Julius-Maximilians-
UNIVERSITÄT
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Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Computational Geometry

Winter semester 2014/15

Convex Hulls in 3D

Lecture #9

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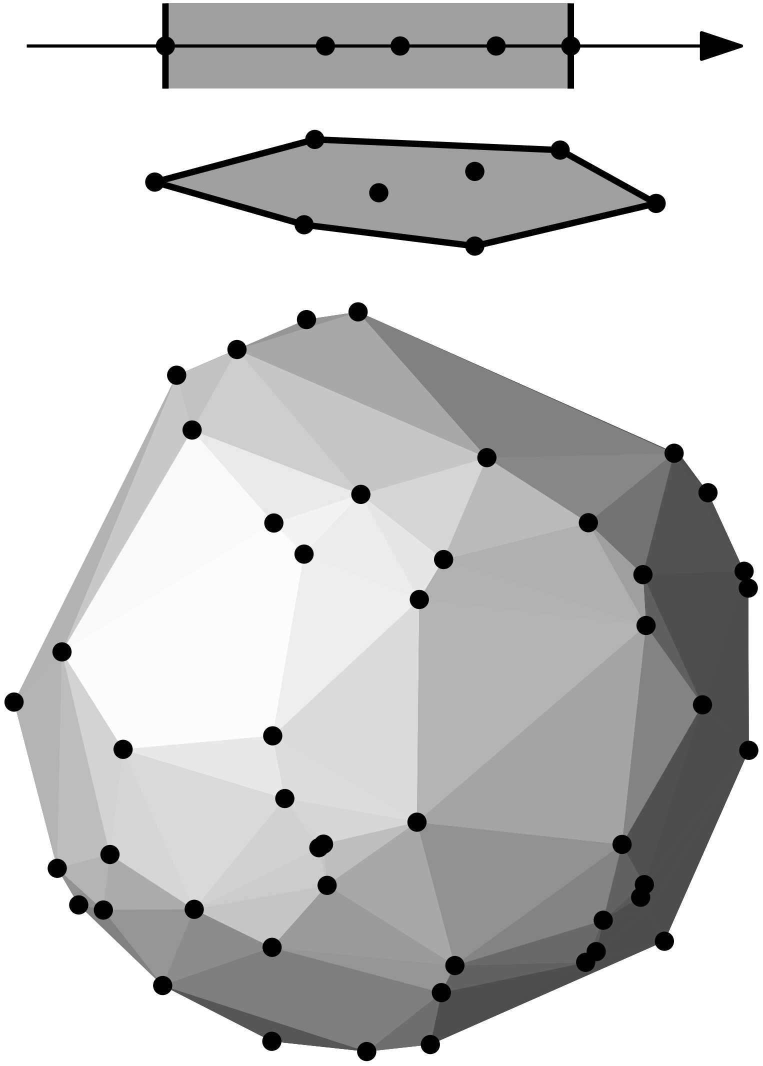
Complexity of the Convex Hull

Given set S of n points in $\mathbb{R}^d$, what is max. #edges on $\partial\mathrm{CH}(S)$?

dim	w-c complexity of CH(S)
1	$2 \in \Theta(1)$
2	$n \in \Theta(n)$
3	$3n - 6 \in \Theta(n)$
d	$\Theta(n^{\lfloor d/2 \rfloor})$

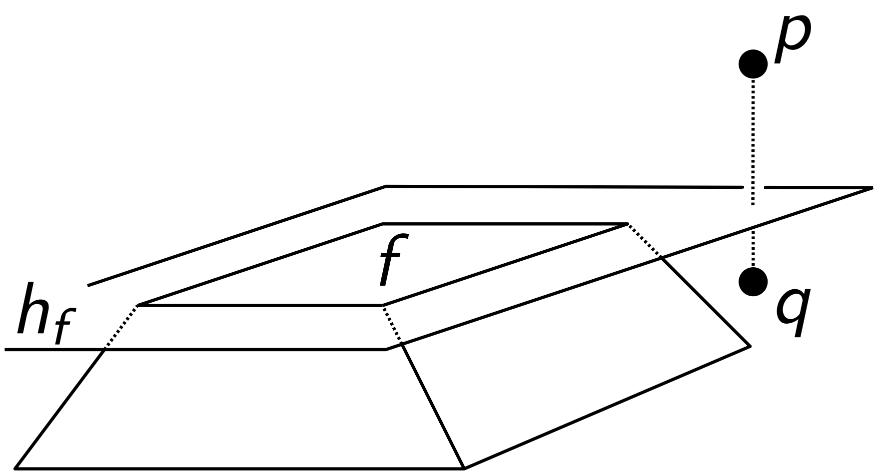
Upper Bound Theorem



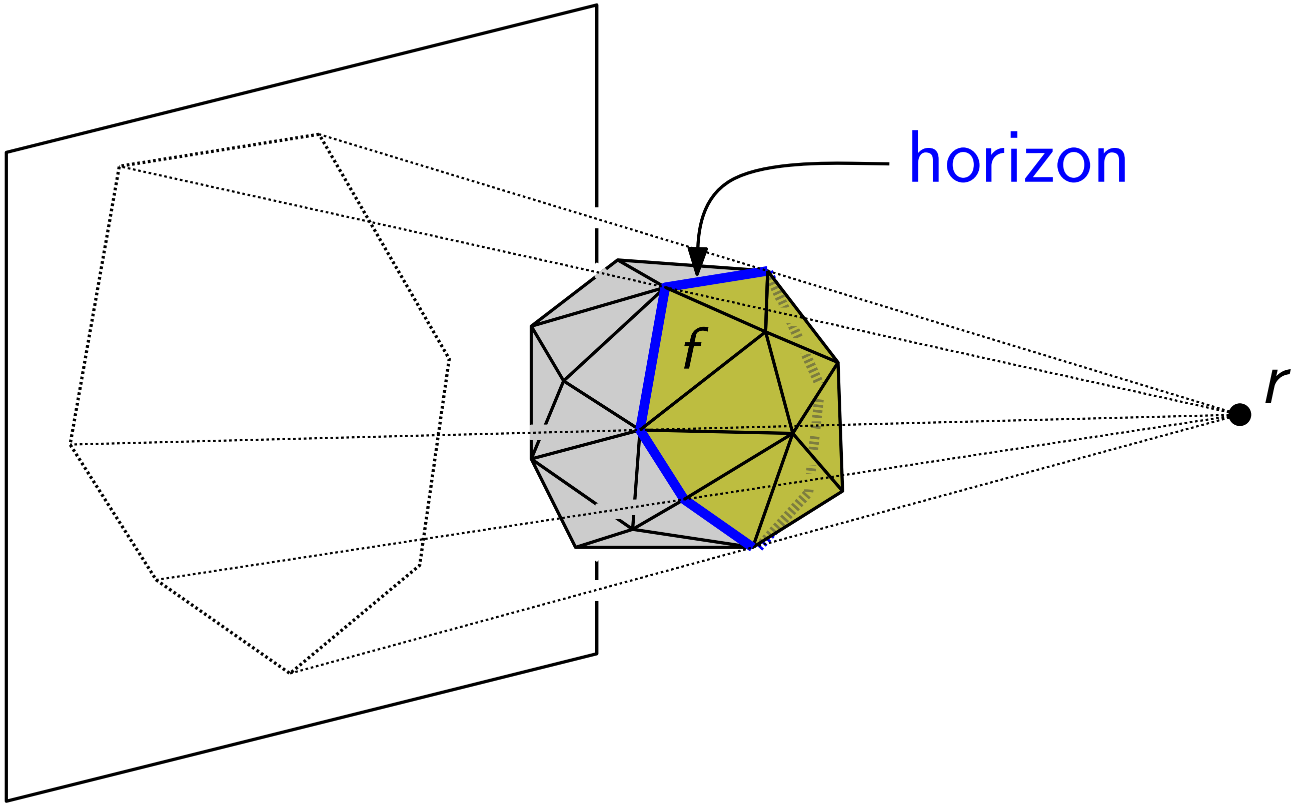
Construction

randomized-incremental!

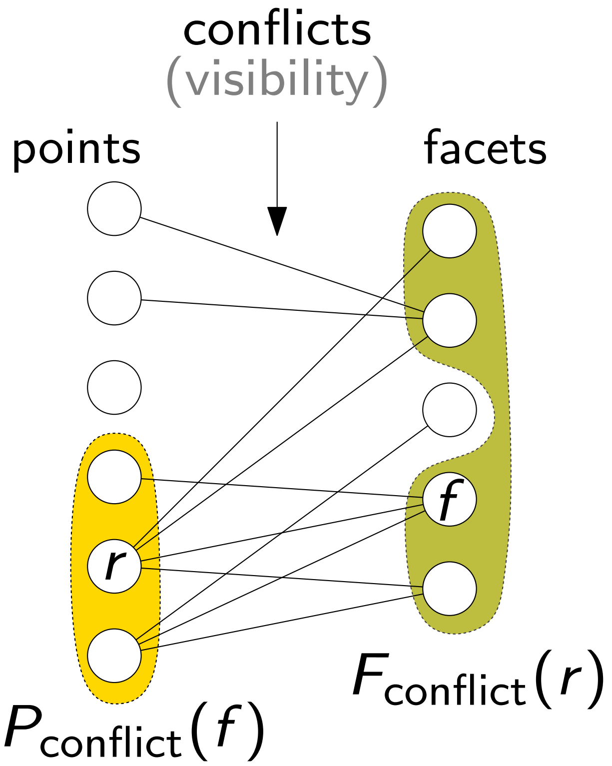
Visibility

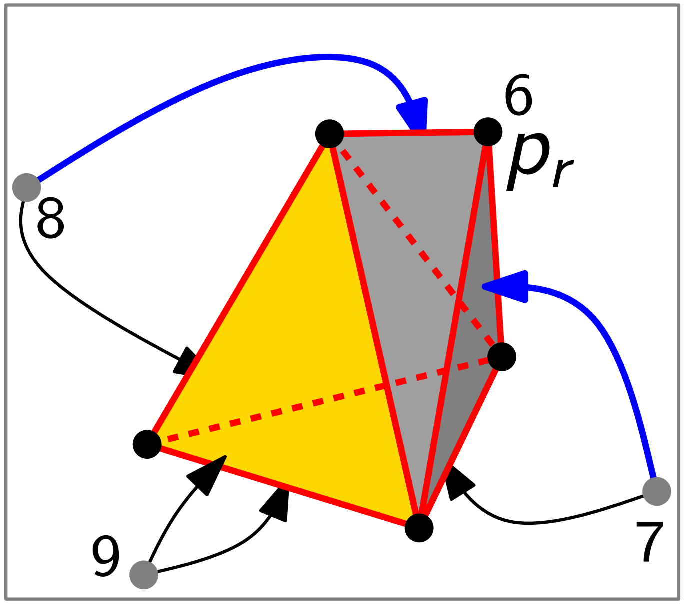


Face f is *visible* from p but not from q.



Define conflict graph G:





```
Rand3dConvexHull($P \subset \mathbb{R}^3$)
    pick non-coplanar set $P' = \{p_1, \ldots, p_4\} \subseteq P$
    $C \leftarrow \mathrm{CH}(P')$
    compute rand. perm. $(p_5, \ldots, p_n)$ of $P \setminus P'$
    initialize conflict graph $G$
    for $r = 5$ to $n$ do
        if $F_{\text{conflict}}(p_r) \neq \emptyset$ then   $\{ p_r \notin C \}$
            delete all facets in $F_{\text{conflict}}(p_r)$ from $C$
            $\mathcal{L} \leftarrow$ list of horizon edges visible from $p_r$
            foreach $e \in \mathcal{L}$ do
                $f \leftarrow C$.create_facet$(e, p_r)$; create vtx for $f$ in $G$
                $(f_1, f_2) \leftarrow$ previously_incident$_C(e)$
                $P(e) \leftarrow P_{\text{conflict}}(f_1) \cup P_{\text{conflict}}(f_2)$
                foreach $p \in P(e)$ do
                    if $f$ visible from $p$ then add edge $(p, f)$ to $G$
            delete vtc $\{p_r\} \cup F_{\text{conflict}}(p_r)$ from $G$
    return $C$
```

Worst-case running time $= O(n^3)$

Analysis

Idea: Bound expected *structural change*,
that is, the total #facets created by the algorithm.

Lemma. The expected #facets created is at most $6n - 20$.

Proof.

$$E[\#\text{facets created}] =$$

$$= 4 + \sum_{r=5}^{n} \underbrace{E[\#\overset{\#\text{edges}}{\text{facets}} \text{ incident to } p_r \text{ in } \mathrm{CH}(P_r)]}_{\deg(p_r,\, \mathrm{CH}(P_r))} \leq \begin{matrix} 6n \\ -20 \end{matrix}$$

For $r > 4$:

$$E[\deg(p_r, \mathrm{CH}(P_r)] = \tfrac{1}{r-4} \sum_{i=5}^{r} \deg(p_i, \mathrm{CH}(P_r))$$

$$\leq \tfrac{1}{r-4} \Big[\underbrace{\Big(\sum_{i=1}^{r} \deg(p_i)\Big)}_{2 \cdot \#\text{ edges of } \mathrm{CH}(P_r)} - 12 \Big]$$

$$\leq \tfrac{1}{r-4} \big[2 \cdot (3r - 6) - 12 \big] \;\leq 6$$

Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

$O(n)$ time

```
Rand3dConvexHull($P \subset \mathbb{R}^3$)
  pick set $P' = \{p_1, \dots, p_4\} \subseteq P$ of 4 non-coplanar pts
  $C \leftarrow \mathrm{CH}(P')$
  compute a random permutation $(p_5, \dots, p_n)$ of $P \setminus P'$
  initialize conflict graph $G$
  for $r = 5$ to $n$ do
      if $F_{\text{conflict}}(p_r) \neq \emptyset$ then
          delete all facets in $F_{\text{conflict}}(p_r)$ from $C$
          $\mathcal{L} \leftarrow$ list of horizon edges visible from $p_r$
          foreach $e \in \mathcal{L}$ do
              $f \leftarrow C$.create_facet($e, p_r$); create vtx for $f$ in $G$
              $(f_1, f_2) \leftarrow$ previously_incident$_C(e)$
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              foreach $p \in P(e)$ do
                  if $f$ visible from $p$ then add edge $(p, f)$ to $G$
          delete vtc $\{p_r\} \cup F_{\text{conflict}}(p_r)$ from $G$
  return $C$
```

Stage r of for-loop (w/o outer foreach loop) takes time $O(|F_{\text{conflict}}(p_r)|) = O(\#\text{facets deleted when adding } p_r)$

This part of for-loop in total:

$$E[\#\text{facets deleted}] = \\ \leq E[\#\text{facets created}] \underset{\text{Lemma}}{=} O(n).$$

Outer foreach-loop:

– in stage r: $O(\sum_{e \in \mathcal{L}} |P(e)|)$

– in total:

$$O\left(\sum_{e \text{ on horizon at some moment}} |P(e)|\right)$$

$$= O(n \log n)$$

using *configuration spaces*, Section 9.5 [De Berg et al.]

Running times – expected vs. worst case

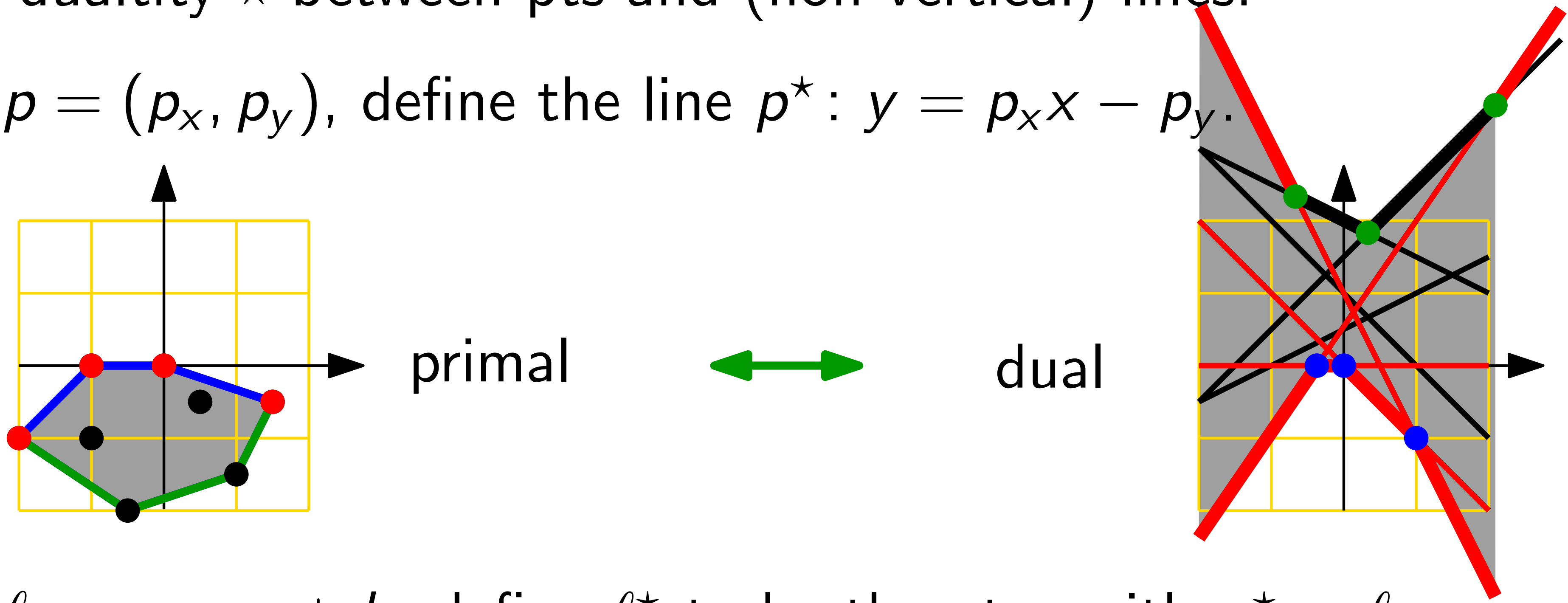
Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

Exercise: Give a simple deterministic algorithm that computes the convex hull in $O(n^2)$ (worst-case) time.

Convex Hulls and Half-~~Space~~ Plane Intersections

Define dualtity $\star$ between pts and (non-vertical) lines:

For $p = (p_x, p_y)$, define the line $p^\star\colon\ y = p_x x - p_y$.



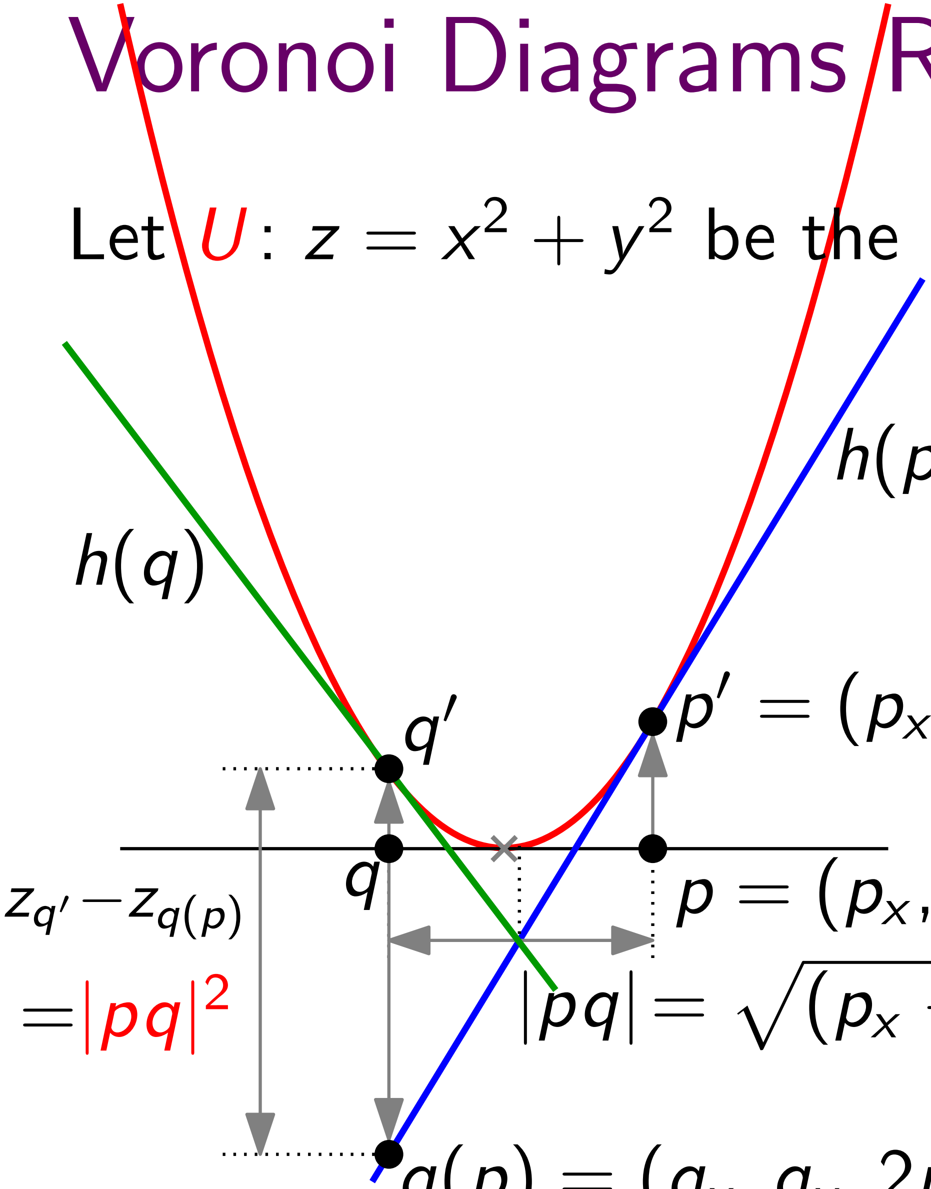
For $\ell\colon\ y = mx + b$, define $\ell^\star$ to be the pt q with $q^\star = \ell$,

that is, $\ell^\star = (m, -b)$.

Observe:
- upper convex hulls of pts ↔ lower envelopes of lines
- can compute intersections of "lower/upper" half planes (spaces) via upper/lower convex hulls

Voronoi Diagrams Revisited

Let $U\colon z = x^2 + y^2$ be the *unit paraboloid* in $\mathbb{R}^3$.



$h(p)\colon z = (2p_x)x + (2p_y)y - (p_x^2 + p_y^2)$

Note that $p' \in h(p)$.

$h(q)$

q'

$p' = (p_x, p_y, p_x^2 + p_y^2)$

q

$p = (p_x, p_y, 0)$

$z_{q'} - z_{q(p)}$

$= |pq|^2$

$|pq| = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}$

$q(p) = (q_x, q_y, 2p_x q_x + 2p_y q_y - (p_x^2 + p_y^2))$

$\Rightarrow$ $h(p)$ and U encode dist. betw. p and any other pt in $z = 0$.

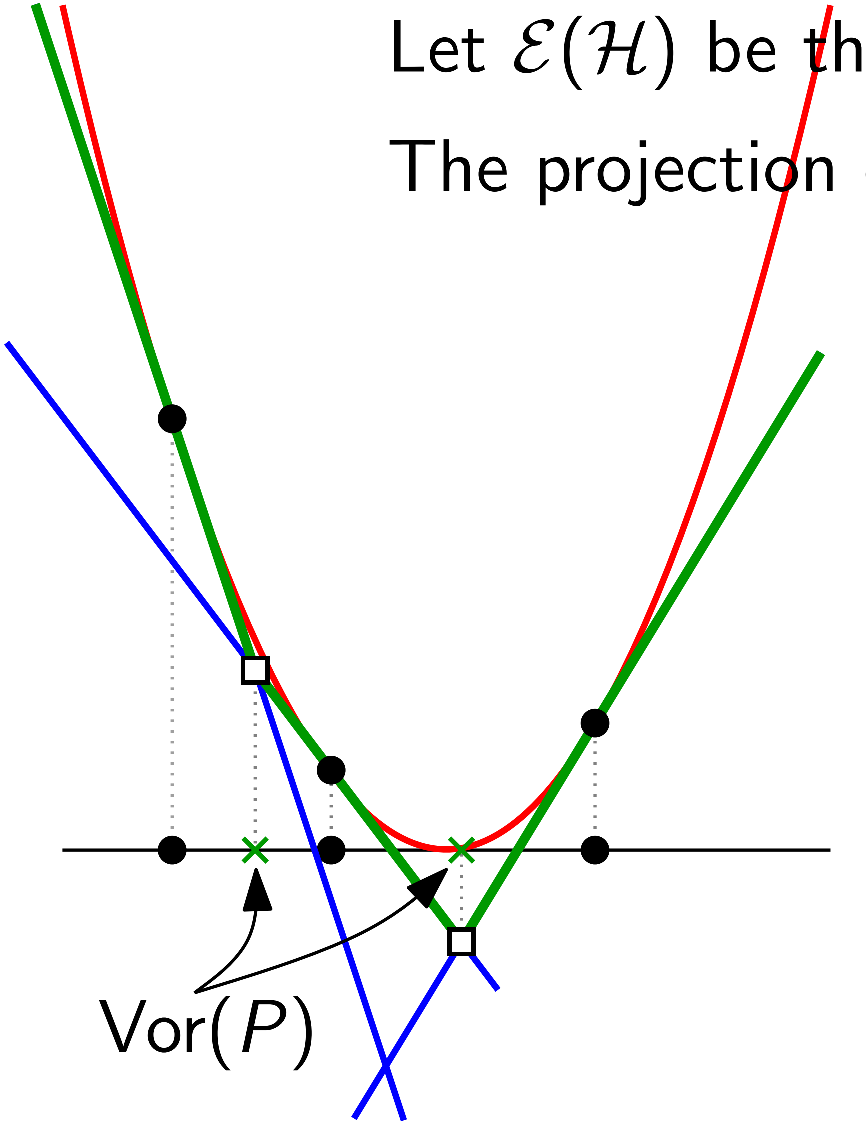
$\Rightarrow$ $h(p) \cap U = \{p'\}$ $\Rightarrow$ $h(p)$ is tangent to U (in p')

The Upper Envelope Strikes Back

Theorem: Let $P \subset \mathbb{R}^2 \times \{0\}$ and $\mathcal{H} = \{h(p) \mid p \in P\}$.

Let $\mathcal{E}(\mathcal{H})$ be the upper envelope of $\mathcal{H}$.

The projection of $\mathcal{E}(\mathcal{H})$ on $z = 0$ is Vor(P).



↓

can compute Vor(P) in $\mathbb{R}^2$
via upper envelope in $\mathbb{R}^3$

↓ exercise 11.10

upper envelope in $\mathbb{R}^3$ is in
one-to-one correspondence to
lower convex hull of the pt set $\mathcal{H}^\star$

↓

use algorithm Rand3dConvexHull!