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Lehrstuhl für
INFORMATIK I
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Computational Geometry

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Seidel's Triangulation Algorithm

Lecture #14

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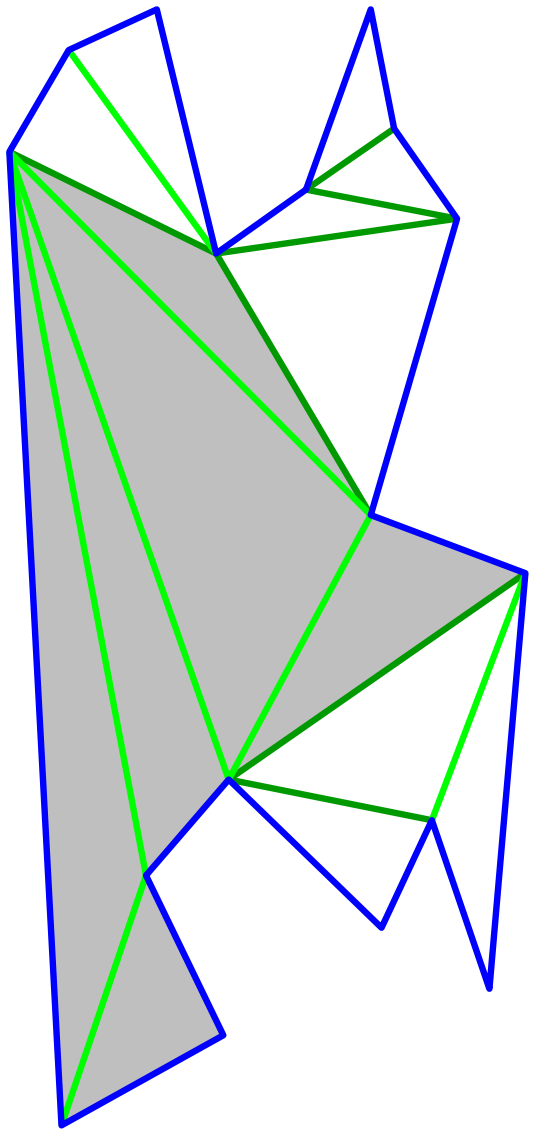
Triangulating a Polygon



Given: Polygon $P = \langle p_1, \ldots, p_n \rangle$
(list of vertices in cw order)

Find: Triangulation of P

i.e., a partition of P into triangles by *diagonals* (segments of type $\overline{p_i p_j} \subset P$)

Approach: *Running time:*

1. Trapezoidize interior of P. $O(?)$
2. Draw diagonals inside trapezoids. $O(n)$
3. Triangulate y-monotone subpolygons $O(n)$

$O(?)$

Lemma 1. Given a trapezoidation, P can be triangulated in linear time.

General Idea

Let S be a set of n non-crossing segments, e.g., polygonal line

WANTED: – trapezoidation $\mathcal{T}(S)$ of S
– point-location data structure $\mathcal{Q}(S)$

Our construction is randomized-incremental:

Trapezoidation(set S of n non-crossing line segments)

$\langle s_1, s_2, \ldots, s_n \rangle \leftarrow$ random ordering of S
$S_0 \leftarrow \emptyset$
for $i = 1$ **to** n **do**
  let $S_i = S_{i-1} \cup \{s_i\}$
  use $\mathcal{T}(S_{i-1})$ and $\mathcal{Q}(S_{i-1})$ to construct $\mathcal{T}(S_i)$ and $\mathcal{Q}(S_i)$

Total cost of one step: – location time
– "threading" time

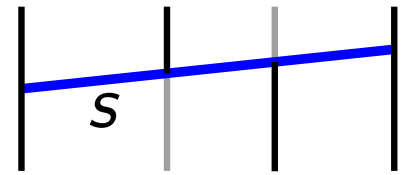
Threading time

We assume general position
(no two points have the same y-coordinate).

Use lexicographic order!

Lemma 2. For $i = 1, \ldots, n$, the expected number of rays of $\mathcal{T}(S_{i-1})$ that are intersected by s_i is at most 4.

Proof. For $s \in S_i$, let $\deg(s, \mathcal{T}(S_i)) = \#$ rays of $\mathcal{T}(S_i)$ that hit the relative interior of s.



$\#$ rays of $\mathcal{T}(S_{i-1})$ intersected by $s_i = \deg(s_i, \mathcal{T}(S_i))$

$\#$ rays in $\mathcal{T}(S_i) \leq 4i$

$\Rightarrow \sum_{s \in S_i} \deg(s, \mathcal{T}(S_i)) \leq 4i$

Ordering of S_i random $\Rightarrow E[\deg(s_i, \mathcal{T}(S_i))] \leq 4$ □

Location Time

Recall: $H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} \in \Theta(\log n)$
More precisely, $\ln n < H_n < 1 + \ln n$ for $n > 1$.

Lemma 3. For any query point q, the expected length of the search path of q in $\mathcal{Q}(S_n)$ is at most $5H_n \in O(\log n)$.

Theorem. Let S be a set of n non-crossing line segments.

- We can build $\mathcal{T}(S)$ and $\mathcal{Q}(S)$ in $O(n \log n)$ expected time.
- The expected size of $\mathcal{Q}(S)$ is $O(n)$.
- The expected time for locating a point in $\mathcal{T}(S)$ via $\mathcal{Q}(S)$ is $O(\log n)$.

Aim: Speed-up construction for simple polygonal chains.

New Approach

Observe: in $\mathcal{Q}(S_i)$,
– point location takes $O(\log i)$ expected time
– threading s_{i+1} takes $O(1)$ expected time

Idea: Exploit polygon structure!
Locate once, then follow polygon.

Problem: This way, we loose the random structure!
$\Rightarrow$ threading becomes more expensive
$\Rightarrow$ $\Theta(n^2)$-time algorithm :-(

Solution:
- insert segments in random order
- every now and then, locate *all* polygon vertices in the current triangulation – *by walking along the polygon!*

The Two Main New Technical Ingredients

Questions:

- How much does the intermediate location information help later?
- How expensive is it to walk along the polygon in the current triangulation?

Lemma 4. Let $1 \leq j \leq k \leq n$ and $q \in \mathbb{R}^2$. Suppose location of q in $\mathcal{Q}(S_j)$ is known, then q can be located in $\mathcal{Q}(S_k)$ in expected time $5(H_k - H_j) \in O(\log k/j)$.

Lemma 5. S as before, $R \subseteq S$ random subset, $r := |R|$.
Let I be the number of intersections between rays of $\mathcal{T}(R)$ and segments in $S \setminus R$. Then $E[I] \leq 4(n - r)$, where the expectation is over all size-r subsets of S.

Logs All Over the Place

Definition. Let the *i-th iterated logarithm* of n be defined by

$$\log^{(i)} n := \begin{cases} n & \text{if } i = 0, \\ \log_2(\log^{(i-1)} n) & \text{if } i > 0. \end{cases}$$

For $n > 0$, let $\log^\star n := \max\{i \mid \log^{(i)} n \geq 1\}$.

For $0 \leq h \leq \log^\star n$, let $N(h) := \lceil n / \log^{(h)} n \rceil$.

Examples.

$$\log^{(0)} 2^{2^{2^2}} = 2^{2^{2^2}}$$

$$\log^{(1)} 2^{2^{2^2}} = \log_2 2^{2^{2^2}} = 2^{2^2}$$

$$\log^{(2)} 2^{2^{2^2}} = \log_2 \log^{(1)} 2^{2^{2^2}} = 2^2$$

$$\log^{(3)} 2^{2^{2^2}} = 2; \quad \log^{(4)} 2^{2^{2^2}} = 1 \Rightarrow \log^\star 2^{2^{2^2}} = 4$$

$N(0) = 1$, $N(1) = \lceil n / \log n \rceil$, $\ldots$, $N(\log^\star n) > n/2$.

The Algorithm

PolygonTrapezoidation((edges along) simple polygonal chain P)

1. $\langle s_1, s_2, \ldots, s_n \rangle :=$ random ordering of the edges of P
2. Compute $\mathcal{T}_1$ and $\mathcal{Q}_1$ for $\{s_1\}$.
   For each vtx v of P, let $\pi(v)$ point to the leaf of $\mathcal{Q}_1$ that contains v.

for $h = 1$ **to** $\log^\star n$ **do** // phase h

3.1 **for** $i = N(h-1)+1$ **to** $N(h)$ **do**
   insert $s_i = v_i w_i$ in $\mathcal{T}_{i-1}$ using $\pi(v_i)$ (node in $\mathcal{Q}_{N(h-1)}$)

3.2 walk along P through $\mathcal{T}_{N(h)}$:
   for each vertex v
   – let Δ be the trapezoid in $\mathcal{T}_{N(h)}$ that contains v
   – let $\pi(v)$ be the node in $\mathcal{Q}_{N(h)}$ corresponding to Δ

4. **for** $i = N(\log^\star n)+1$ **to** n **do**
   insert $s_i = v_i w_i$ in $\mathcal{T}_{i-1}$ using $\pi(v_i)$ (node in $\mathcal{Q}_{N(\log^\star n)}$)

return $(\mathcal{T}_n, \mathcal{Q}_n)$

Time Complexity

Step 1: Random permutation $O(n)$

Step 2: Setting up $\mathcal{T}_1$ and $\mathcal{Q}_1$ $O(n)$

Step 3: Phases 1 to $\log^\star n$ $(\log^\star n) \cdot$

Step 3.2: Walking the polygon Lemma 5 $\Rightarrow$ $O(n)$

Step 3.1: Inserting $s_i = v_i w_i$ using $\mathcal{Q}_{N(h-1)}$

– threading cost: Lem. 2 $\Rightarrow$ expected $O(1)$ per segm.

– locating cost: Know the location of v_i in $\mathcal{Q}_{N(h-1)}$.
Lem. 4 $\Rightarrow$ expected location cost
$O(\log i / N(h-1)) \subseteq O(\log^{(h)} n)$

$\Big\} \cdot N(h) = O(n)$

Step 4: Inserting s_i (for $N(\log^\star n) < i \leq n$) using $\mathcal{Q}_{N(\log^\star n)}$ $O(n)$

– threading cost: Lem. 2 $\Rightarrow O(1)$

– locating cost: Lem. 4 $\Rightarrow O(\log n / \underbrace{N(\log^\star n)}_{> n/2}) = O(1)$

$\Big\} \cdot O(n) =$

$O(n \log^\star n)$

The Results

Theorem. Let S be the edge set of a polygonal chain, $|S| = n$.

- We can build $\mathcal{T}(S)$ and $\mathcal{Q}(S)$ in $O(n \log^\star n)$ expected time.
- The expected size of $\mathcal{Q}(S)$ is $O(n)$.
- The expected time for locating a point in $\mathcal{T}(S)$ via $\mathcal{Q}(S)$ is $O(\log n)$.

Theorem. Let S be the edge set of a plane straight-line graph with k connected components, $|S| = n$.

- We can build $\mathcal{T}(S)$ and $\mathcal{Q}(S)$ in $O(n \log^\star n + k \log n)$ expected time.
- The expected size of $\mathcal{Q}(S)$ is $O(n)$.
- The expected time for locating a point in $\mathcal{T}(S)$ via $\mathcal{Q}(S)$ is $O(\log n)$.