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Lehrstuhl für
INFORMATIK I
Effiziente Algorithmen und
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Institut für Informatik

Computational Geometry

Winter semester 2014/15

Arrangements and Duality

Lecture #12

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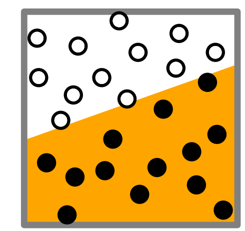
Chair for Computer Science I

Rendering und Ray-Tracing

- Determine which object in the scene is visible through a given pixel (given the viewing direction of the observer).
- Compute the intensity of the light that the visible object emits in the direction of the pixel.
- How to color a pixel that is intersected by the boundary of an object?



  - As the object or the background?
    Problem: object boundaries look "pixelated".
  - Better: if the object occupies $x\,\%$ of the pixel area (and the background $(100-x)\,\%$, color the pixel with ratio

$$\text{object color} \;/\; \text{background color} = x/(100-x).$$

Supersampling: – send many rays through the pixel

– use ratio $\dfrac{\#\,\text{rays that hit object}}{\#\,\text{rays that hit background}} = \dfrac{|\bullet|}{|\circ|}$

Supersampling

- distribute rays regularly:
  leads to artefacts that can be visible
- distribute rays randomly:
  may lead to better results, but –
  how to measure the quality of a distribution?
  (if the distribution *is* bad, simply produce a new one!)

Def. *discrepancy* Δ(distribution S, object o) $= |\mu(o) - \mu_S(o)|$,

and

$\Delta(S) = \sup_{o \text{ object}} \Delta(S, o)$.

simpler:

$\Delta(S) = \sup_{h \in \mathcal{H}} \Delta(S, h)$.

where

$\mu(o) = \text{area}(Q \cap o)$
$Q = [0, 1] \times [0, 1] \mathrel{\hat{=}} \text{pixel}$
$\mu_S(o) = |S \cap o| / |S|$

$\mathcal{H}$ = set of all closed half-planes whose boundary intersects Q

Halfplane discrepancy

$\Delta(S) = \sup_{h \in \mathcal{H}} \Delta(S, h)$. **Problem:** is uncountable :-(

Strategy: – choose candidate set $\mathcal{H}' \subseteq \mathcal{H}$ which contains a halfplane h that maximizes $\Delta(S, h)$

– find $h \in \mathcal{H}'$ with $\Delta(s, h)$ maximum.

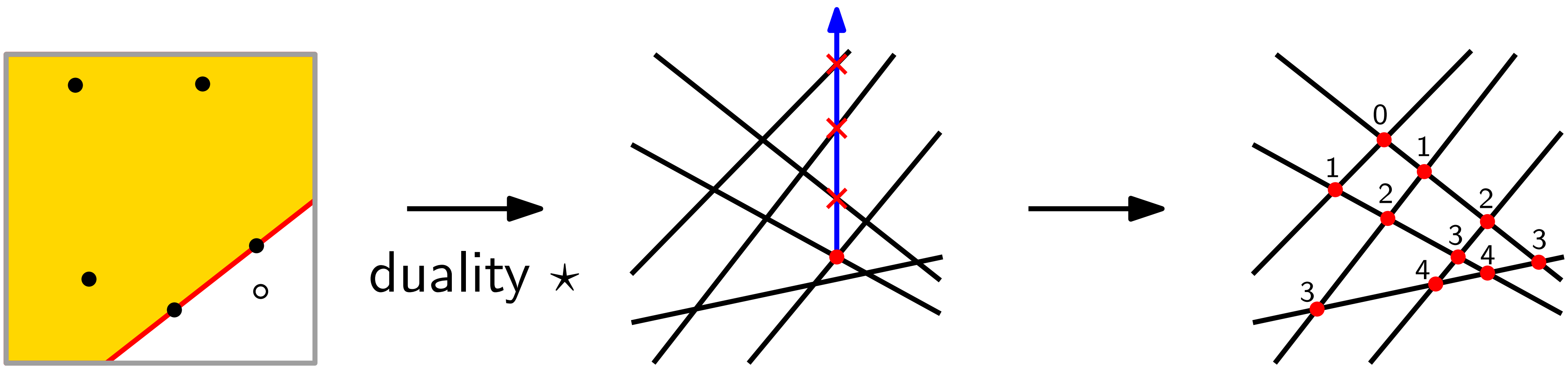
Lemma. The set $\mathcal{H}' = \mathcal{H}_1 \cup \mathcal{H}_2$ contains a halfplane h that maximizes $|S \cap h|$, where
$\mathcal{H}_1 = \{h \in \mathcal{H} : \{p\} := |\partial h \cap S| = 1$ and $\mu(h)$ is locally extreme among all halfplanes that touch $p\}$,
$\mathcal{H}_2 = \{h \in \mathcal{H} : |\partial h \cap S| = 2\}$.
With $n = |S|$, we have $|\mathcal{H}_1| \in \Theta(n)$, $|\mathcal{H}_2| \in \Theta(n^2)$.

Thm. The halfplane discrepancy of a set of n pts in the unit square can be computed in $O(n^2)$ time. TODO!

Idea

For each $h \in \mathcal{H}_1$, we can compute $\Delta(S, h)$ in $O(n)$ time (by brute force). $\Rightarrow$ $O(n^2)$ time

But how to compute the discrepancy of the $\Theta(n^2)$ many half planes in $\mathcal{H}_2$ in $O(n^2)$ total time? In particular, we need $|h \cap S|$!



- Compute arrangement $\mathcal{A}$ of the lines in $S^\star$
- For each vertex $h^\star$ of $\mathcal{A}$, compute its *depth* in $\mathcal{A}$, that is, the number of lines that lie strictly above $h^\star$ $(= |S \cap \operatorname{int}(h)|)$ and the degree of $h^\star$ in $\mathcal{A}$ $(= 2|S \cap \partial h|)$

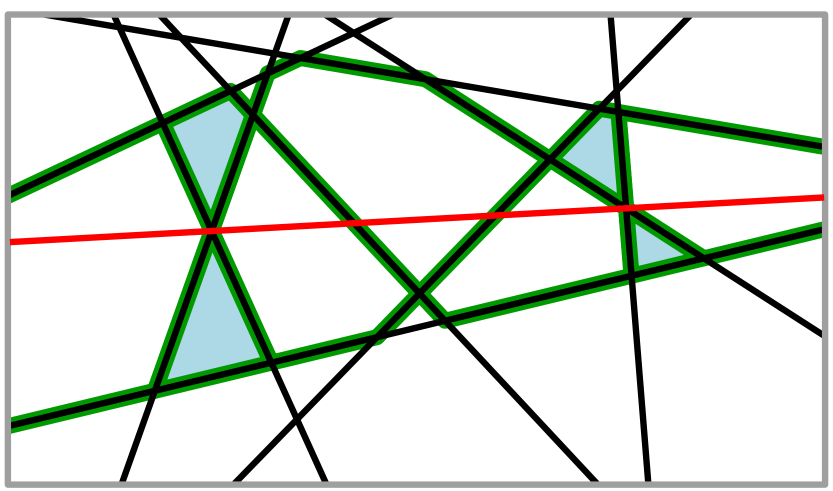
Arrangements of Lines

How can we compute the arrangement $\mathcal{A}(\mathcal{L})$ of a set $\mathcal{L}$ of n lines in $O(n^2)$ time? – This would be optimal!

- line-sweep algorithm? $O(n^2 \log n)$ time...
- incrementally?

  If we can insert line i in $O(i)$ time, we get a runtime of $O(n^2)$.

  – compute doubly-connected edge list (DCEL) of $\mathcal{A}(\mathcal{L})$
  – use sufficiently large rectangle R (how?) $\Rightarrow$ edges bounded



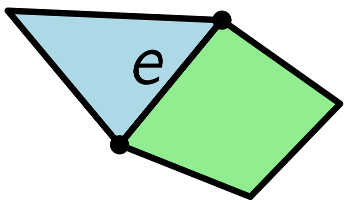
Effort for inserting ℓ_i is bounded by the complexity of the *zone* of ℓ_i, that is, the set of faces whose *closure* intersects ℓ_i

The Zone Theorem

Theorem. The complexity of the zone of a line ℓ in an arrangement $\mathcal{A}(\mathcal{L})$ of a set $\mathcal{L}$ of m lines is $O(m)$.

Proof. W.l.o.g. $\ell = x$-axis.

We (initially) assume that no line in $\mathcal{L}$ is horizontal.



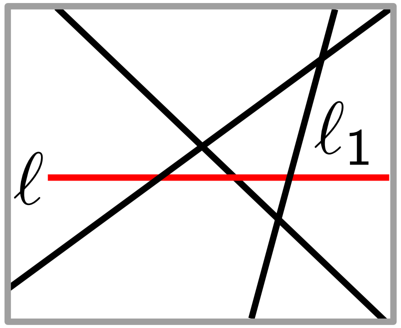
Every edge e of $\mathcal{A}$ bounds two faces.

We say that e *left-bounds* the face to its right.

Show: The number of left-bounding edges in zone(ℓ) is $\leq 5m$.

We use induction over m. The base case ($m = 1$) is trivial.

Let $m > 1$. Let ℓ_1 be the rightmost line that is intersected by ℓ.



Induction hypothesis $\Rightarrow$ zone(ℓ) in $\mathcal{A}(\mathcal{L} \setminus \{\ell_1\})$ has $\leq 5(m-1)$ left-bounding edges.

Now we re-insert ℓ_1 into $\mathcal{A}(\mathcal{L} \setminus \{\ell_1\})$.

Proof cont'd

Let v / w be the first intersection pt on ℓ_1 above / below ℓ. (if such pts exist)

$\Rightarrow$ $\overline{vw}$ is a new left-bounding edge.

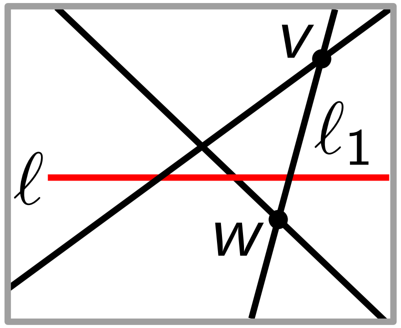
Additionally, ℓ_1 splits existing left-bounding edges in v and w.

$\Rightarrow$ # of left-bounding edges increases by ≤ 3 (less if v or w don't exist).

If ℓ_1 is not unique, let ℓ_1 be an arbitrary line through the rightmost intersection pt on ℓ.

$\Rightarrow$ ℓ_1 causes ≤ 5 new left-bounding edges (argue similarly as above)

$\Rightarrow$ total # left-bounding edges $\leq 5(m-1)+5=5m$.



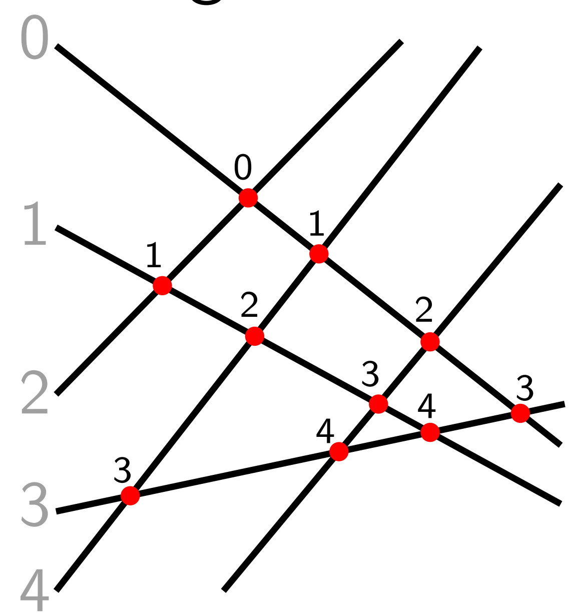
If lines in $\mathcal{L}$ are horizontal, then slighly rotating them only increases the complexity of the zone.
$\Rightarrow$ Our above bound holds in this case, too.

$\square$

Back to Discrepancy

Theorem. Given a set $\mathcal{L}$ of n lines, we can compute a DCEL of the arrangement induced by $\mathcal{L}$ in $O(n^2)$ time.

It remains to compute, for each vertex of the arrangement, its depth (and degree).



- Sort lines in $S^{\star}$ according to slope
  $\Rightarrow$ depth at $x = -\infty$ corresponds to rank in this order (minus 1)
- Traverse each line from left to right.
  Adjust its depth by $+1$ (-1) if the intersecting line comes from below (above).