(w \neq y\) be a neighbor of \(x\) on \(K\).

Let \(N(w) = \{x, w_1, \ldots, w_t, y\}\) be the neighborhood of \(w\).

\(|L_w| \geq 3 \Rightarrow \) there are \(\gamma, \delta \in L_w \setminus \{\alpha\}\)

\(L'_{w_i} := L_{w_i} \setminus \{\gamma, \delta\}. \quad G' = G - w\) is nearly triangulated.

\(\Rightarrow \) \(G'\) with lists \(L'\) fulfills IH \(\Rightarrow\) 5-list coloring of \(G'\)

Color \(w\) with \(\{\gamma, \delta\} \setminus \text{color}(v)\) \(\Rightarrow\) 5-list coloring of \(G\) 

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