Julius-Maximilians-
UNIVERSITÄT WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Computational Geometry

Winter semester 2014/15

Convex Hulls in 3D

Lecture #9

Prof. Dr. Alexander Wolff

Chair for Computer Science I

Complexity of the Convex Hull

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3	
d	

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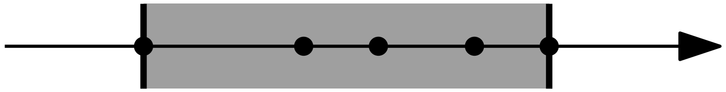

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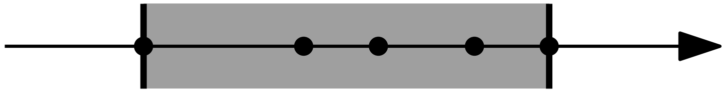

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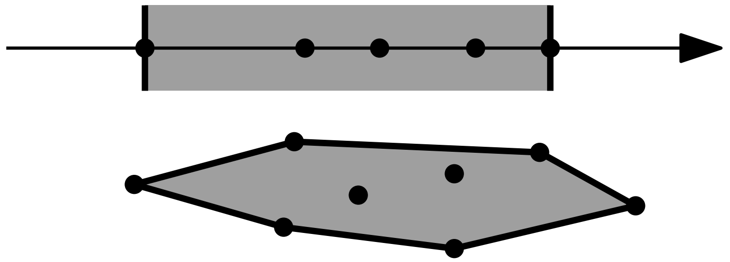

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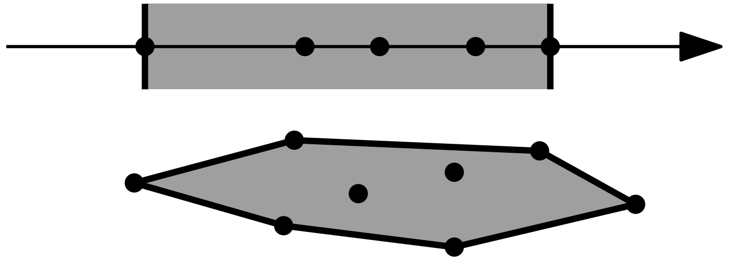

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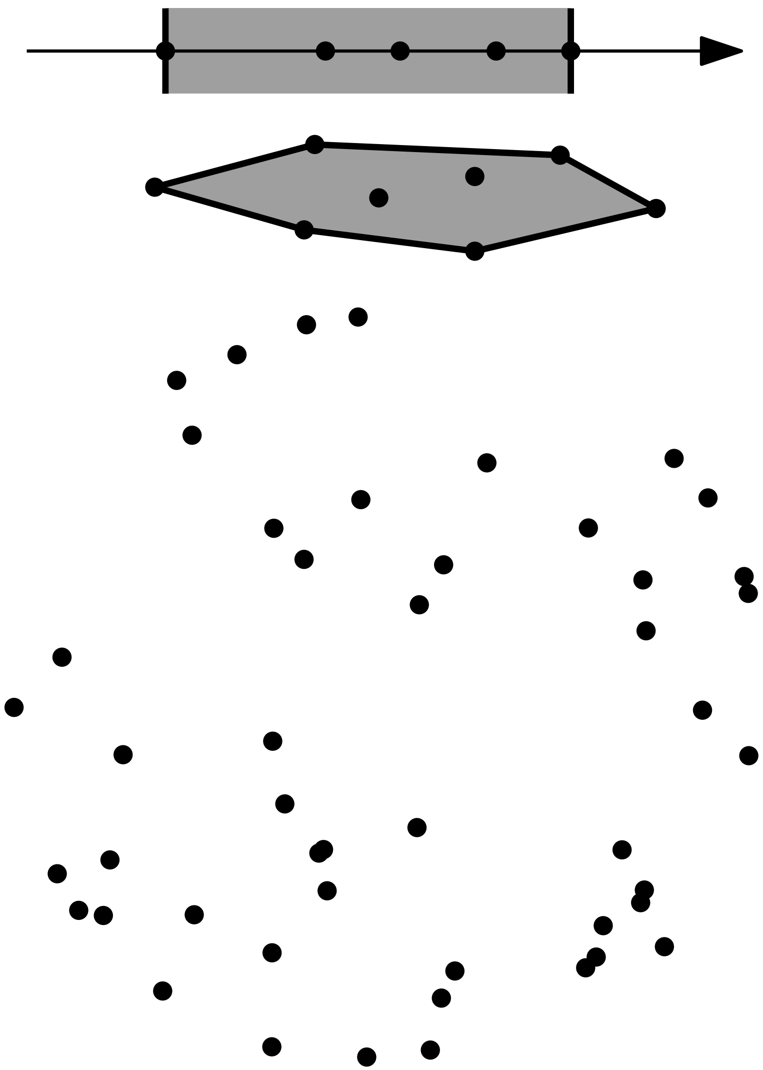

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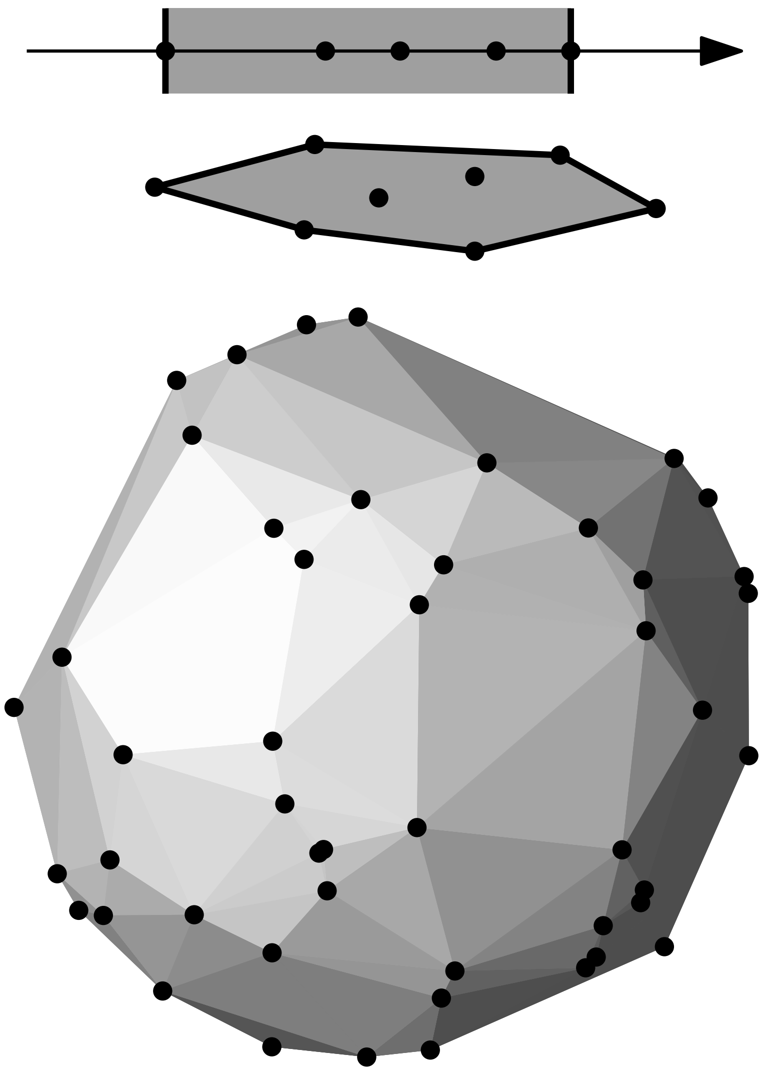

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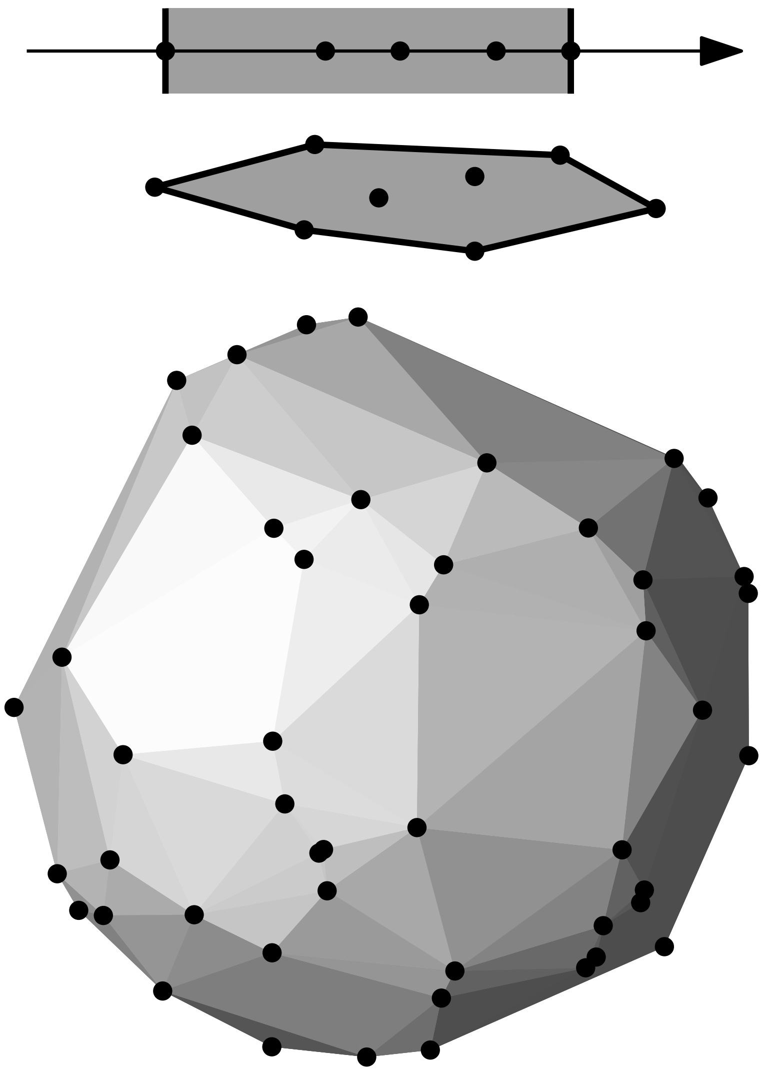

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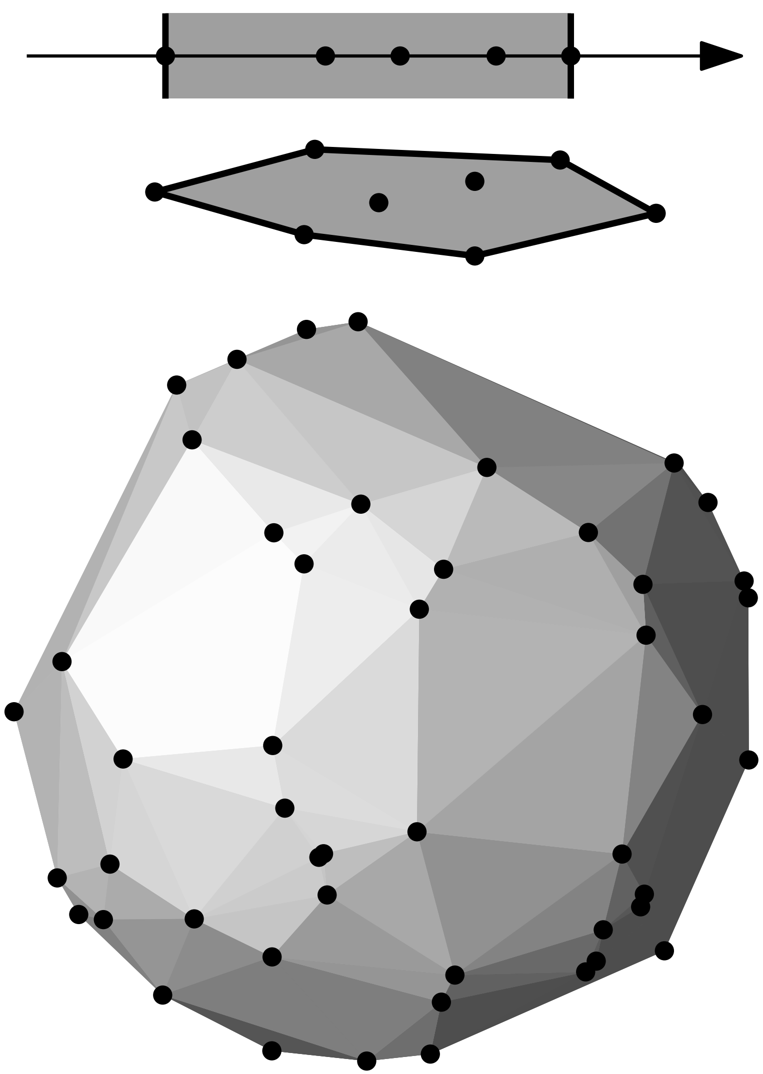

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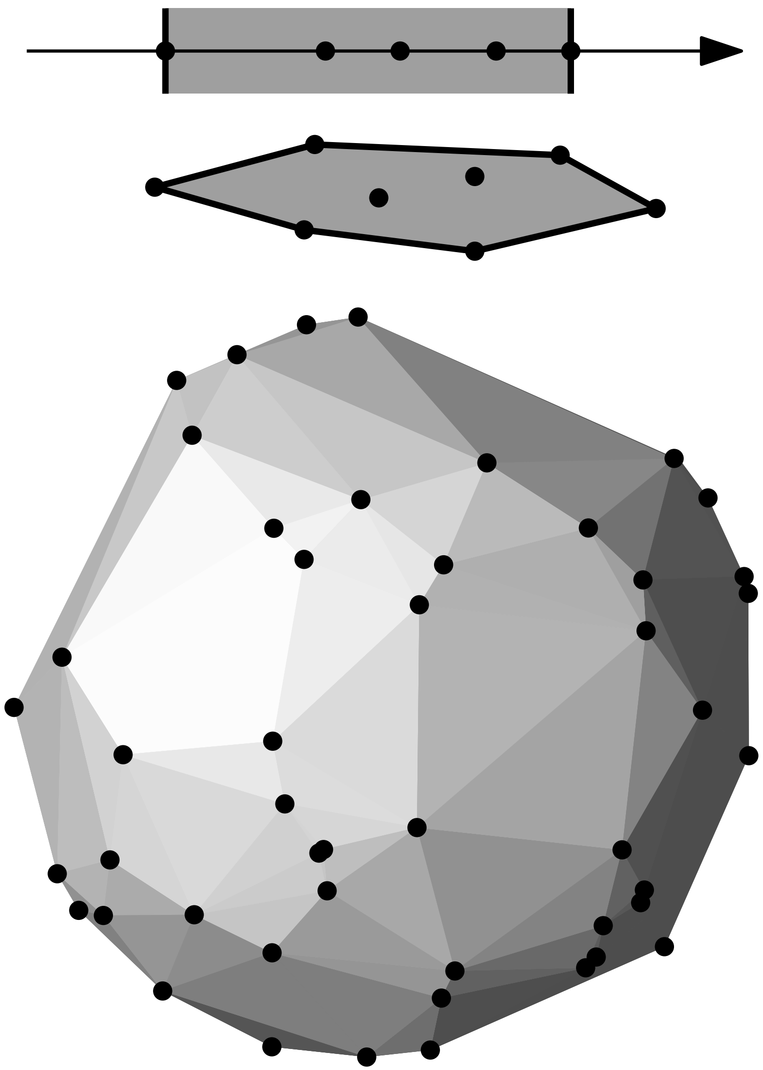

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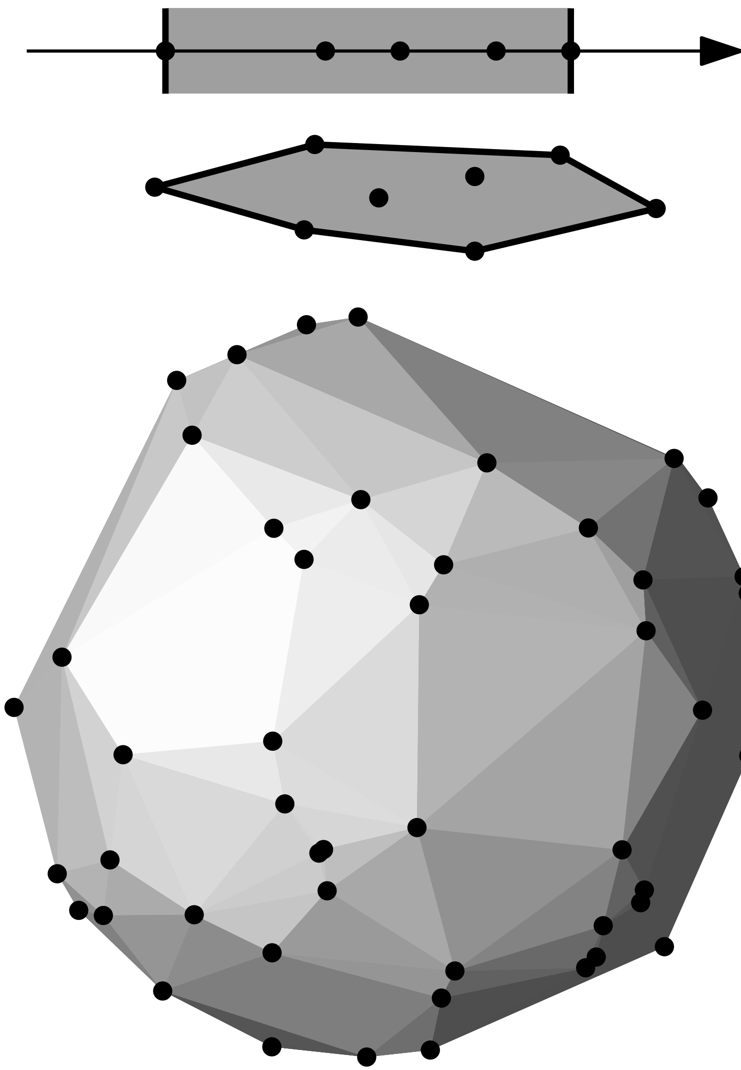

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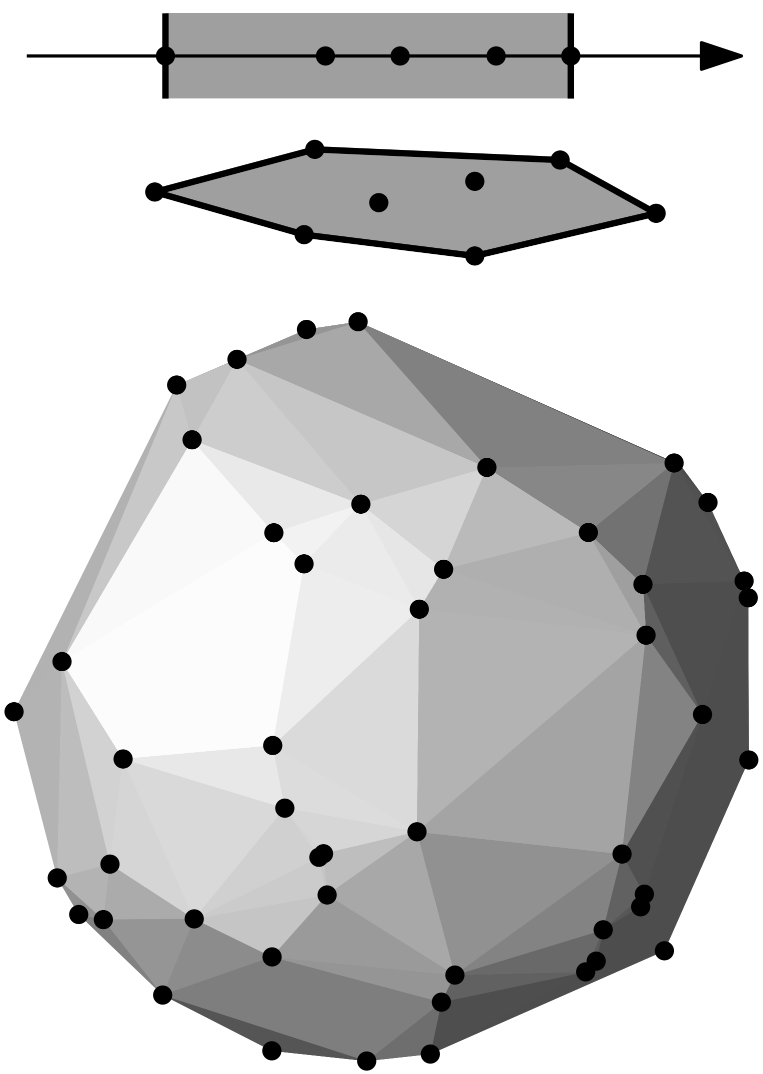

Construction?

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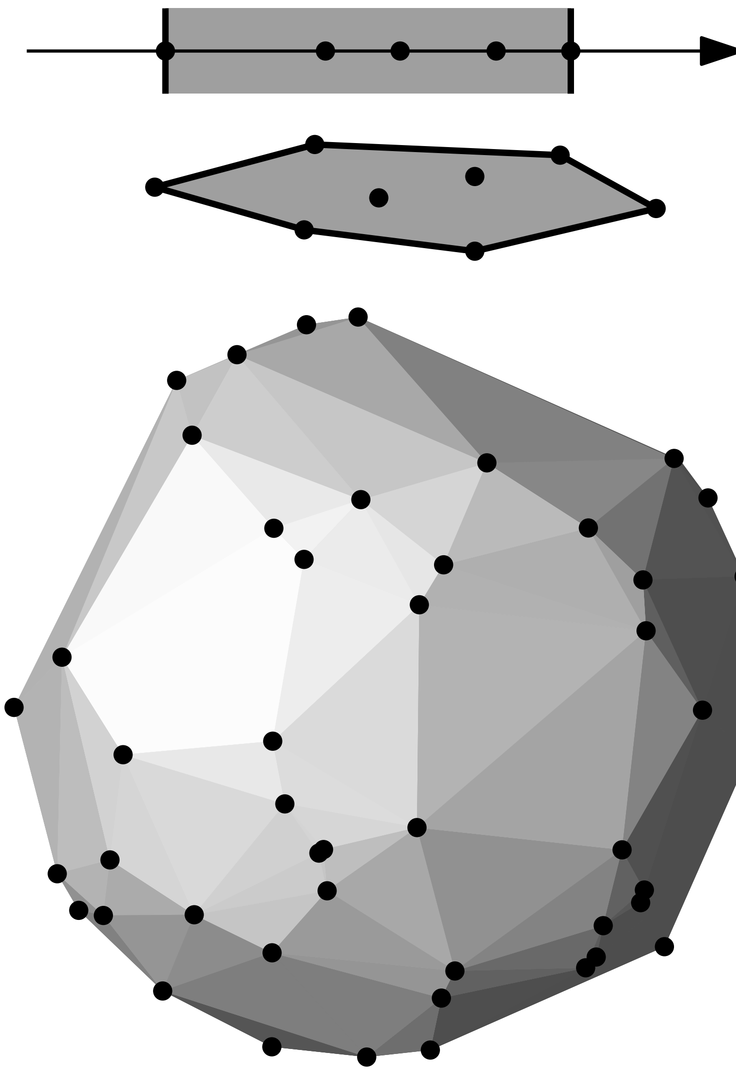
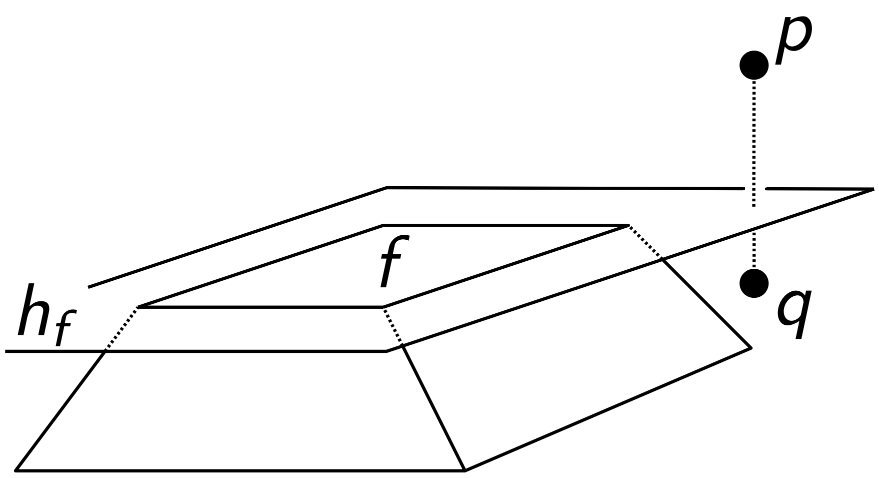


Construction

randomized-incremental!

Visibility



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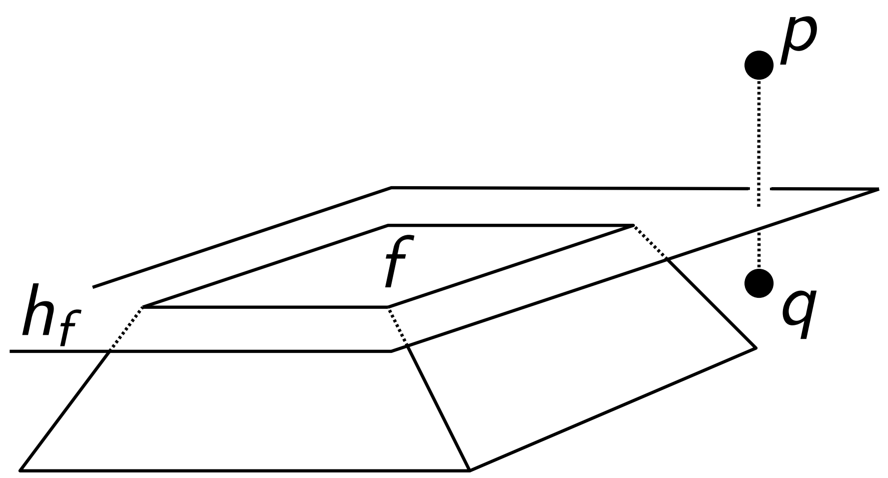

Face f is *visible* from p but not from q.

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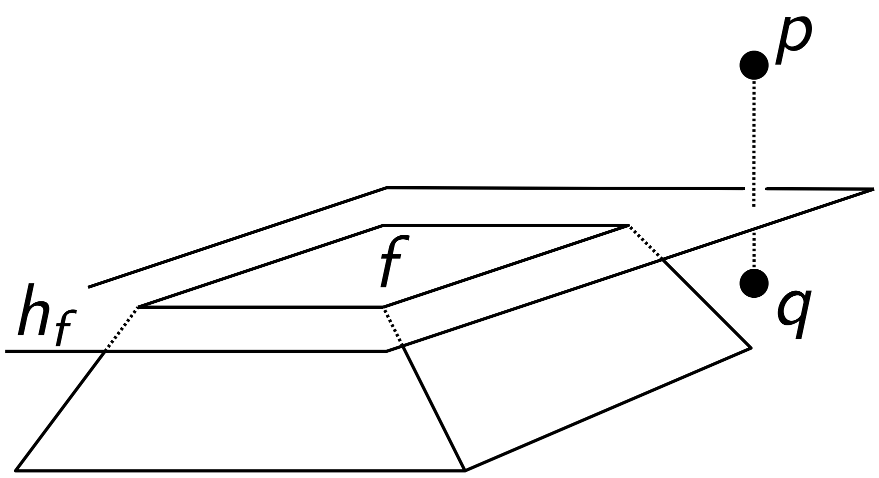

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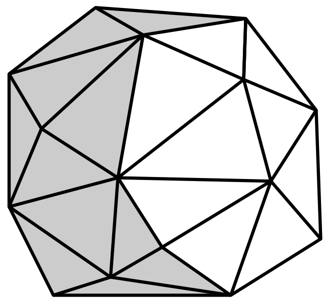

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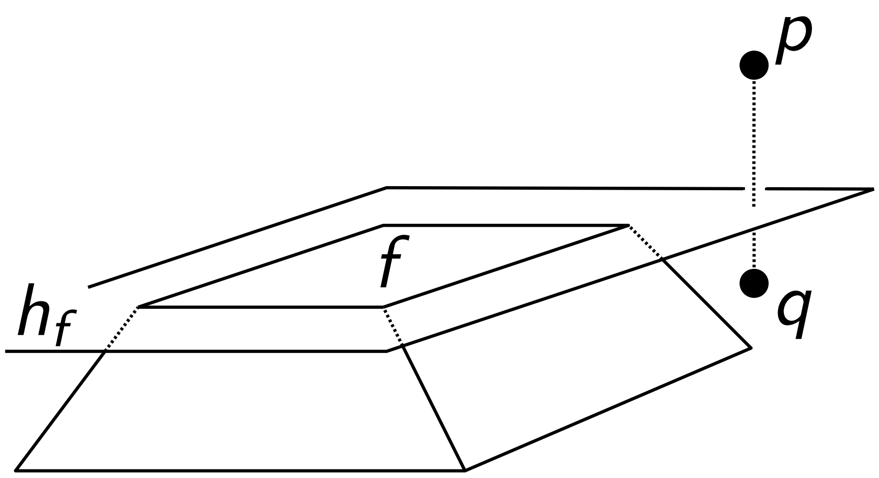

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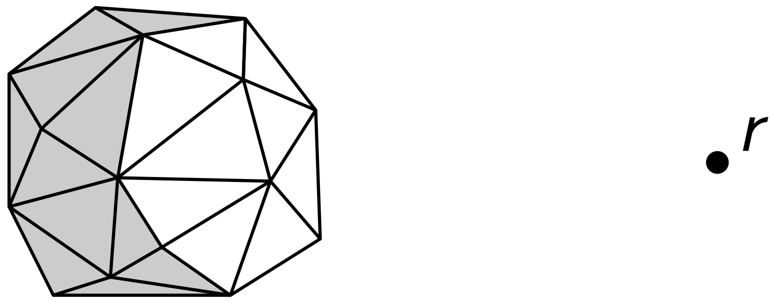

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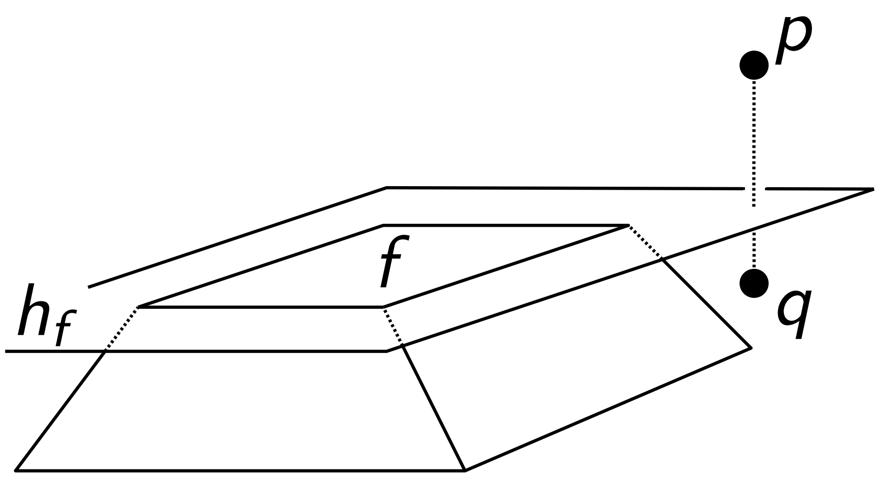

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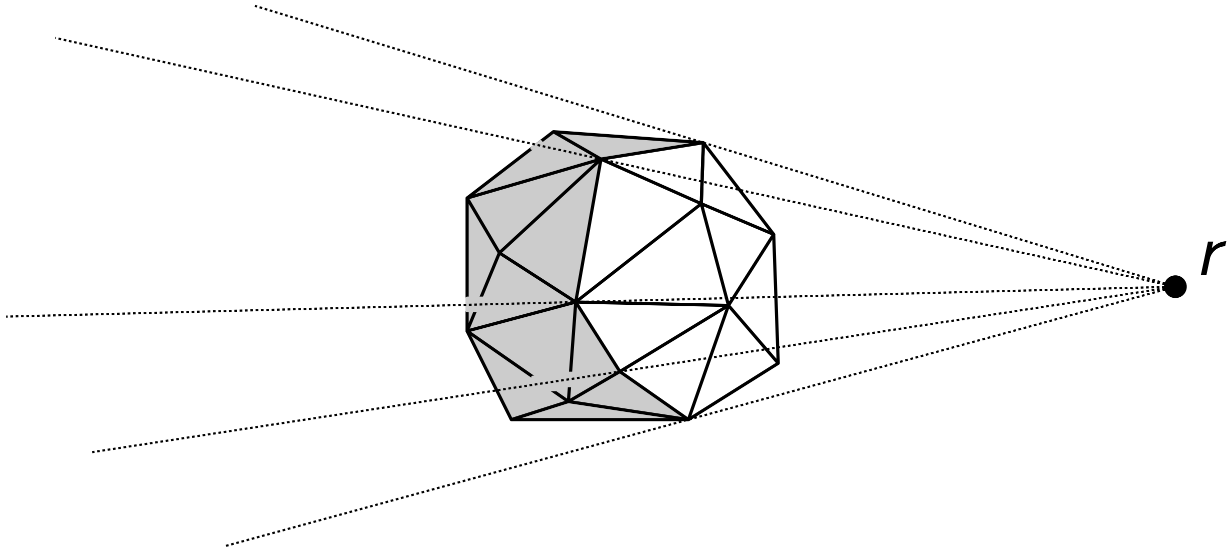

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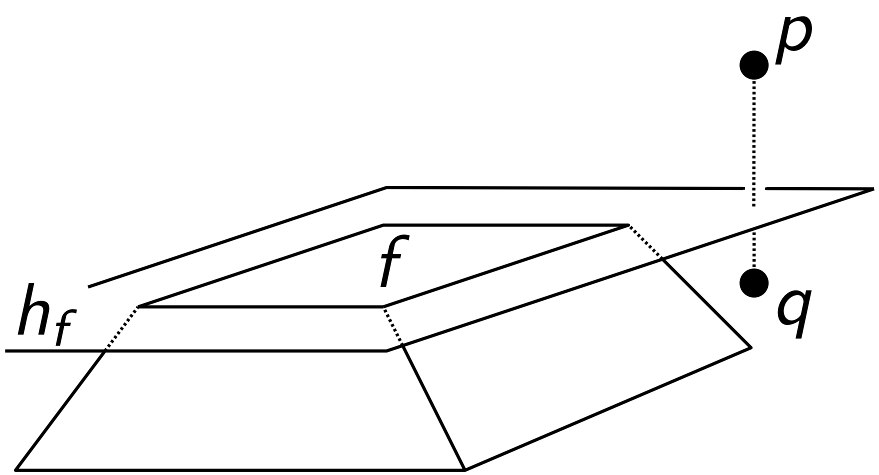

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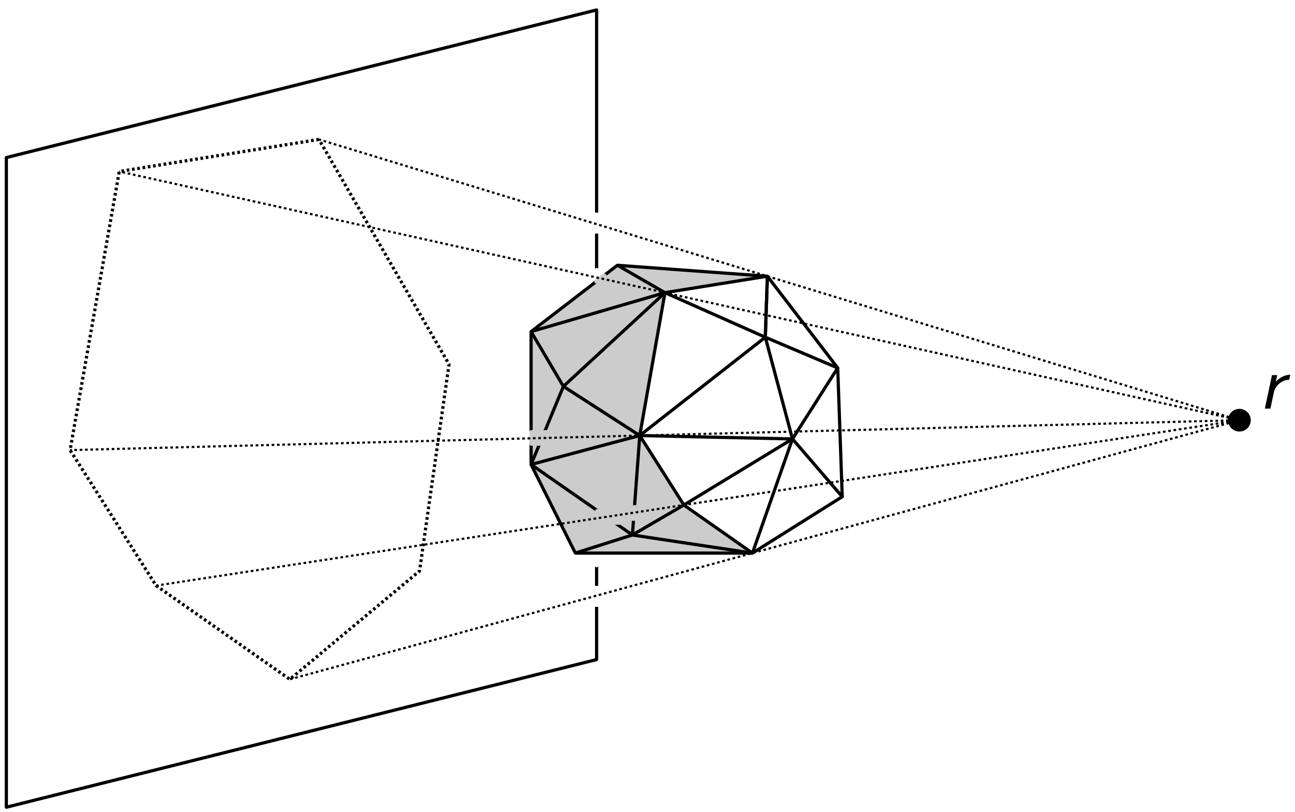

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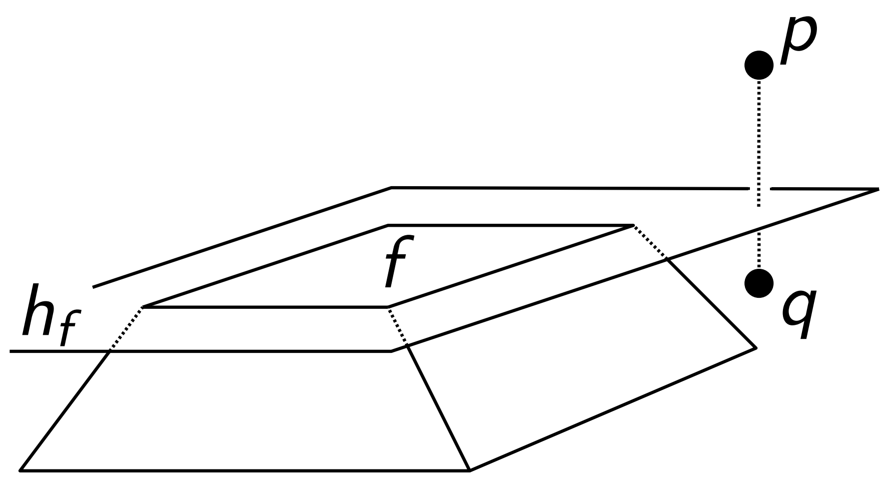

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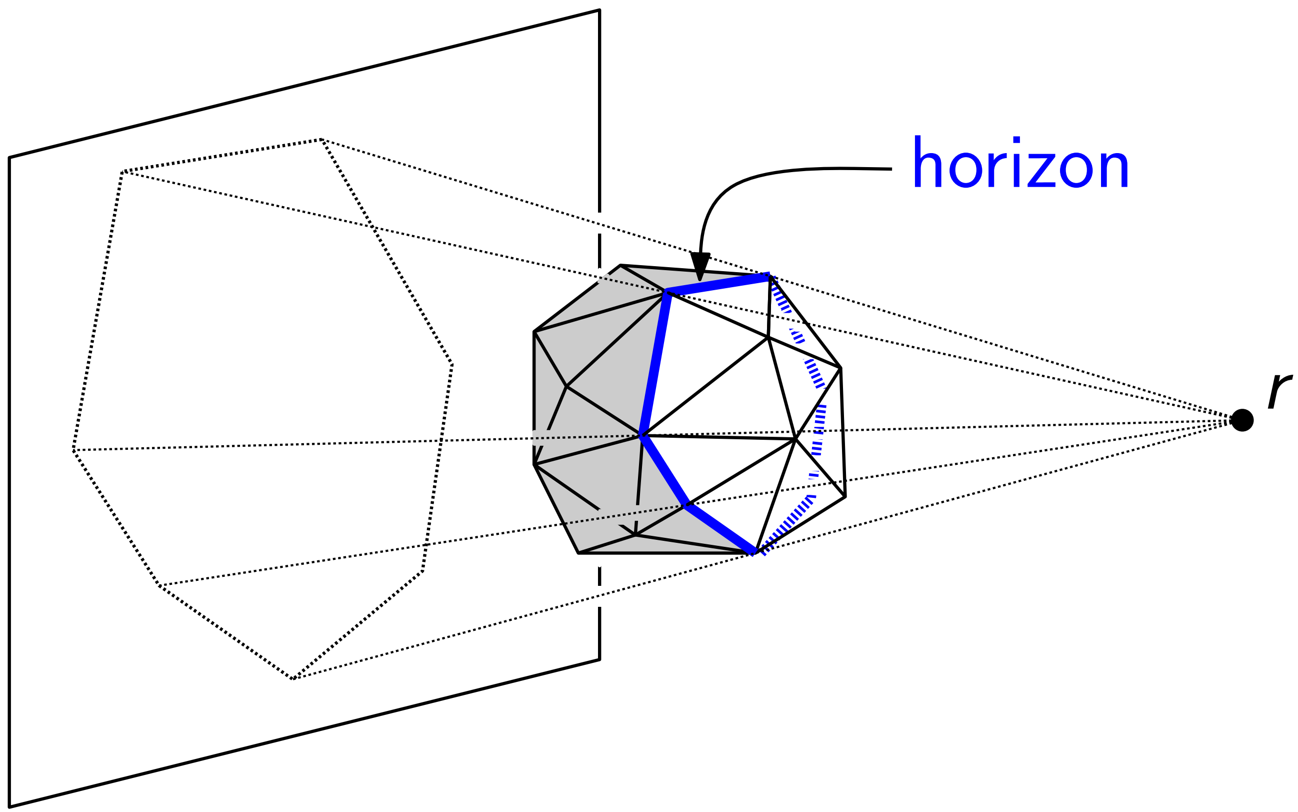

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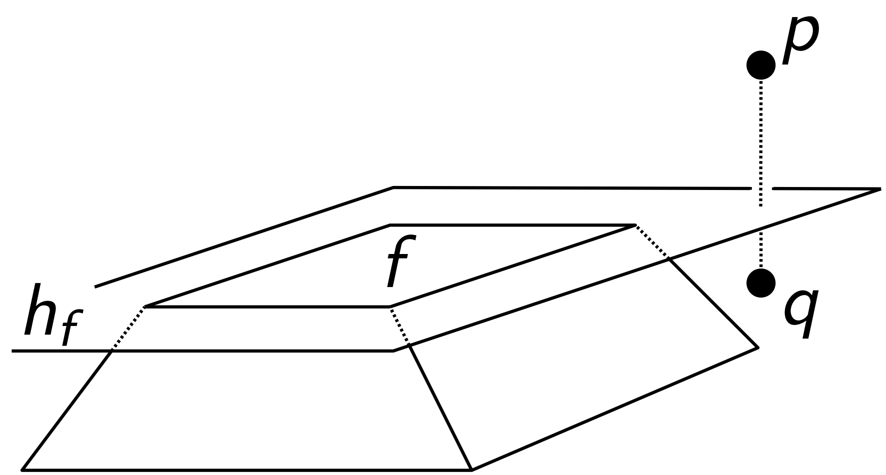

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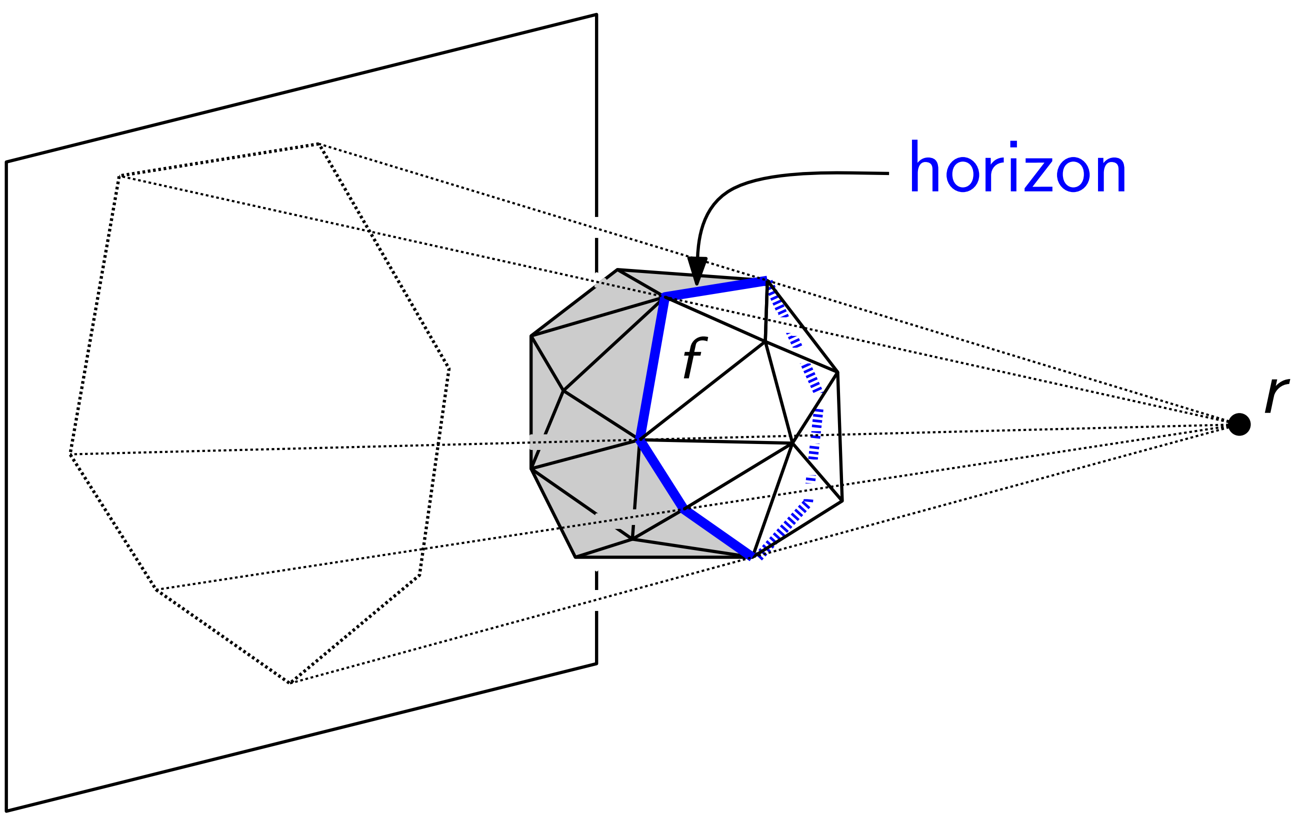
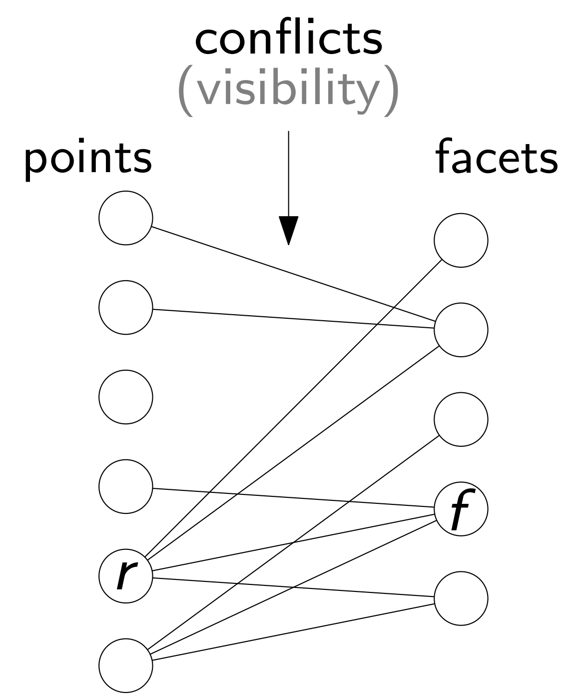


Define conflict graph G:



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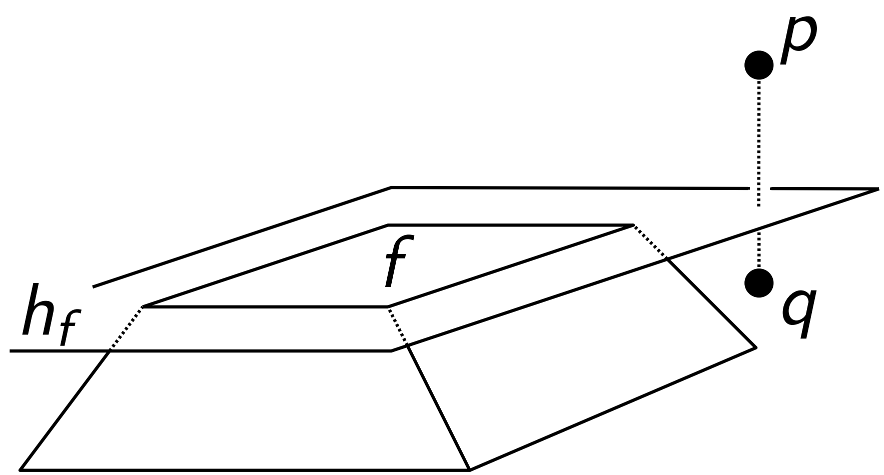

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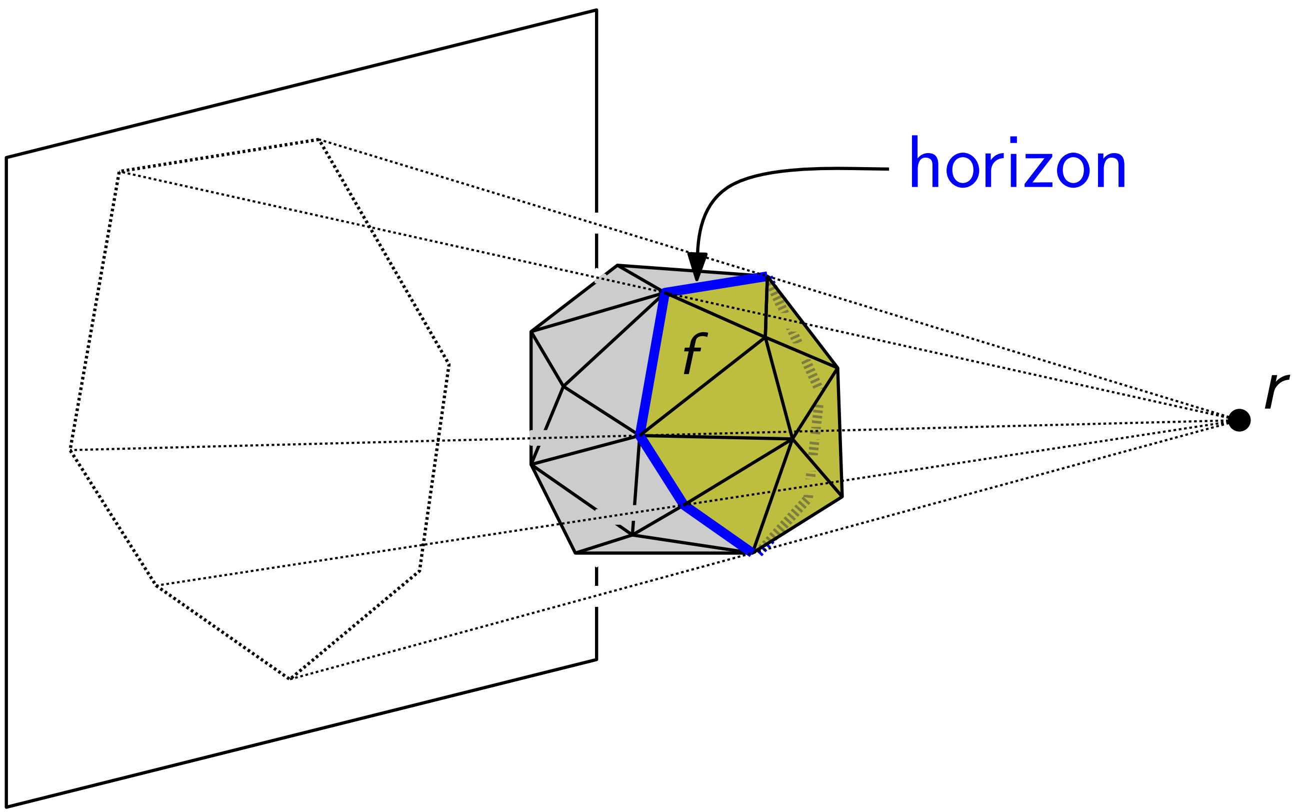

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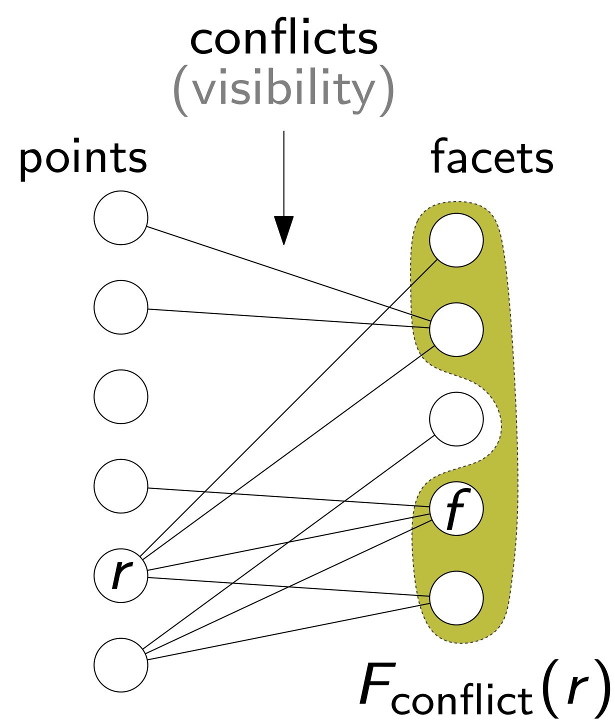

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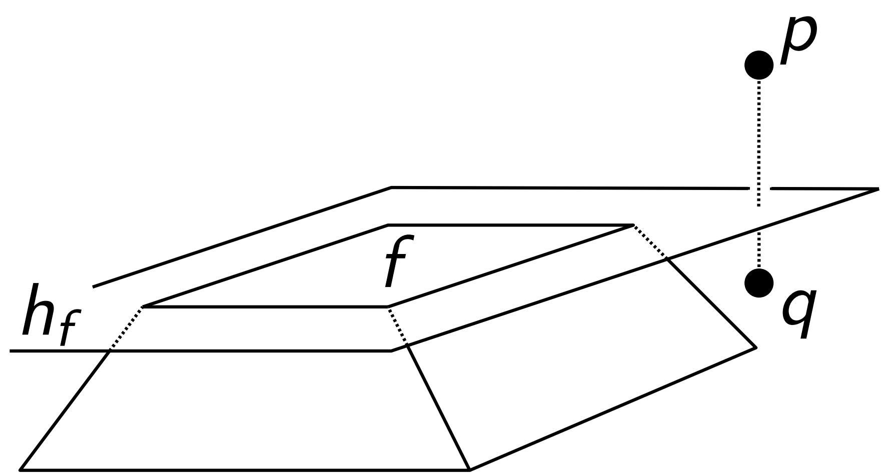

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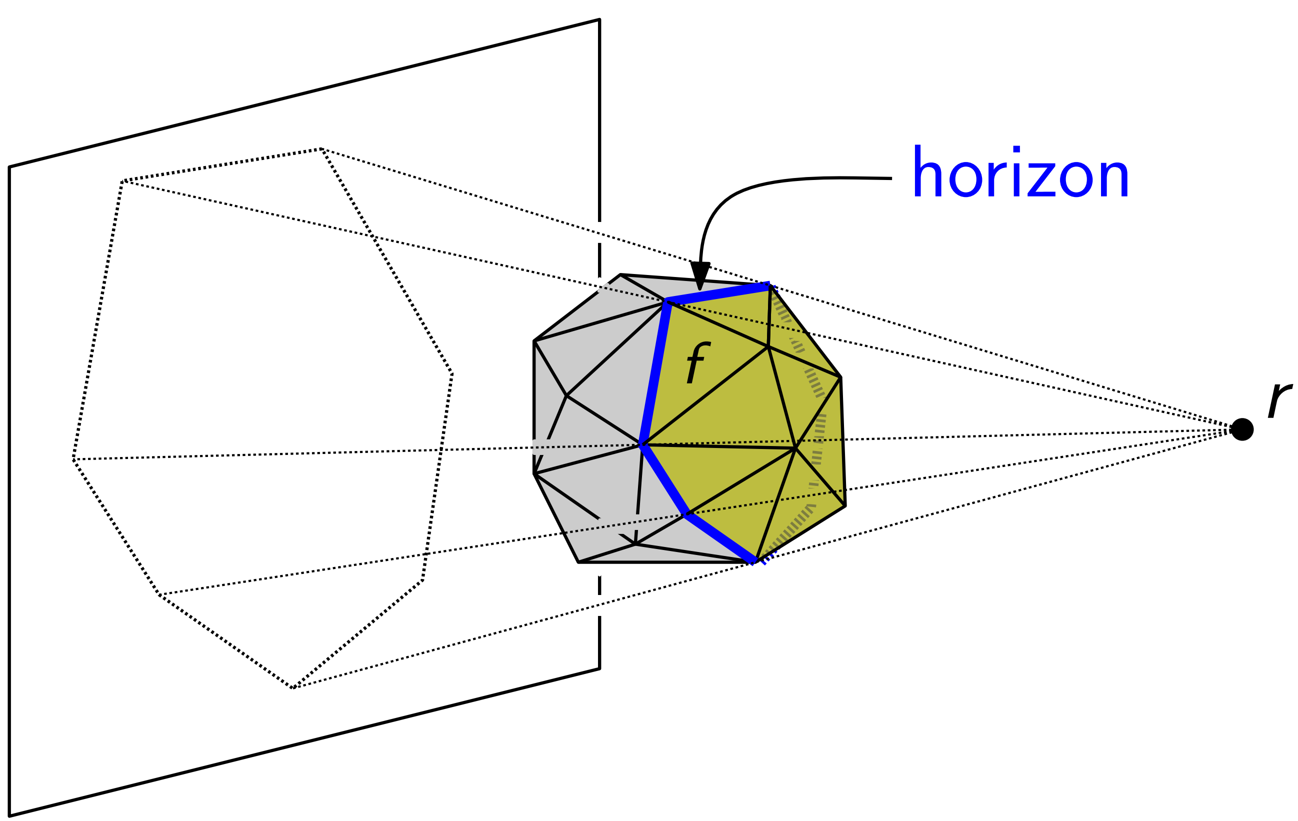

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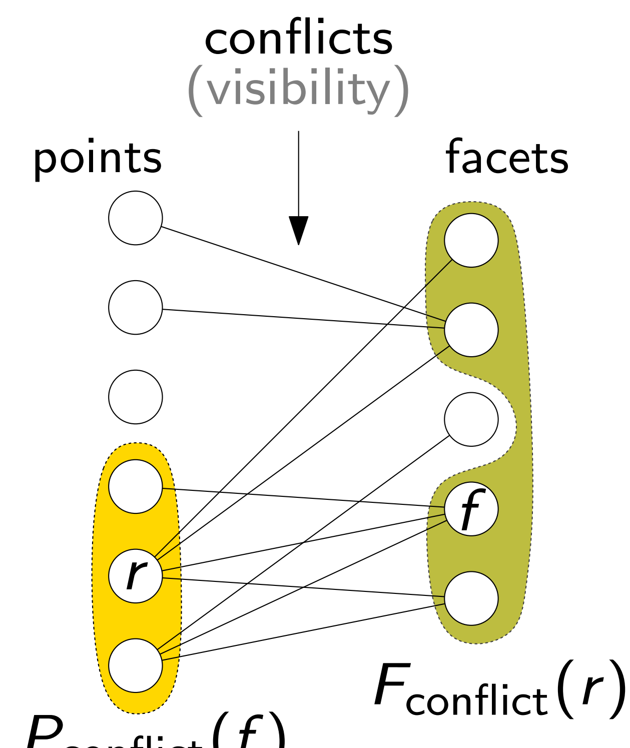
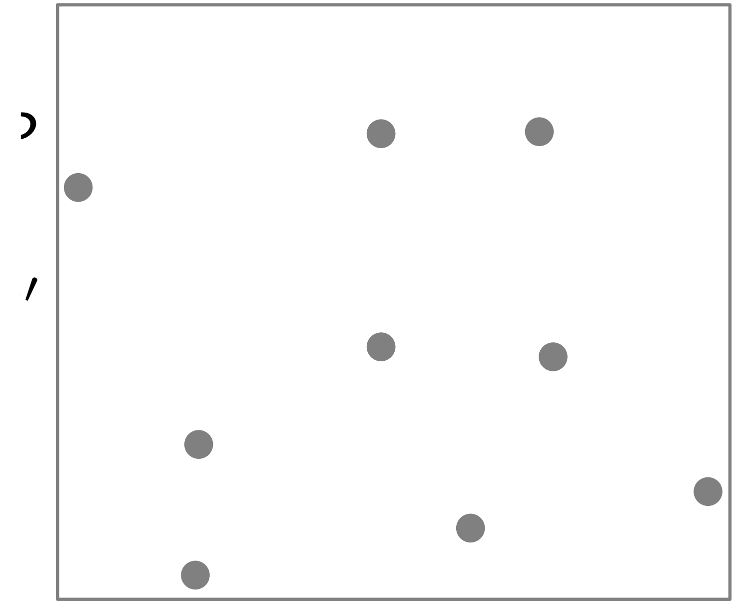


Rand3dConvexHull($P \subset \mathbb{R}^3$)



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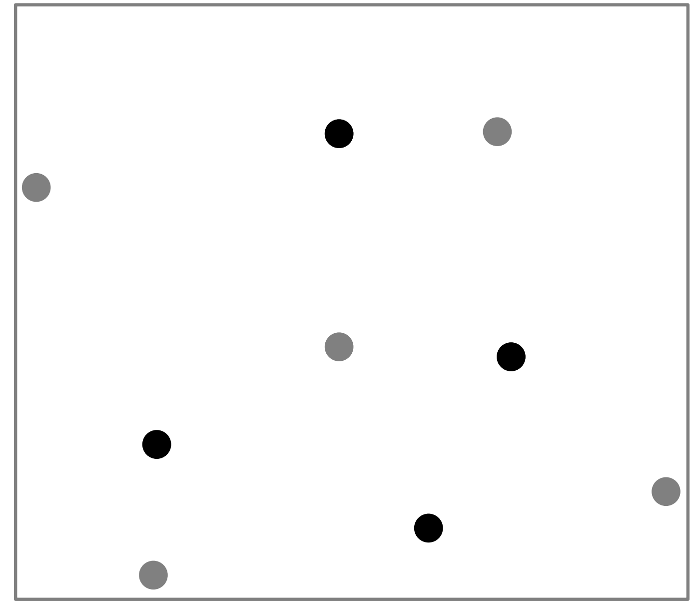

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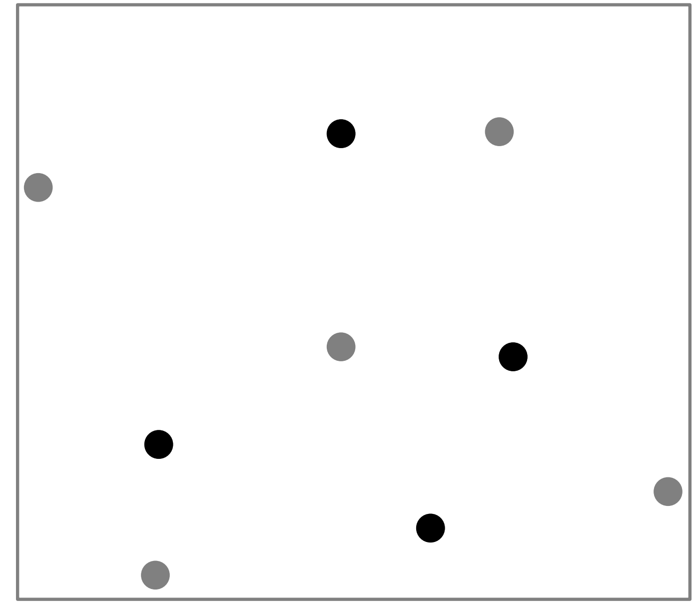

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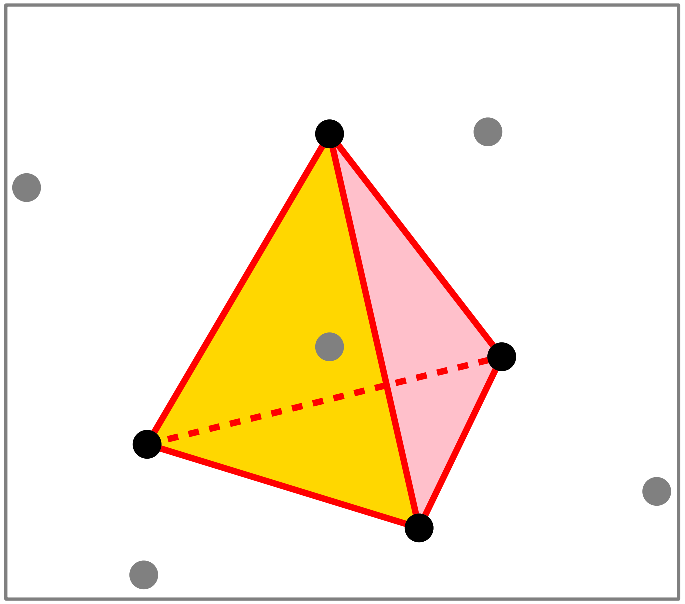

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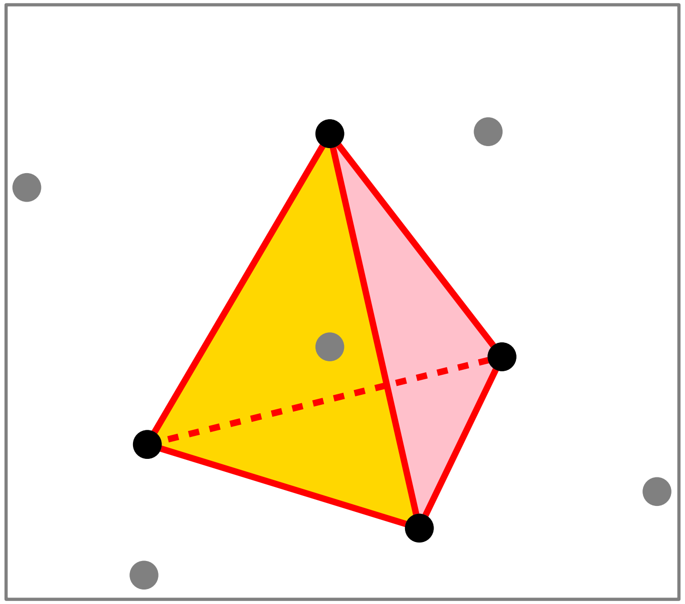

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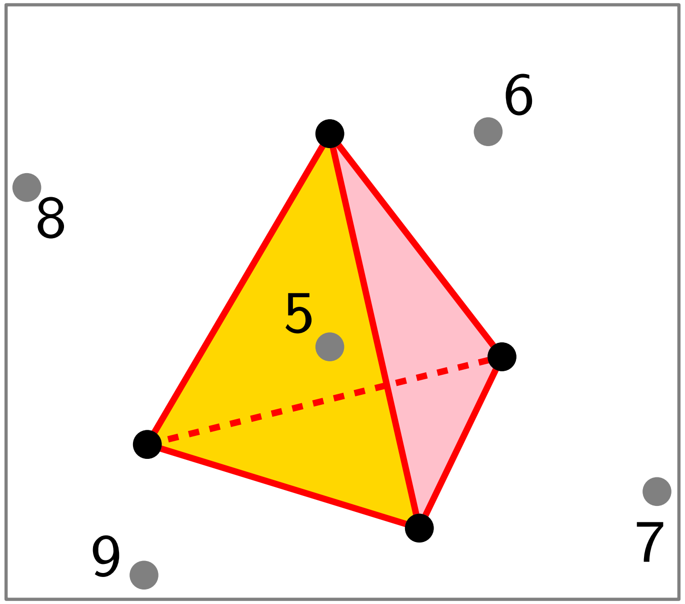

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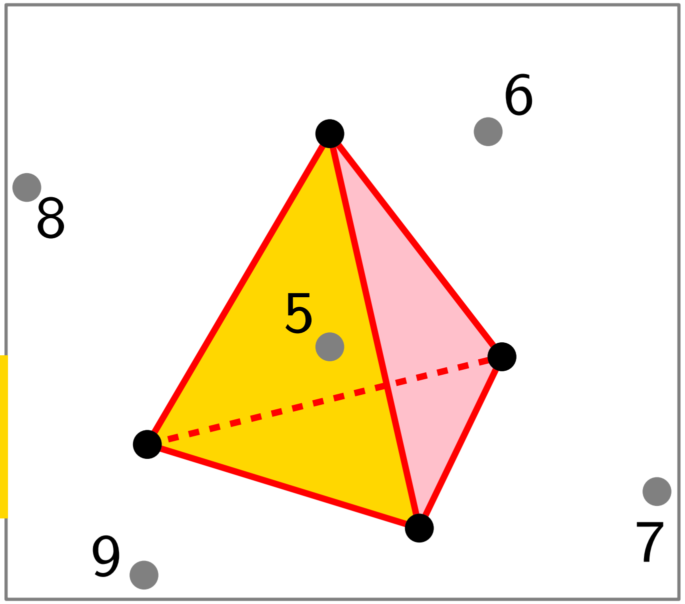

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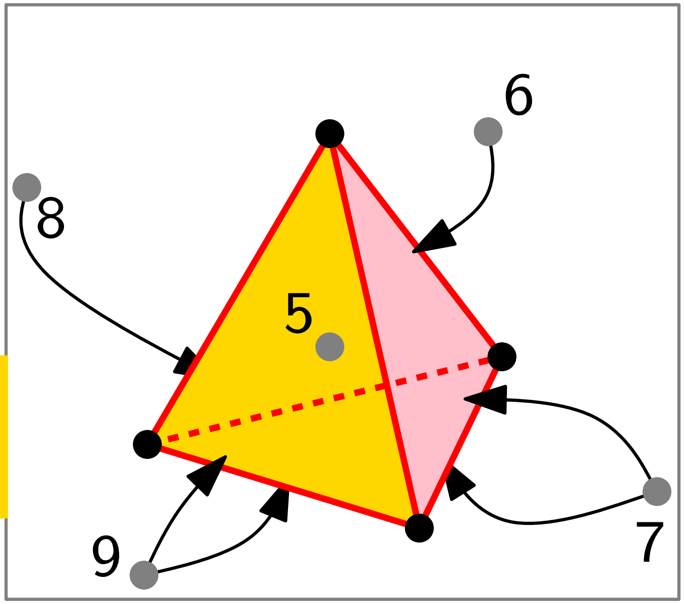

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for $r = 5$ **to** n **do**

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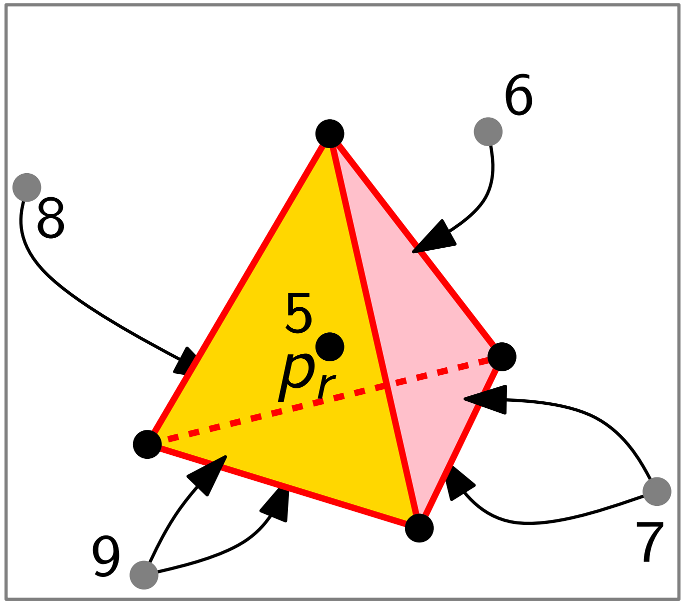
return C



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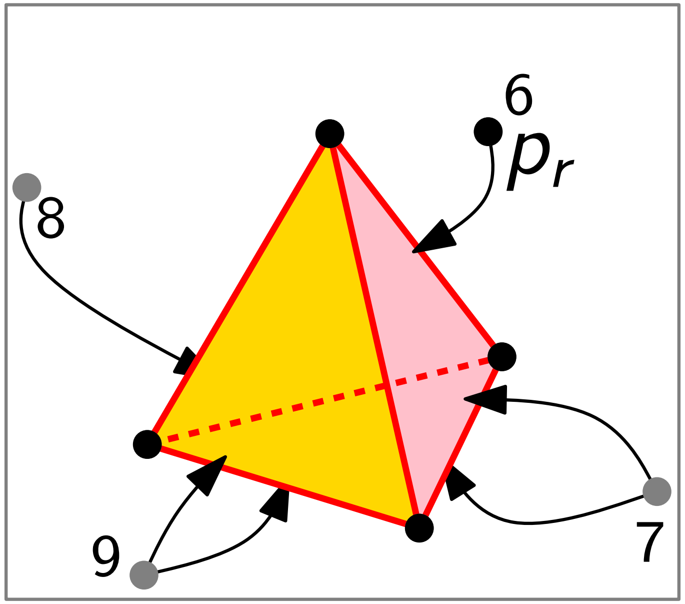

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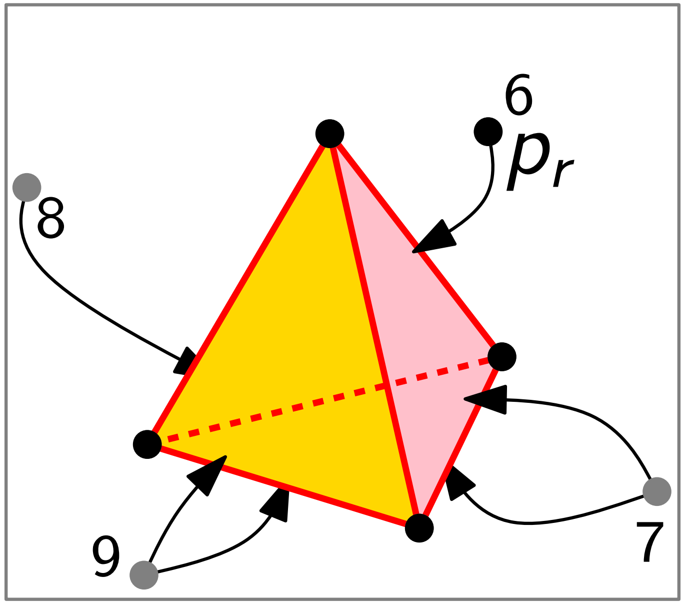

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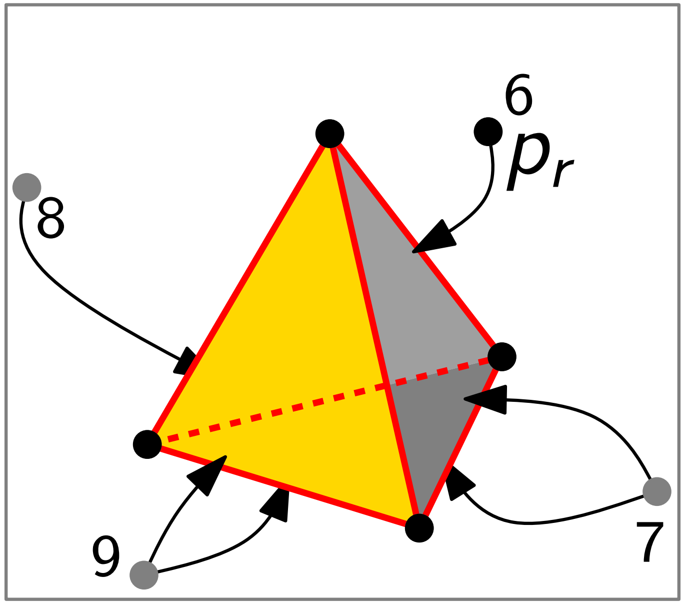

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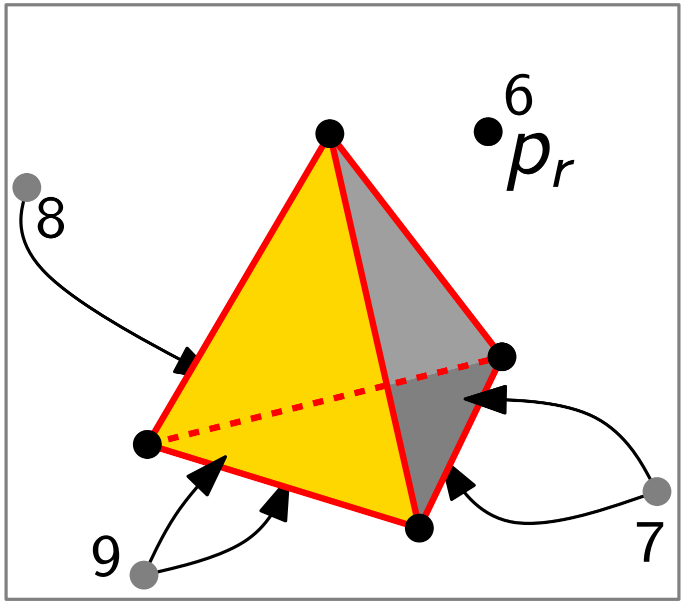

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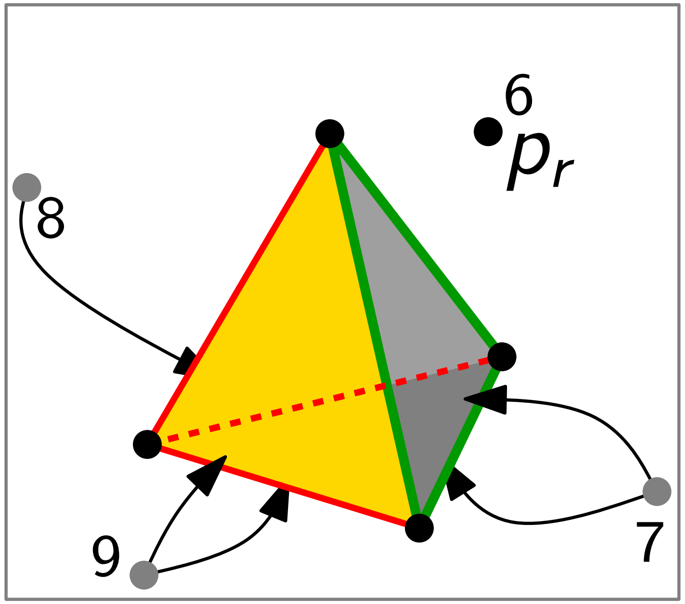

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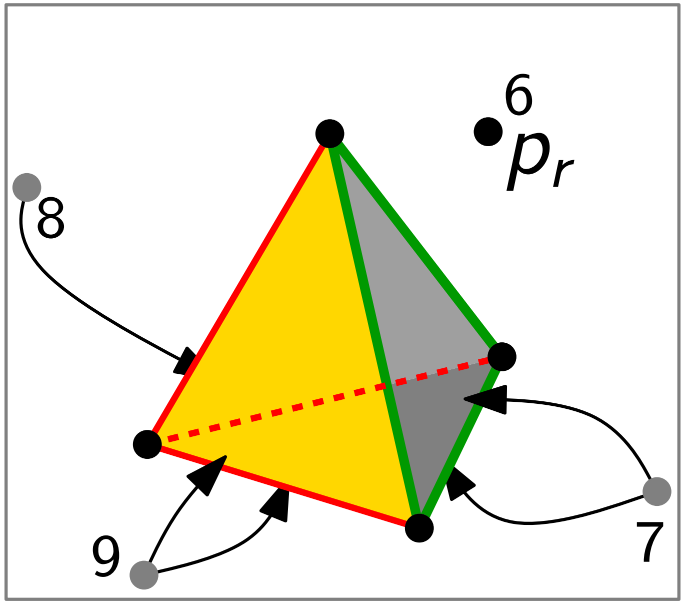

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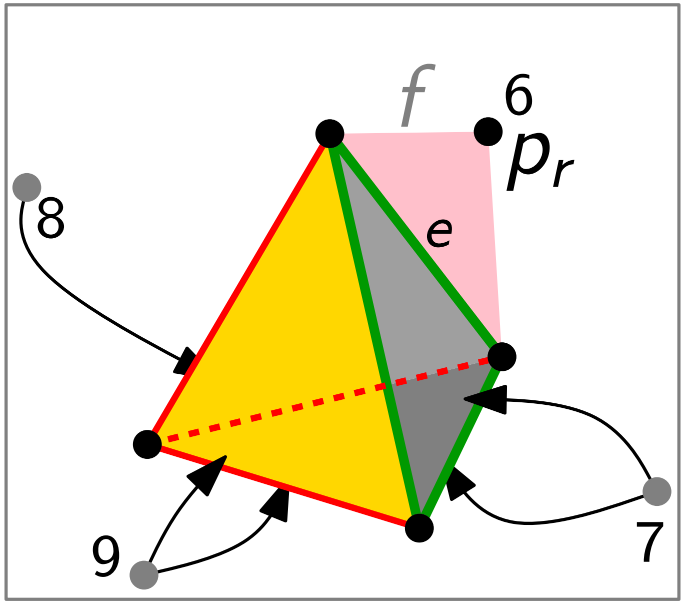

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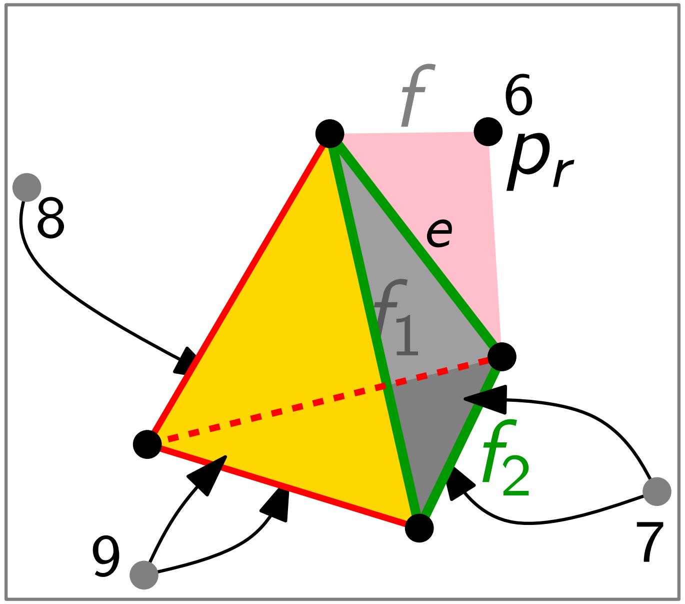

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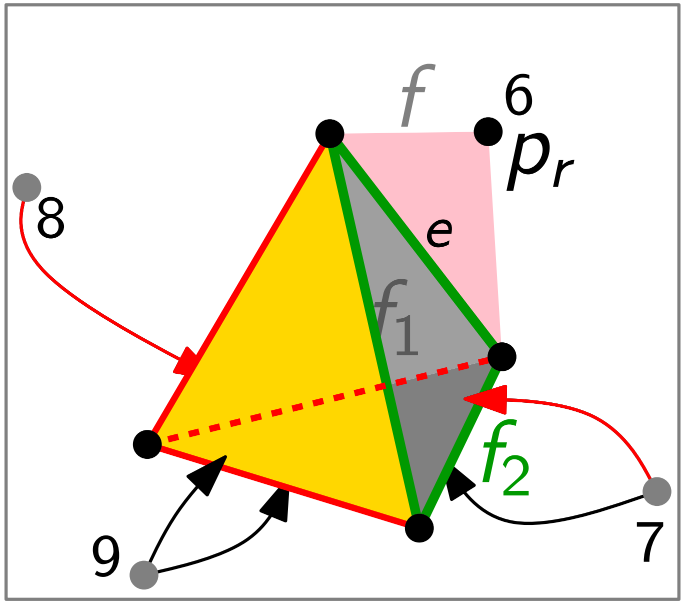

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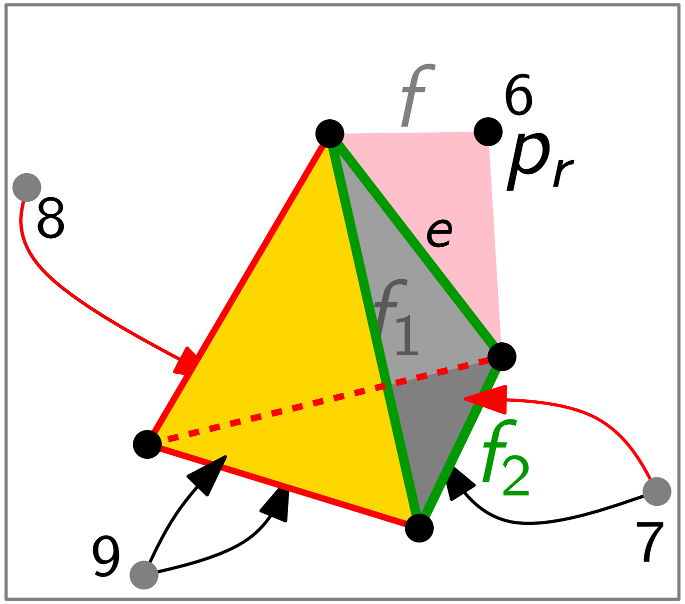

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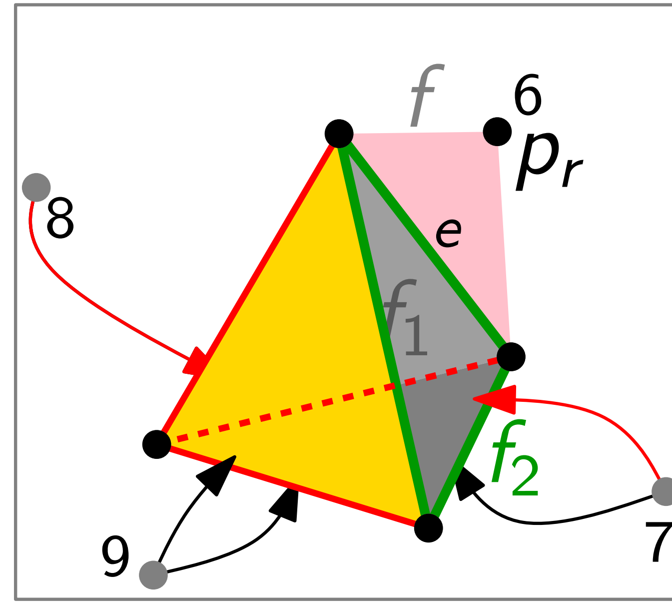

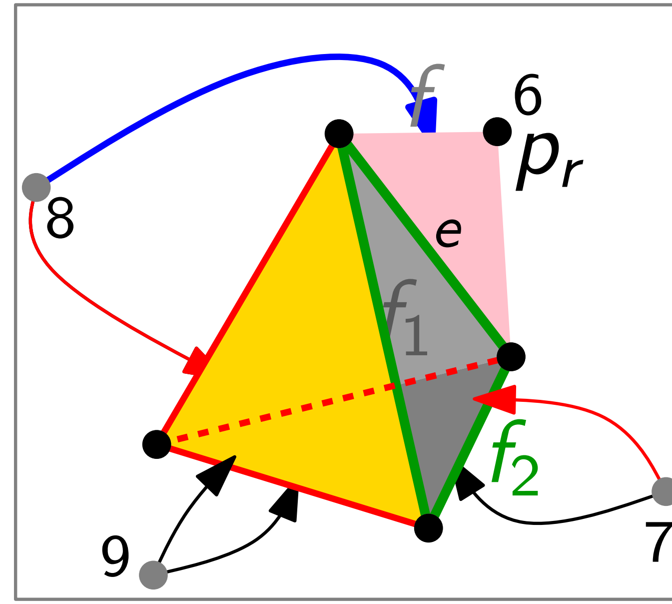

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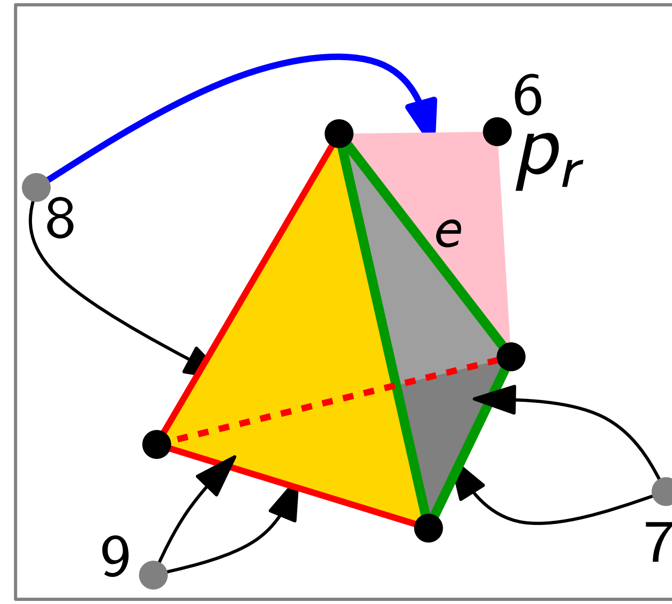

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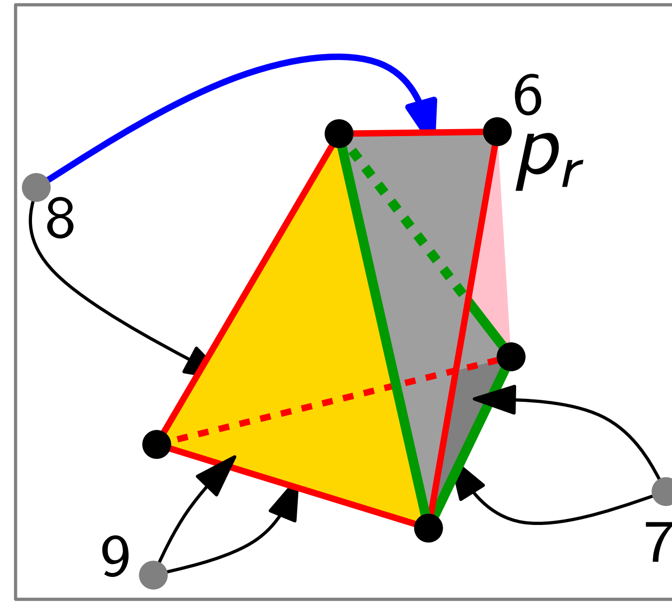

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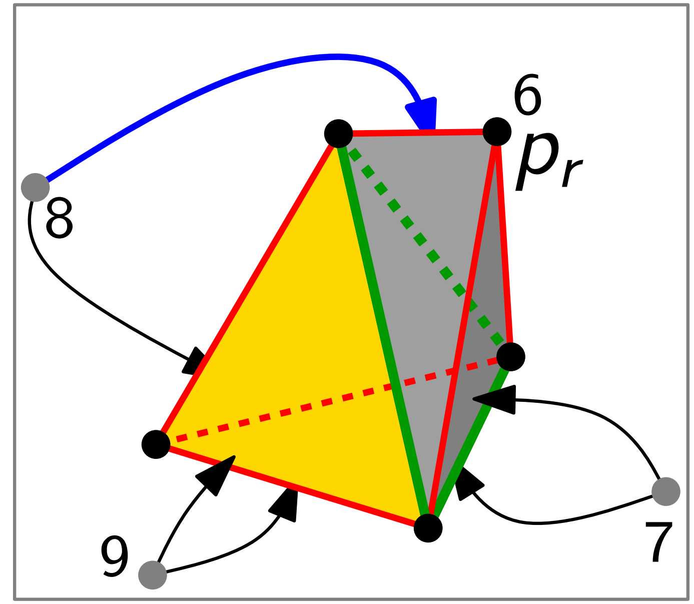

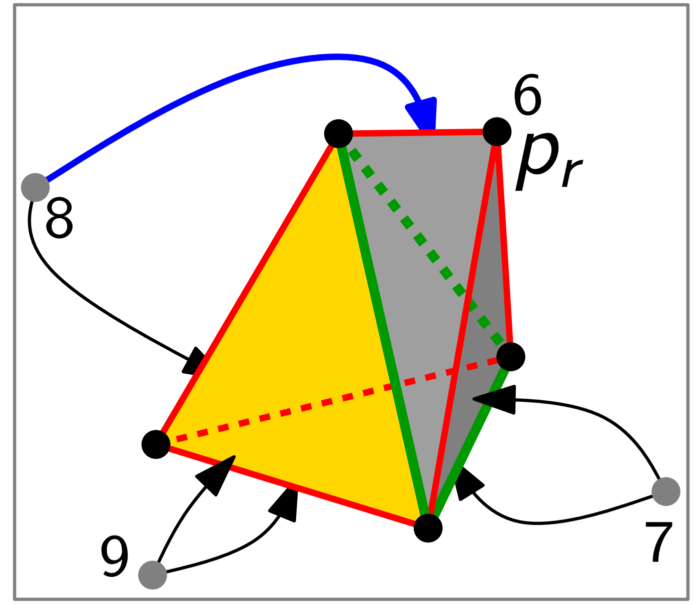

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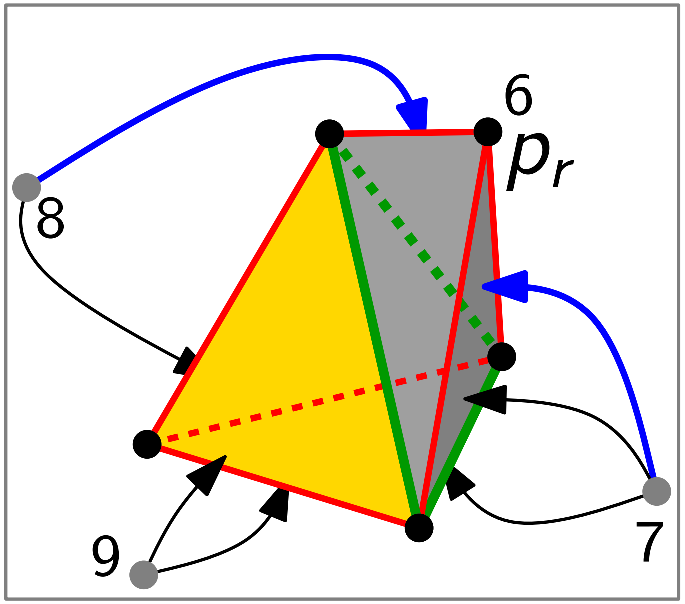

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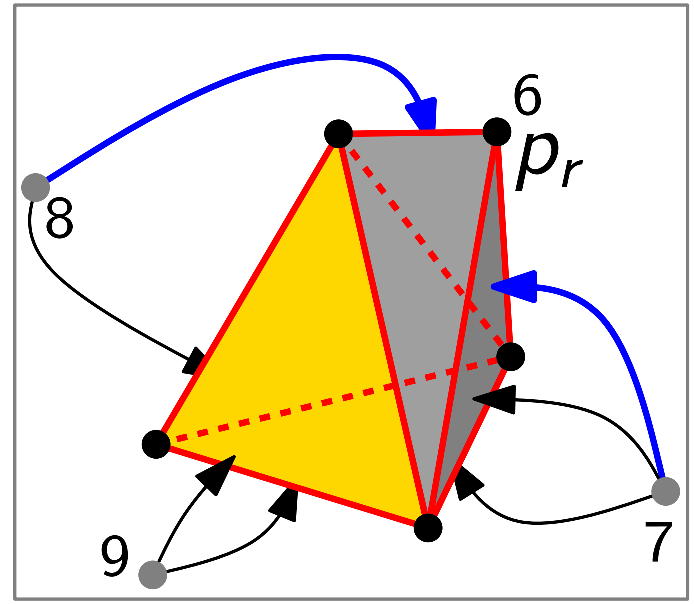

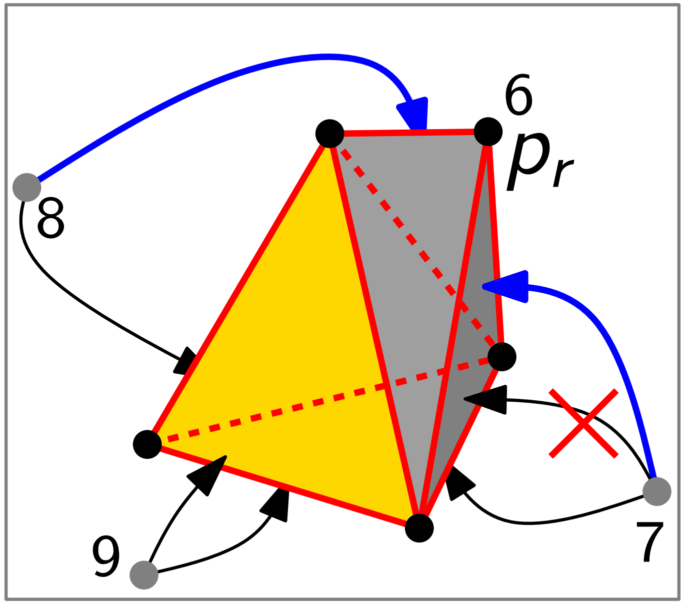

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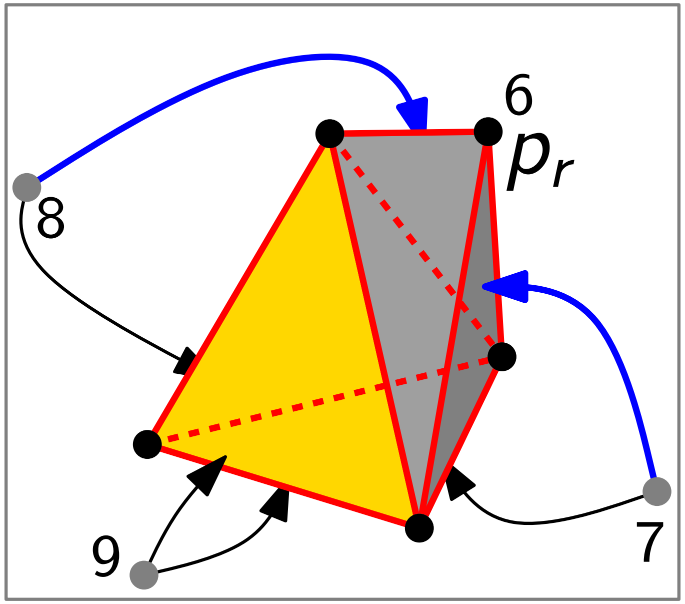

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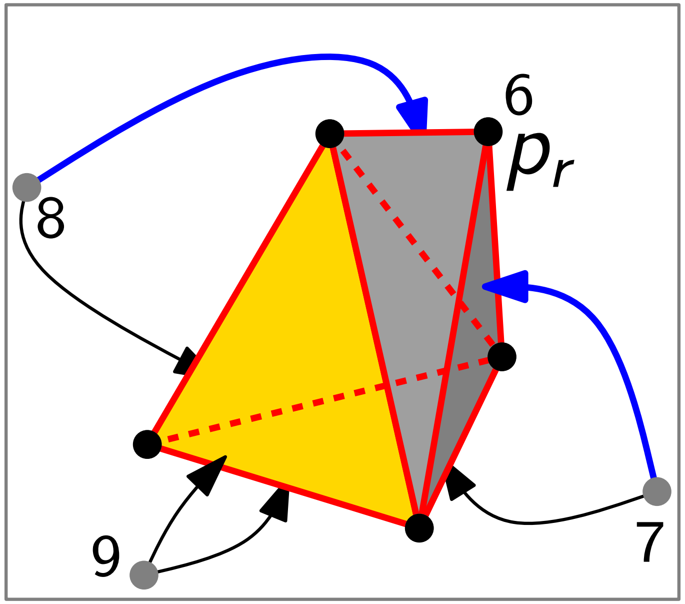

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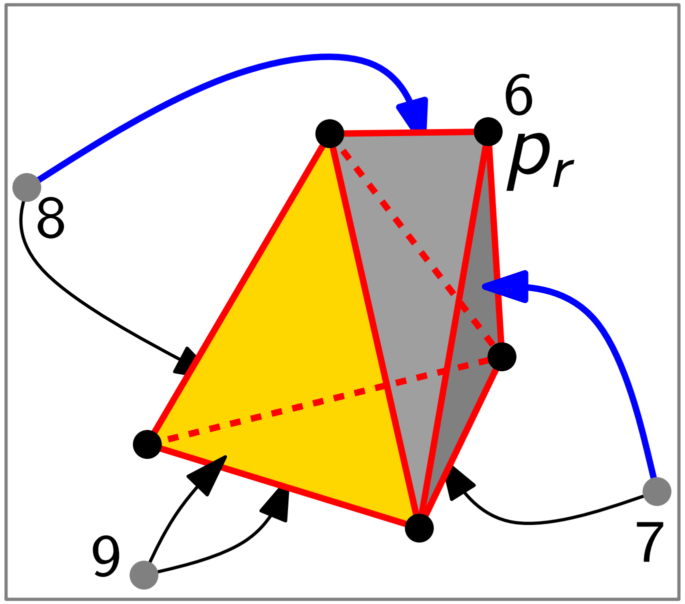

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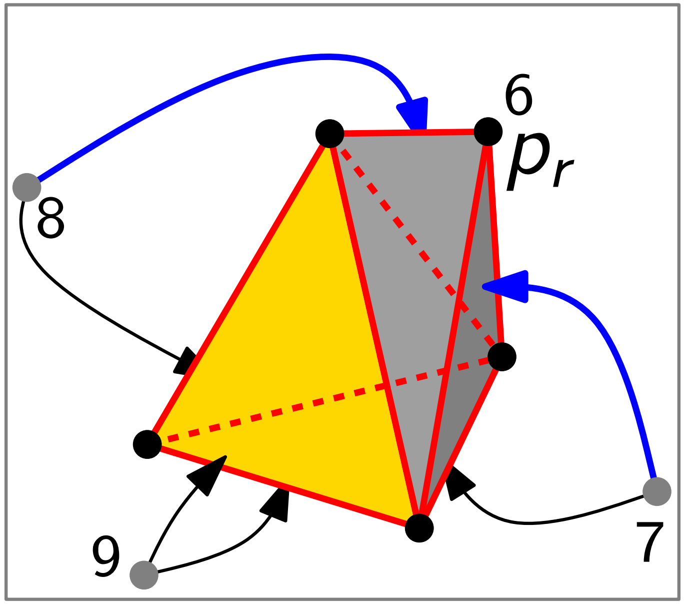

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#edges

$$= 4 + \sum_{r=5}^{n} E[\underbrace{\#\text{facets incident to } p_r \text{ in } \mathrm{CH}(P_r)}_{\deg(p_r,\, \mathrm{CH}(P_r))}]$$

For $r > 4$:

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Analysis

Idea: Bound expected *structural change*,
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$$\leq \frac{1}{r-4} \left[2 \cdot (3r - 6) - 12 \right]$$

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$$\leq \frac{1}{r-4} \big[2 \cdot (3r - 6) - 12 \big] \leq 6$$

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Running Time

Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

```
Rand3dConvexHull(P ⊂ ℝ³)
  pick set P' = {p_1, ..., p_4} ⊆ P of 4 non-coplanar pts
  C ← CH(P')
  compute a random permutation (p_5, ..., p_n) of P \ P'
  initialize conflict graph G
  for r = 5 to n do
    if F_conflict(p_r) ≠ ∅ then
      delete all facets in F_conflict(p_r) from C
      ℒ ← list of horizon edges visible from p_r
      foreach e ∈ ℒ do
        f ← C.create_facet(e, p_r); create vtx for f in G
        (f_1, f_2) ← previously_incident_C(e)
        P(e) ← P_conflict(f_1) ∪ P_conflict(f_2)
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Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

Rand3dConvexHull($P \subset \mathbb{R}^3$)

pick set $P' = \{p_1, \ldots, p_4\} \subseteq P$ of 4 non-coplanar pts
$C \leftarrow \mathrm{CH}(P')$
compute a random permutation $(p_5, \ldots, p_n)$ of $P \setminus P'$
initialize conflict graph G
for $r = 5$ **to** n **do**
    if $F_{\text{conflict}}(p_r) \neq \emptyset$ **then**
        delete all facets in $F_{\text{conflict}}(p_r)$ from C
        $\mathcal{L} \leftarrow$ list of horizon edges visible from p_r
        foreach $e \in \mathcal{L}$ **do**
            $f \leftarrow C$.create_facet(e, p_r); create vtx for f in G
            $(f_1, f_2) \leftarrow$ previously_incident$_C(e)$
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            foreach $p \in P(e)$ **do**
                if f visible from p **then** add edge (p, f) to G
        delete vtc $\{p_r\} \cup F_{\text{conflict}}(p_r)$ from G
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Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

Stage r of for-loop (w/o outer foreach loop)

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Theorem: The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

Stage r of for-loop (w/o outer foreach loop) takes time $O(|F_{\text{conflict}}(p_r)|) =$

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This part of for-loop in total:
$E[\#\text{facets deleted}] =$
$\leq E[\#\text{facets created}] = O(n)$.
Lemma

Running Time

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This part of for-loop in total:
$E[\#\text{facets deleted}] =$
$\leq E[\#\text{facets created}] = \underset{\text{Lemma}}{O(n)}$.

Outer foreach-loop:

Running Time

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This part of for-loop in total:
$E[\#\text{facets deleted}] =$
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Lemma

Outer foreach-loop:
– in stage r: $O(\sum_{e \in \mathcal{L}} |P(e)|)$

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Stage r of for-loop (w/o outer foreach loop) takes time $O(|F_{\text{conflict}}(p_r)|) =$ $O(\#\text{facets deleted when adding } p_r)$

This part of for-loop in total:
$E[\#\text{facets deleted}] =$
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Outer foreach-loop:
– in stage r: $O(\sum_{e \in \mathcal{L}} |P(e)|)$
– in total:

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$O(n)$ time

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          ℒ ← list of horizon edges visible from p_r
          foreach e ∈ ℒ do
              f ← C.create_facet(e, p_r); create vtx for f in G
              (f_1, f_2) ← previously_incident_C(e)
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Outer foreach-loop:
– in stage r: $O(\sum_{e \in \mathcal{L}} |P(e)|)$
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Exercise: Give a simple deterministic algorithm that computes the convex hull in $O(n^2)$ (worst-case) time.

Convex Hulls and Half-Space Intersections

Convex Hulls and Half-~~Space~~ Plane Intersections

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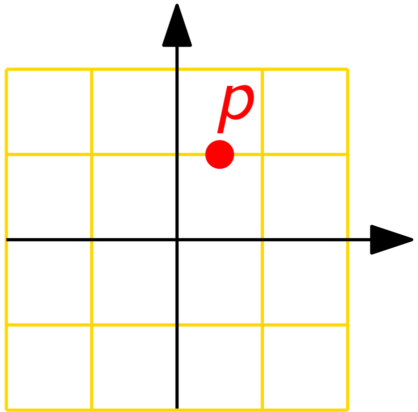

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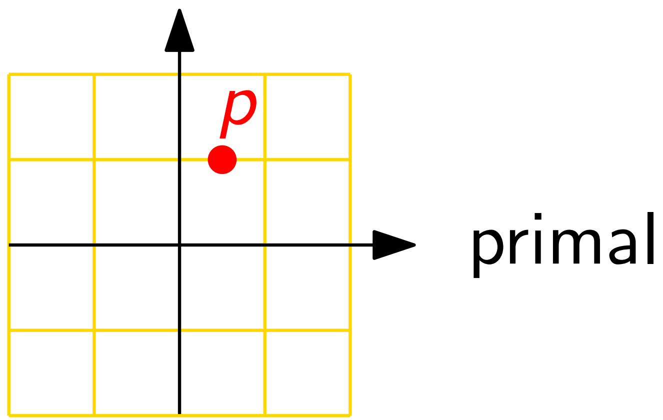

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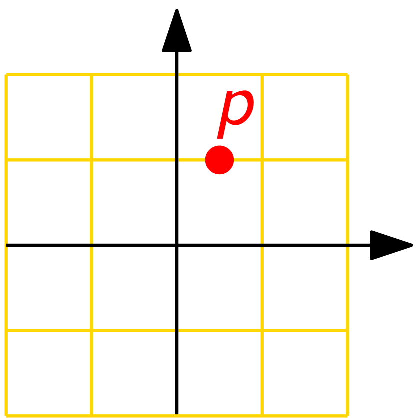

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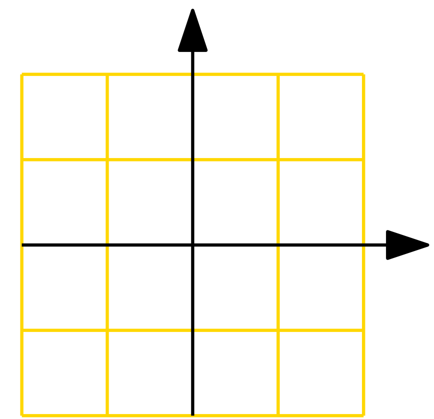

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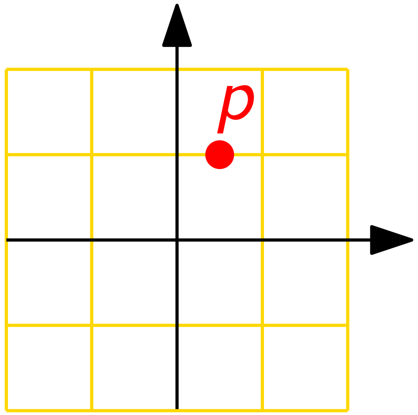

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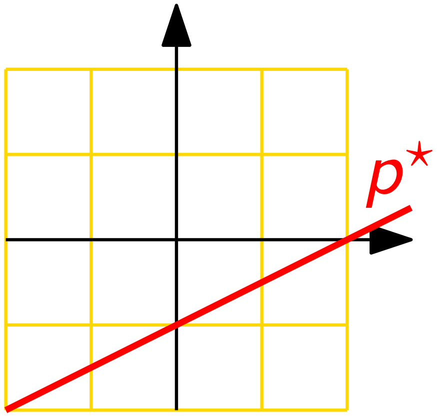

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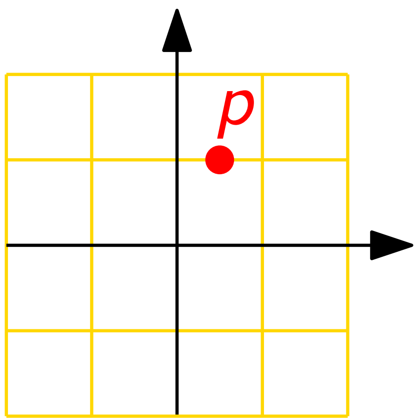

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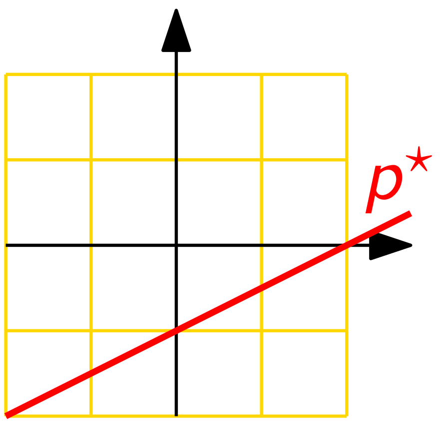

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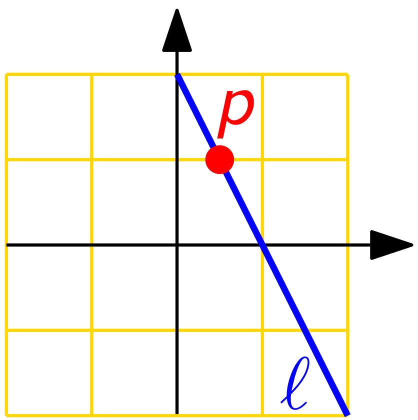

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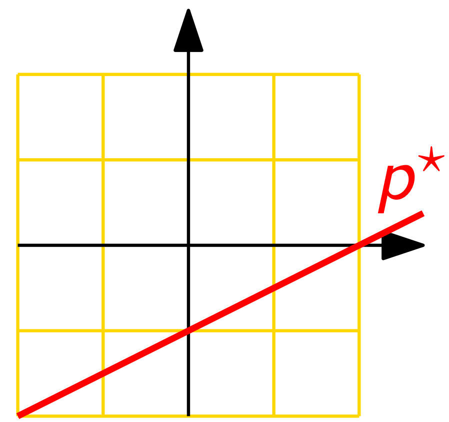

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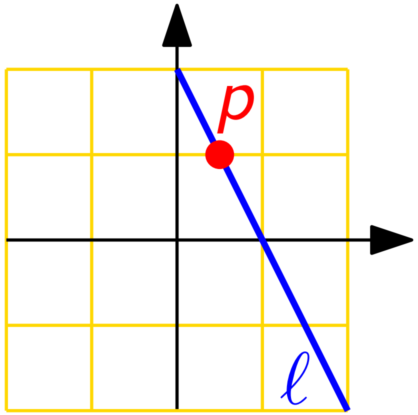

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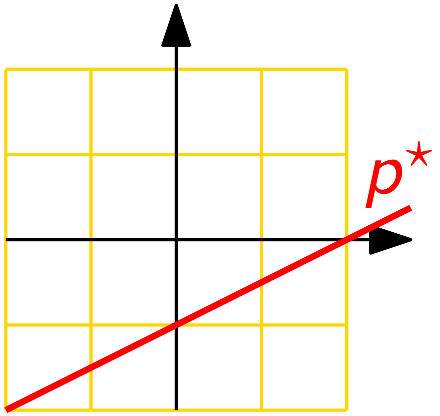

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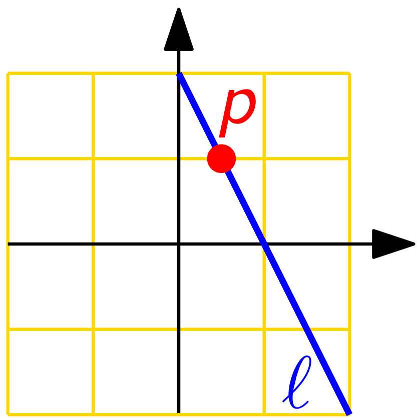

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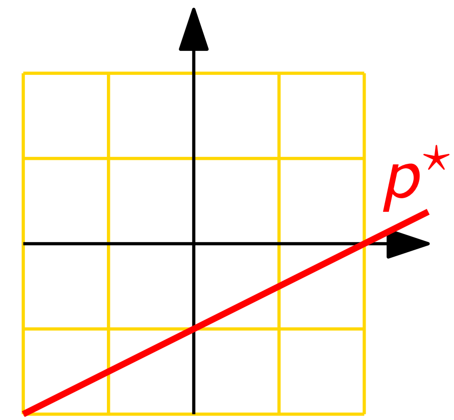

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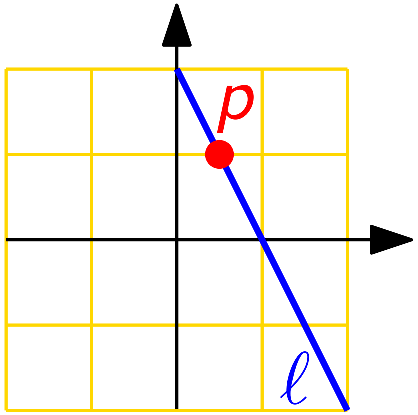

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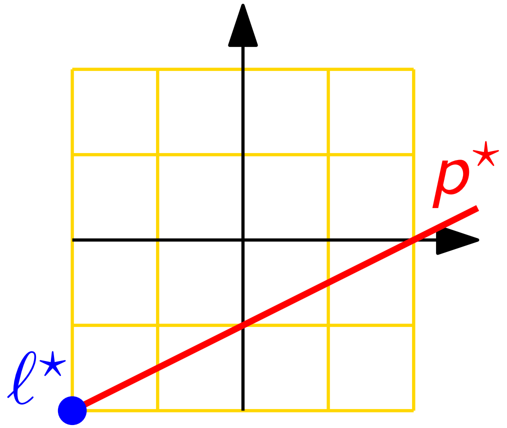

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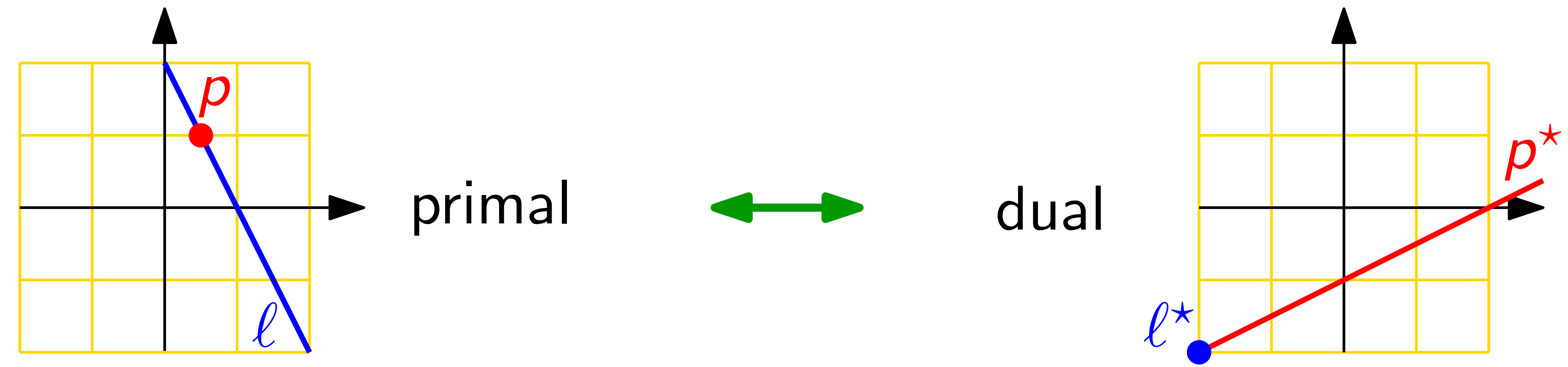

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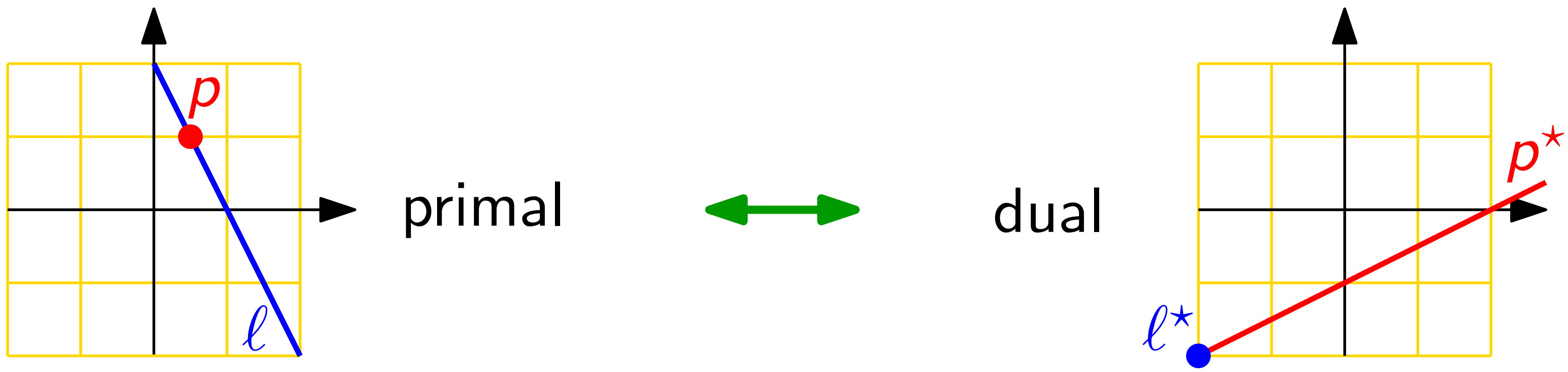

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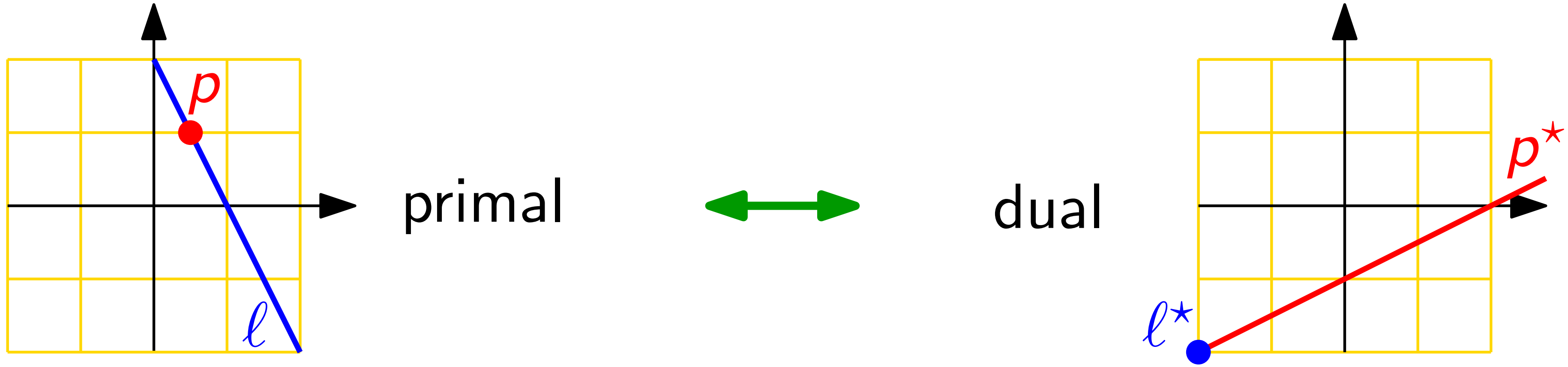

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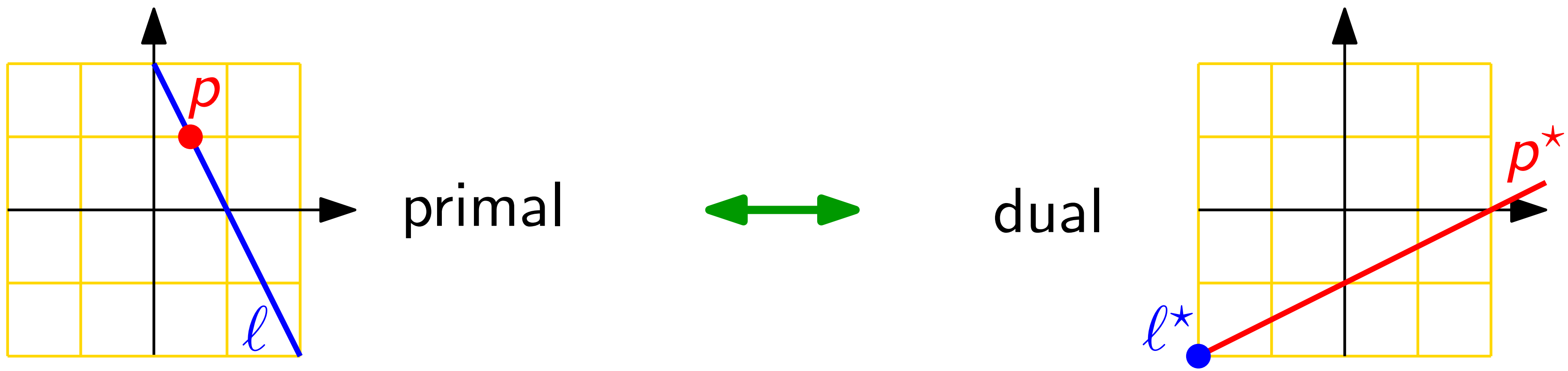

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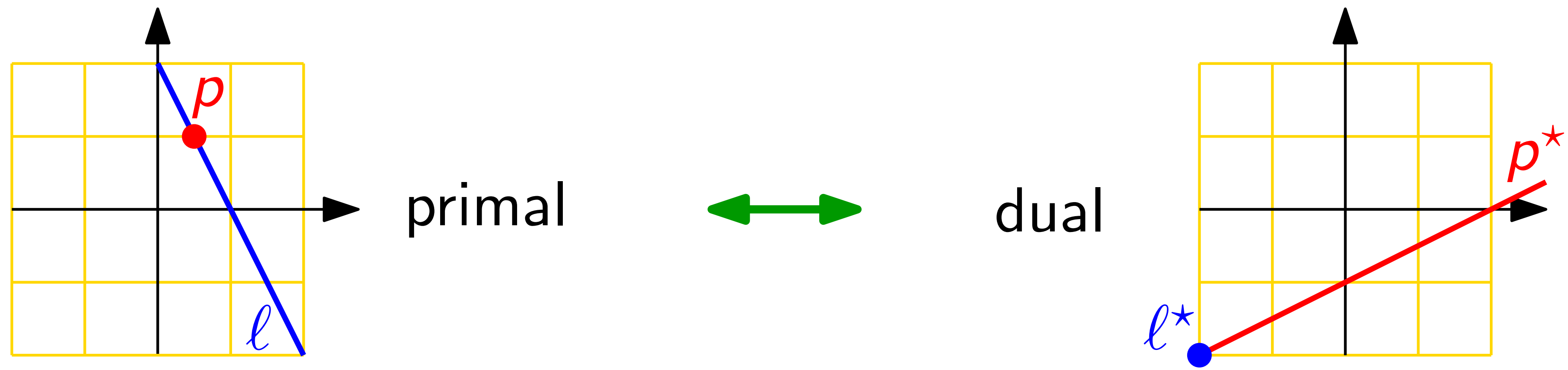

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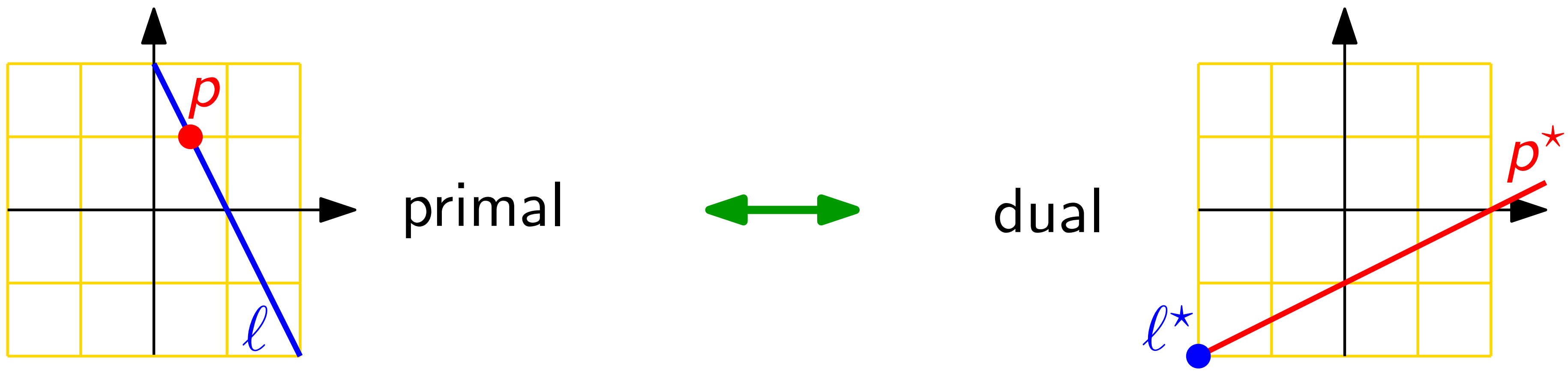

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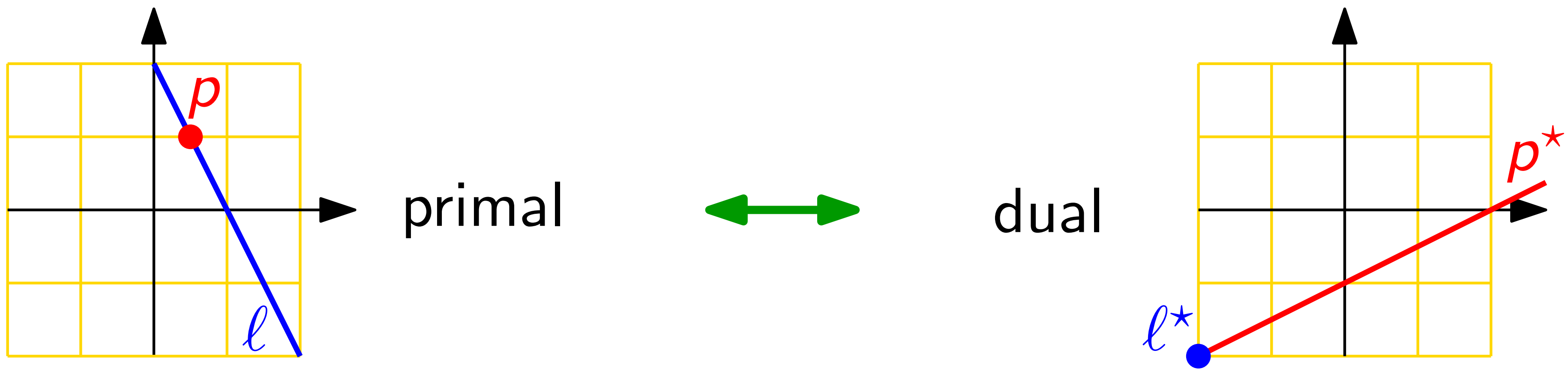

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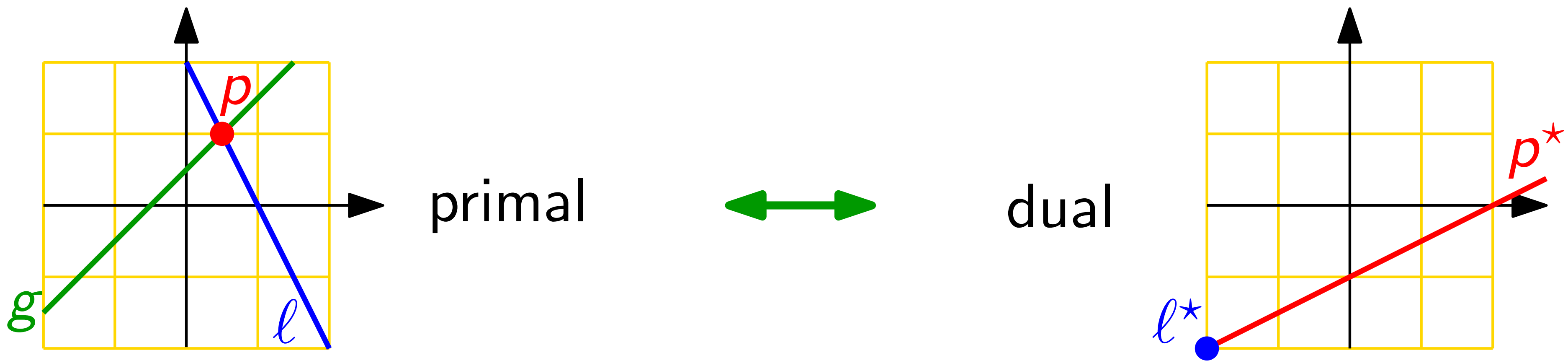

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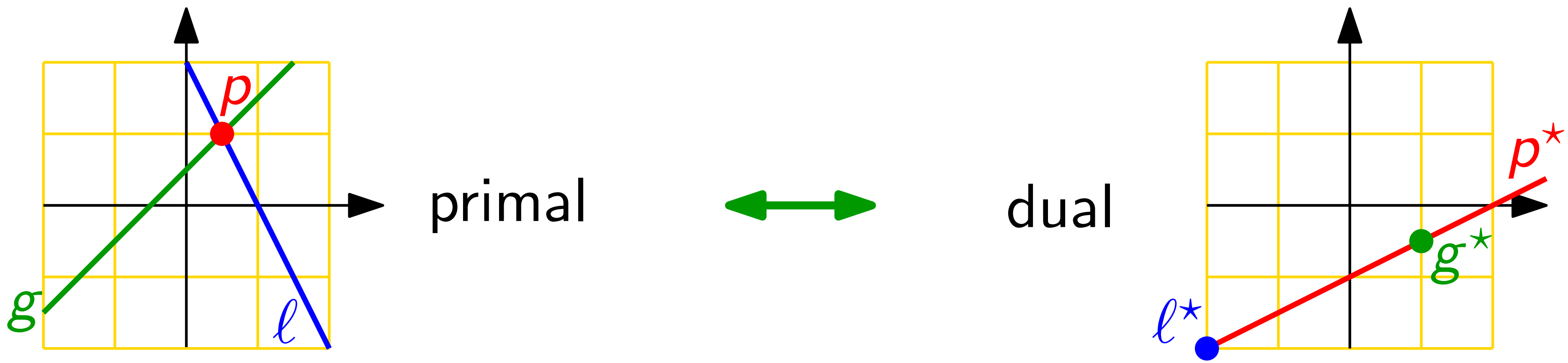

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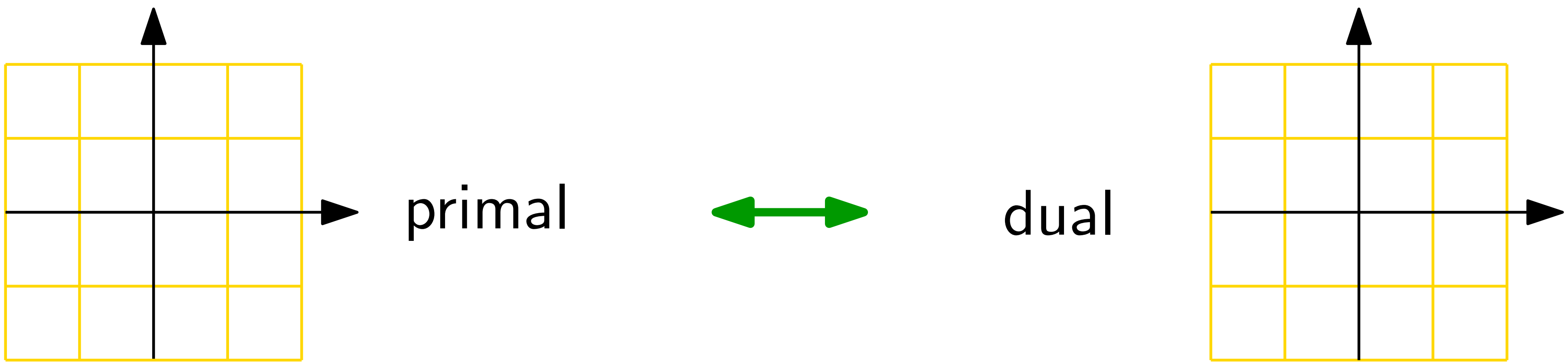

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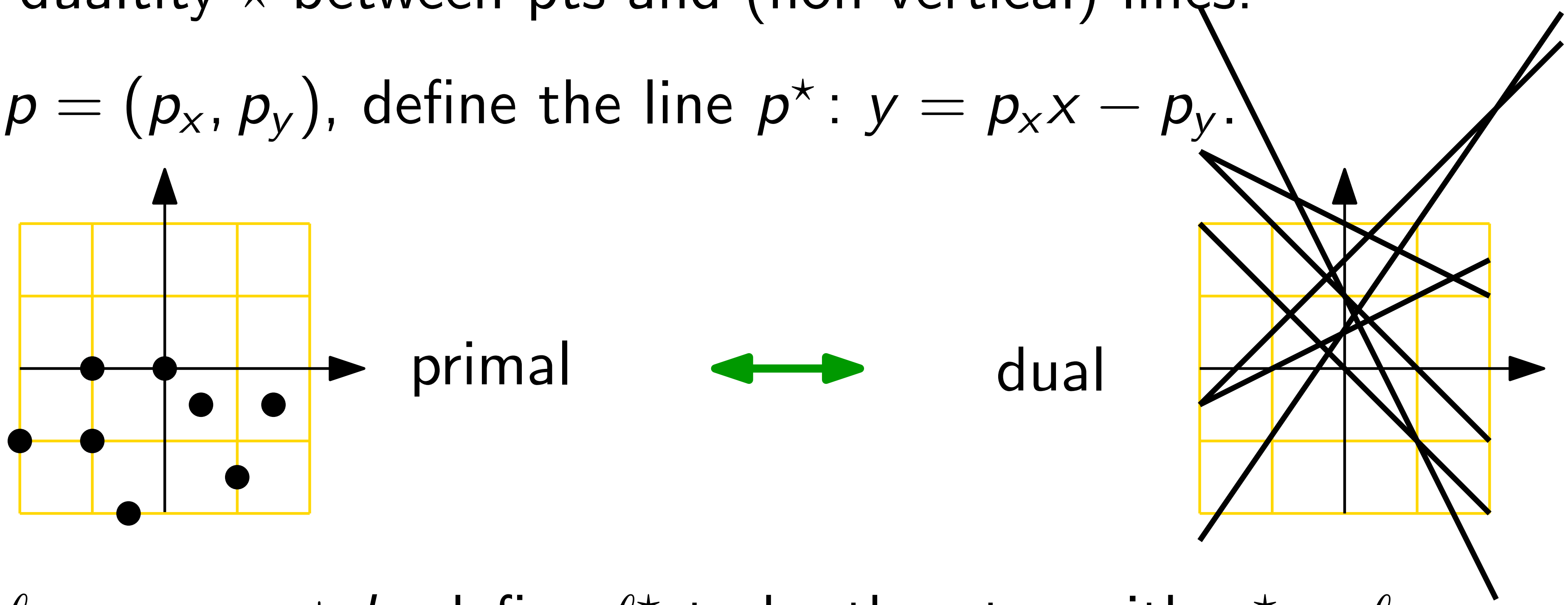

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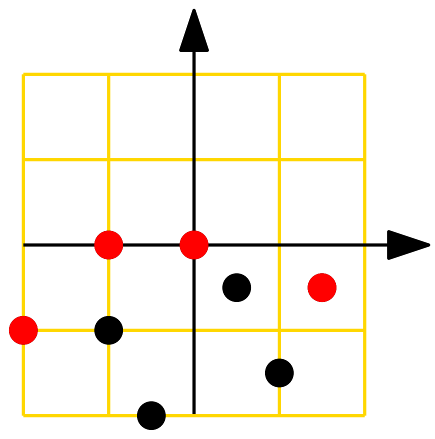

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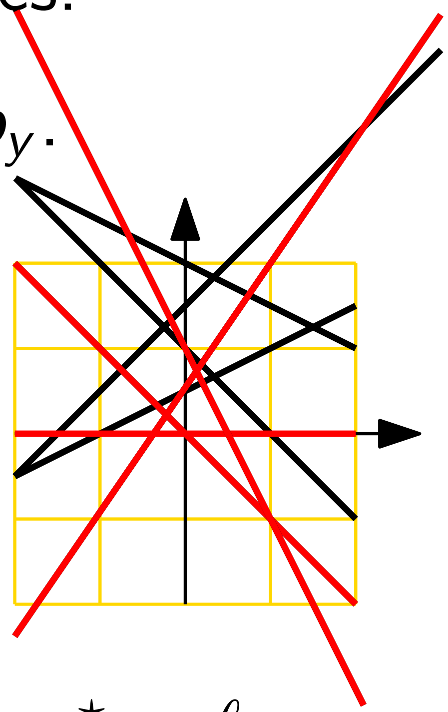

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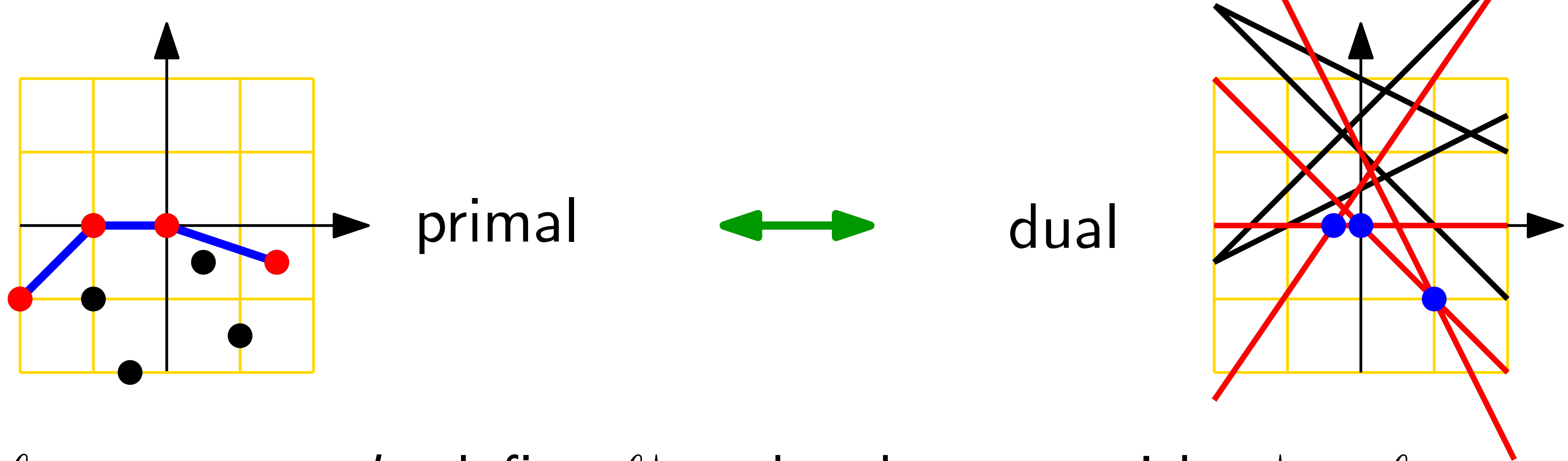

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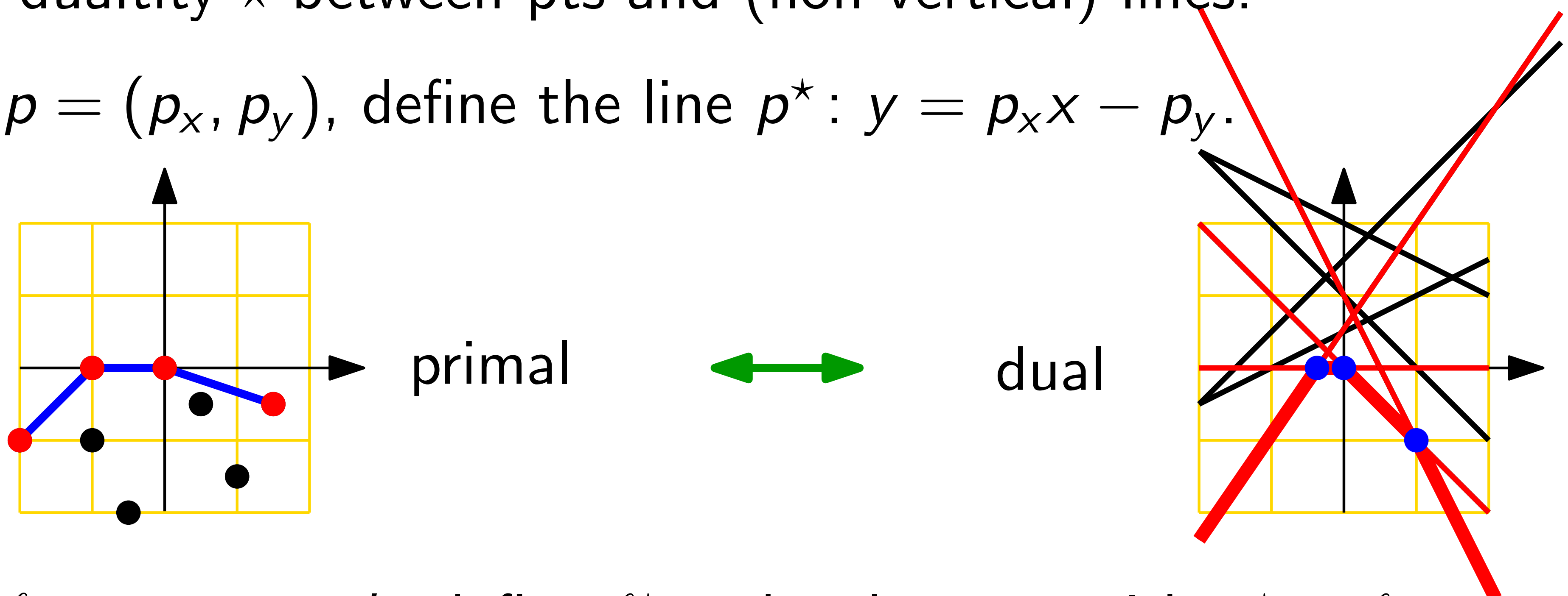

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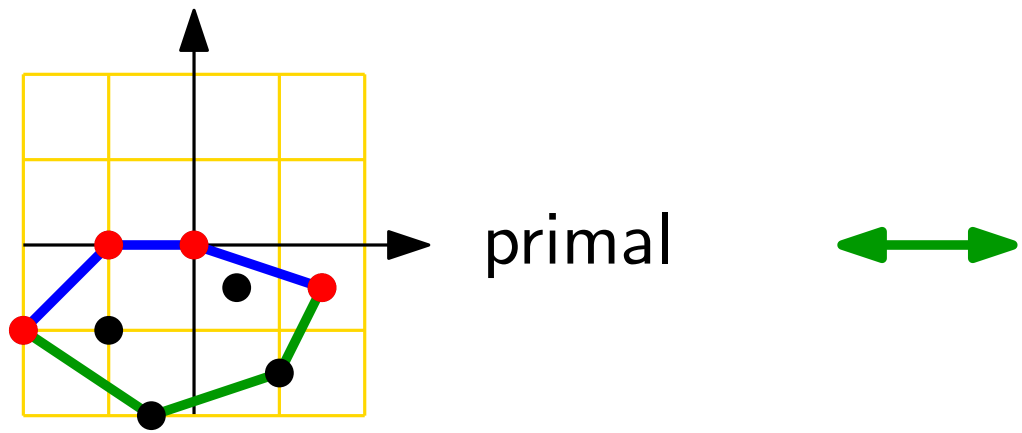

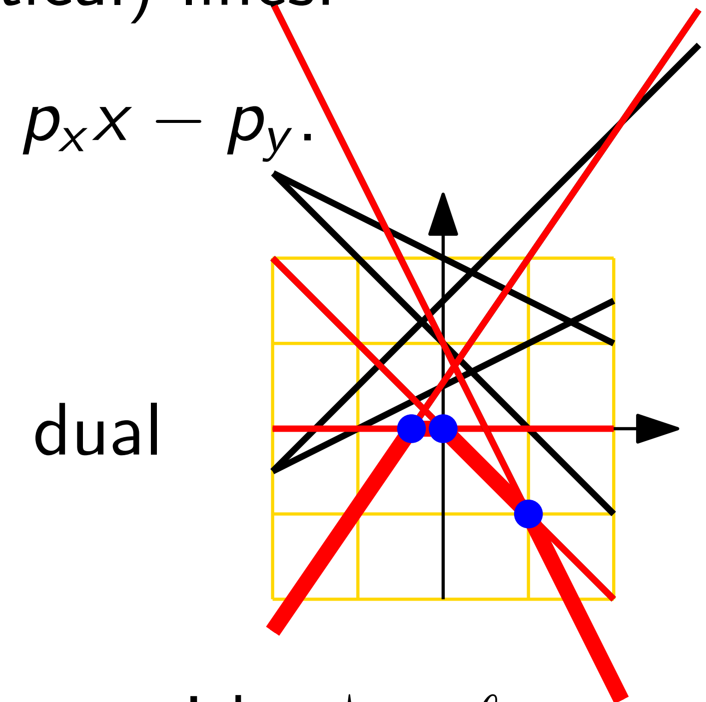

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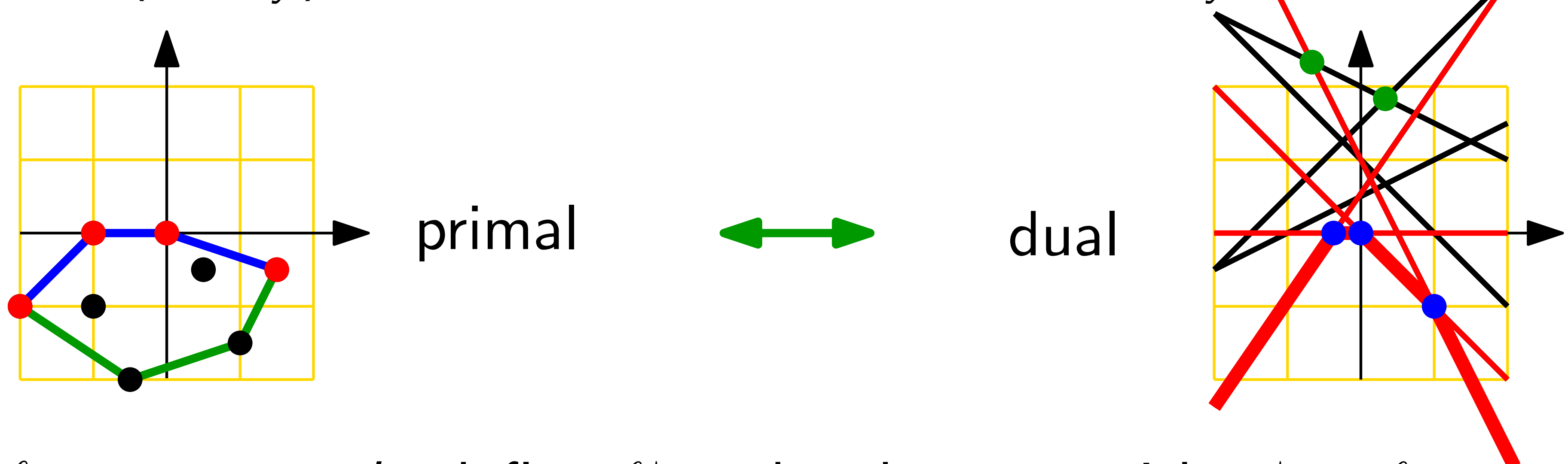

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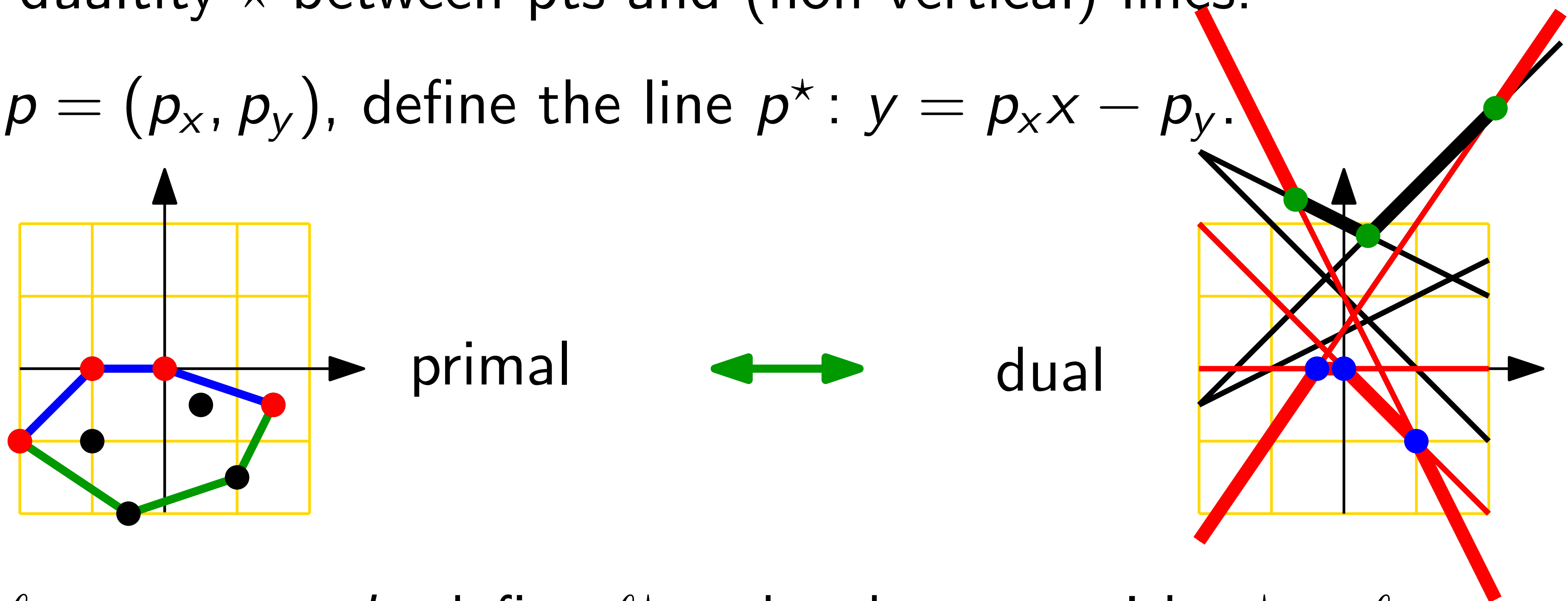

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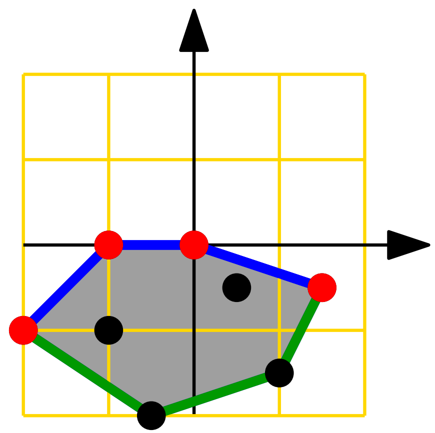
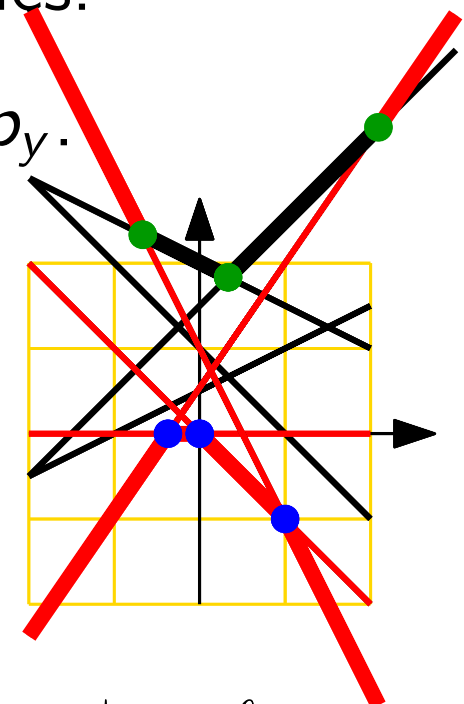


primal $\longleftrightarrow$ dual



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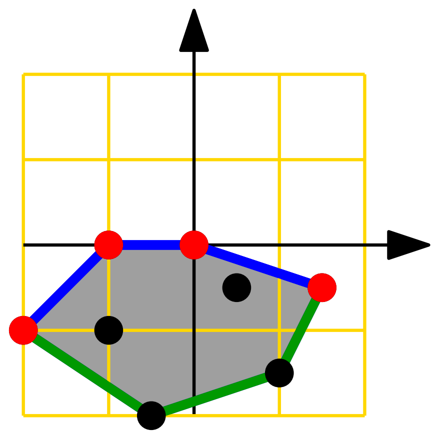

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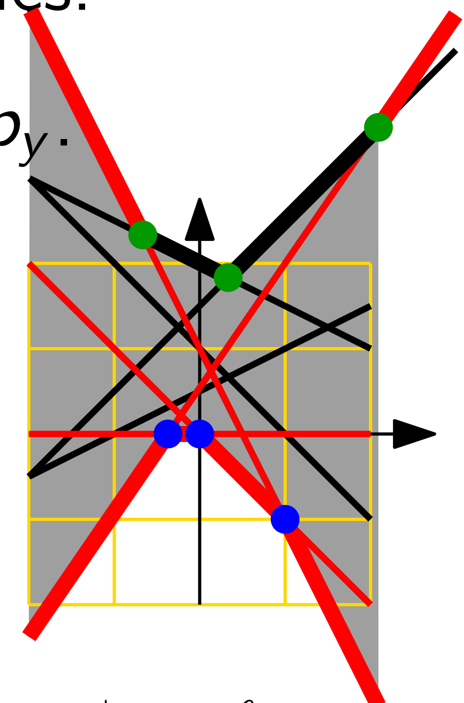

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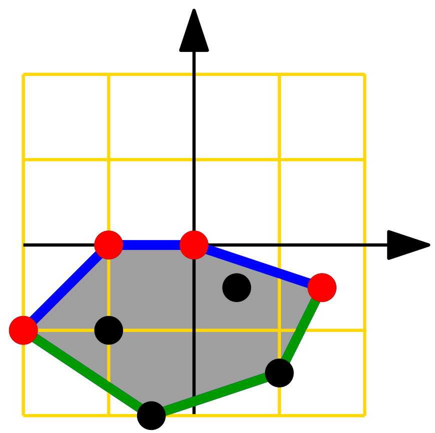

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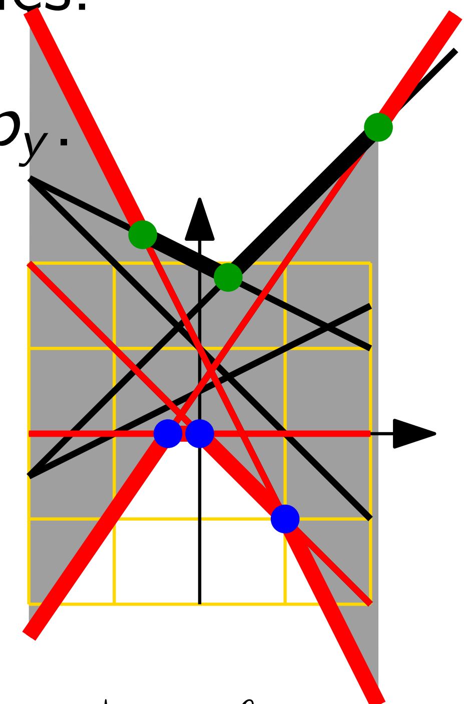

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Observe: upper convex hulls of pts $\leftrightarrow$ lower envelopes of lines

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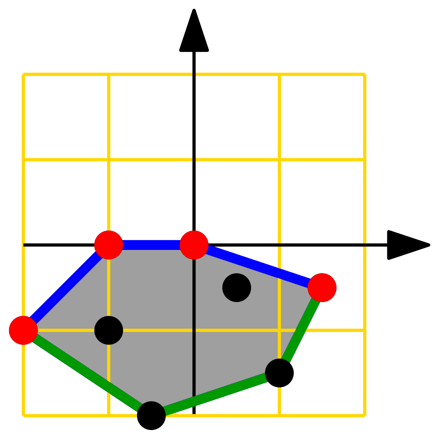

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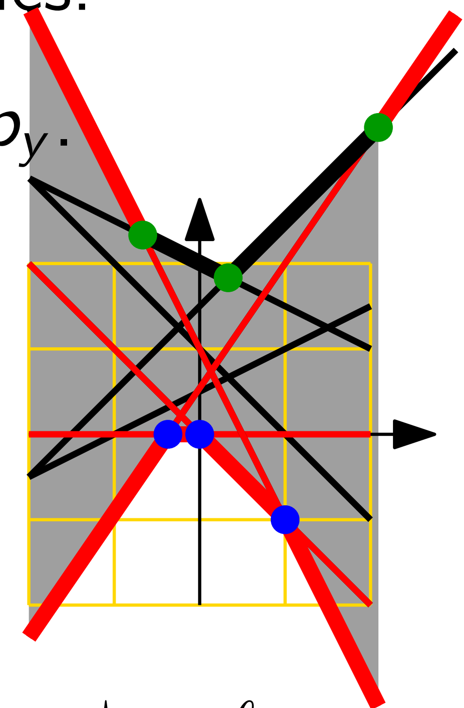

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Observe:

- upper convex hulls of pts $\leftrightarrow$ lower envelopes of lines
- can compute intersections of "lower/upper" half planes (spaces) via upper/lower convex hulls

Voronoi Diagrams Revisited

Let $U\colon z = x^2 + y^2$ be the *unit paraboloid* in $\mathbb{R}^3$.

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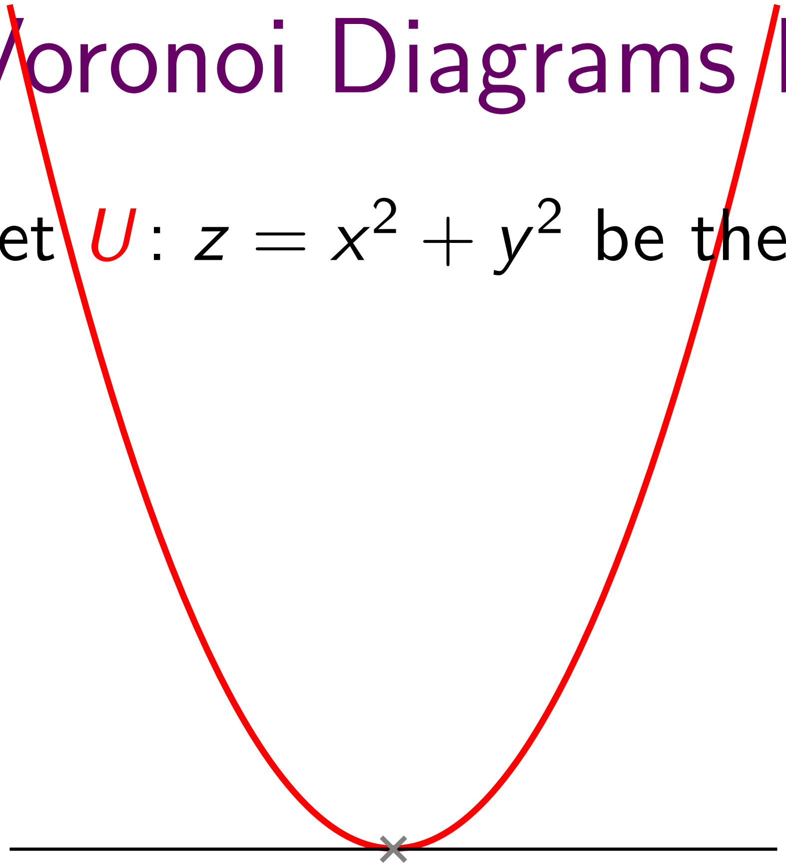

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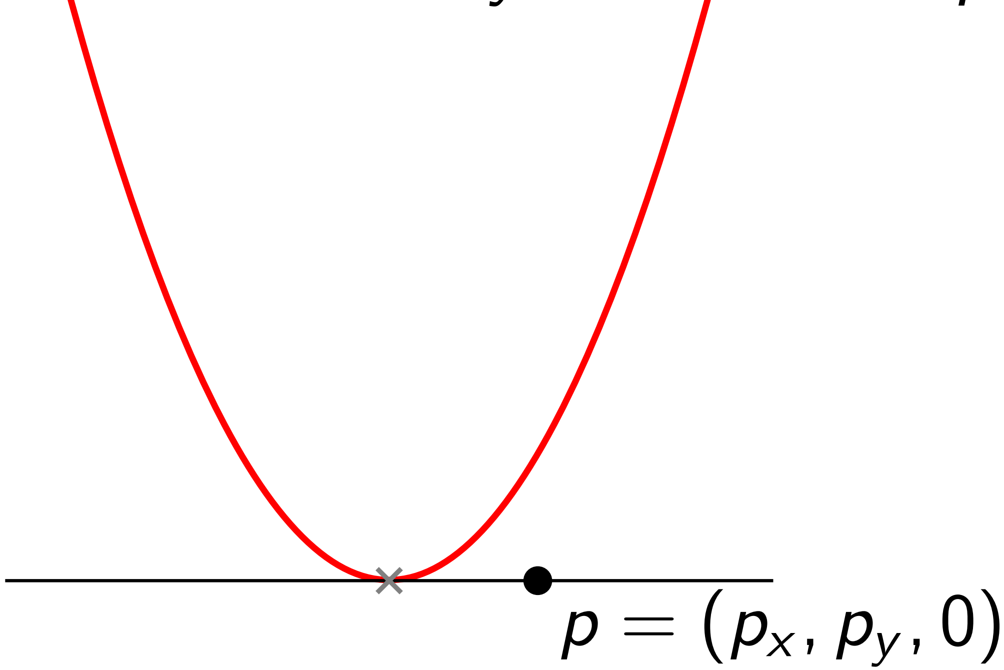

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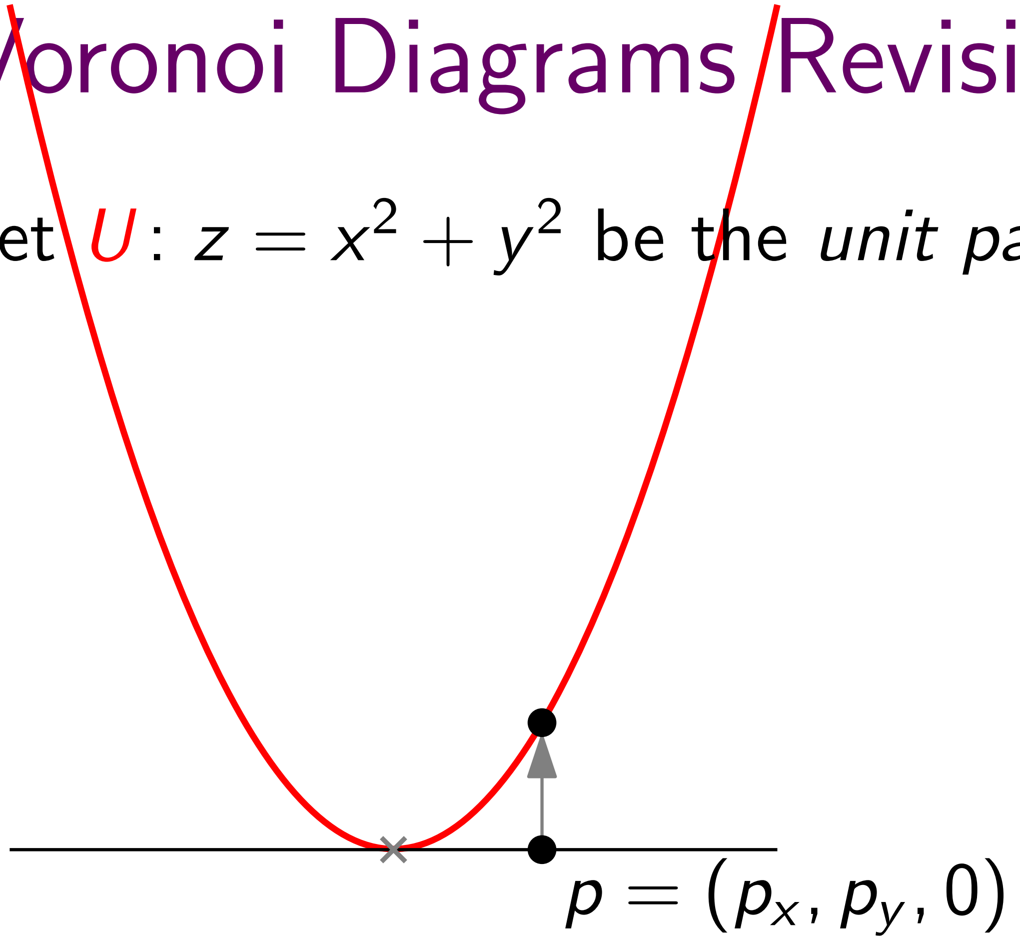

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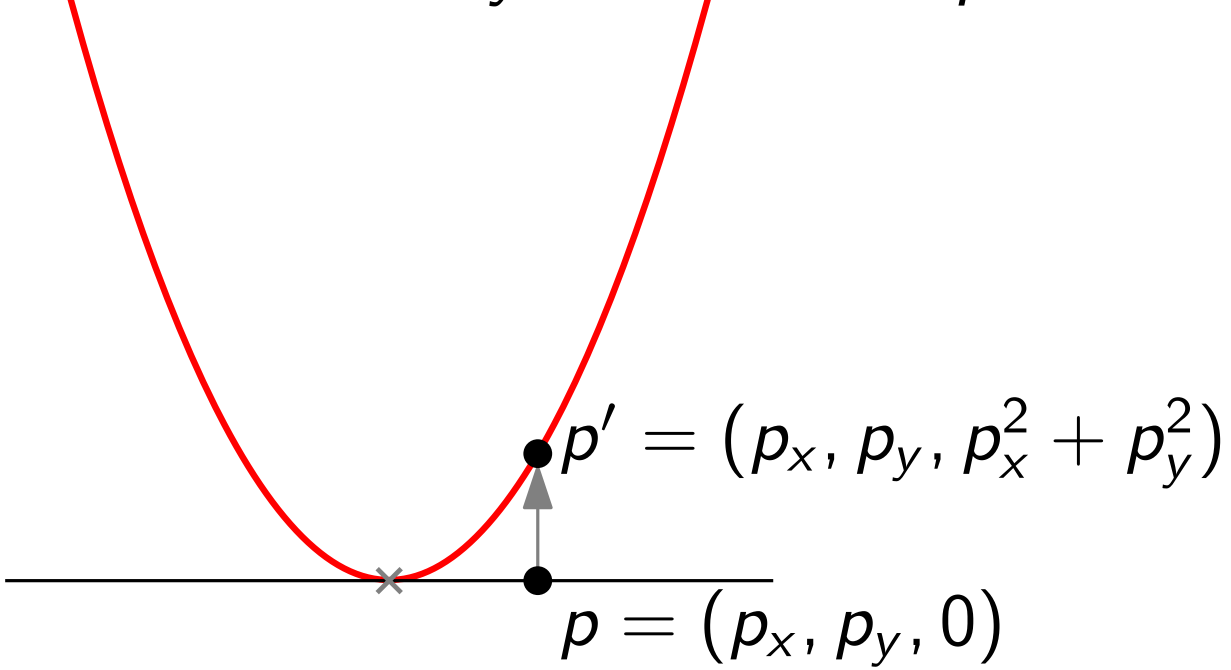

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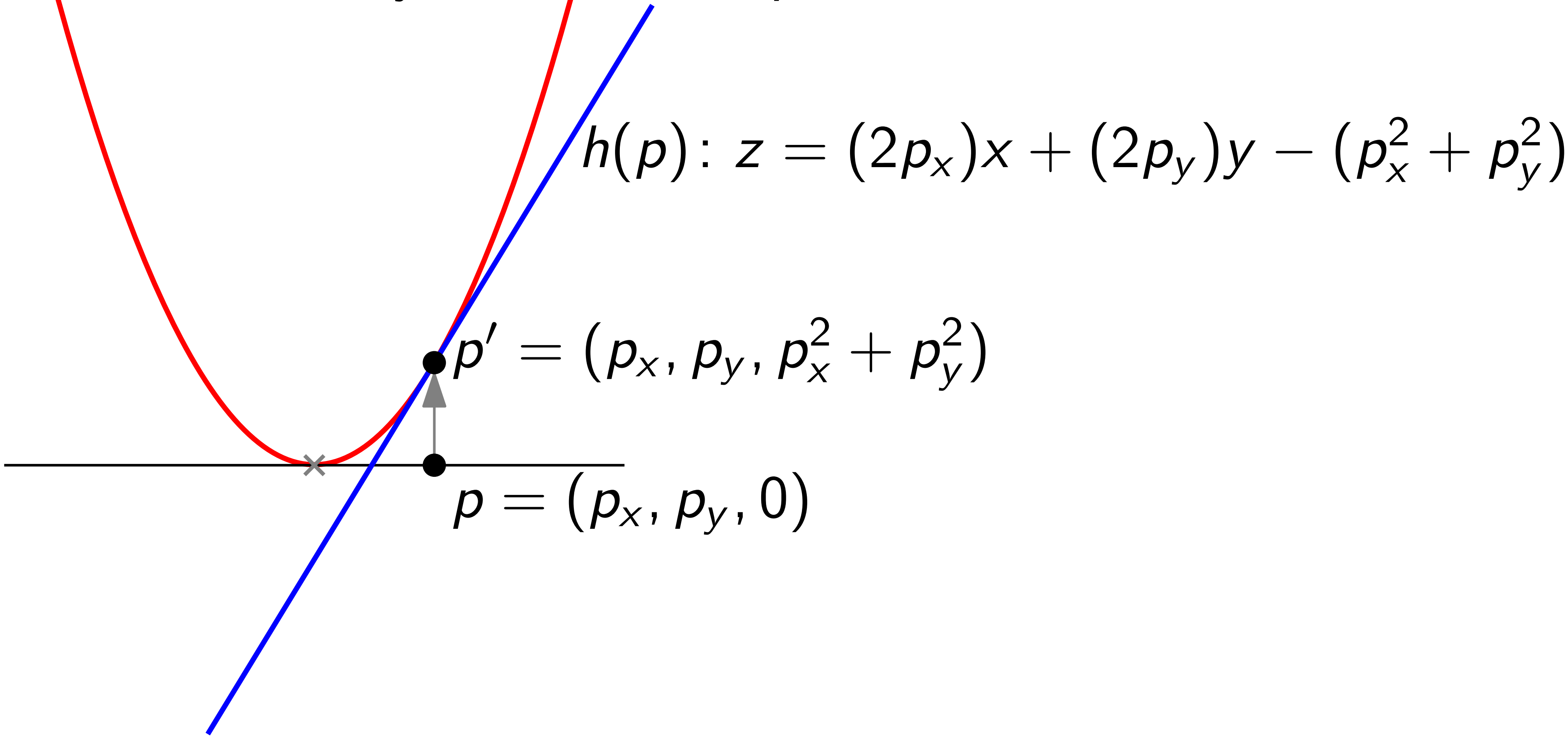

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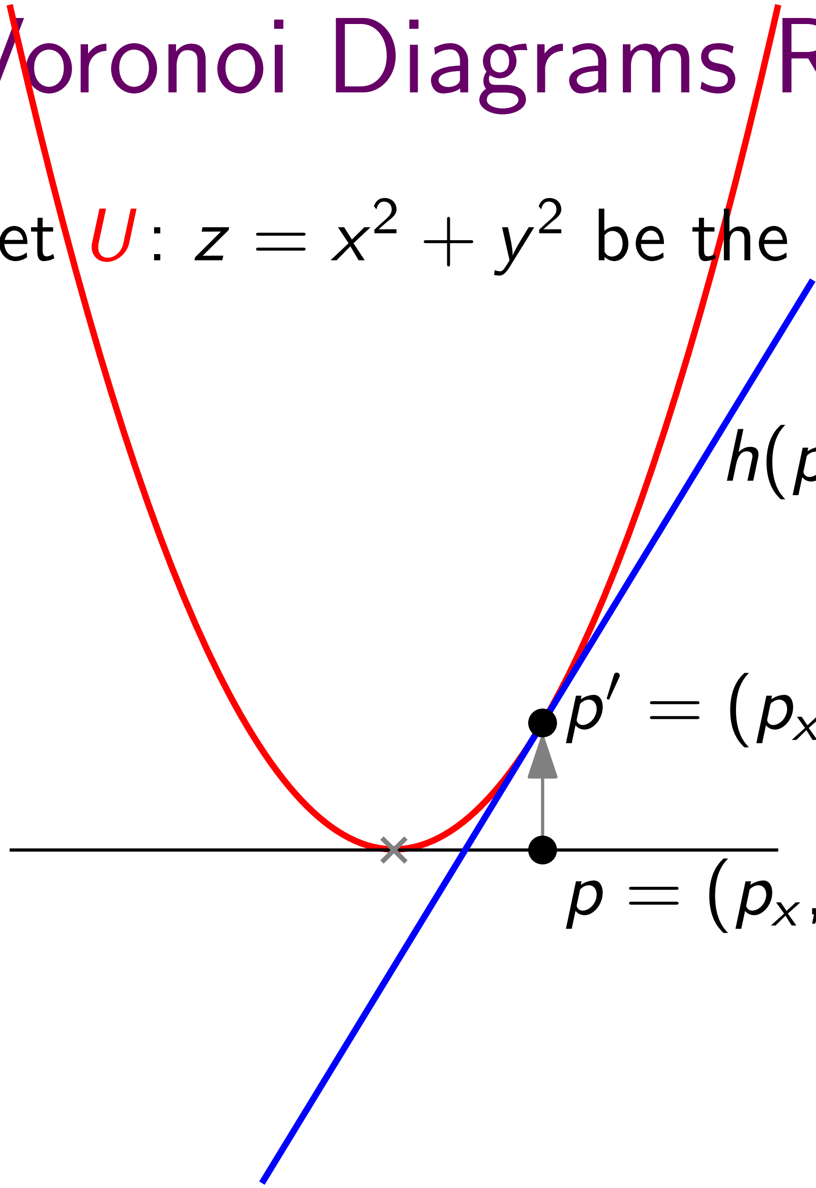

$$h(p)\colon z = (2p_x)x + (2p_y)y - (p_x^2 + p_y^2)$$

Note that $p' \in h(p)$.

$$p' = (p_x, p_y, p_x^2 + p_y^2)$$

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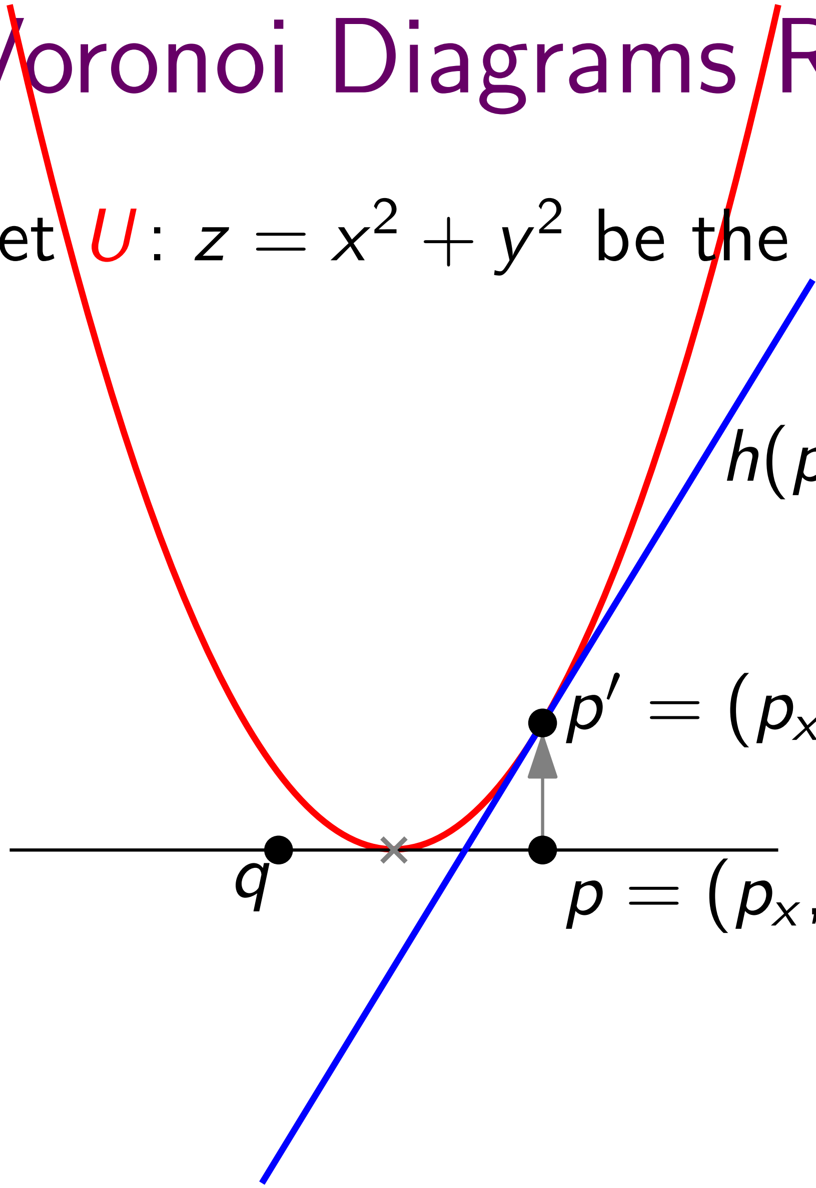

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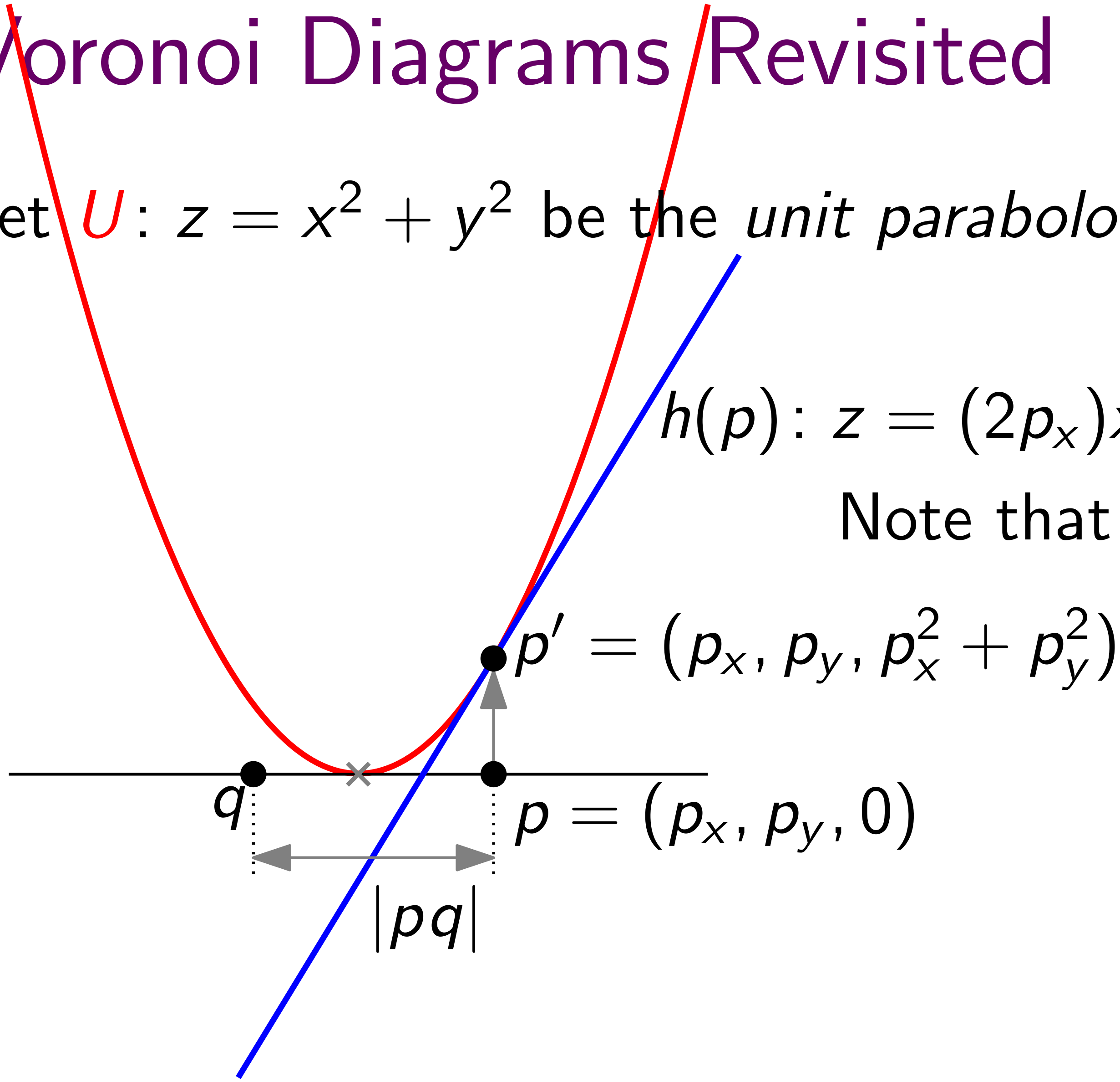

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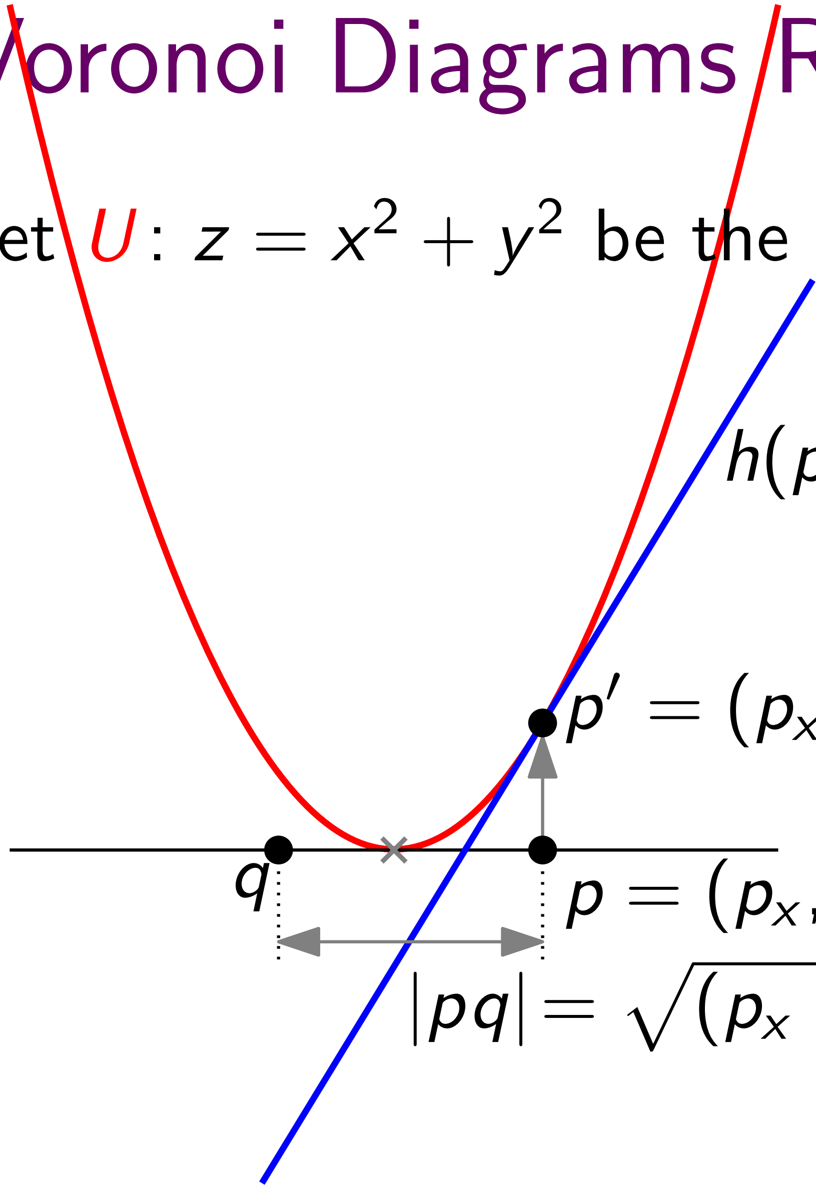

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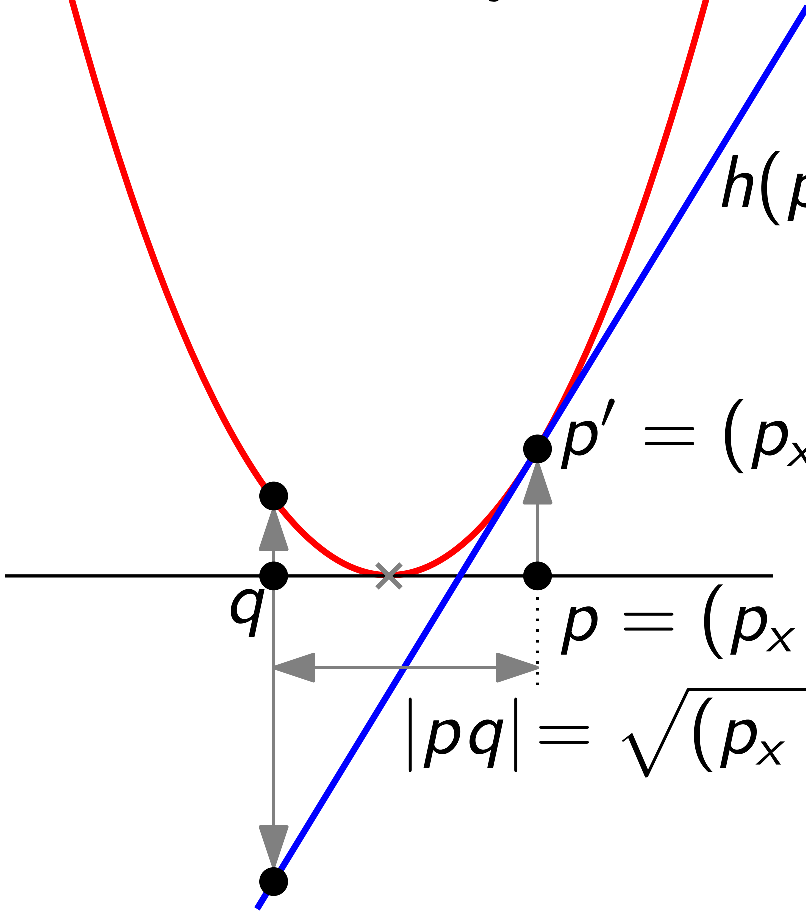

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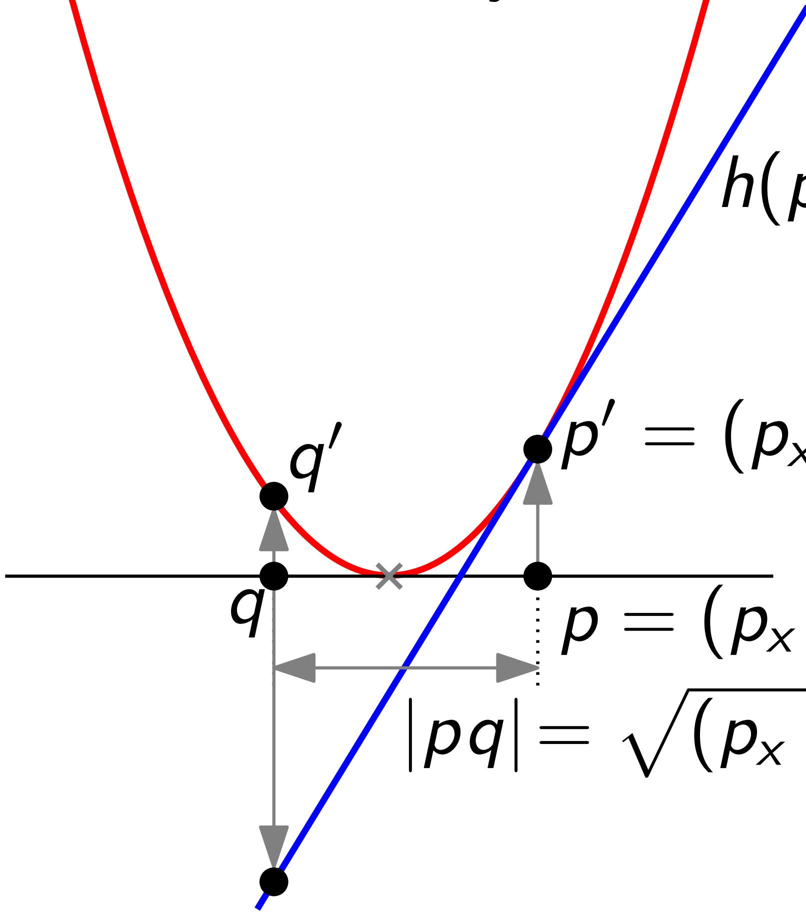

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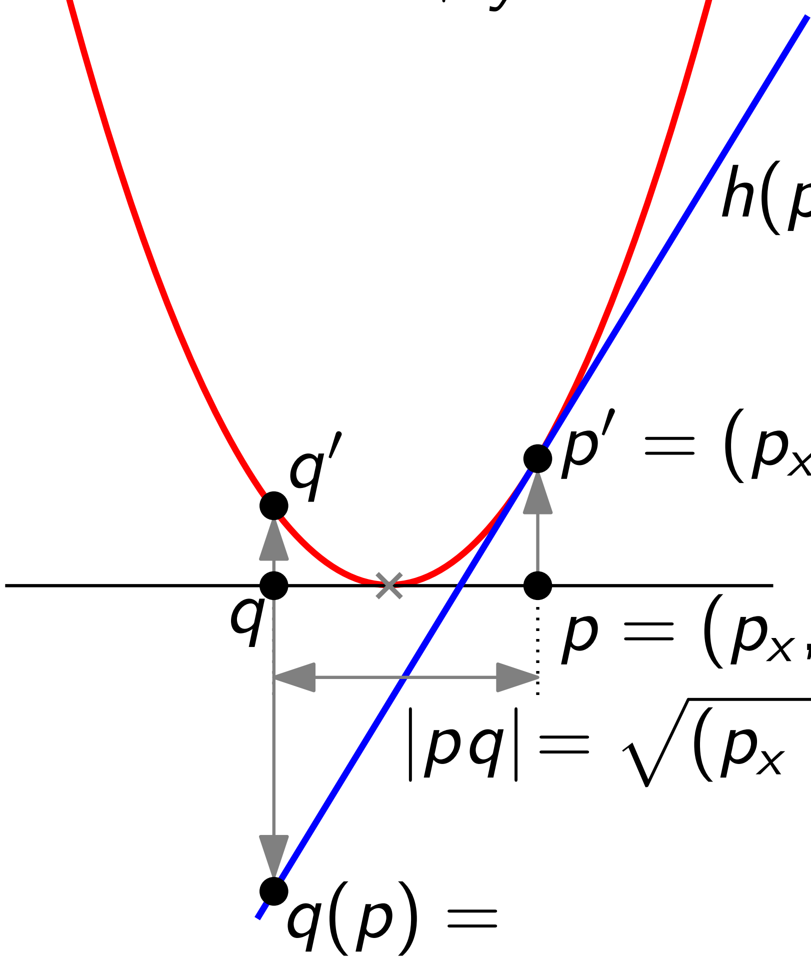

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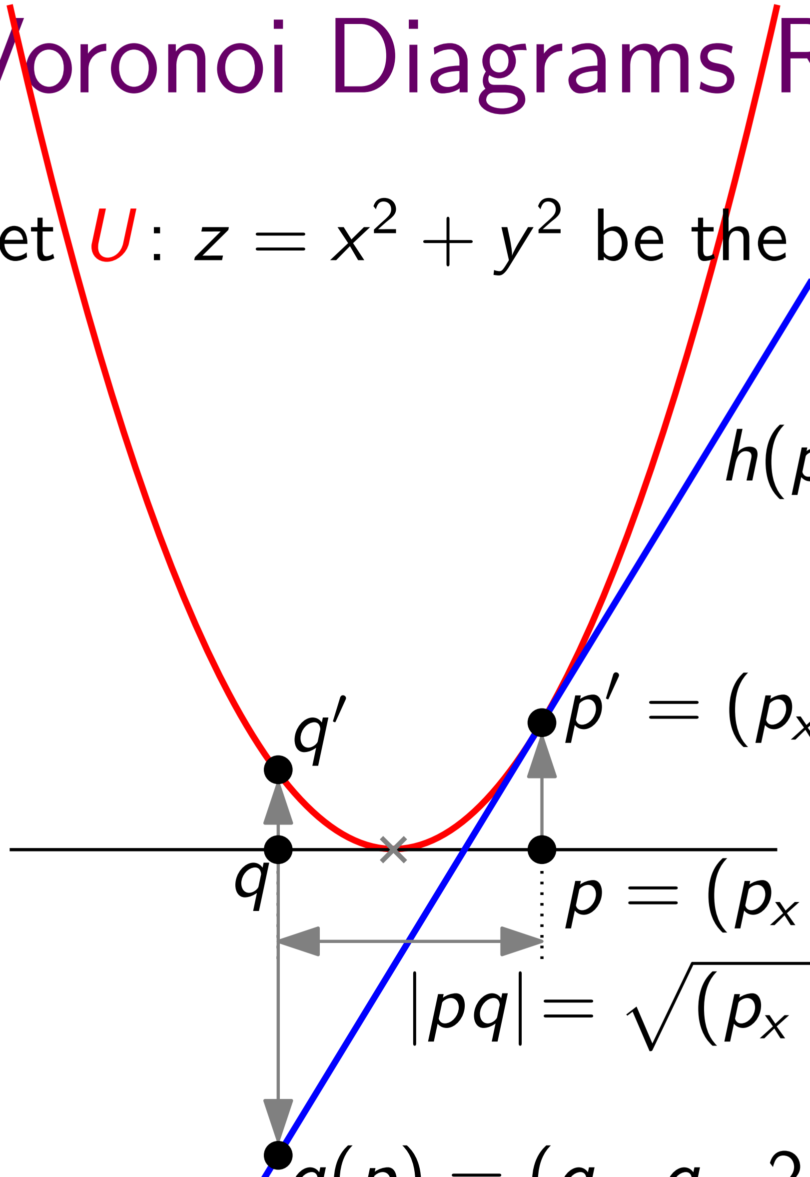

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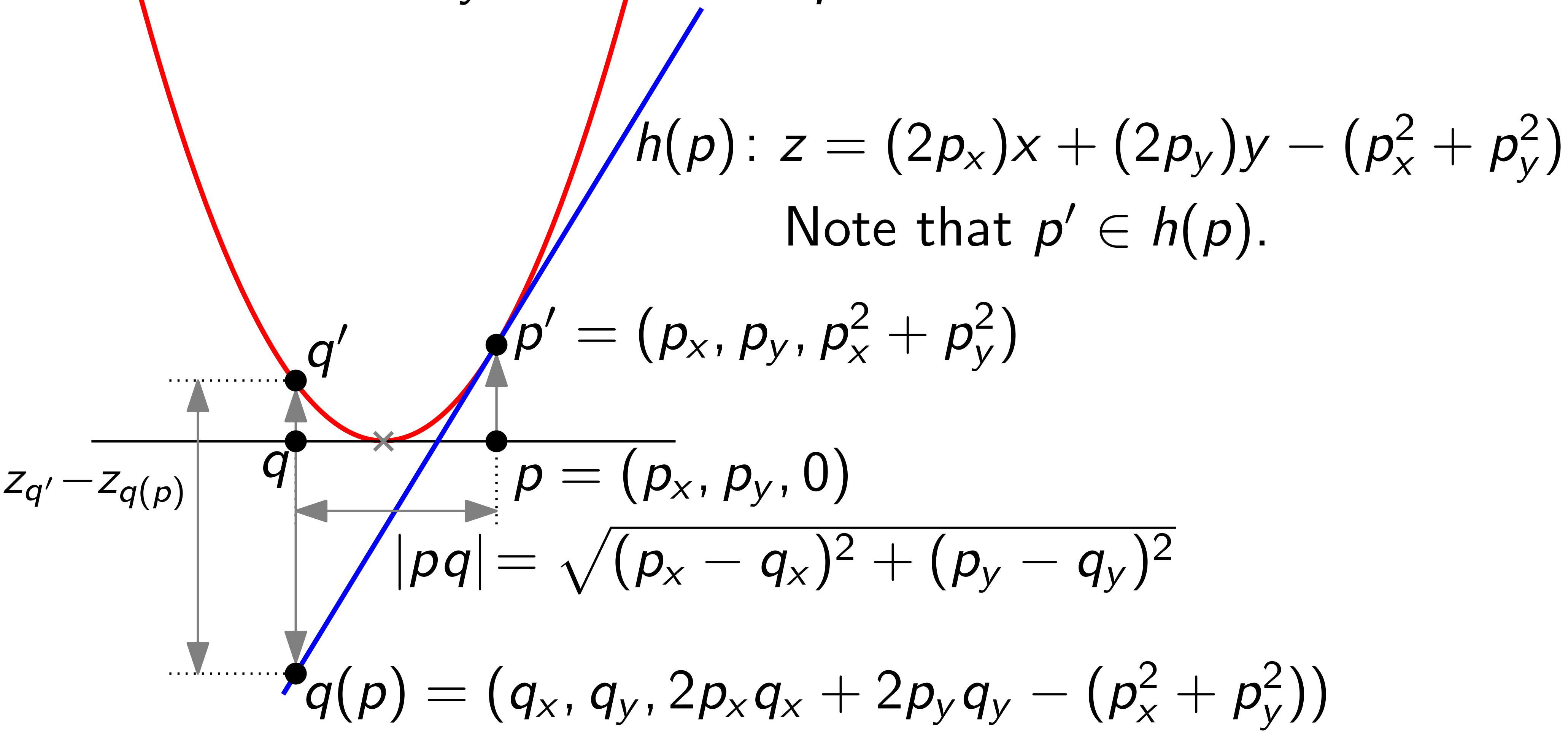

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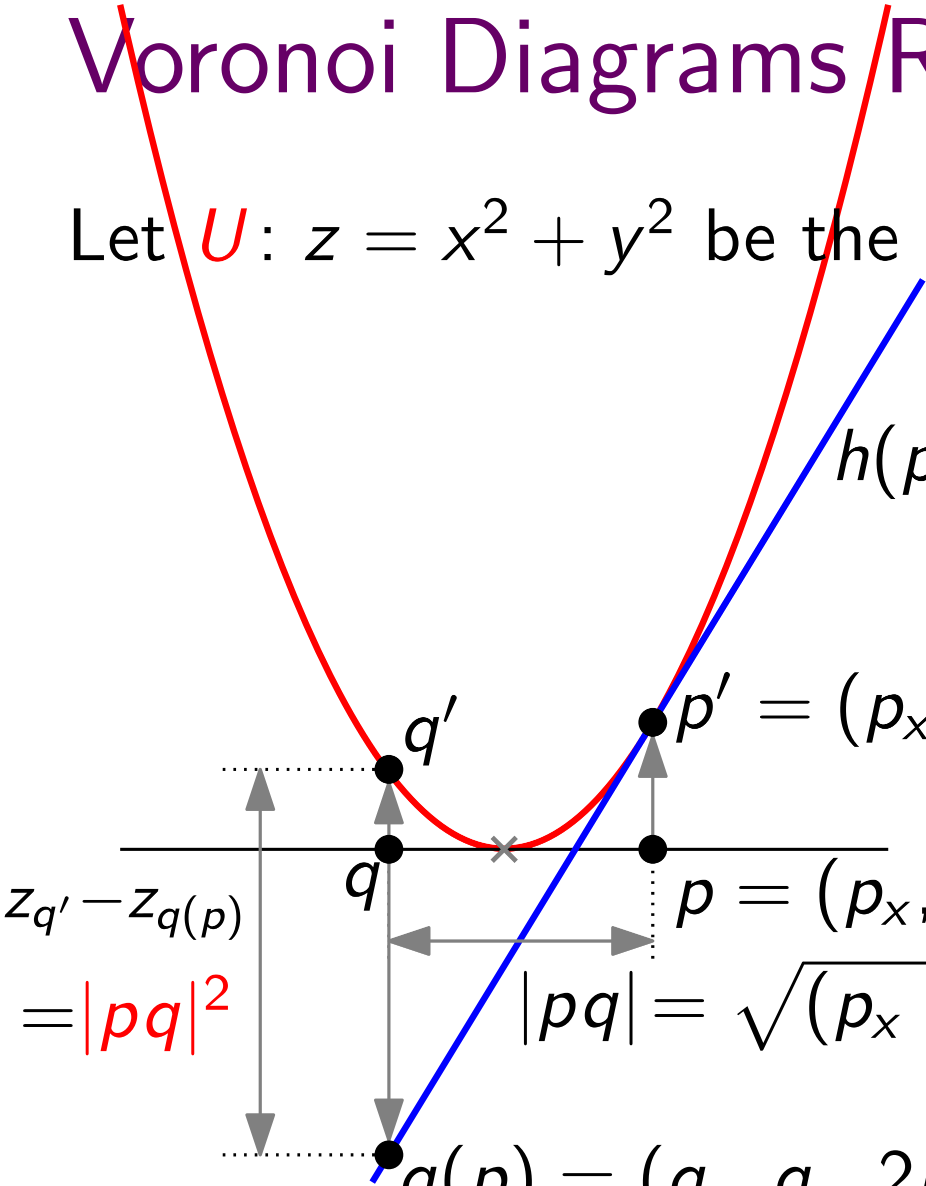

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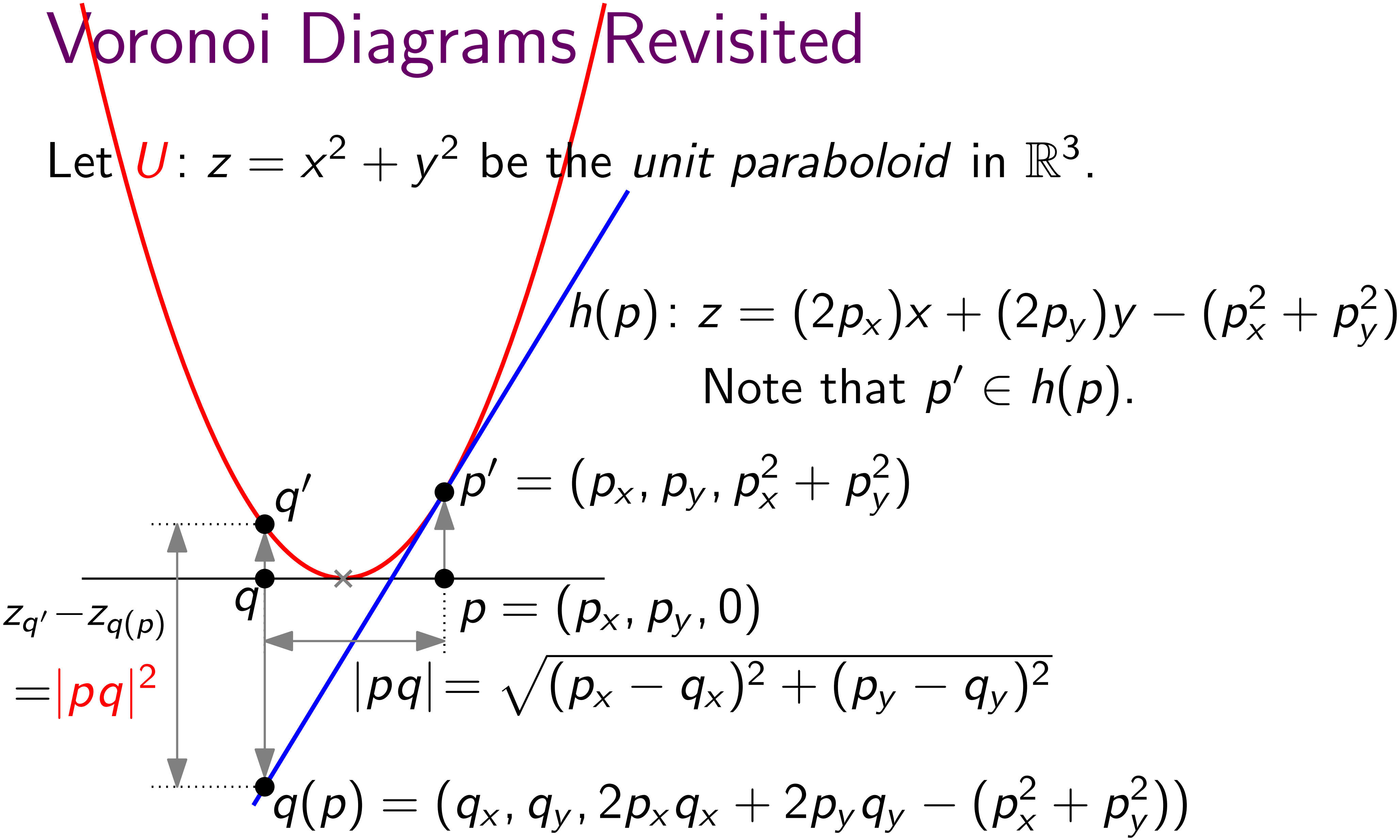

$\Rightarrow$ $h(p)$ and U encode dist. betw. p and any other pt in $z = 0$.

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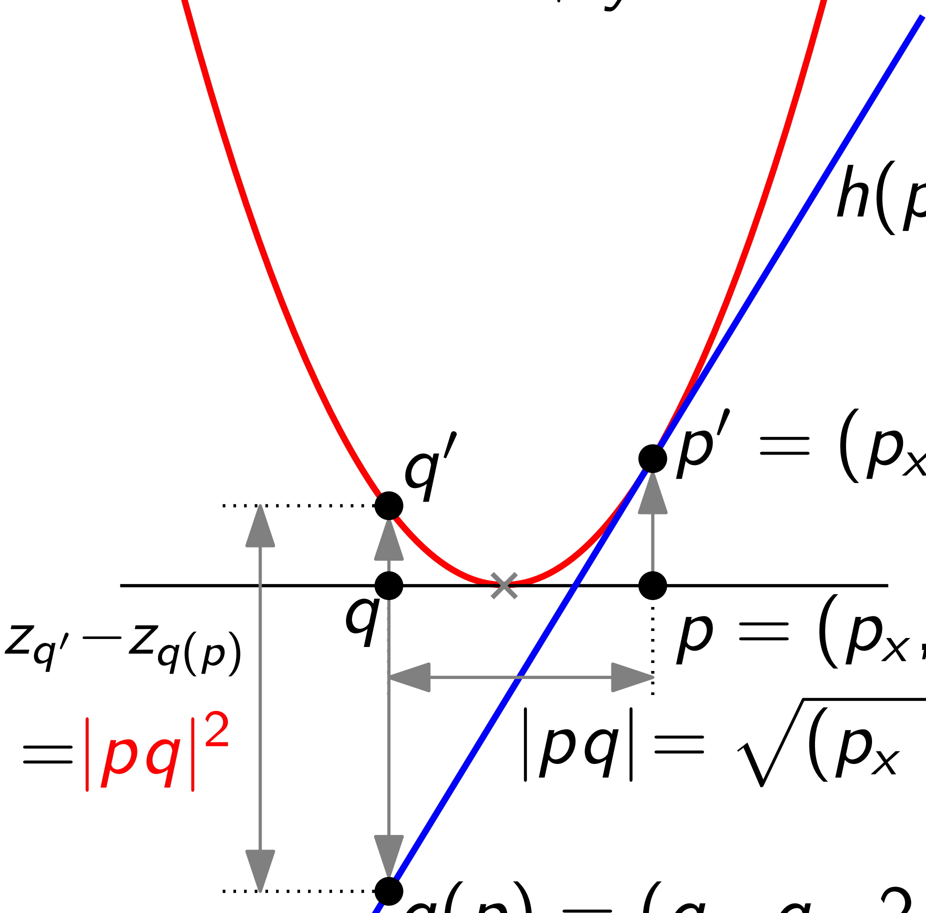

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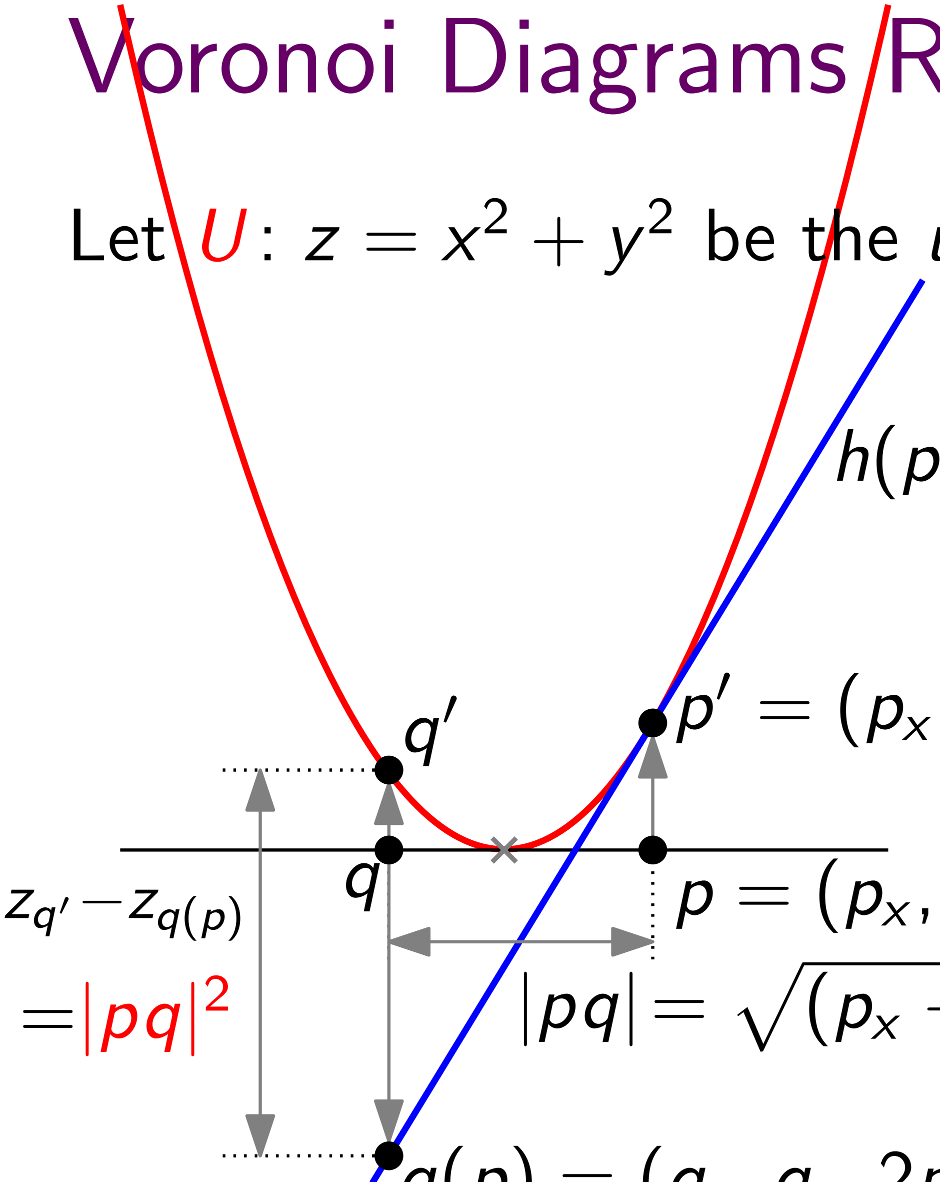

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$\Rightarrow$ $h(p) \cap U = \{p'\}$ $\Rightarrow$ $h(p)$ is tangent to U (in p')

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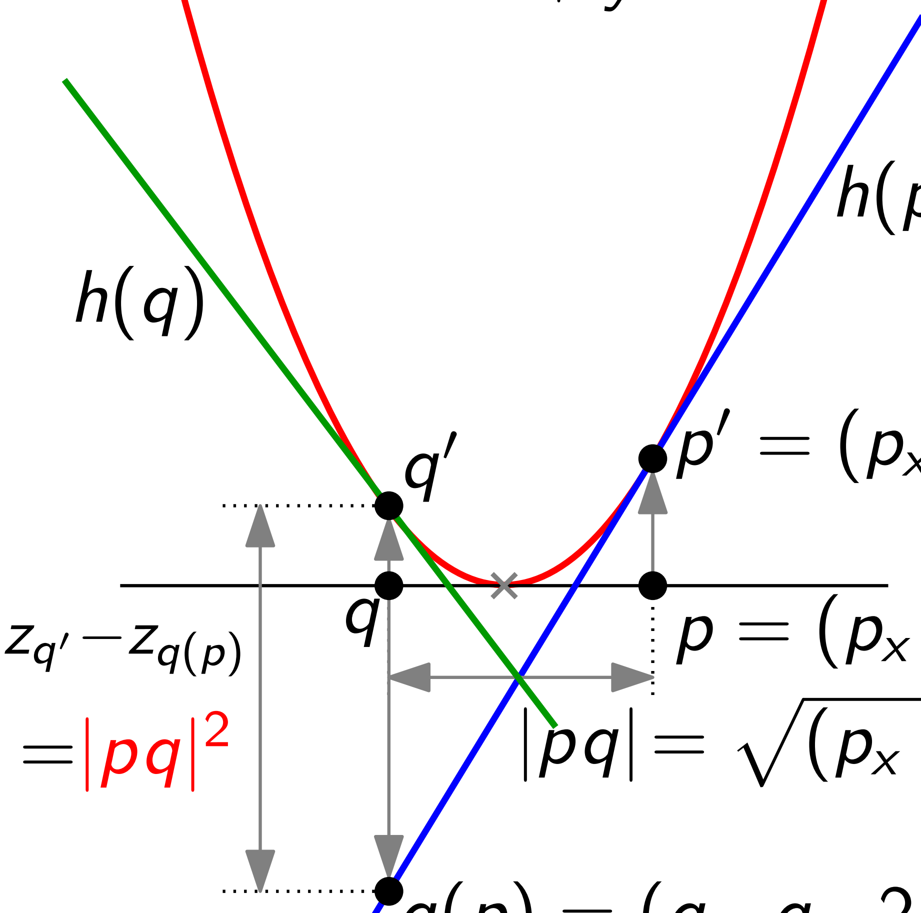

$h(p)\colon z = (2p_x)x + (2p_y)y - (p_x^2 + p_y^2)$

Note that $p' \in h(p)$.

$p' = (p_x, p_y, p_x^2 + p_y^2)$

$p = (p_x, p_y, 0)$

$|pq| = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}$

$q(p) = (q_x, q_y, 2p_x q_x + 2p_y q_y - (p_x^2 + p_y^2))$

$\Rightarrow$ $h(p)$ and U encode dist. betw. p and any other pt in $z = 0$.

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Voronoi Diagrams Revisited

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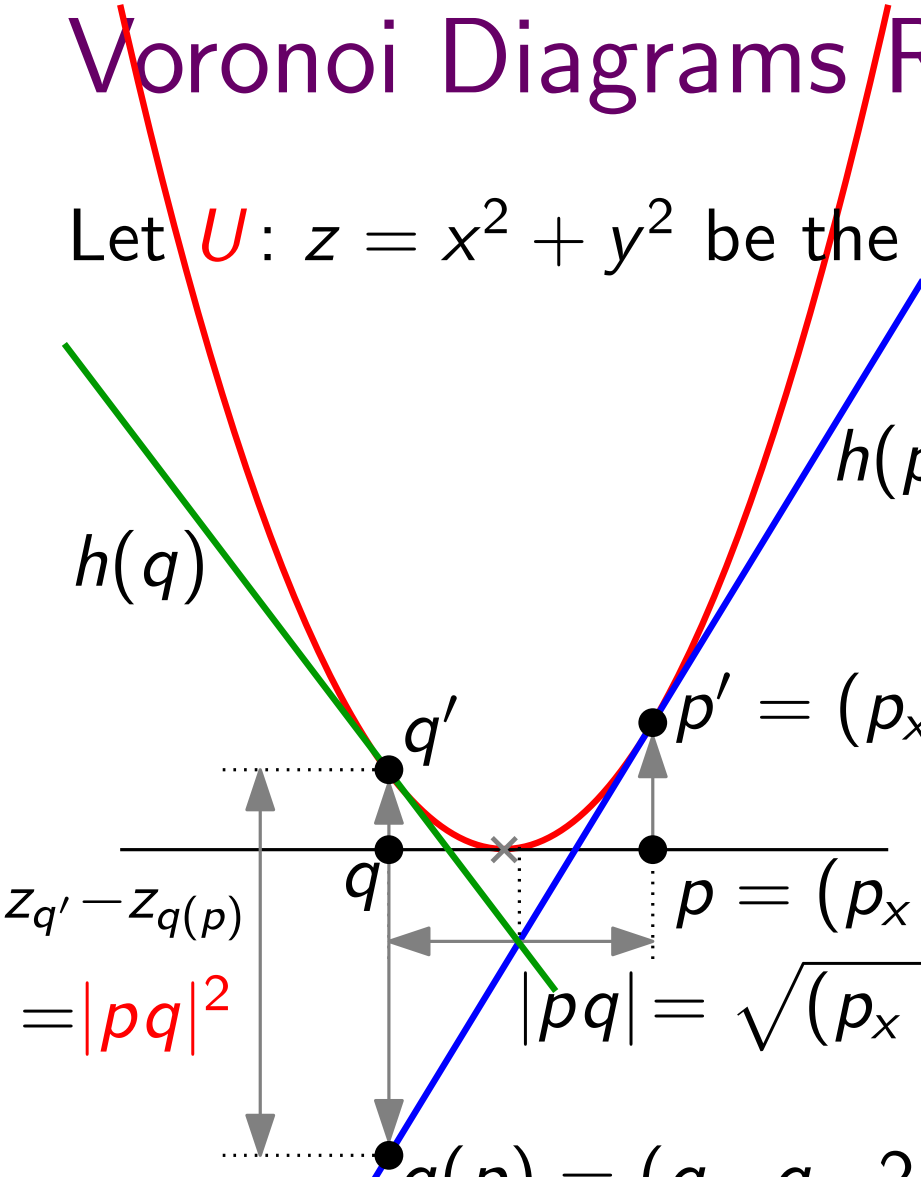

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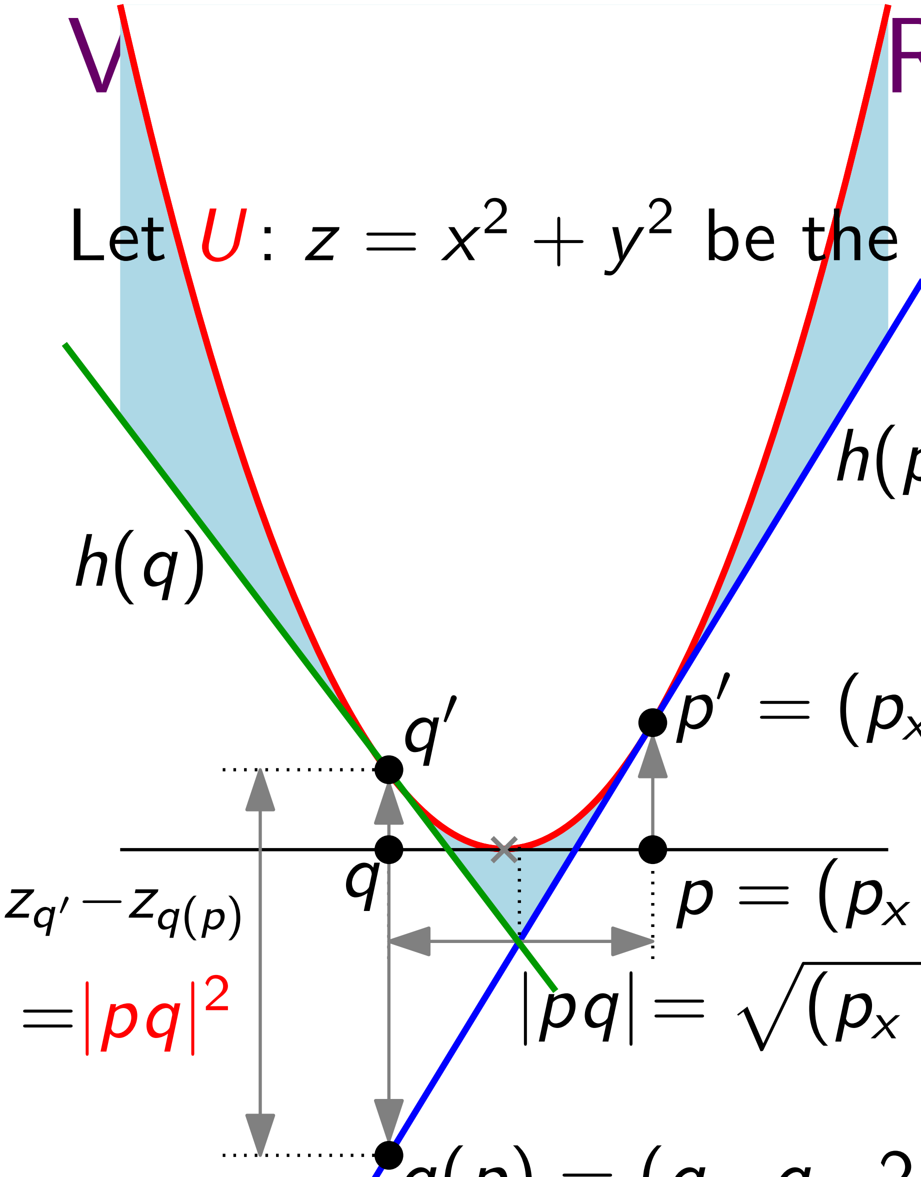

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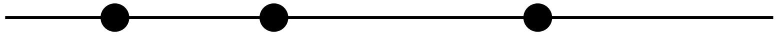

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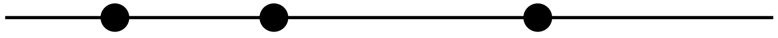

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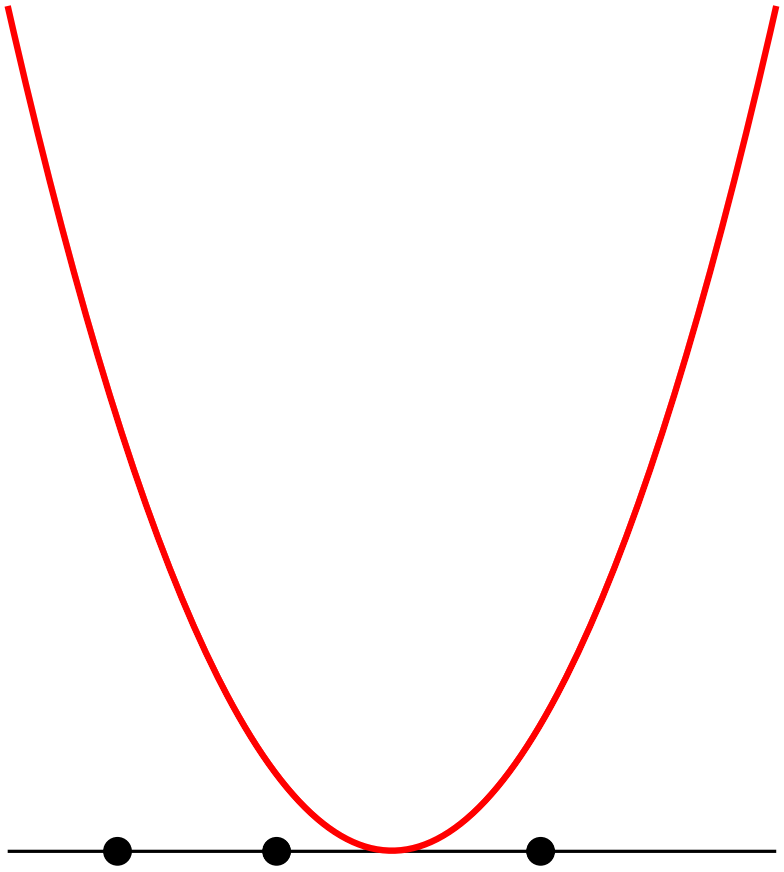

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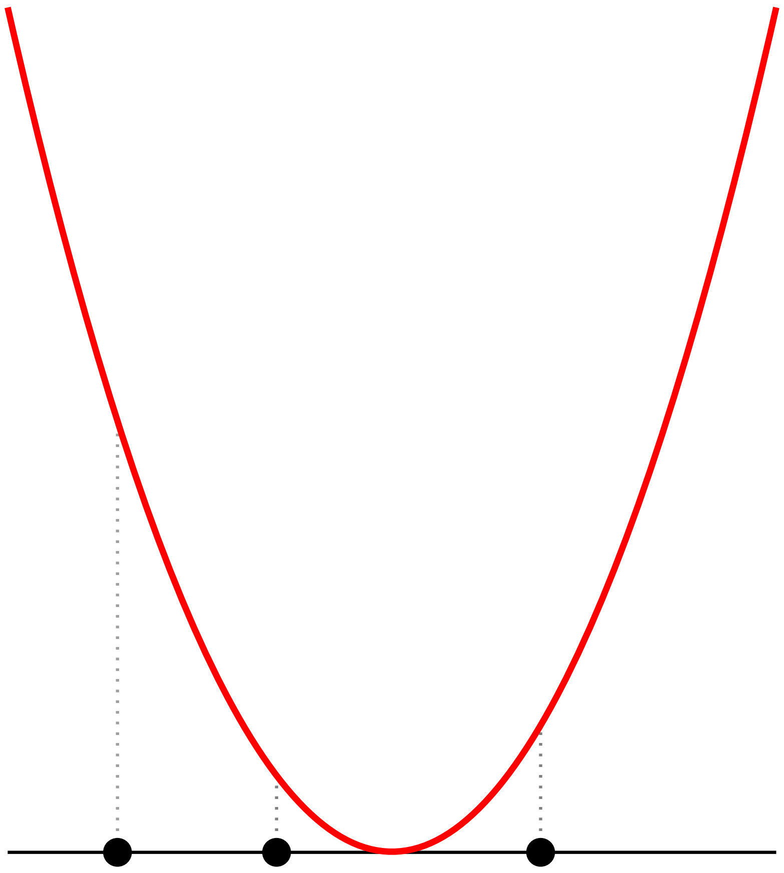

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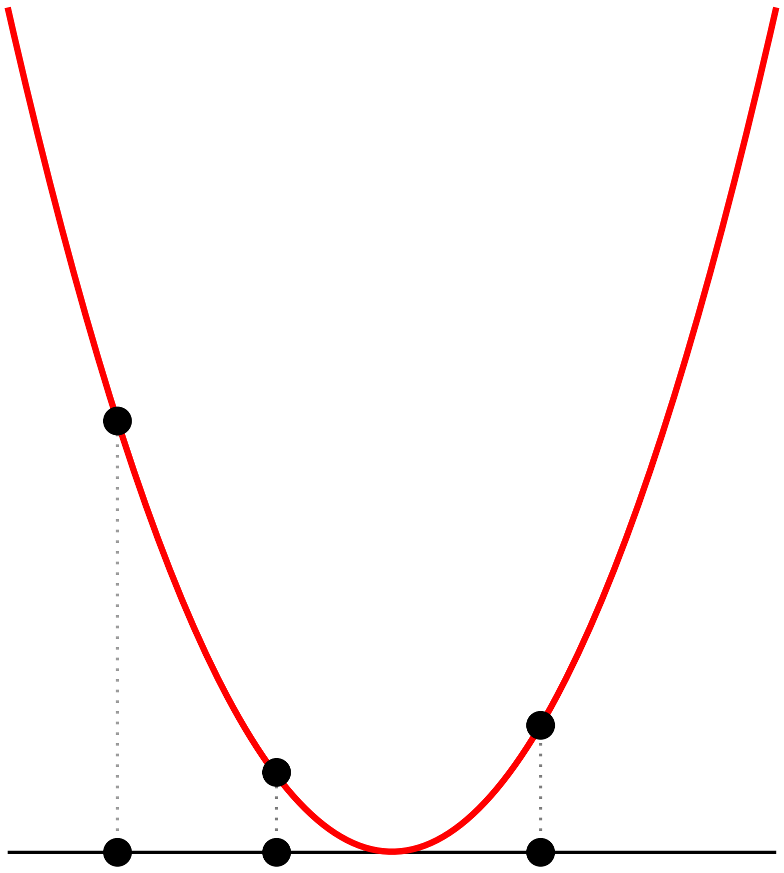

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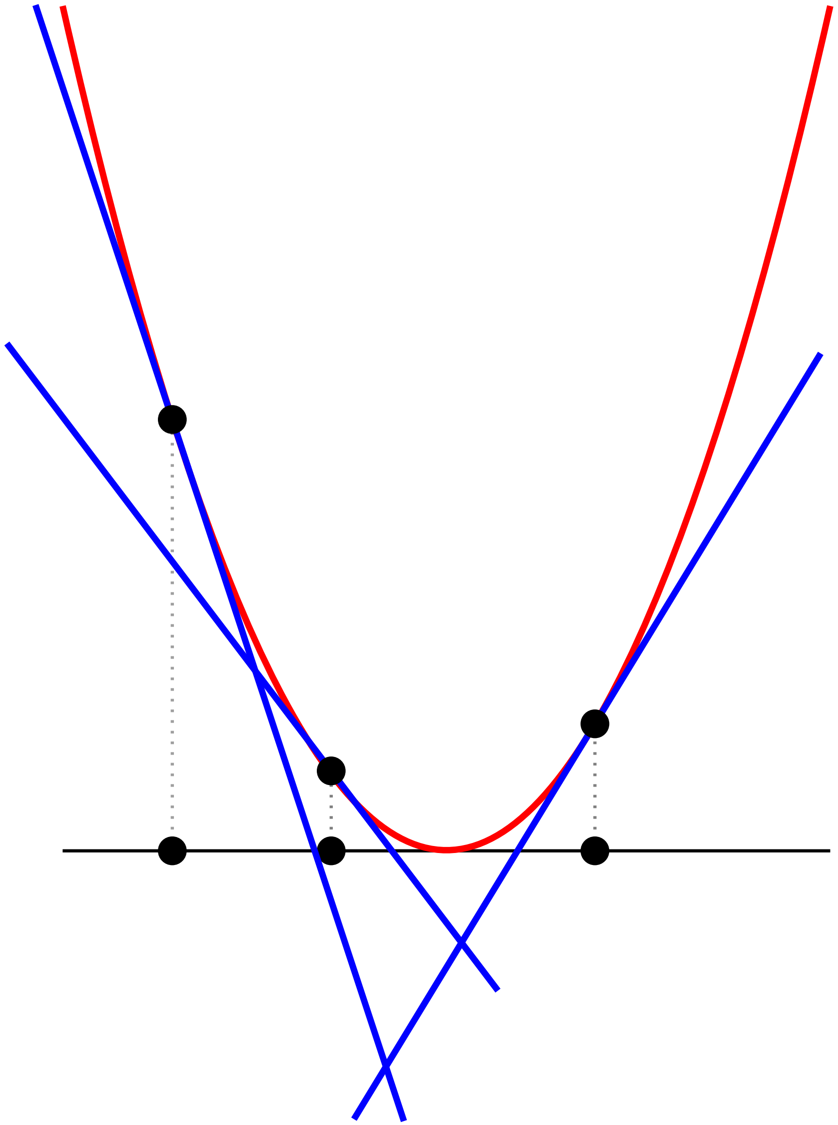

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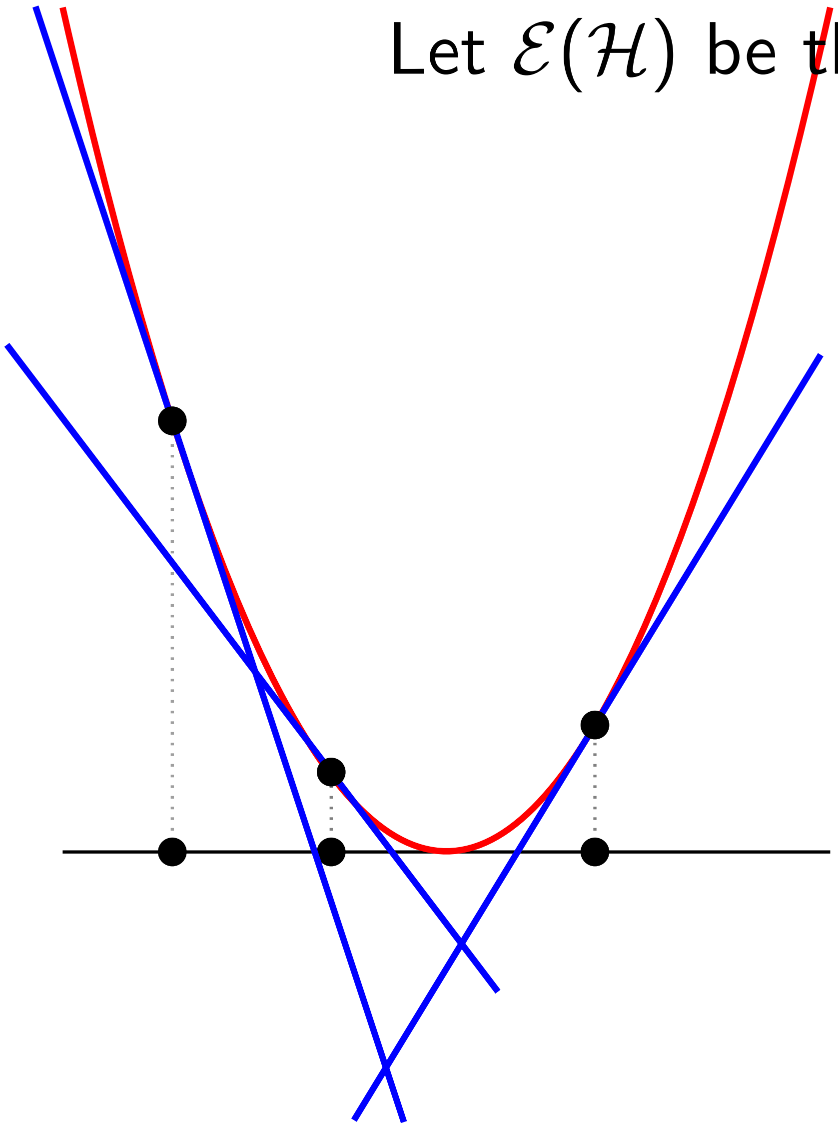

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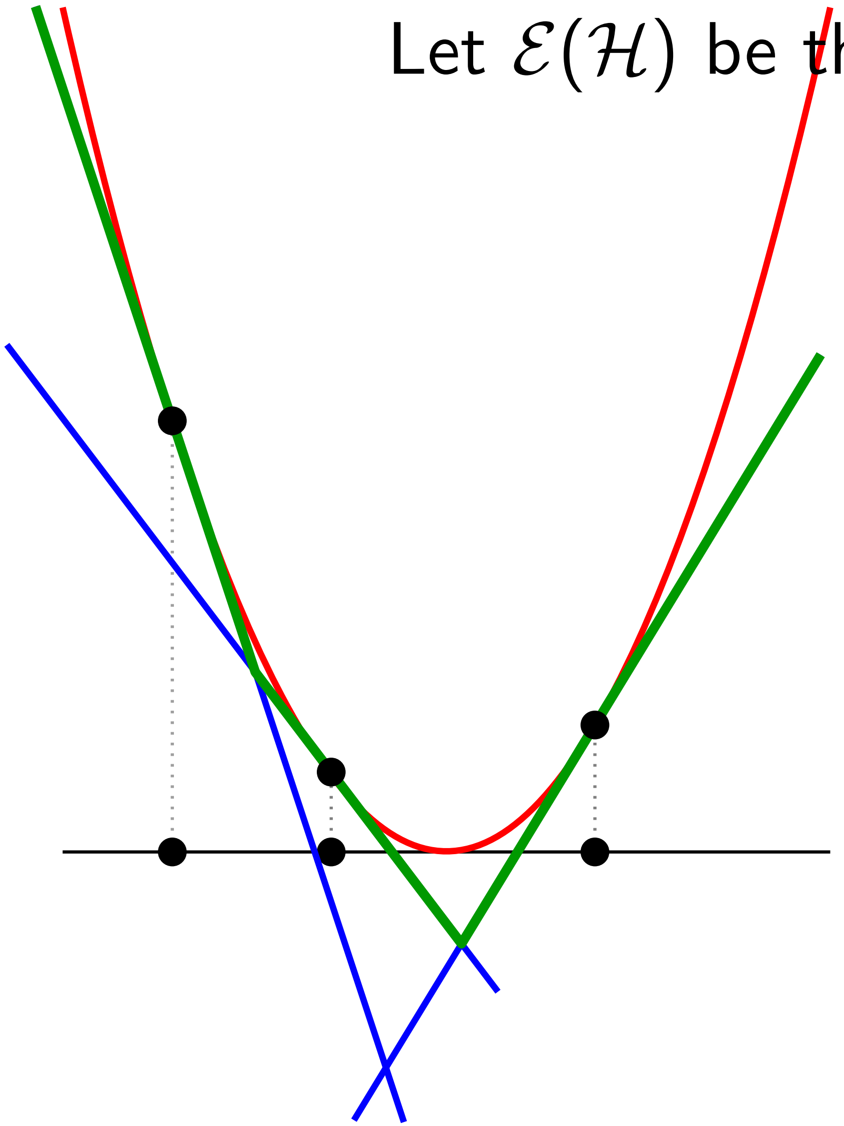

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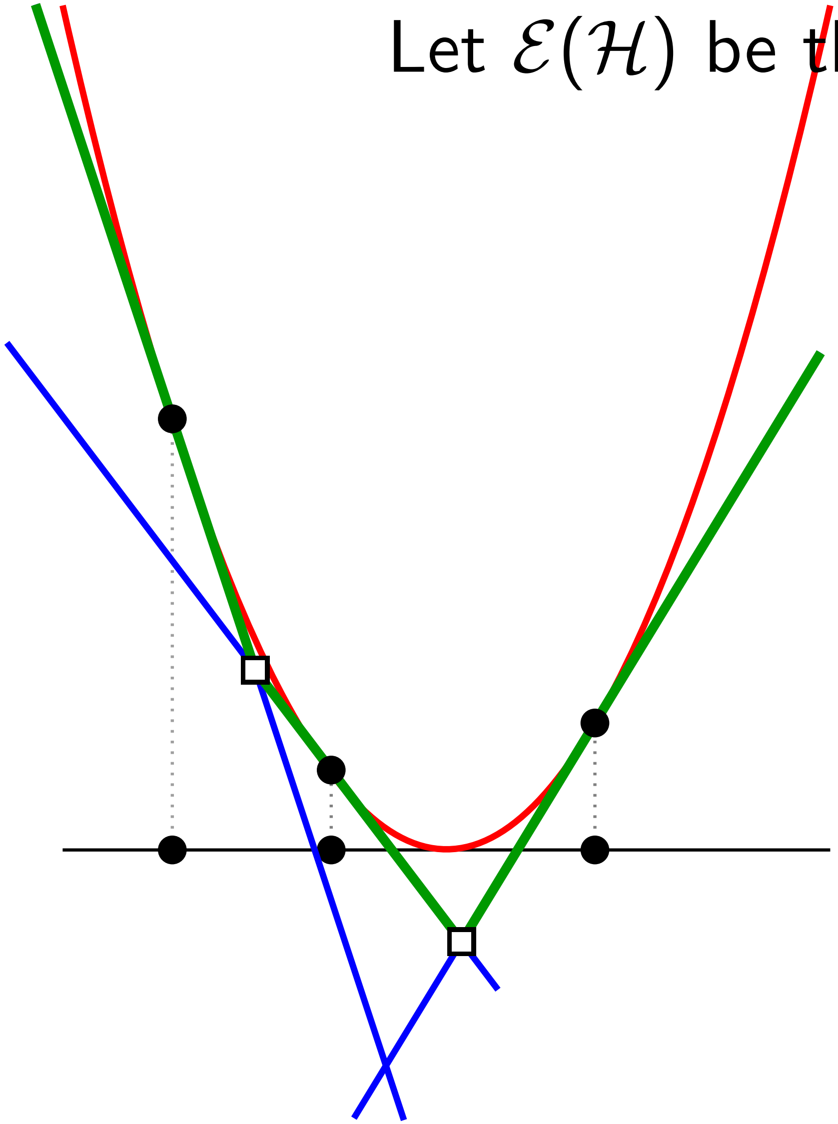

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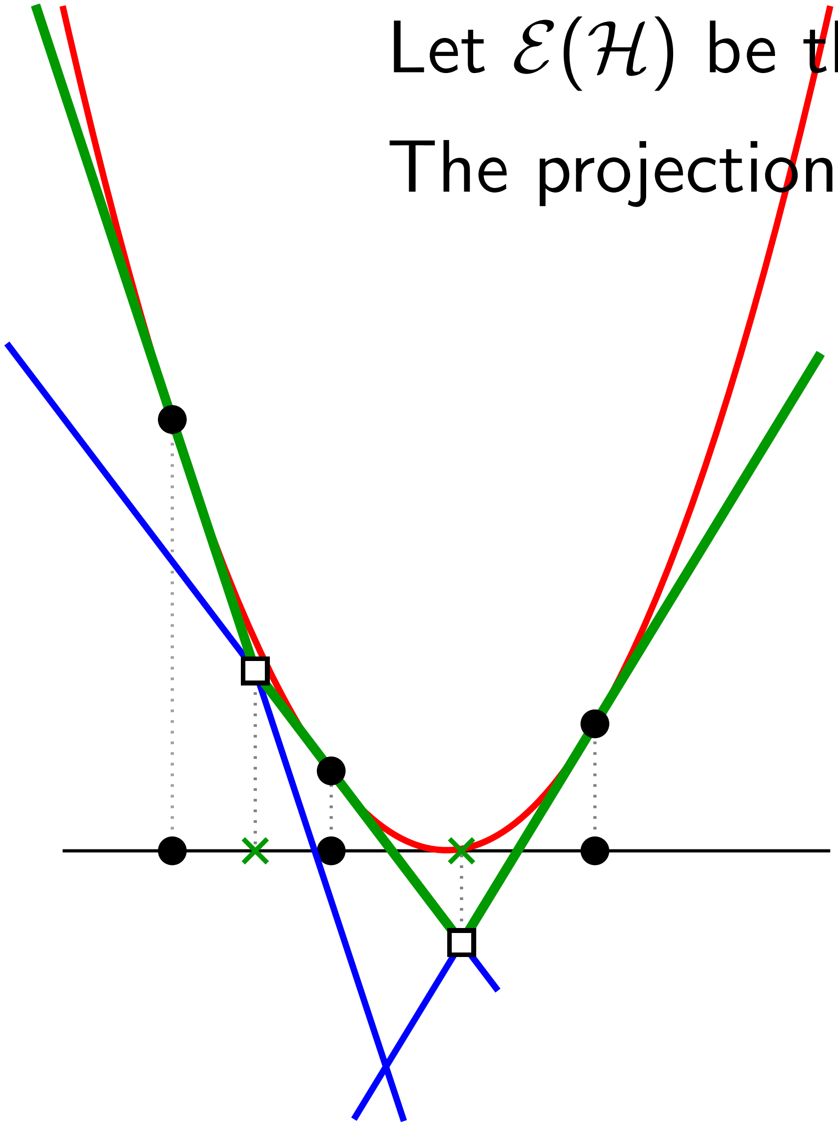

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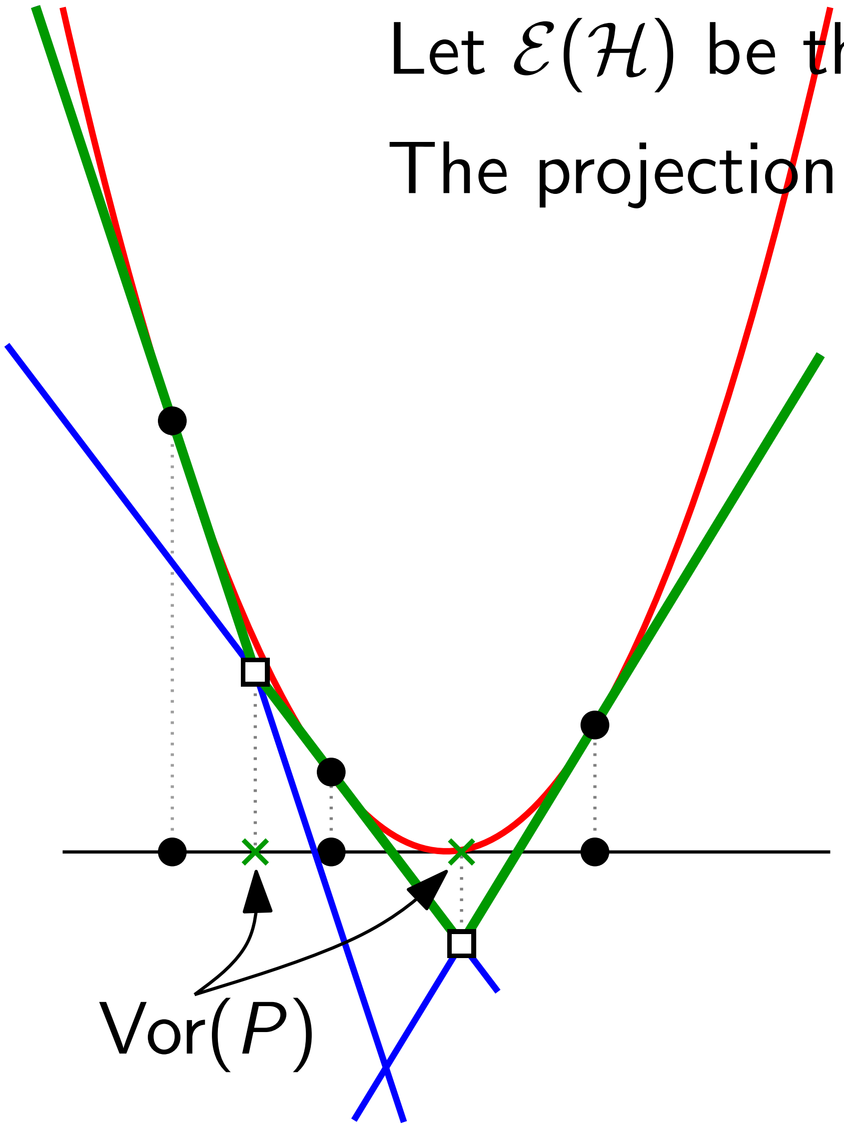

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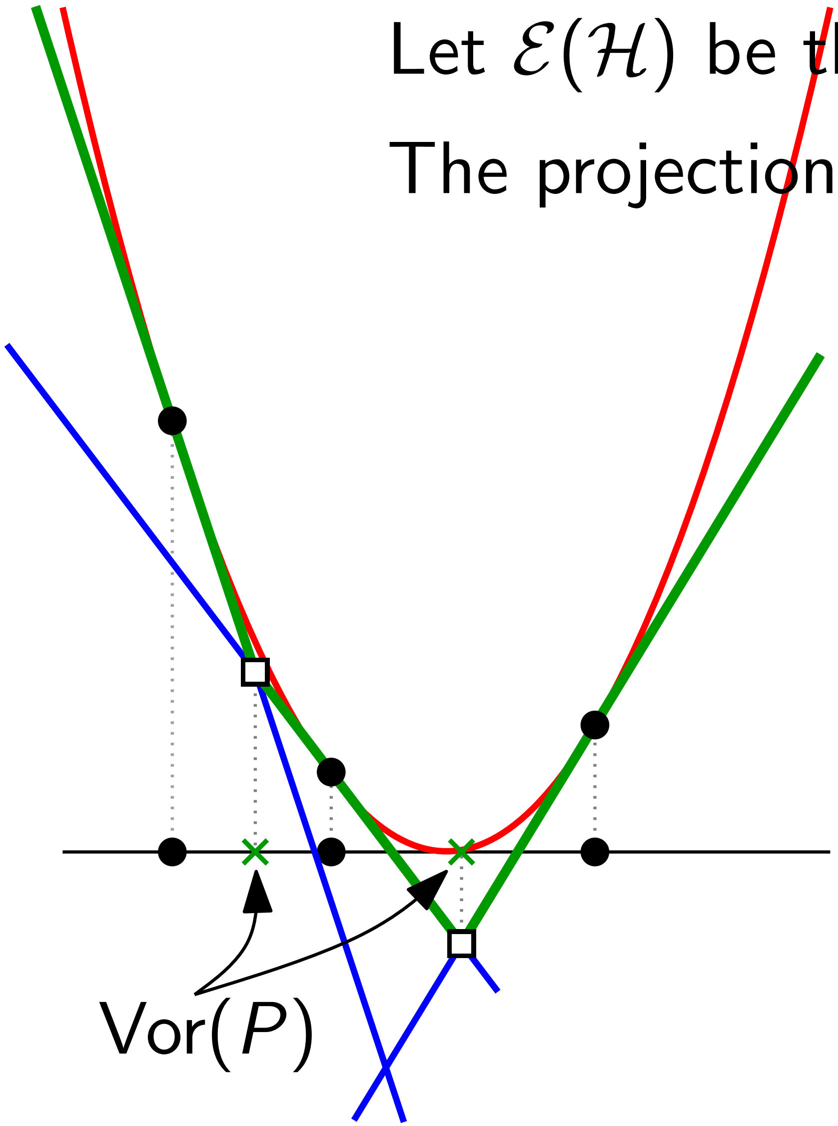

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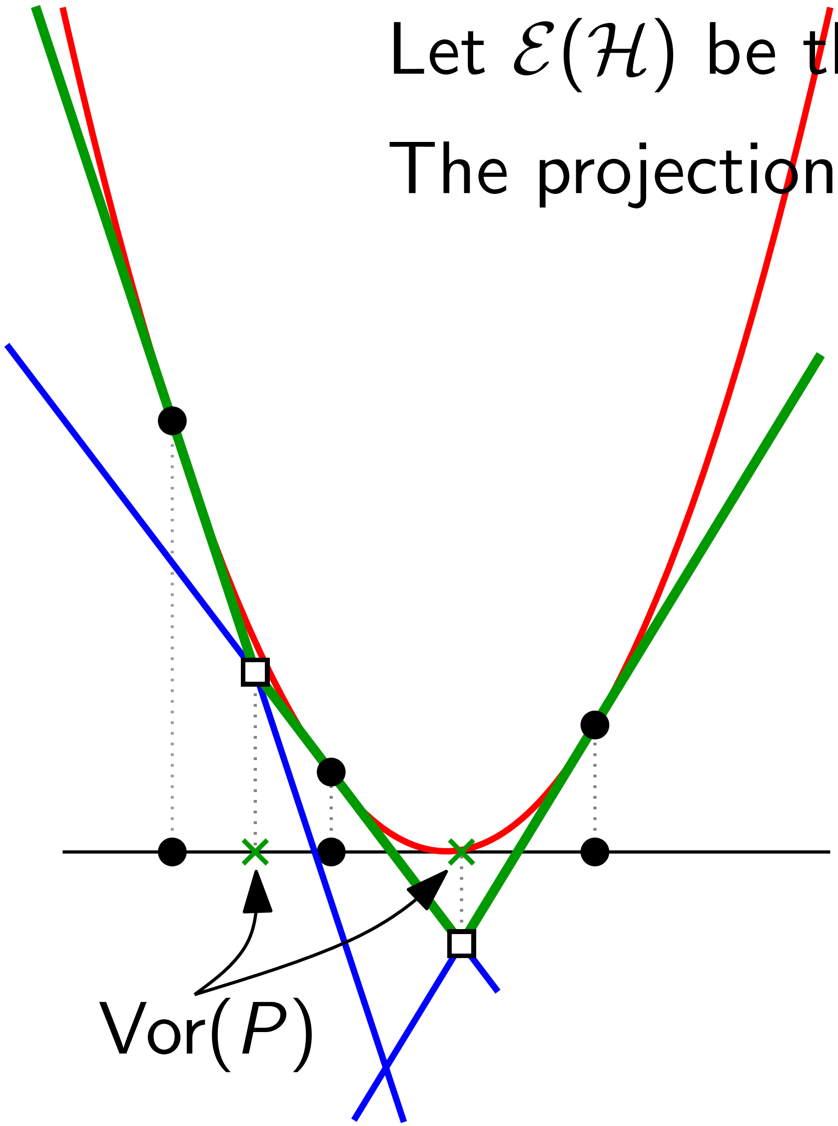

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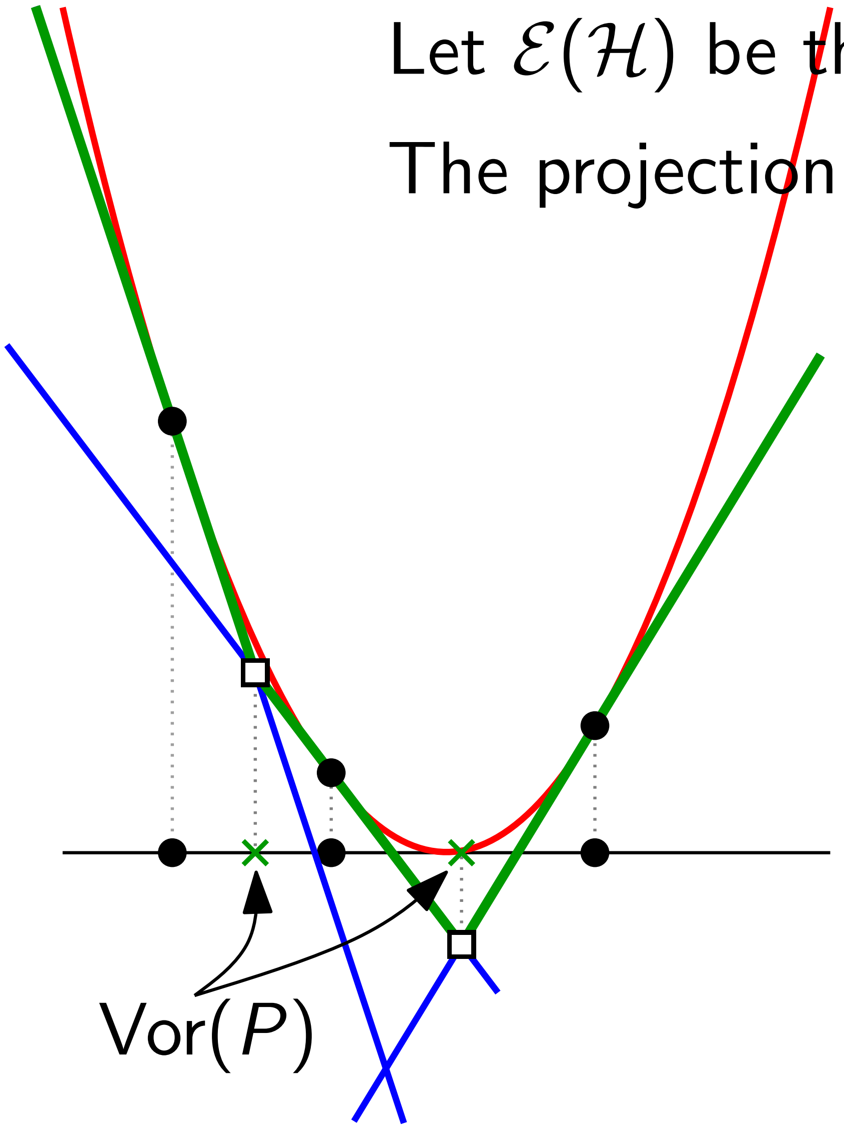

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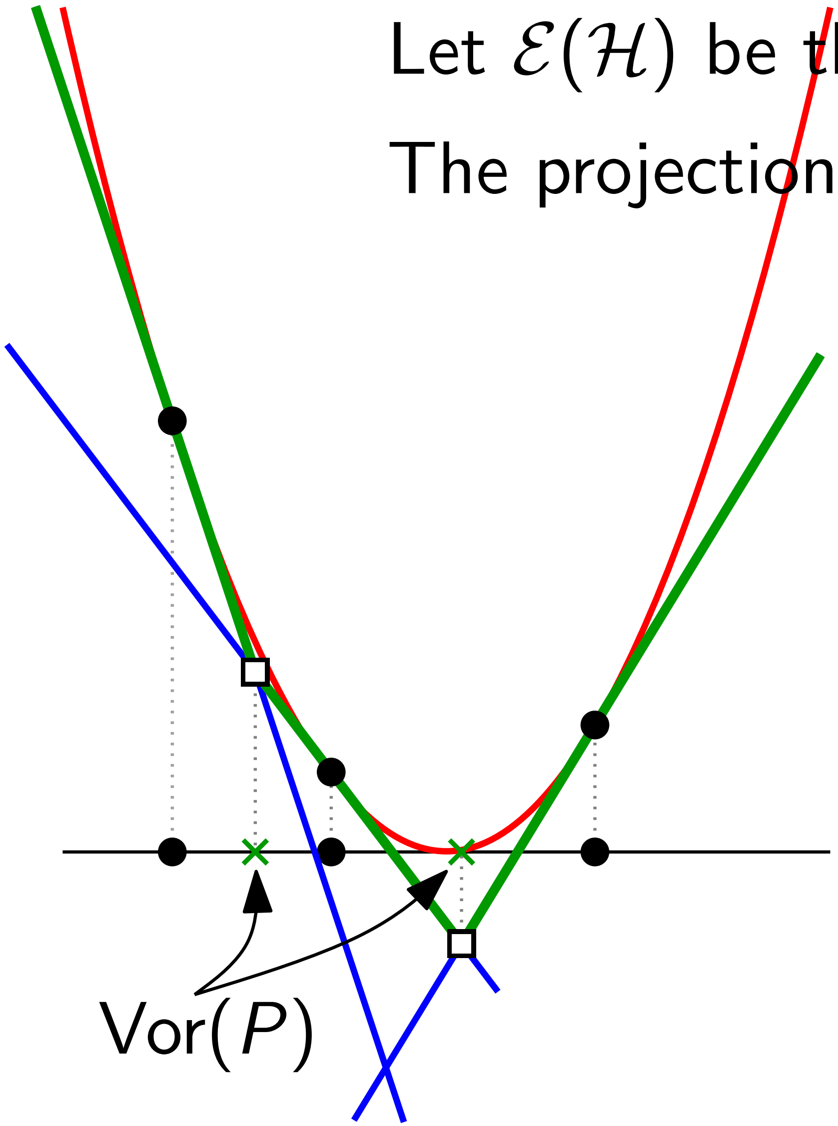

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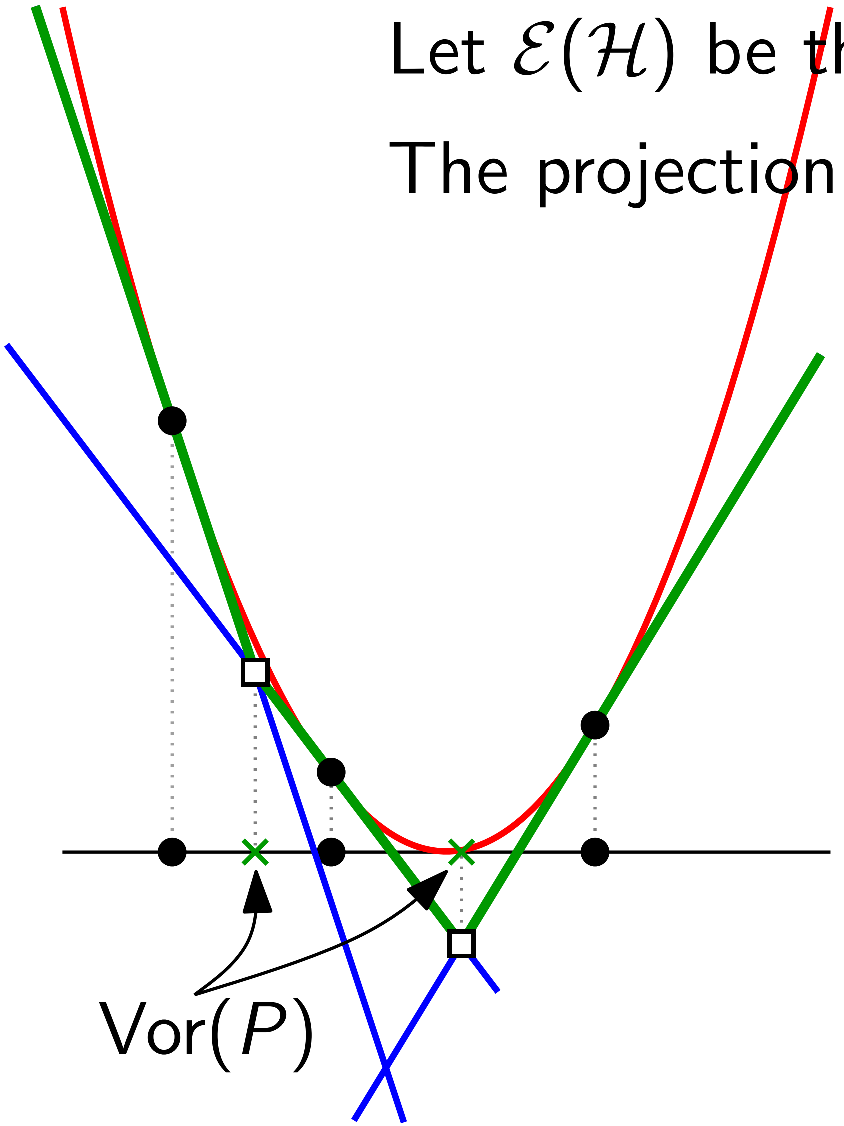

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use algorithm Rand3dConvexHull!