Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Computational Geometry

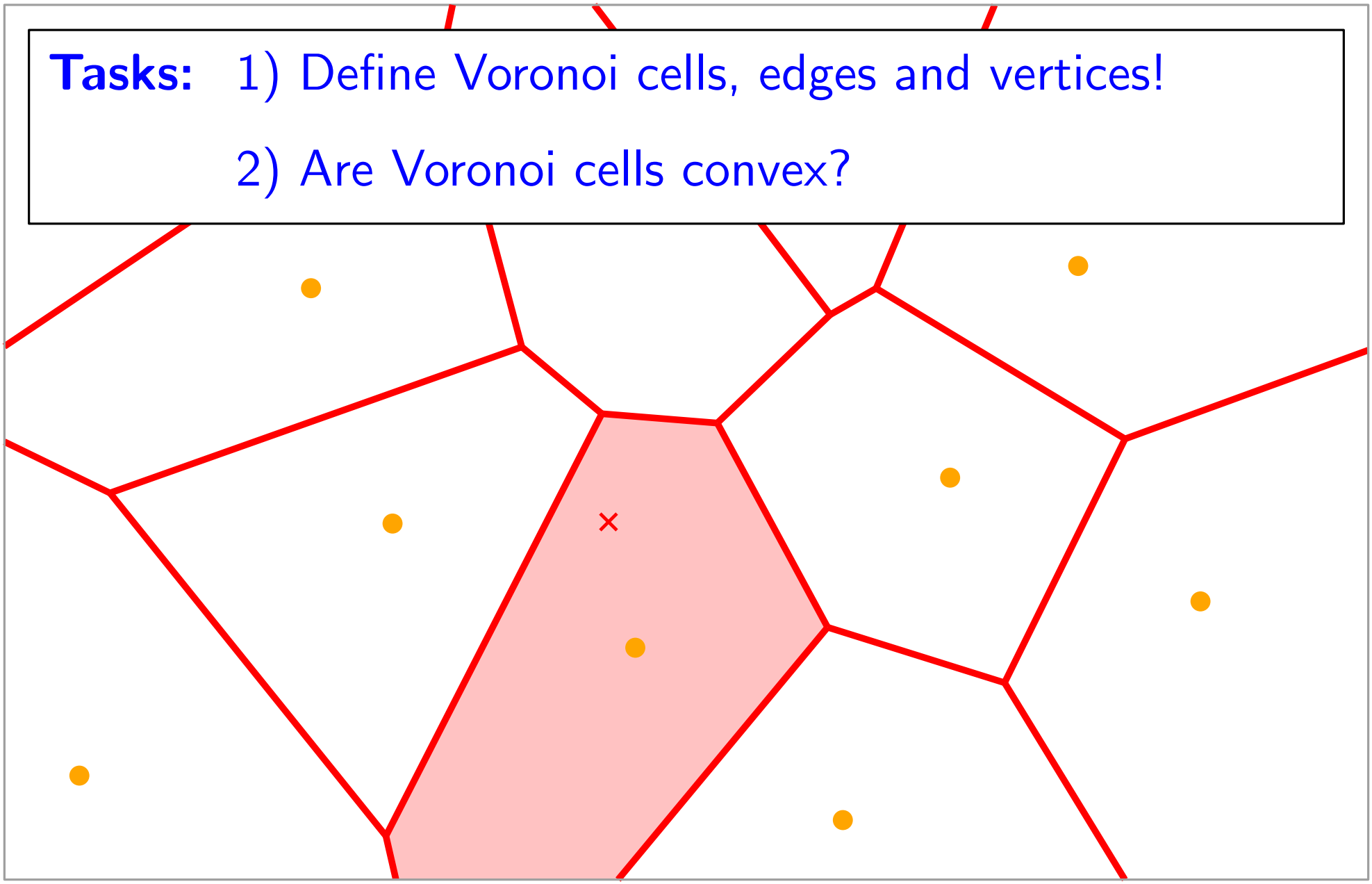
Winter term 2014/15

The Post-Office Problem

Lecture #7

Prof. Dr. Alexander Wolff *Chair for Computer Science I*

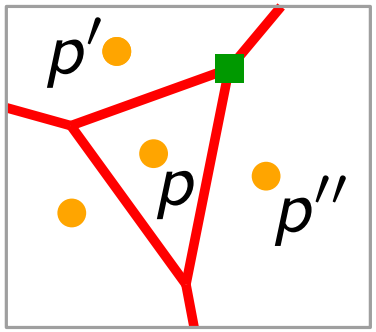
The Post-Office Problem



The Voronoi diagram

Let P be a set of points in the plane and let $p, p', p'' \in P$.

[Voronoi diagram]



$\mathrm{Vor}(P)$ → subdivision of $\mathbb{R}^2$

$\mathrm{Vor}(P)$ → geometric graph

[Voronoi cell]

$$\mathcal{V}(\{p\}) = \mathcal{V}(p) = \{x \in \mathbb{R}^2 : |xp| < |xq| \text{ for all } q \in P \setminus \{p\}\}$$
$$= \bigcap_{q \neq p} h(p, q)$$

[Voronoi edge]

$$\mathcal{V}(\{p, p'\}) = \{x : |xp| = |xp'| \text{ and } |xp| < |xq| \ \ \forall q \neq p, p'\}$$
$$= \text{rel-int}\big(\partial\mathcal{V}(p) \cap \partial\mathcal{V}(p')\big), \text{ i.e., w/o the endpts}$$

[Voronoi vertex]

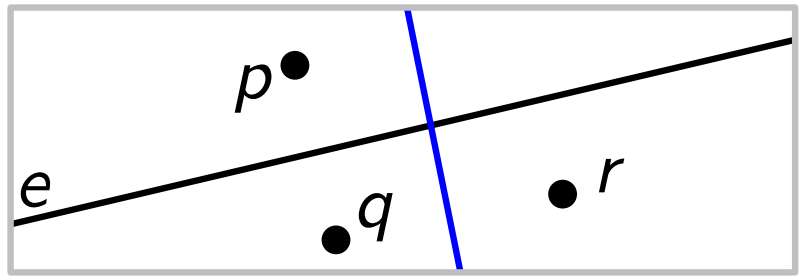
$$\mathcal{V}(\{p, p', p''\}) = \partial\mathcal{V}(p) \cap \partial\mathcal{V}(p') \cap \partial\mathcal{V}(p'')$$

Overall Shape of Vor(P)

Theorem. Let $P \subset \mathbb{R}^2$ be a set of n pts (called *sites*).
If all sites are collinear, Vor(P) consists of $n-1$ parallel lines. Otherwise, Vor(P) is connected and its edges are line segments or half-lines.

Proof. Assume that P is not collinear.

– Assume that Vor(P) contains an edge e that is a full line, say, $e = b(p, q)$.



Let $r \in P$ be not collinear with p and q.

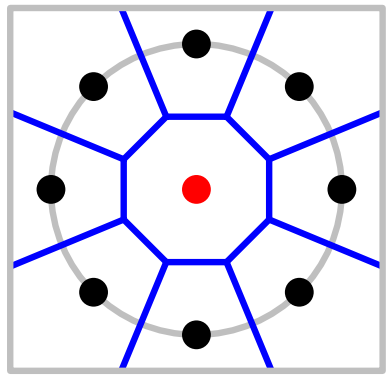
Then $b(q, r)$ is not parallel to e.

$\Rightarrow e \cap h(r, q)$ is closer to r than to p or q.

$\Rightarrow e$ is bounded on at least one side. □

Complexity

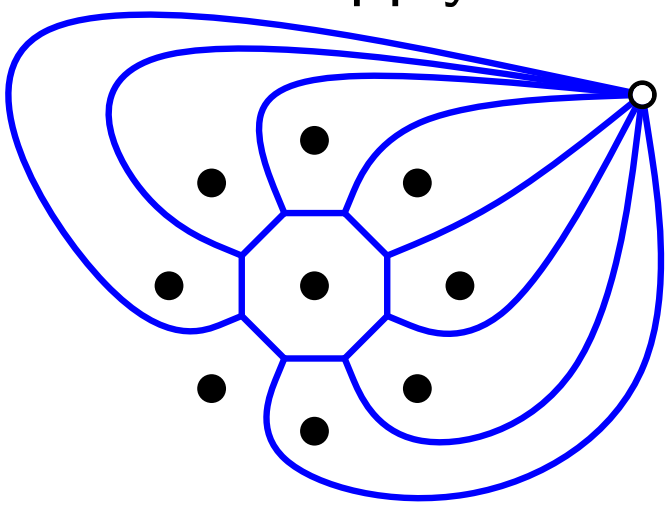
Task: Construct a set P of point sites such that $\mathrm{Vor}(P)$ has a cell of linear complexity!



Theorem. Given a set $P \subset \mathbb{R}^2$ of n sites, $\mathrm{Vor}(P)$ consists of at most $2n - 5$ vertices and $3n - 6$ edges.

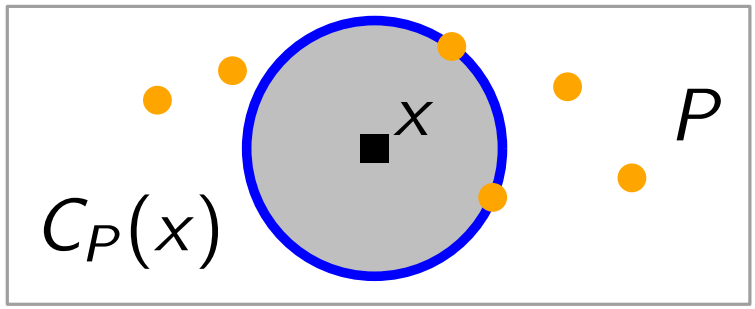
Proof. *Problem:* unbounded edges!

$\Rightarrow$ can't apply Euler directly, but...



Characterization of Voronoi vtc and edges

$C_P(x) :=$ largest circle centered at x w/o sites in its interior



Theorem: (i) x Voronoi vtx $\Leftrightarrow$ $|C_P(x) \cap P| \geq 3$

(ii) $b(p, p')$ contains a Voronoi edge $\Leftrightarrow$
$\exists x \in b(p, p') \colon C_P(x) \cap P = \{p, p'\}$

Computation

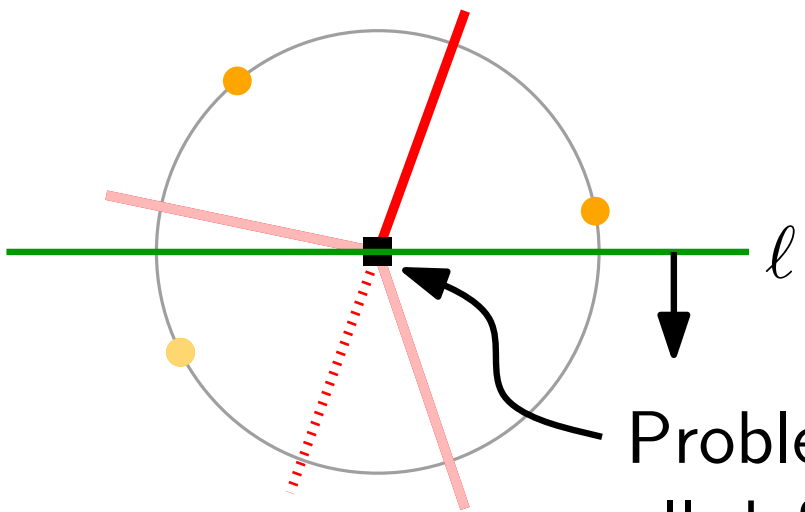
Brute force: For each $p \in P$, compute $\mathcal{V}(p) = \underbrace{\bigcap_{p' \neq p} h(p, p')}_{O(n \log n) \text{ time}}$.

[Chapter 4]

in total: $O(n^2 \log n)$ time

– but the complexity of $\mathrm{Vor}(P)$ is *linear!*

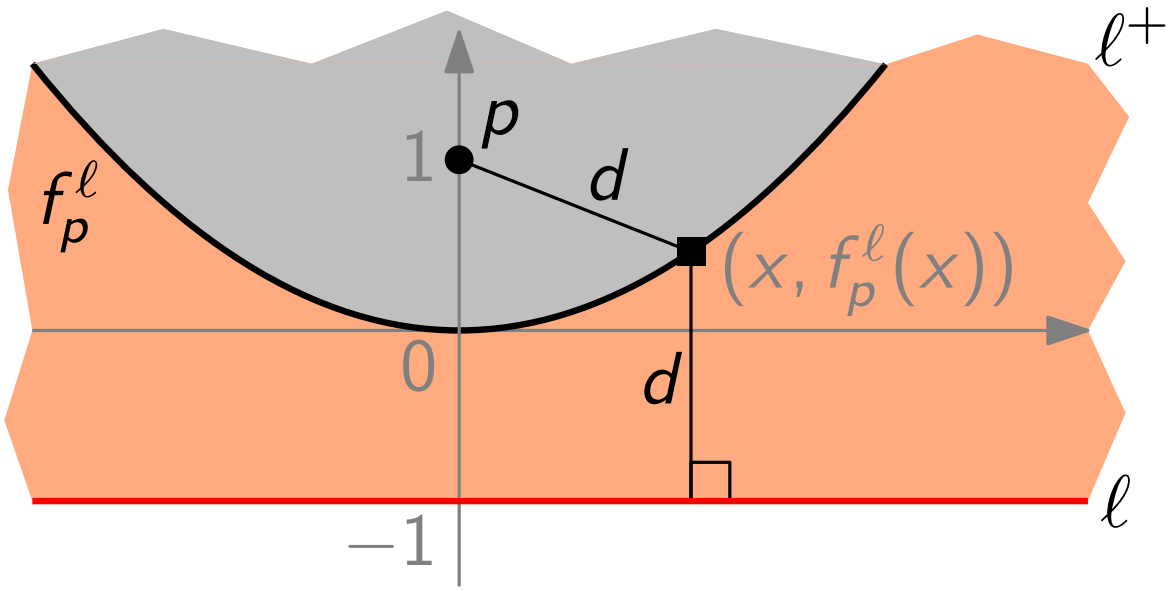
Sweep?



Problem: We don't know all defining sites yet :(

Sweep?

Which part of the plane above ℓ is fixed by what we've seen?



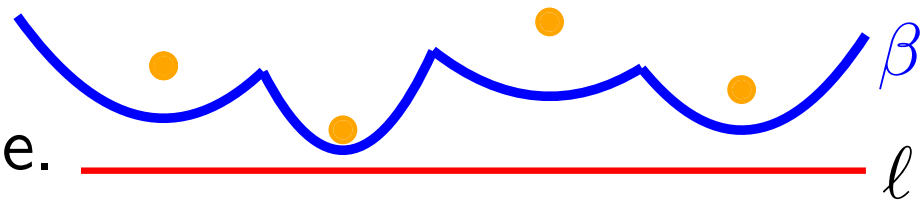
Solution:

f_p^ℓ is the parabola with focus p and directrix ℓ.

Task: Compute f_p^ℓ for $p = (0, 1)$ and $\ell\colon y = -1$!

Definition. *beachline* $\beta \equiv$ lower envelope of $(f_p^\ell)_{p \in P \cap \ell^+}$

Observation. β is x-monotone.



The beachline β

Question: What does β have to do with Vor(P)?

Answer: "Breakpoints" of β trace out the Voronoi edges!

Lemma. New arcs appear only through *site events* on β, that is, whenever ℓ hits a new site.

Corollary. β consists of at most $2n - 1$ arcs.

Definition. *Circle event:* ℓ reaches lowest pt of a circle through three sites above ℓ whose arcs are consecutive on β.

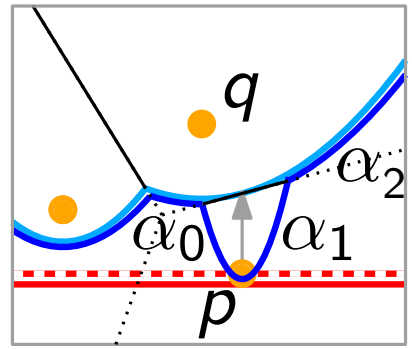
Lemma. Arcs disappear from β only at circle events.

Lemma. The Voronoi vtc correspond 1:1 to circle events.

Fortune's Sweep

```
VoronoiDiagram(P ⊂ ℝ²)
 Q ← new PriorityQueue(P)    // site events sorted acc. y-coord.
 T ← new BalancedBinarySearchTree()    // sweep status (β)
 D ← new DCEL()    // to-be Vor(P)
 while not Q.empty() do
     p ← Q.ExtractMax()
     if p site event then
         HandleSiteEvent(p)
     else
         α ← arc on β that will disappear
         HandleCircleEvent(α)
 treat remaining internal nodes of T (≡ unbnd. edges of Vor(P))
 return D
```

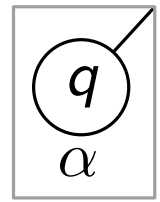
Handling Events

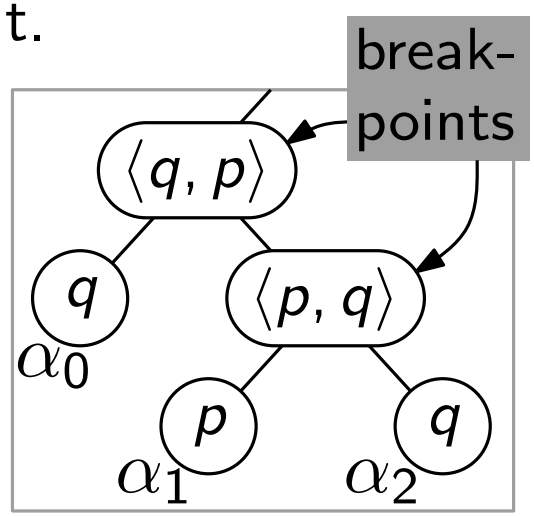


HandleSiteEvent(point p)

- Search in $\mathcal{T}$ for the arc α vertically above p.
  If α has pointer to circle event in $\mathcal{Q}$, delete this event.
- Split α into α_0 and α_2.
  Let α_1 be the new arc of p.

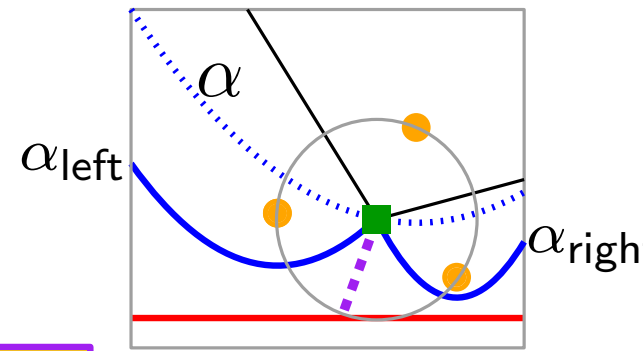
In $\mathcal{T}$:





- Add Vor-edges $\langle q, p \rangle$ and $\langle p, q \rangle$ to DCEL.
- Check $\langle \cdot, \alpha_0, \alpha_1 \rangle$ and $\langle \alpha_1, \alpha_2, \cdot \rangle$ for circle events.

HandleCircleEvent(arc α)



- $\mathcal{T}$.delete(α); update breakpts
- Delete all circle events involving α from $\mathcal{Q}$.
- Add Vor-vtx $\alpha_{\text{left}} \cap \alpha_{\text{right}}$ and Vor-edge $\langle \alpha_{\text{left}}, \alpha_{\text{right}} \rangle$ to DCEL.
- Check $\langle \cdot, \alpha_{\text{left}}, \alpha_{\text{right}} \rangle$ and $\langle \alpha_{\text{left}}, \alpha_{\text{right}}, \cdot \rangle$ for circle events.

Running time? $O(\log n)$ per event...

Running Time?

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  treat remaining internal nodes of T (≡ unbnd. edges of Vor(P))
  return D
```

Summary

Theorem. Given a set P of n pts in the plane, Fortune's sweep computes $\mathrm{Vor}(P)$ in $O(n \log n)$ time and $O(n)$ space.

Steven Fortune
Bell Labs

Steven Fortune. A sweepline algorithm for Voronoi diagrams. *Proc. 2nd Annual ACM Symposium on Computational Geometry.* Yorktown Heights, NY, pp. 313–322. 1986.