Julius-Maximilians-
UNIVERSITÄT WÜRZBURG
Chair for INFORMATICS I
Efficient Algorithms and Knowledge-Based Systems
Institute for Informatics

Computational Geometry

Winter term 2014/15

Point Localization

or

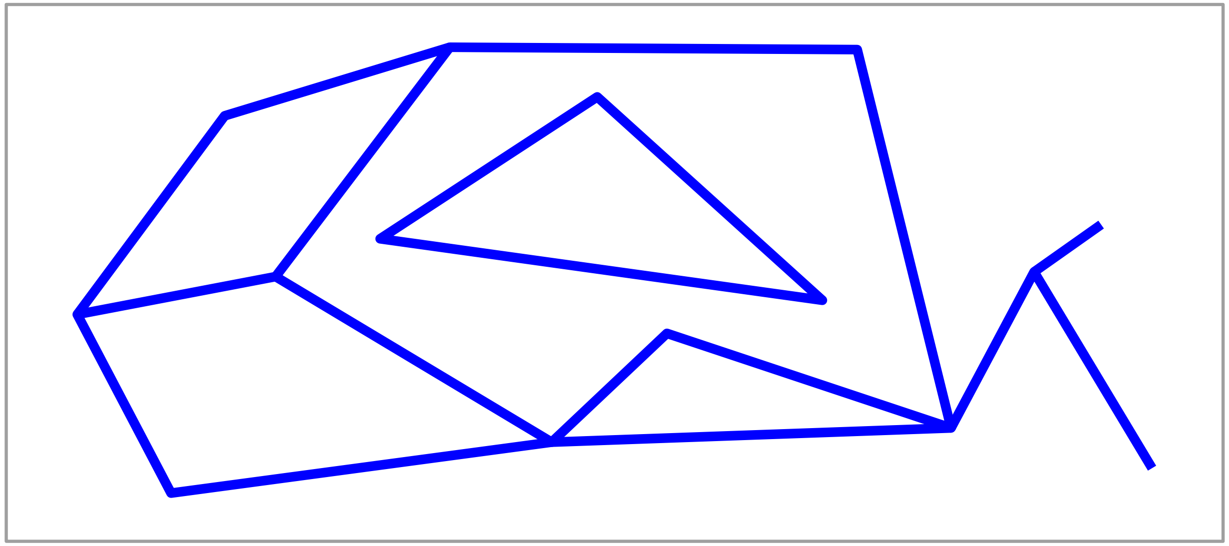
Where am I?

Lecture #6

Prof. Dr. Alexander Wolff

Chair for Computer Science I

What's the Problem?



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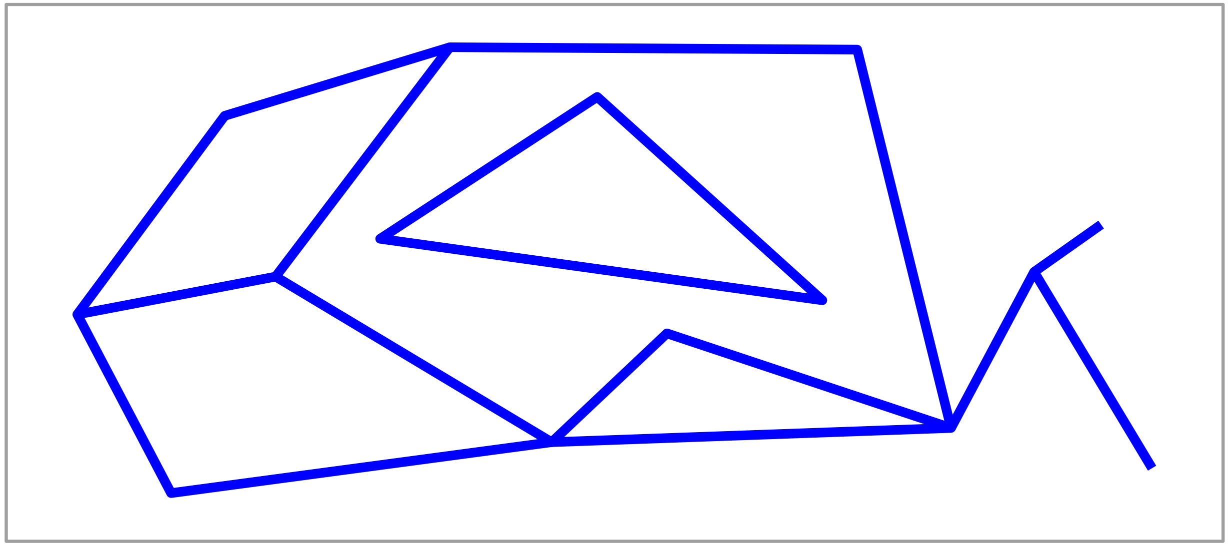

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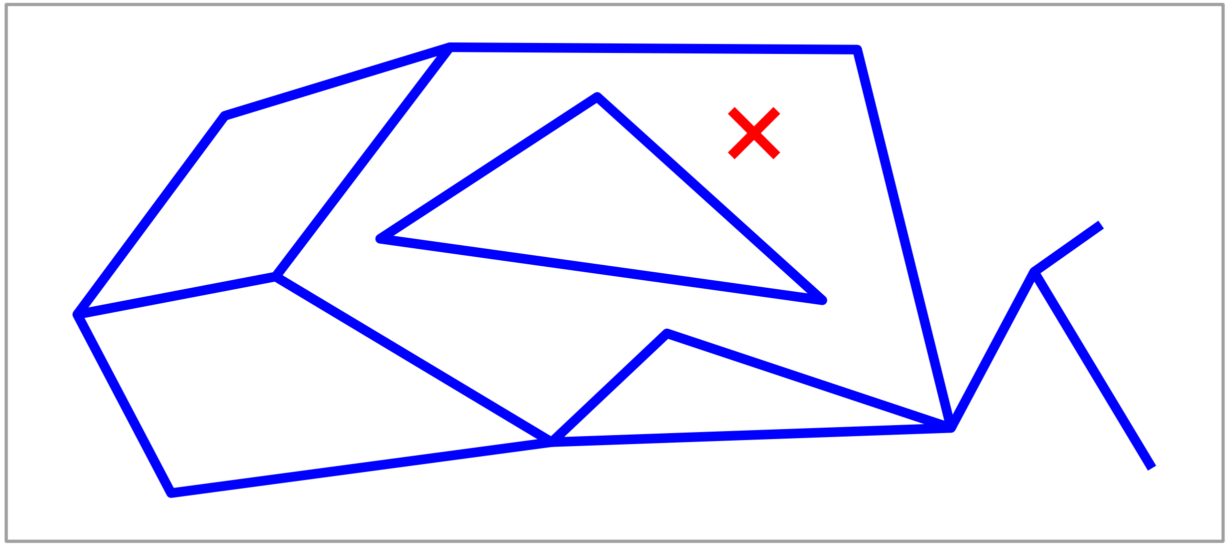

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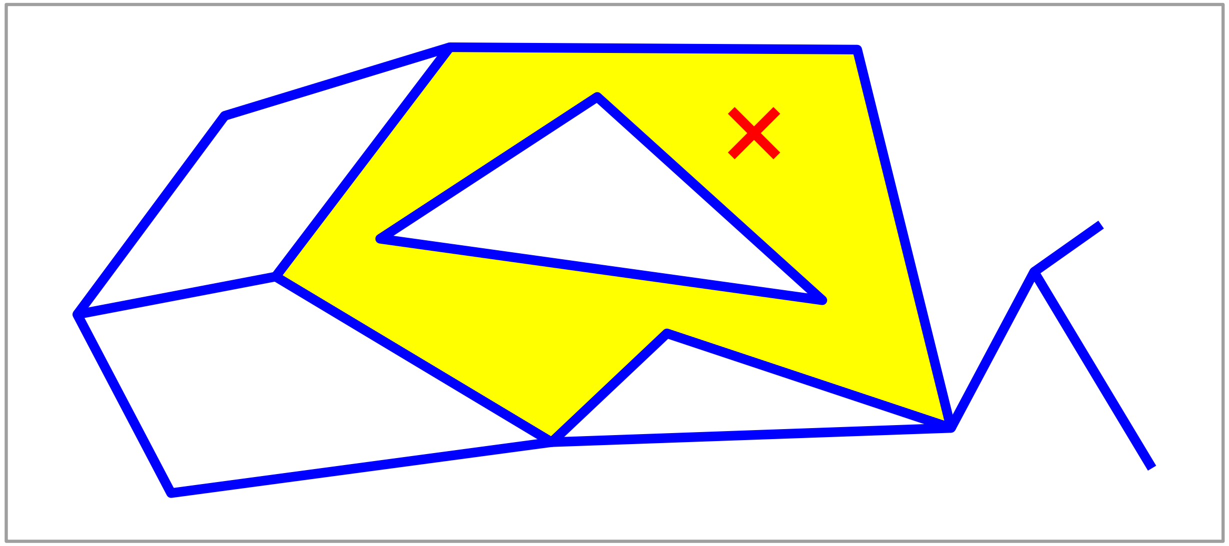

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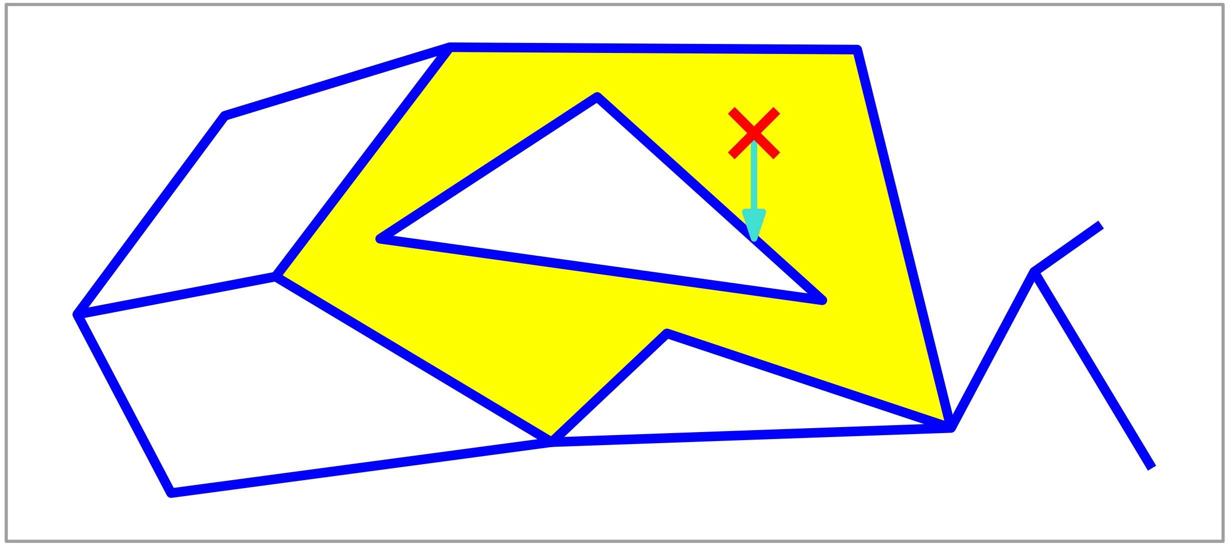

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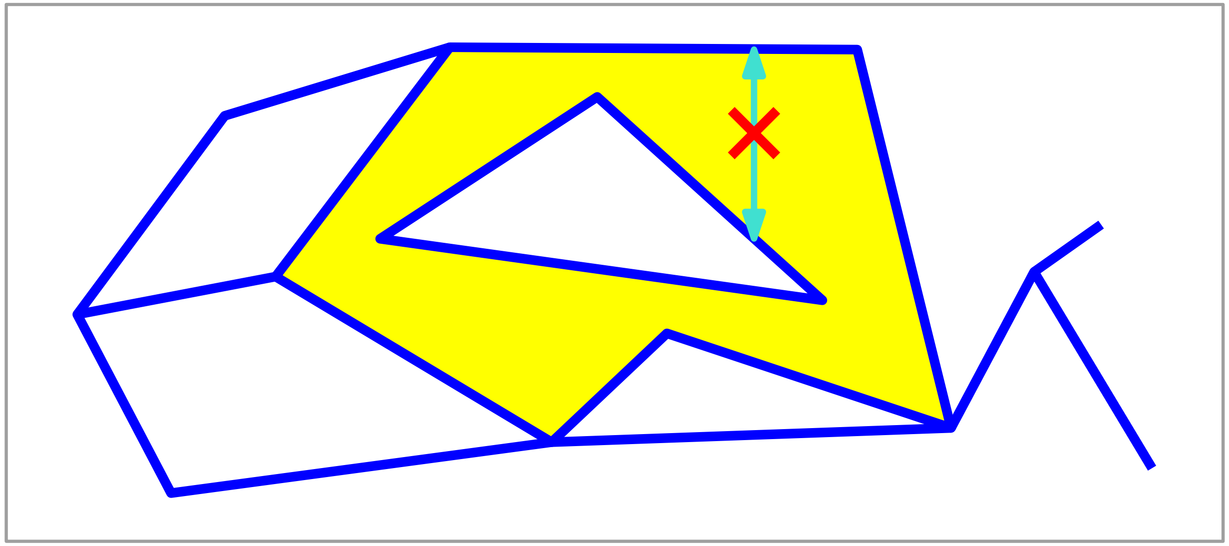

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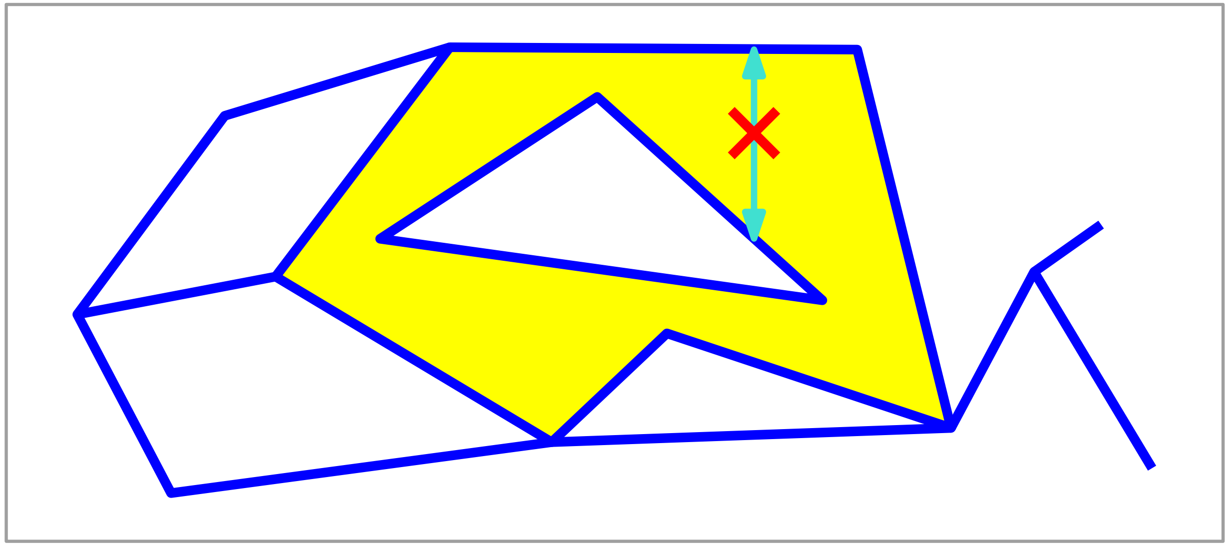

[2 min]

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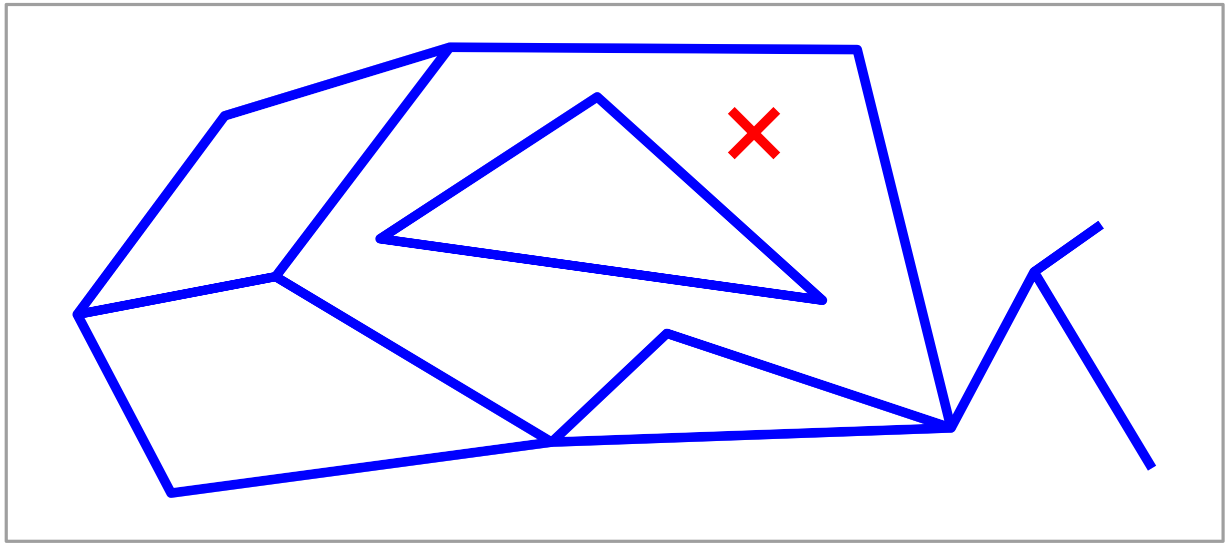

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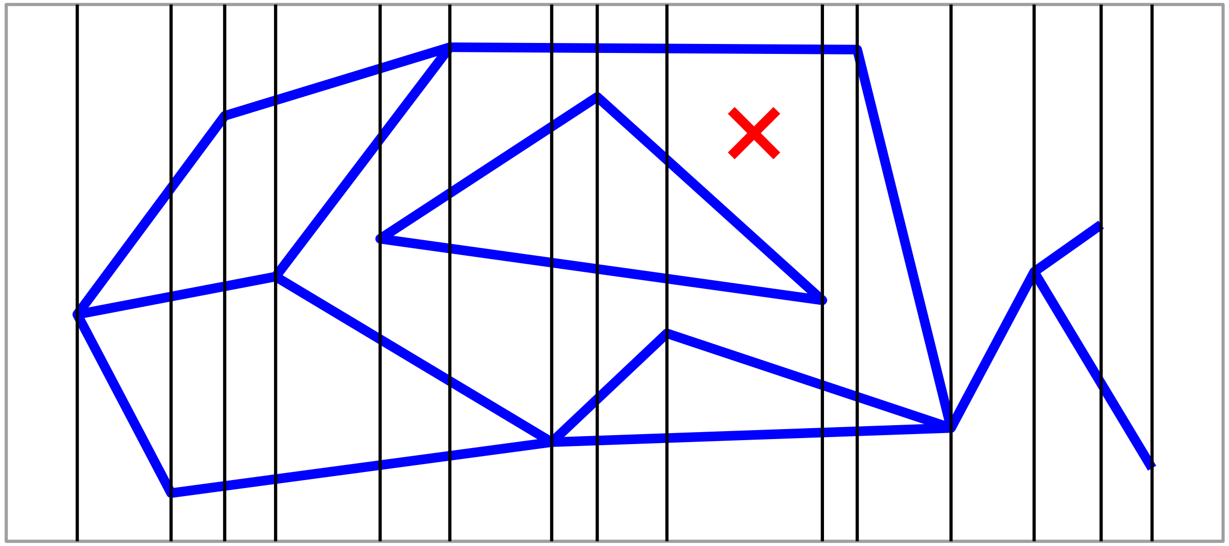

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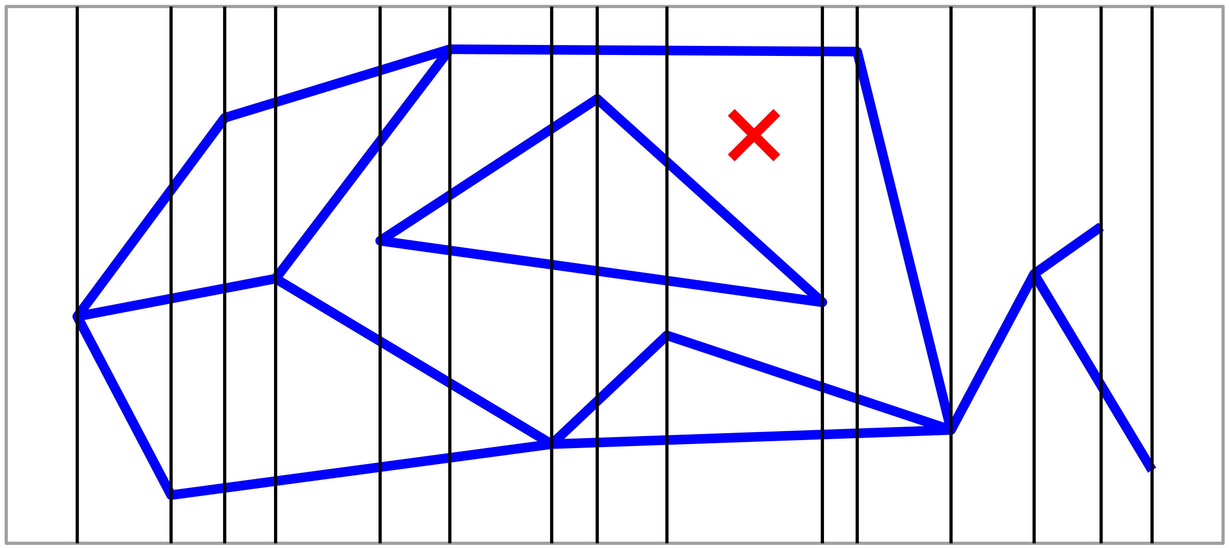

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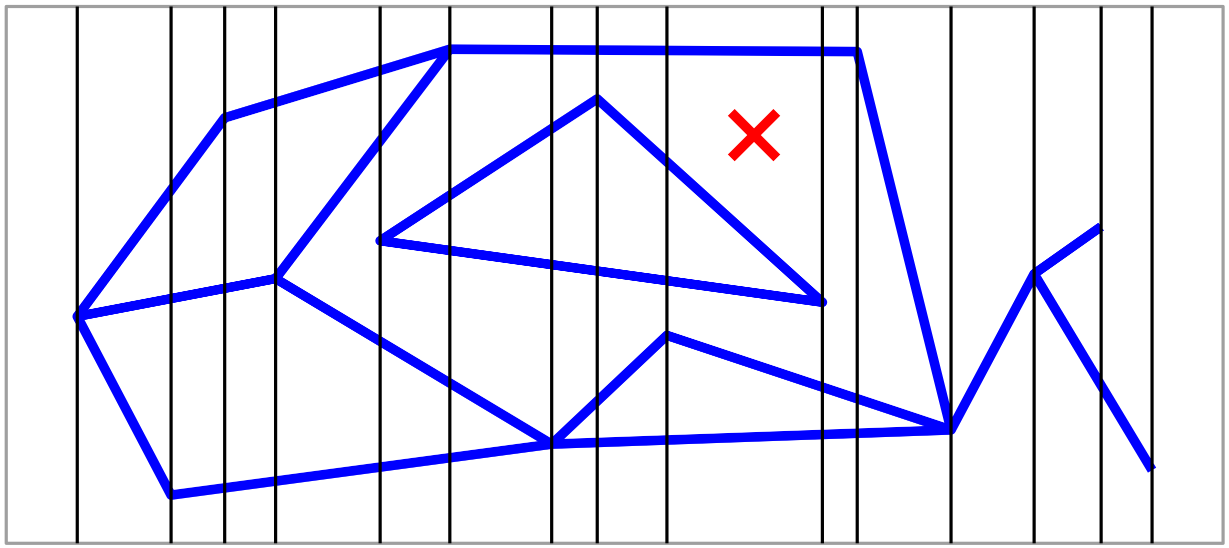

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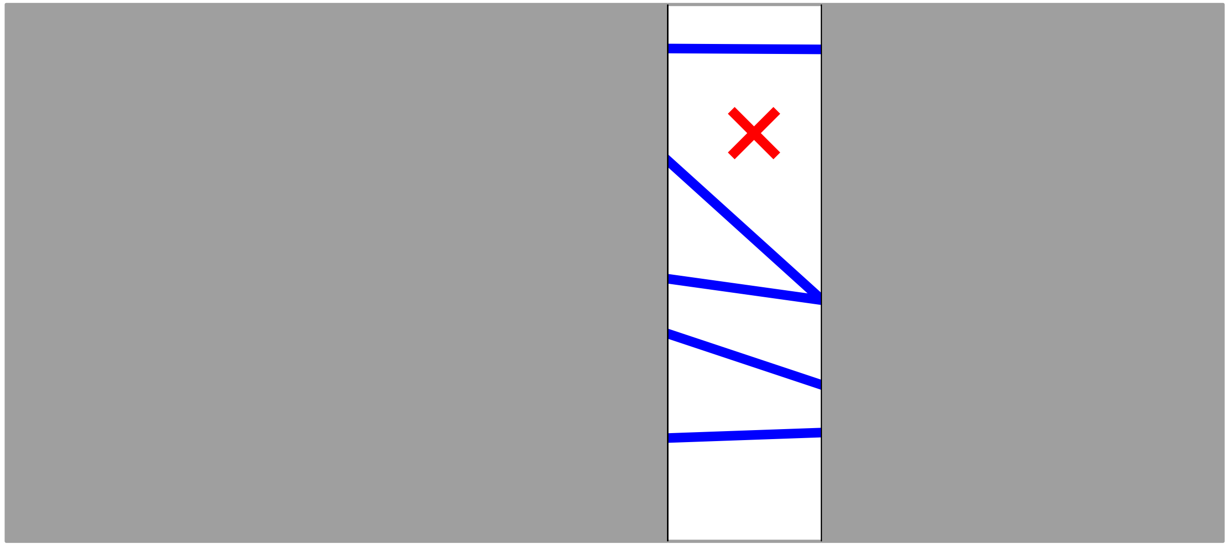

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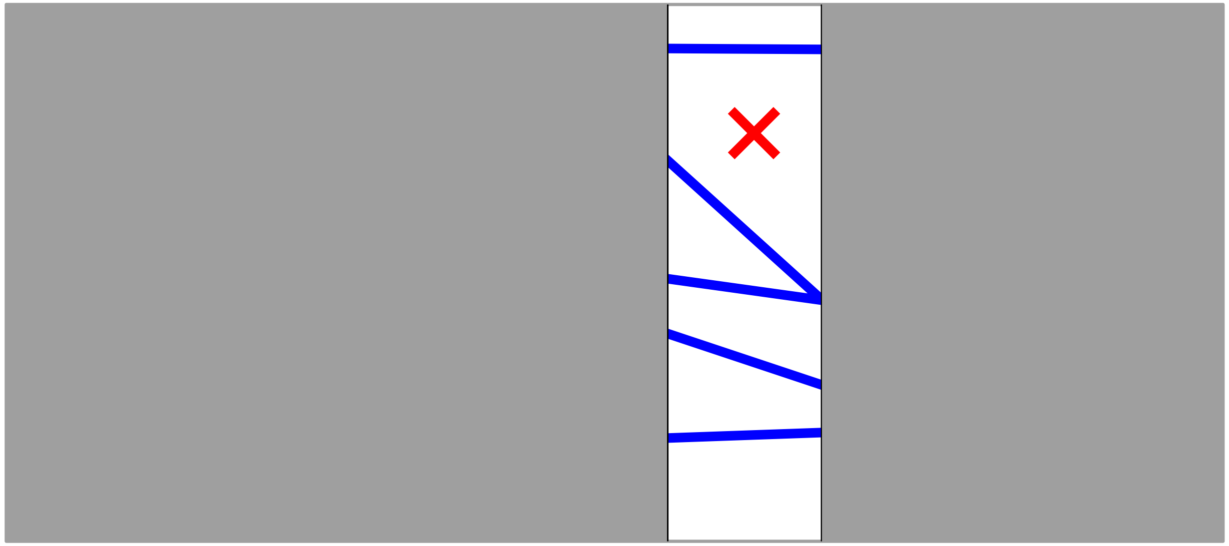

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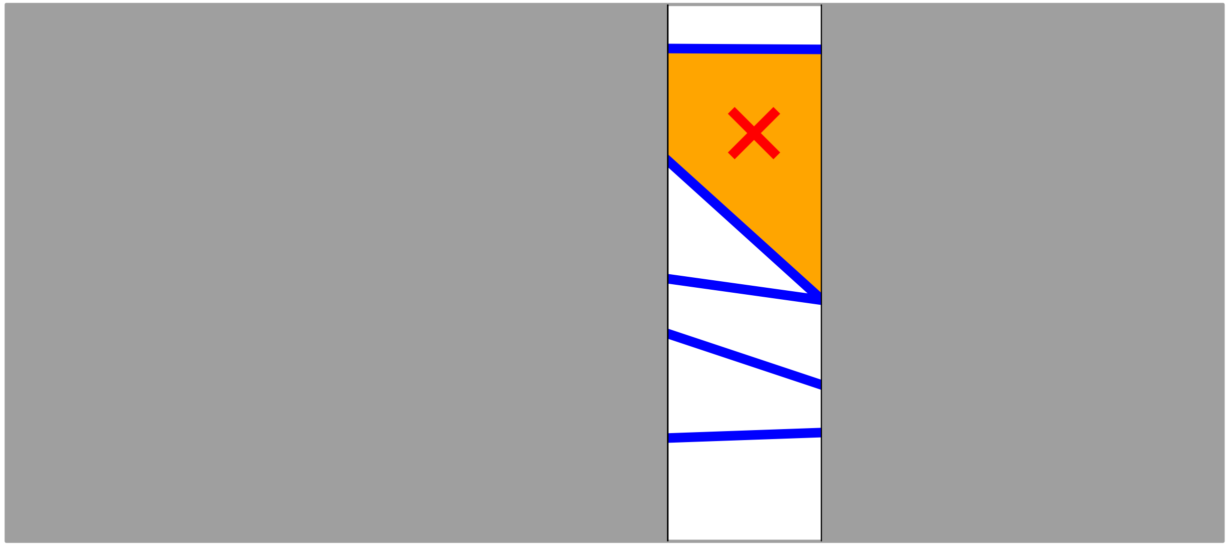

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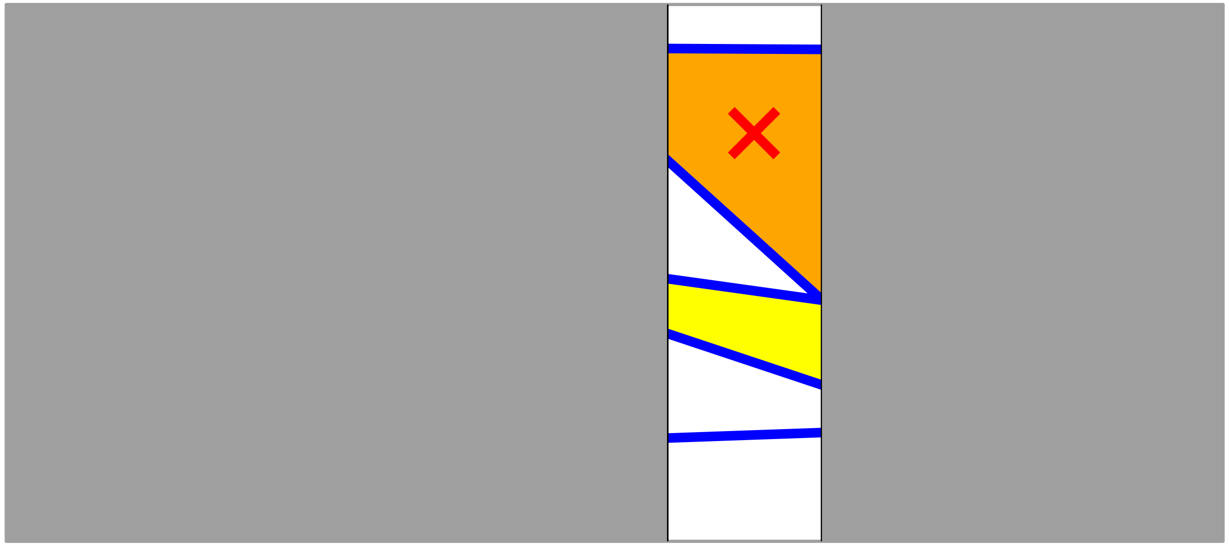

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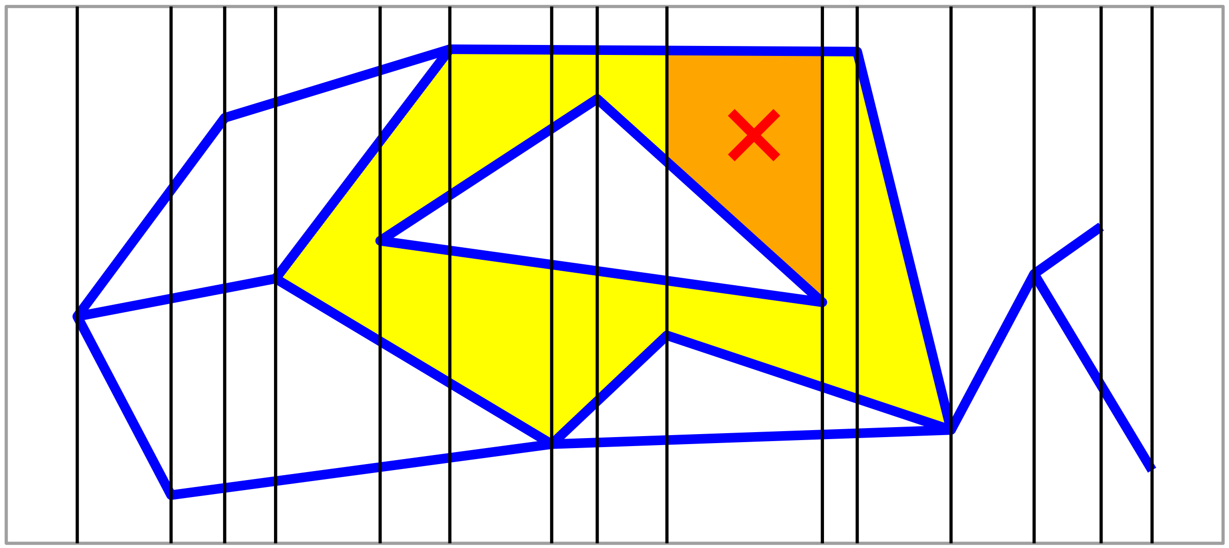

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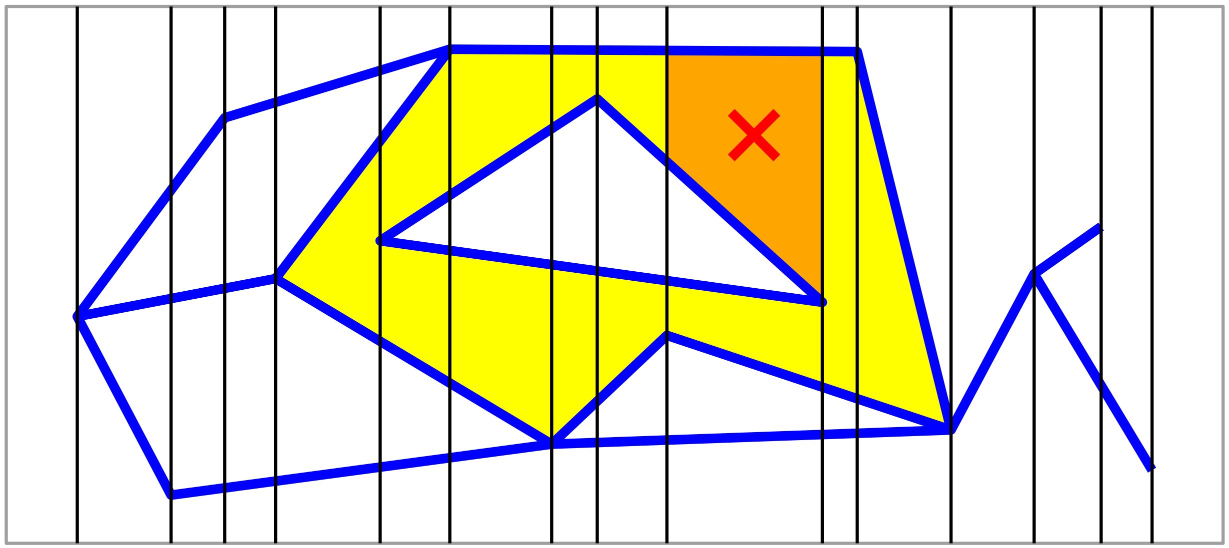

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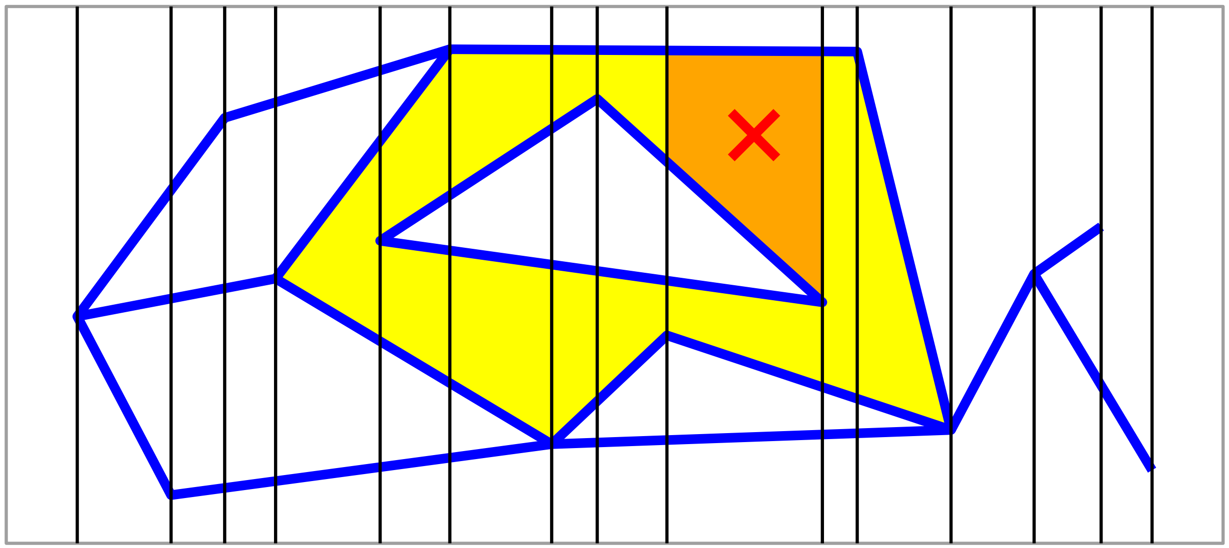

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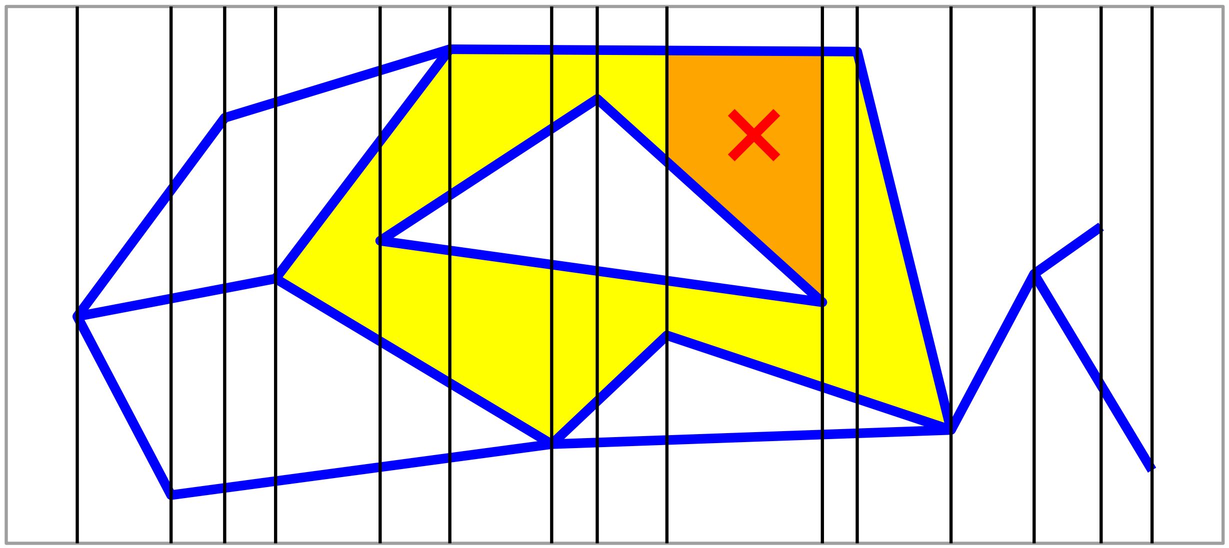

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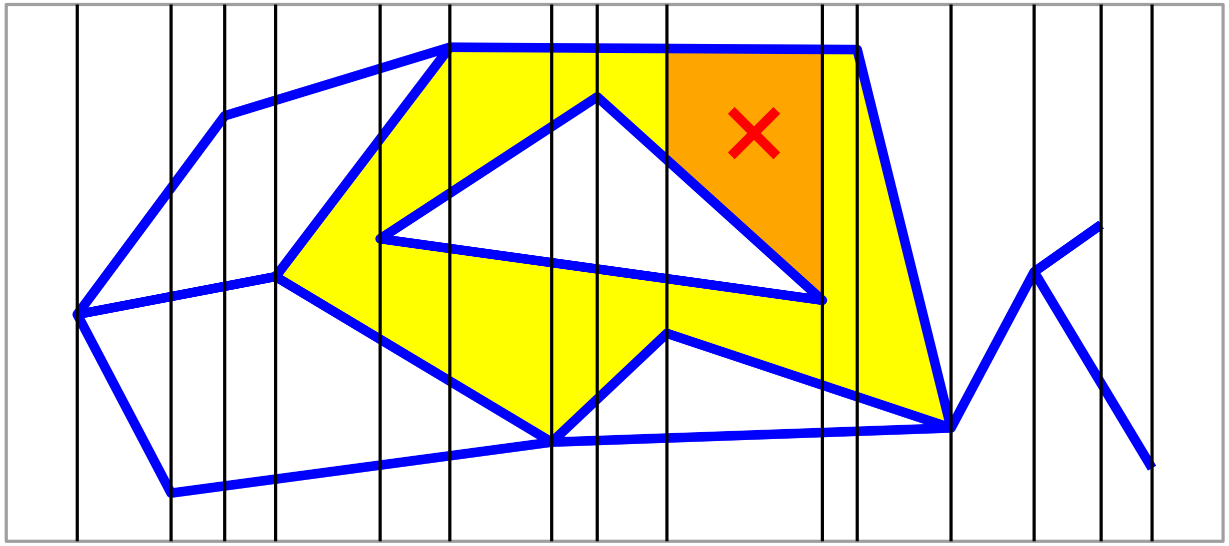

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But: Space? $\Theta(n^2)$ **Task:** Give lower-bound example!

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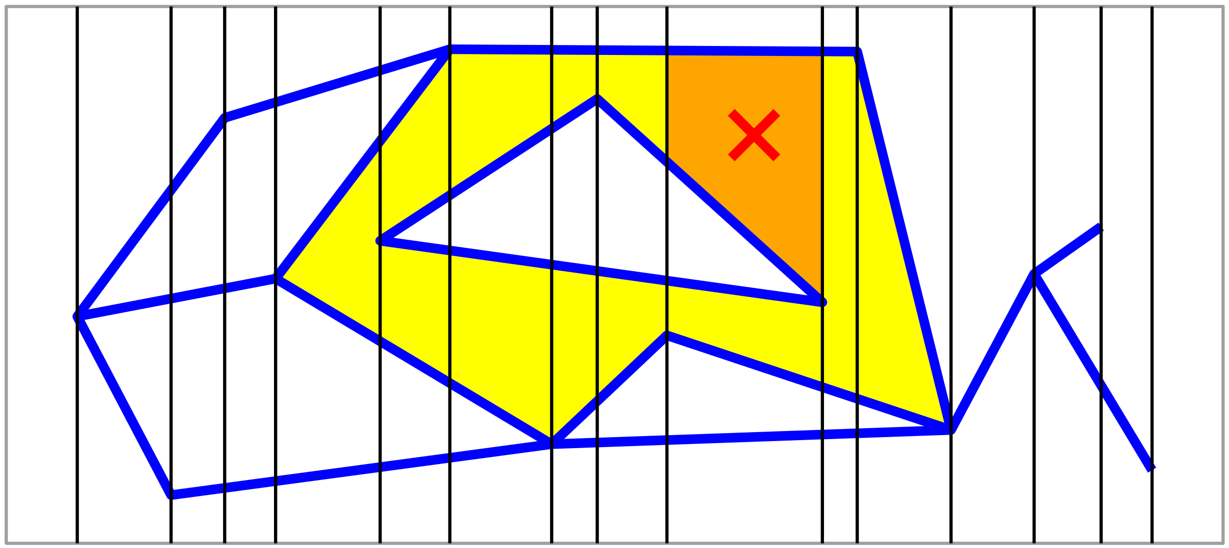

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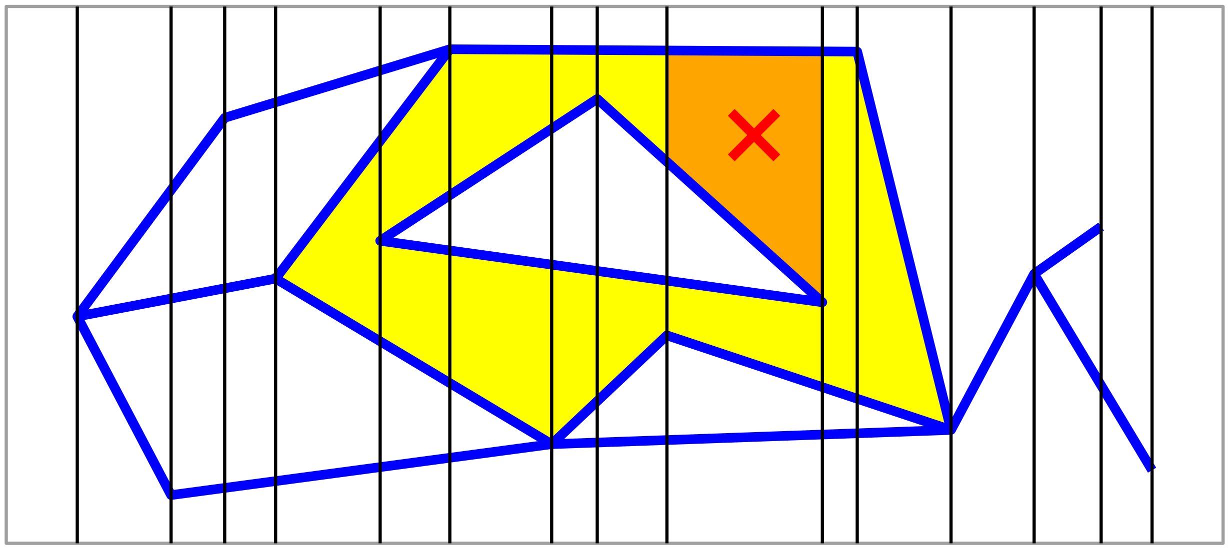

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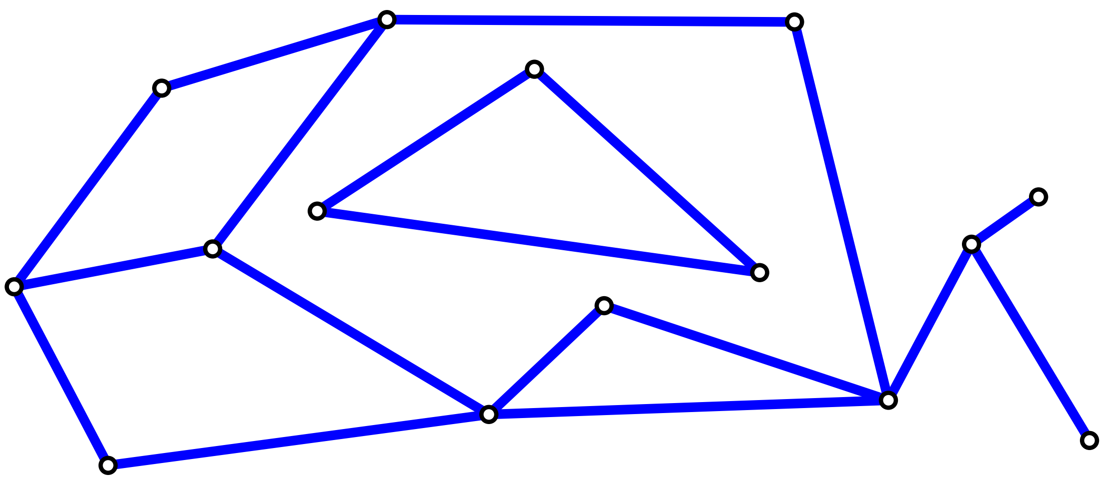

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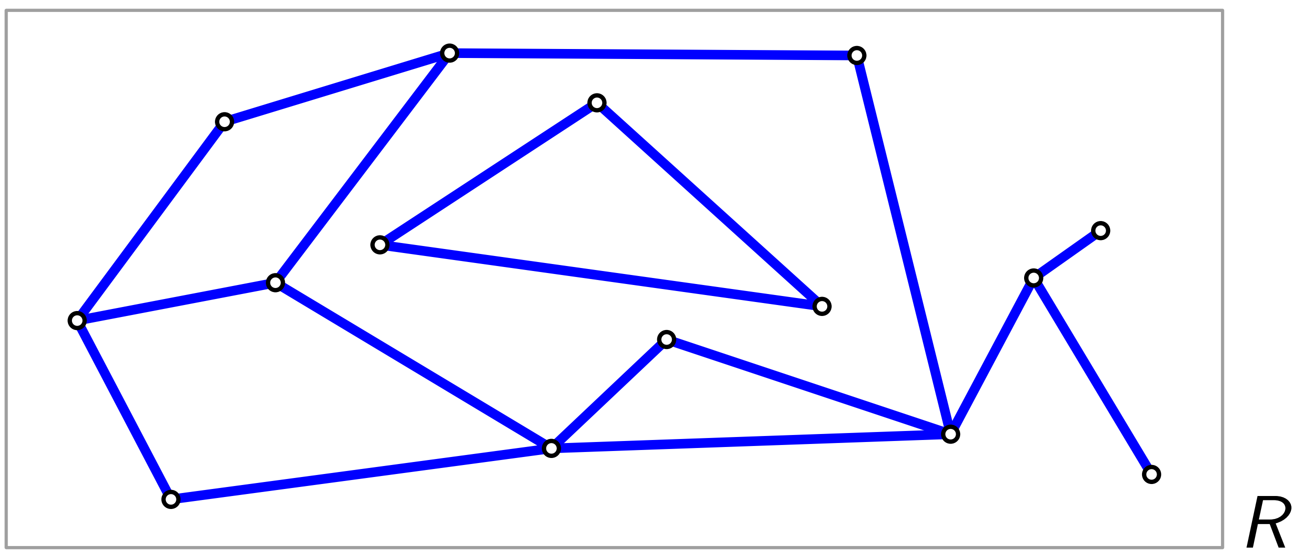

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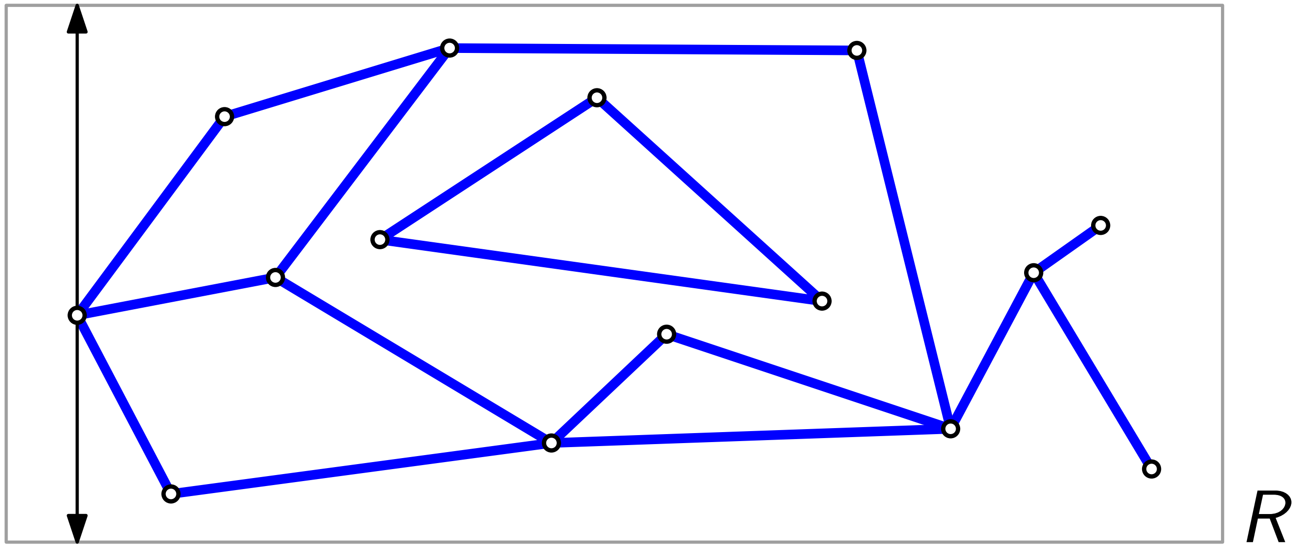

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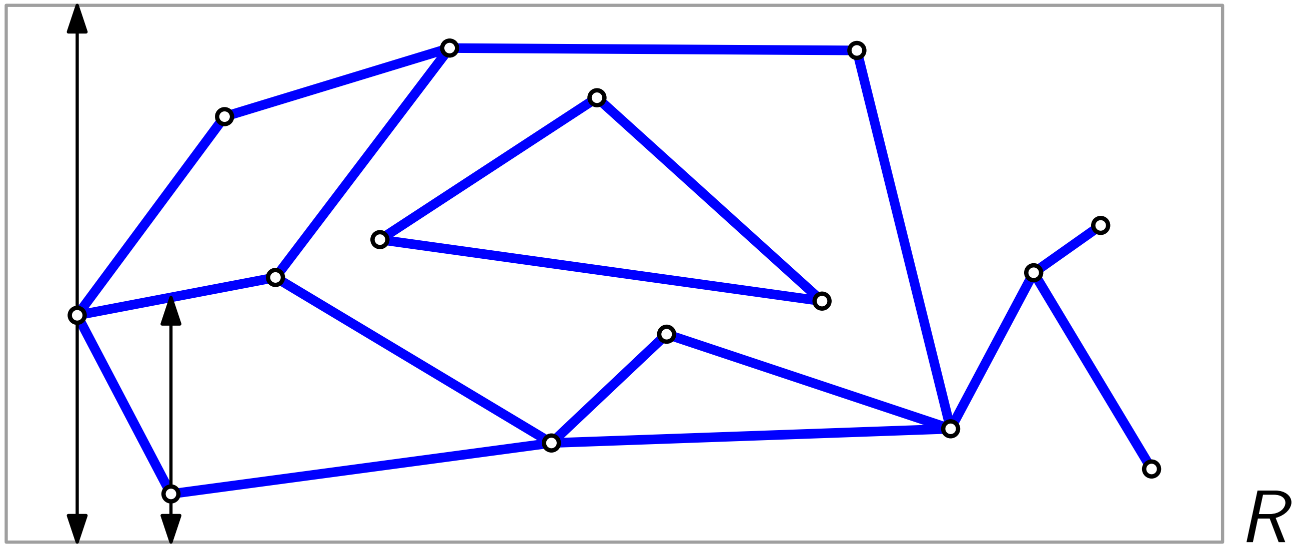

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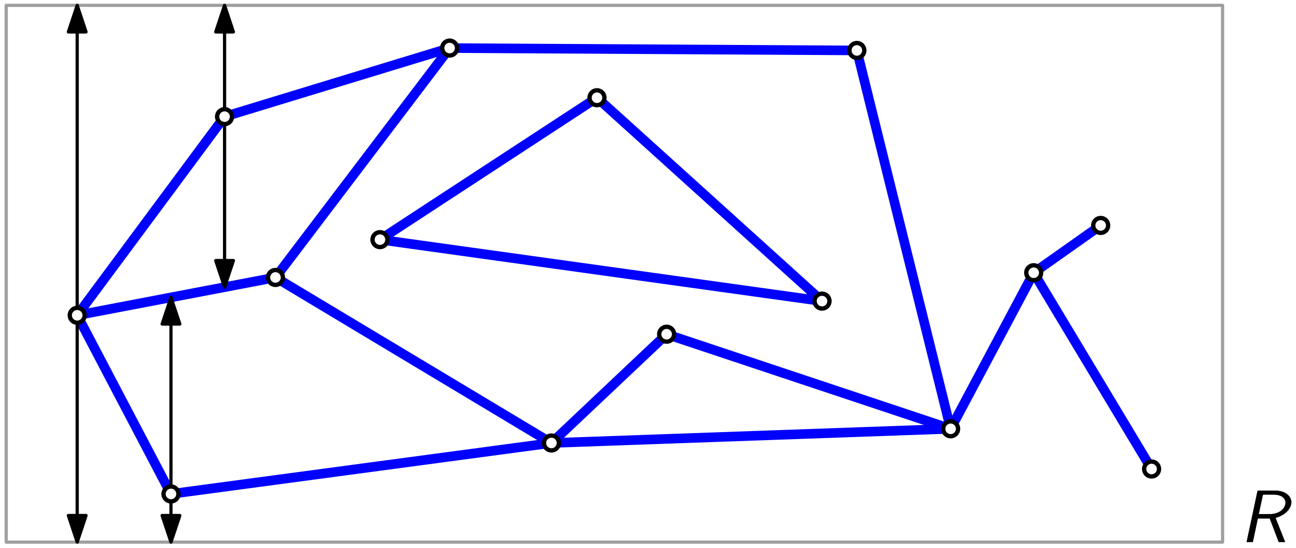

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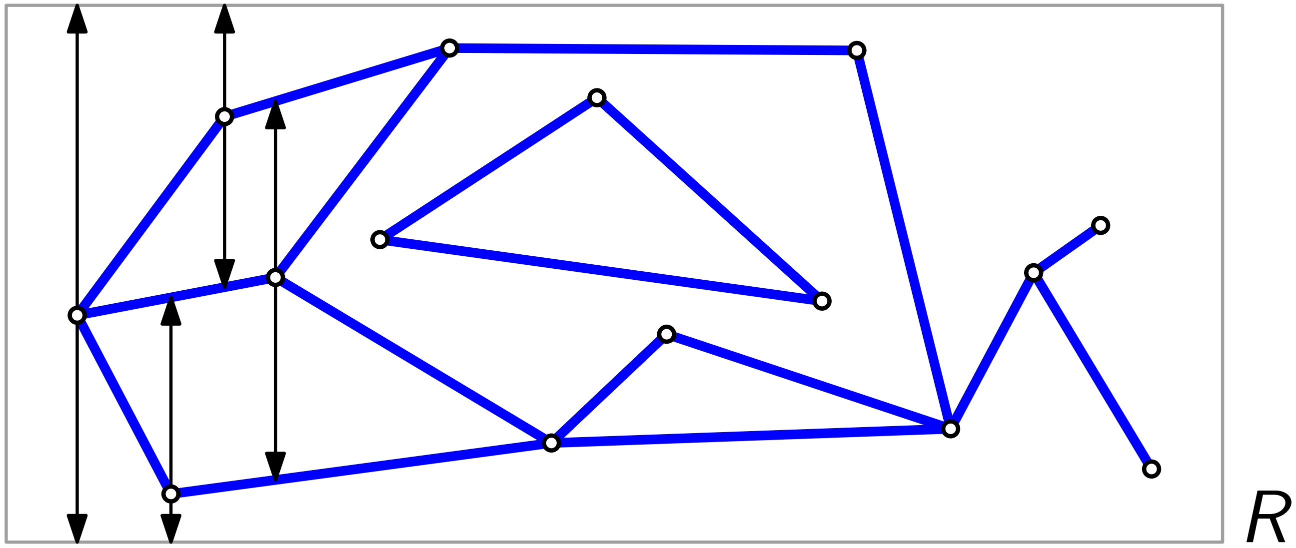

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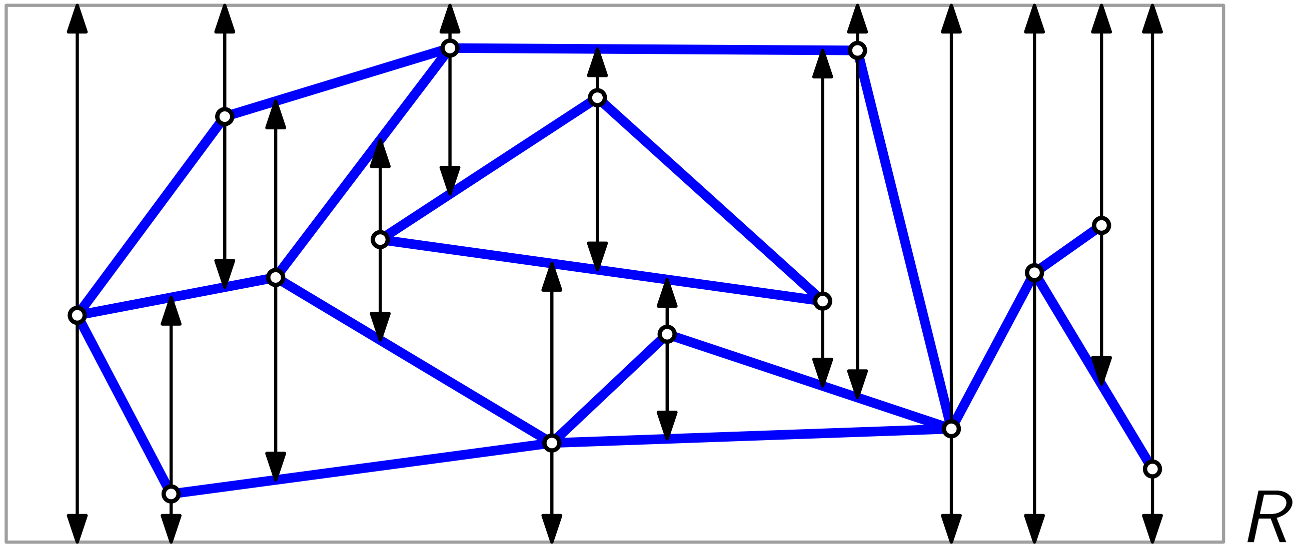

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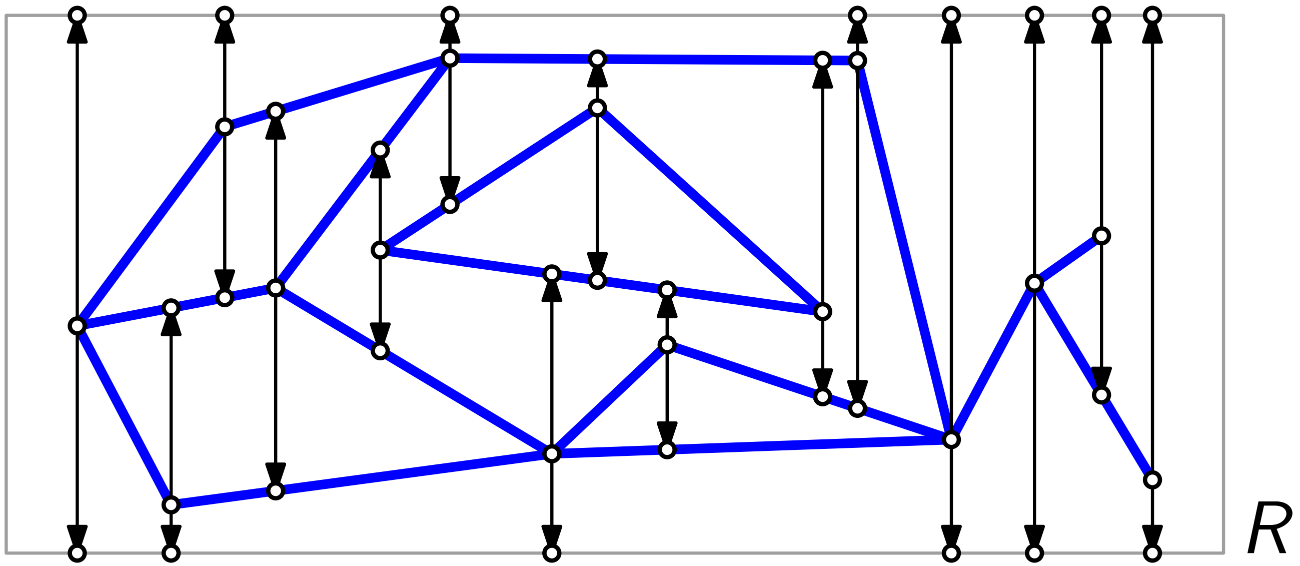

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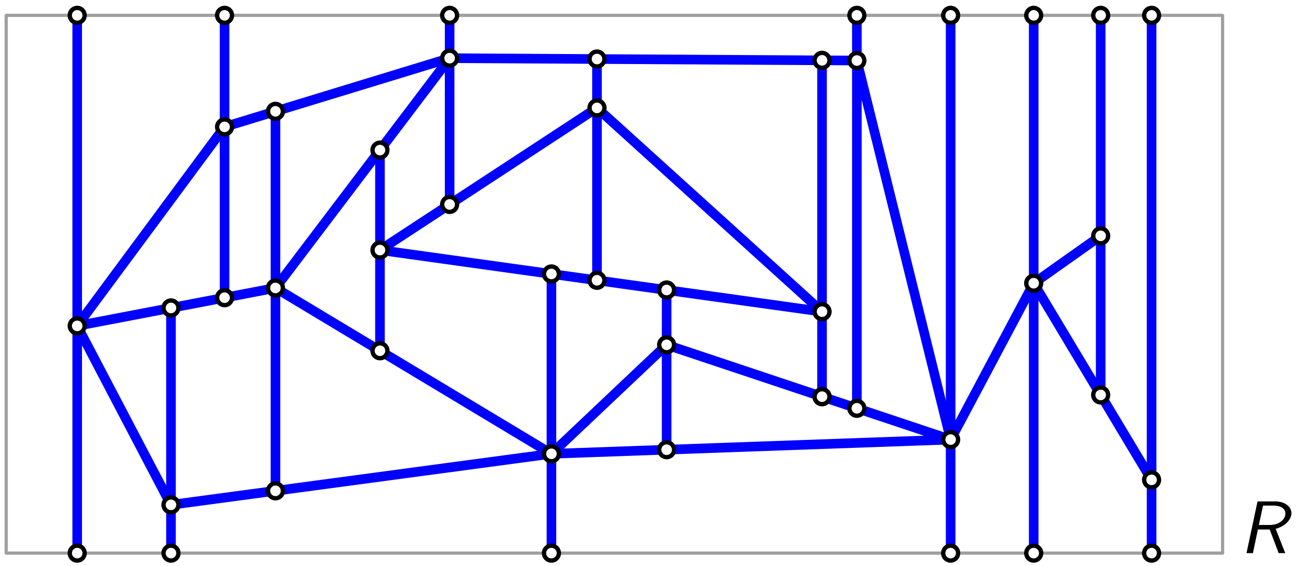

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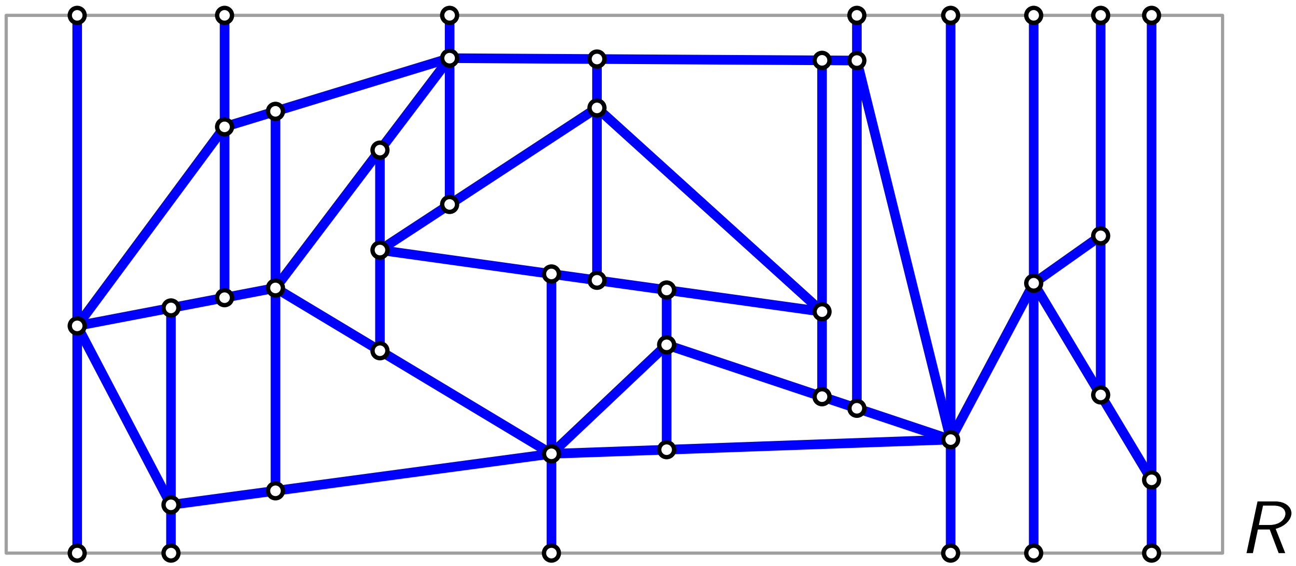
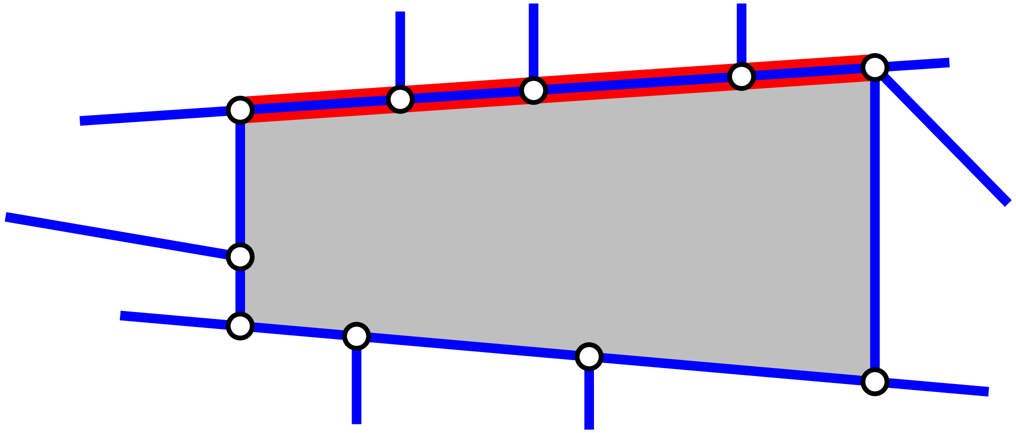


Assumption: $\mathcal{S}$ is in *general position*, that is,
no two vertices have the same x-coordinates.

Notation

Definition: A *side* of a face of $\mathcal{T}(\mathcal{S})$ is a segment of maximum length contained in the boundary of the face.



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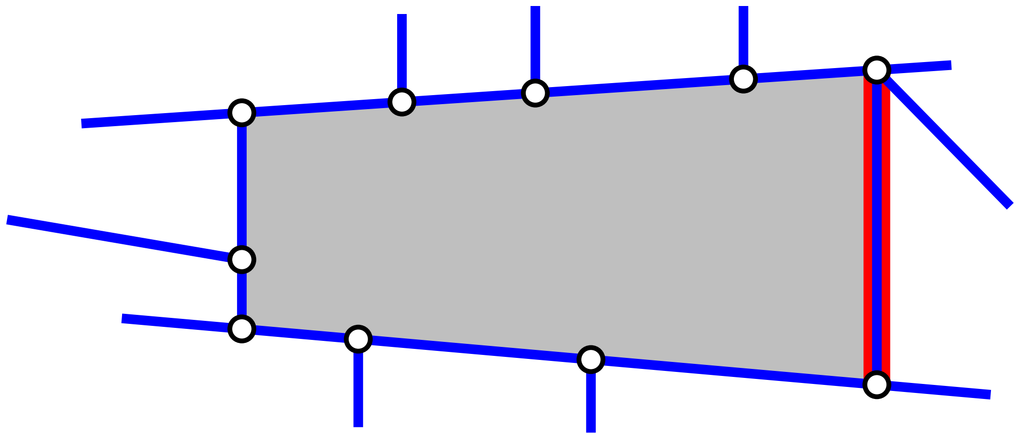

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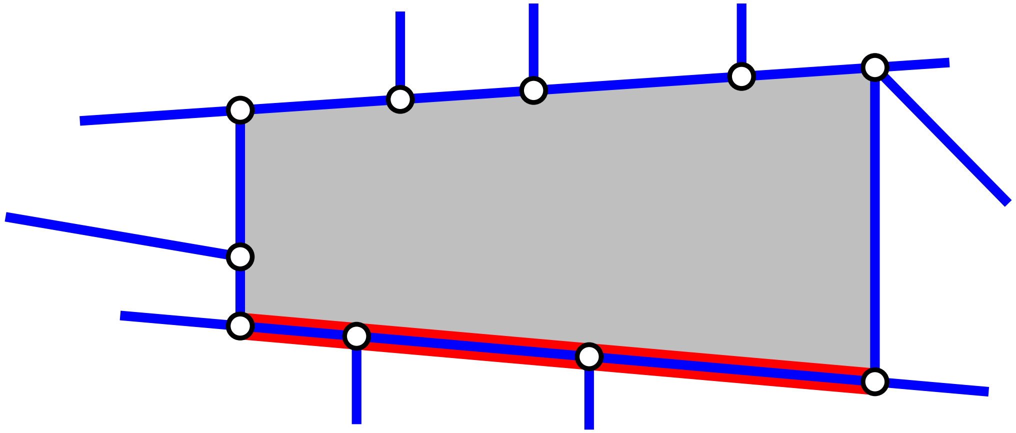

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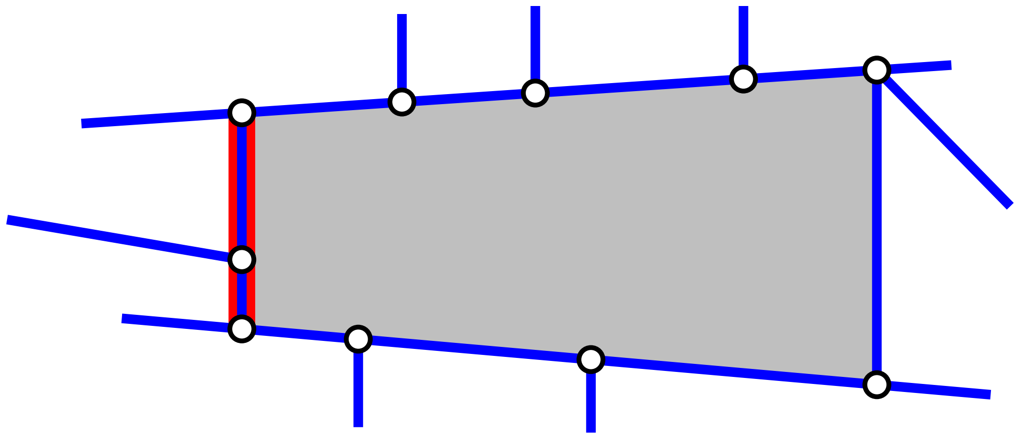

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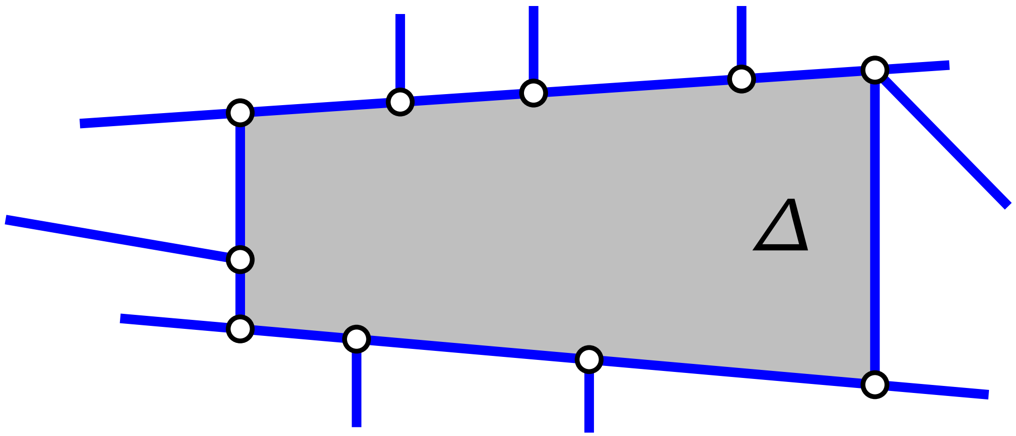

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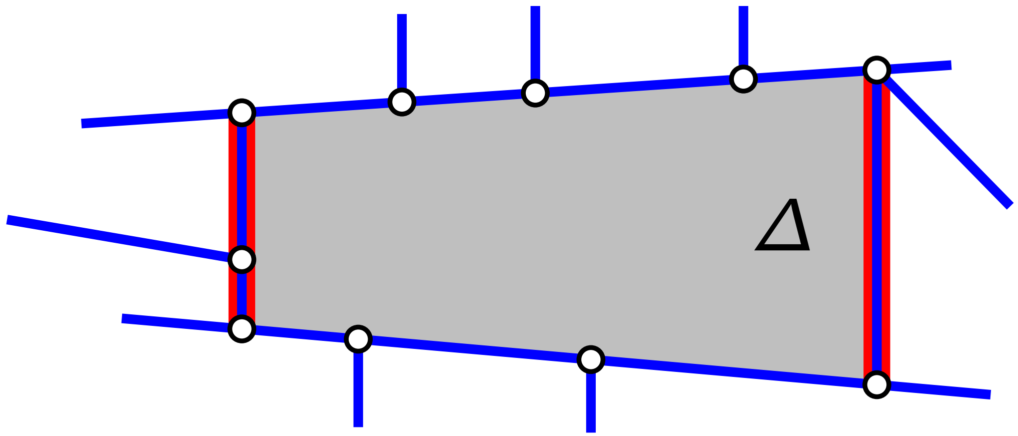

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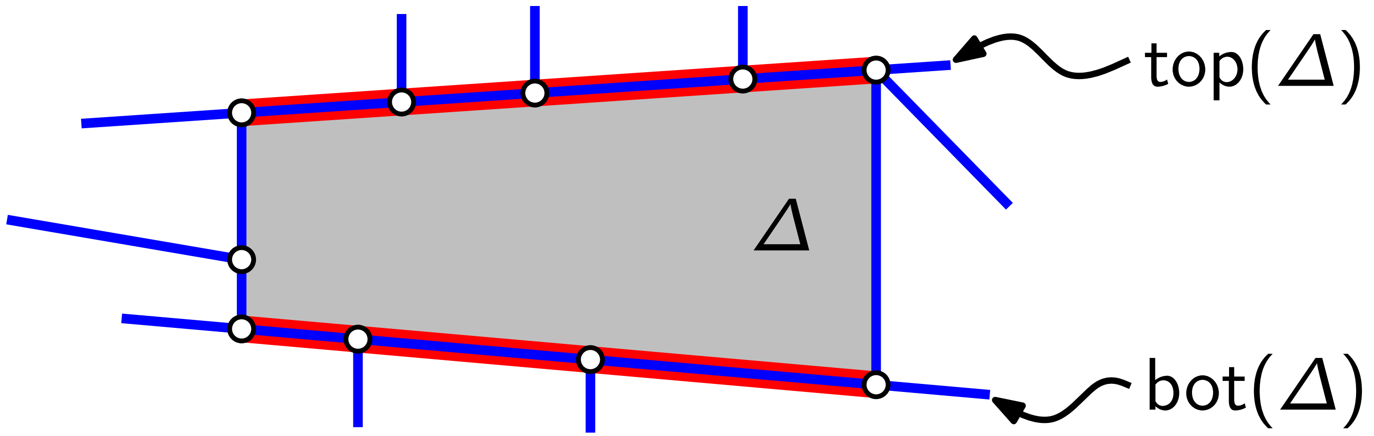

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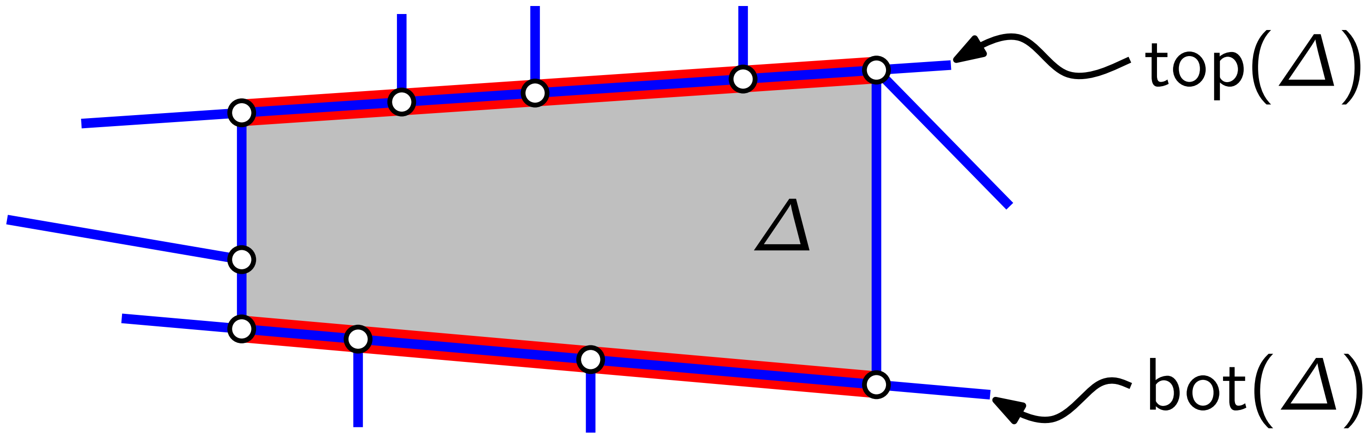

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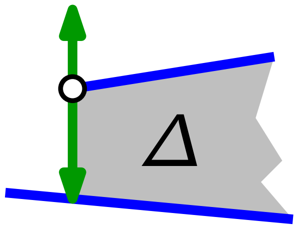

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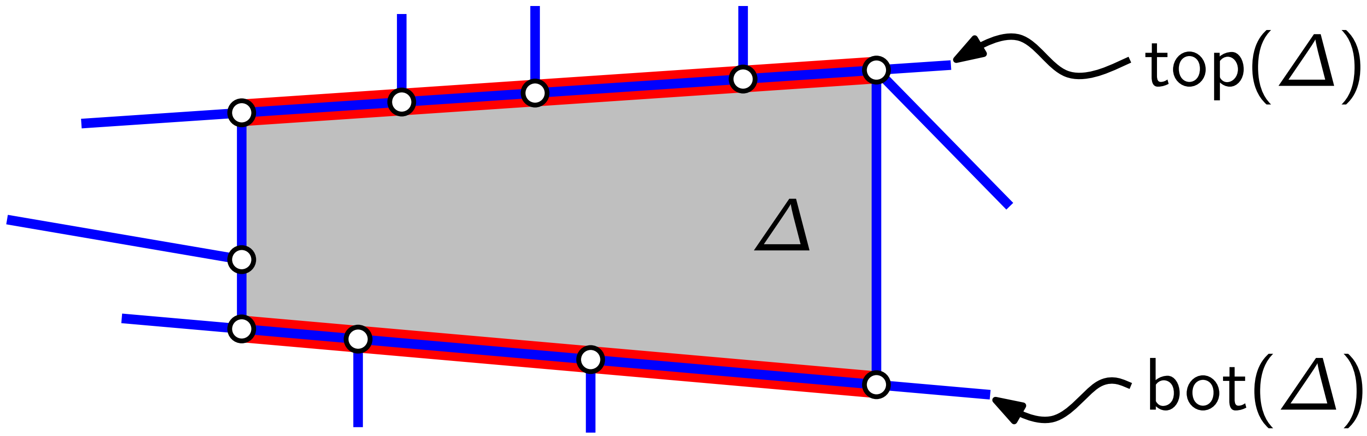

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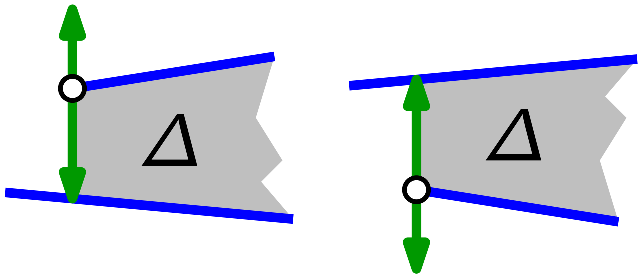

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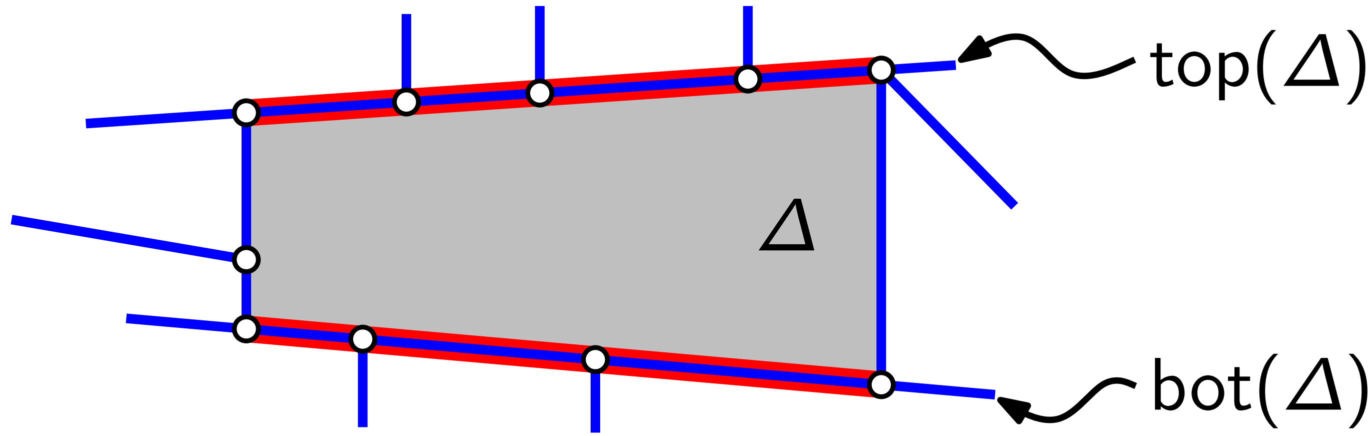

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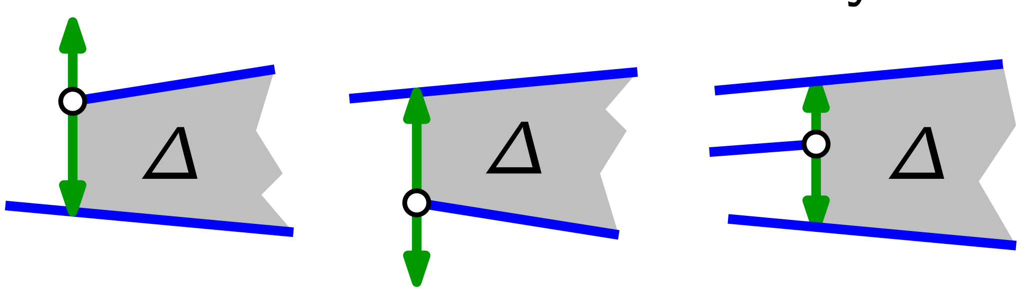

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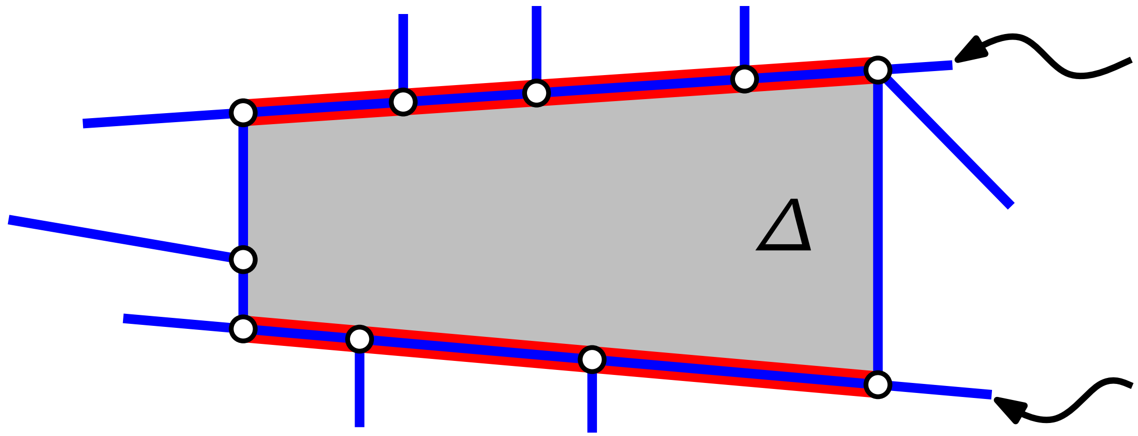
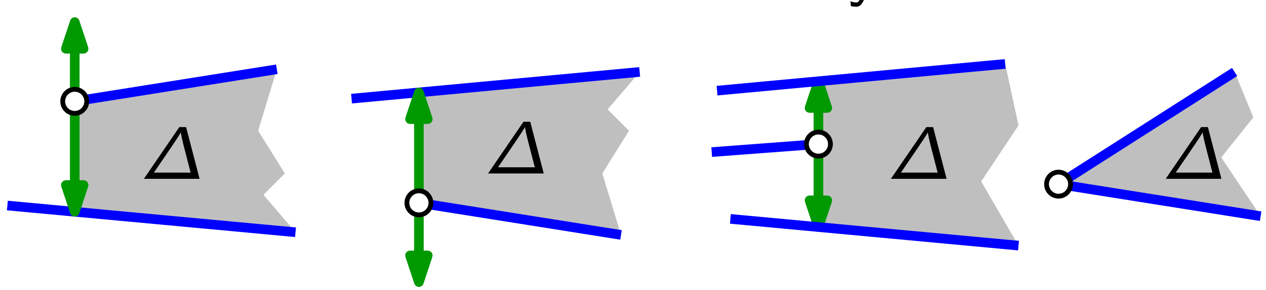


Observation: $\mathcal{S}$ in gen. pos. $\Rightarrow$ each face Δ of $\mathcal{T}(\mathcal{S})$ has:
– one or two vertical sides
– exactly 2 non-vertical sides

Left side:



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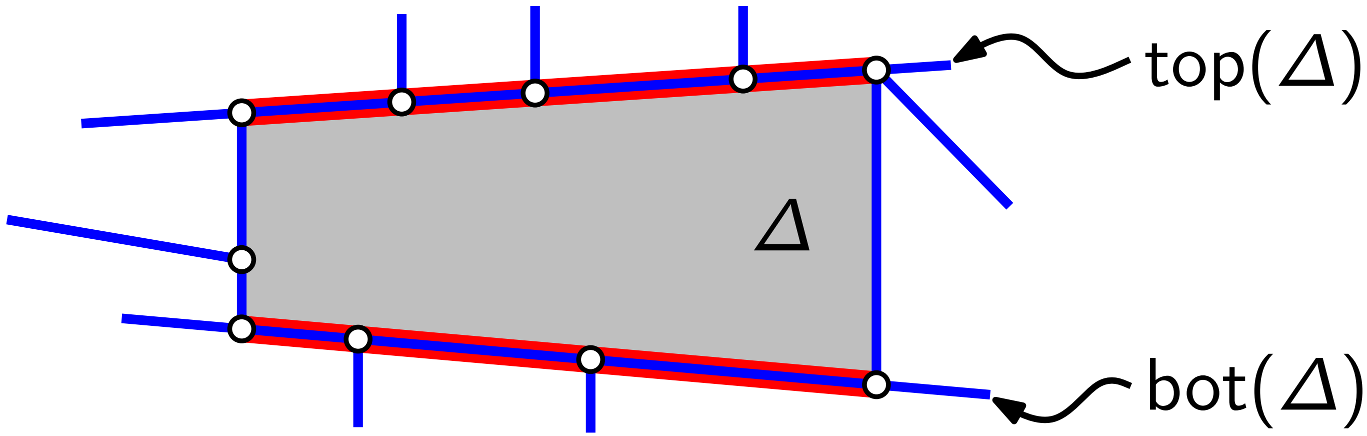

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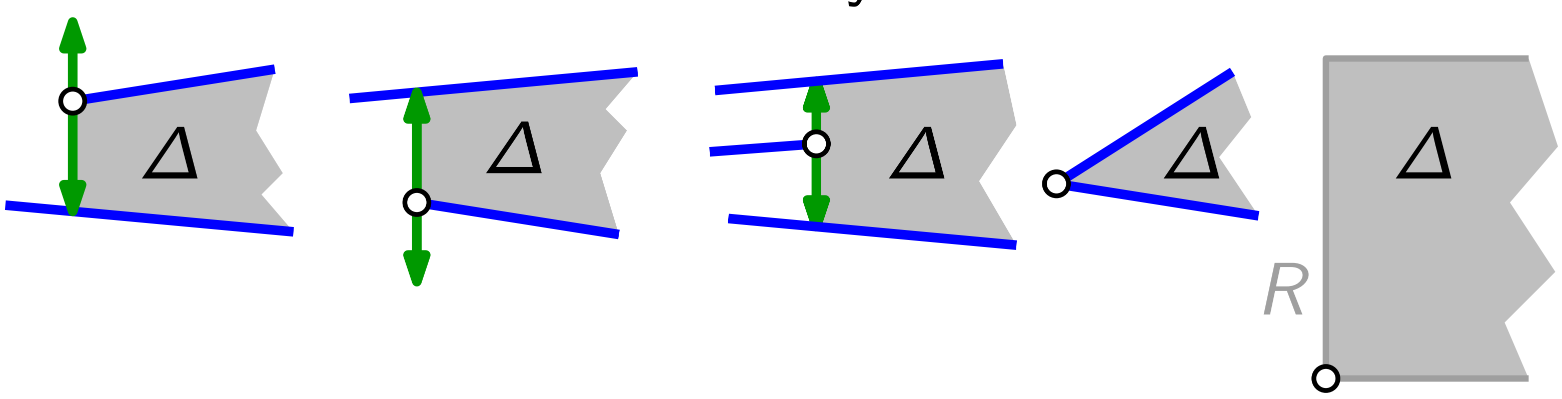

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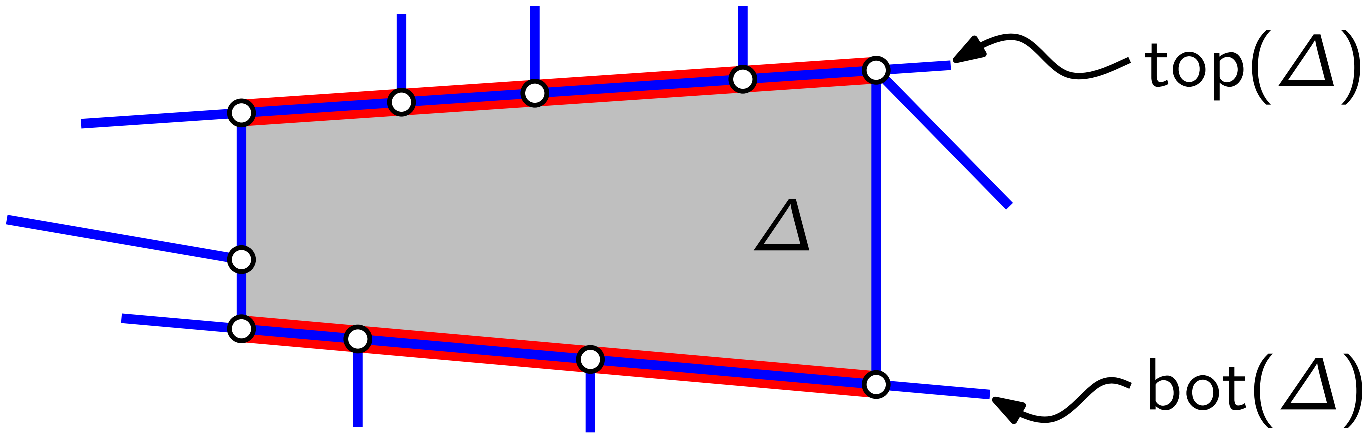

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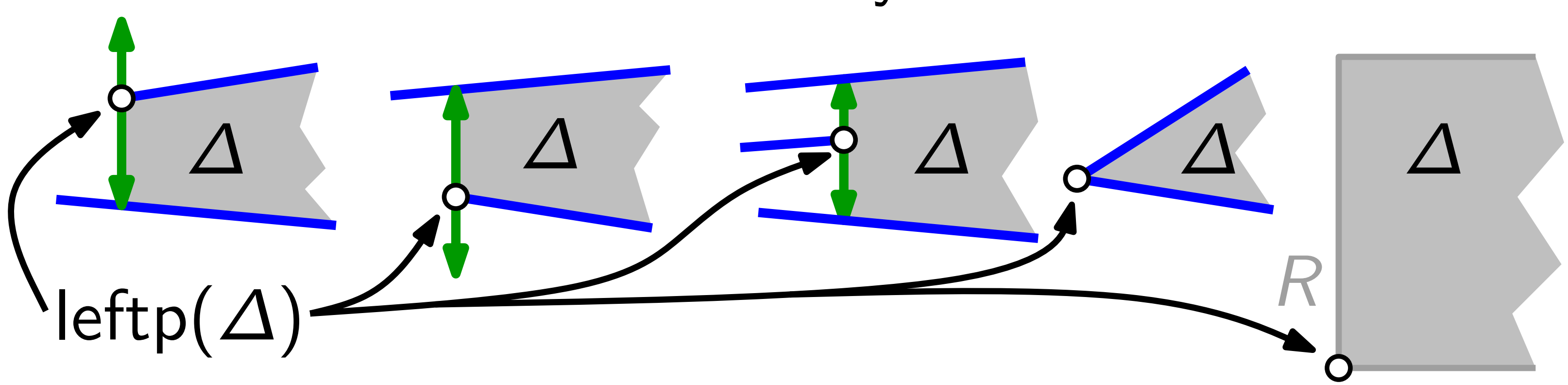

Complexity of $\mathcal{T}(\mathcal{S})$

Observe: A face Δ of $\mathcal{T}(\mathcal{S})$ is uniquely defined by top(Δ), bot(Δ), leftp(Δ), and rightp(Δ).

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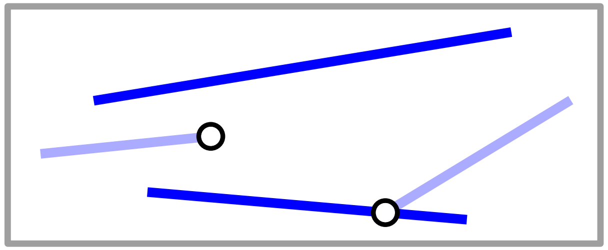

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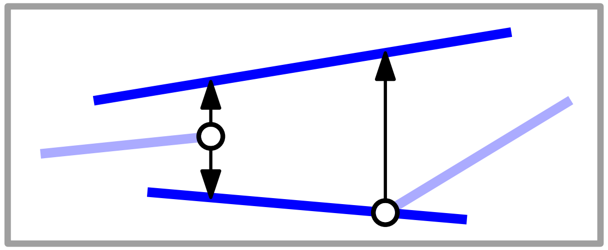

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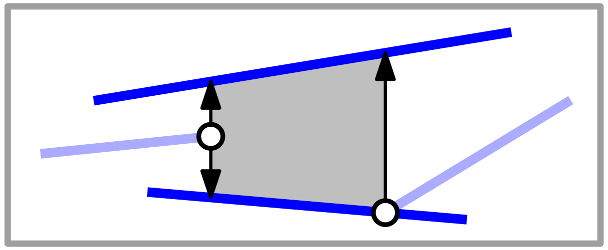

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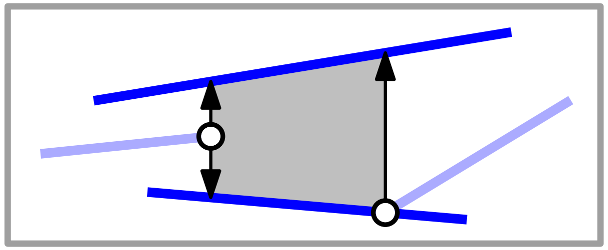

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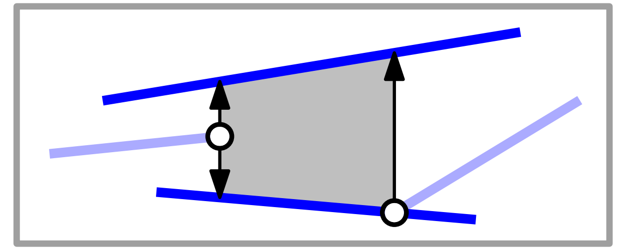

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– endpts of segments in $\mathcal{S}$

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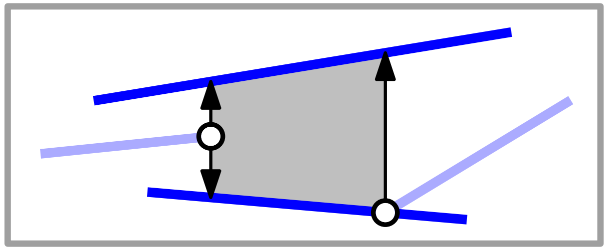

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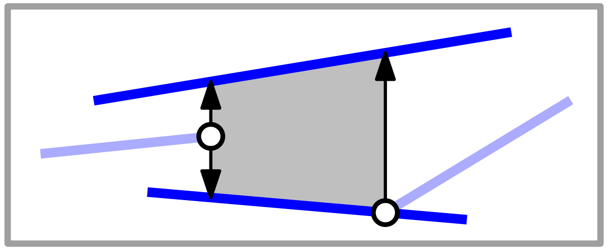

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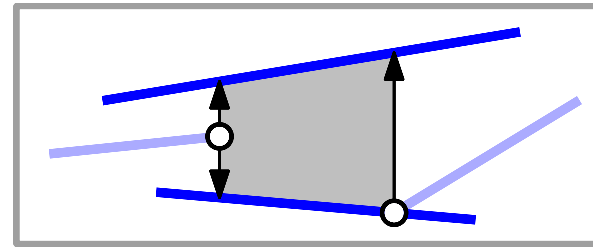

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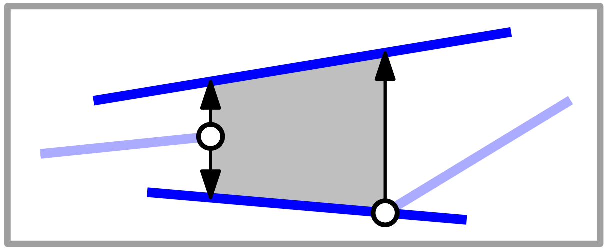

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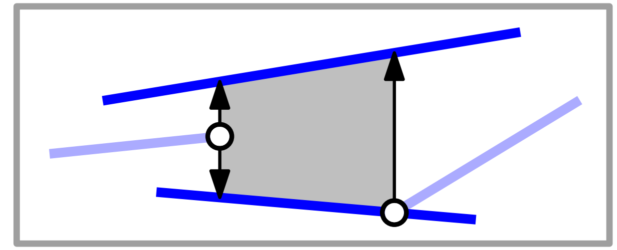

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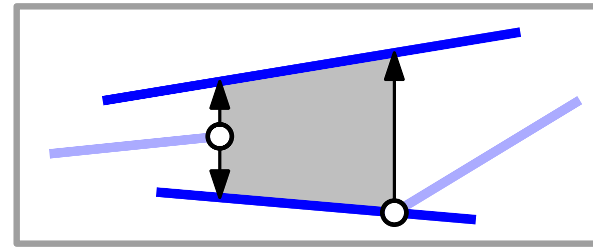
Complexity of $\mathcal{T}(\mathcal{S})$



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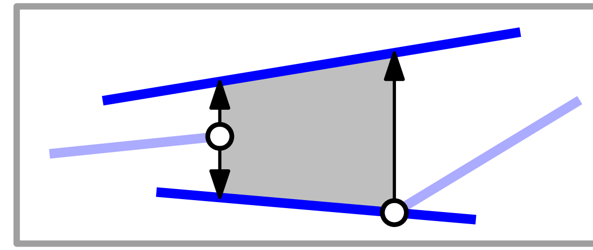

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$\Big\} \leq 6n + 4$

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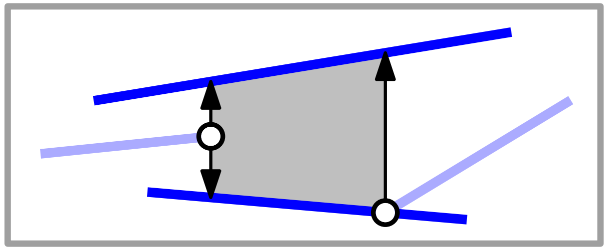

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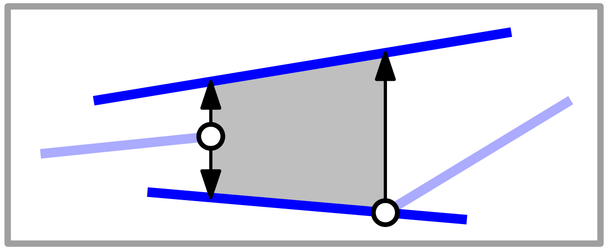

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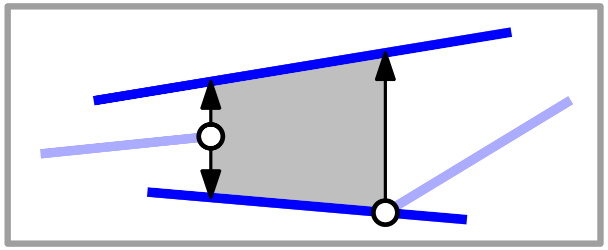

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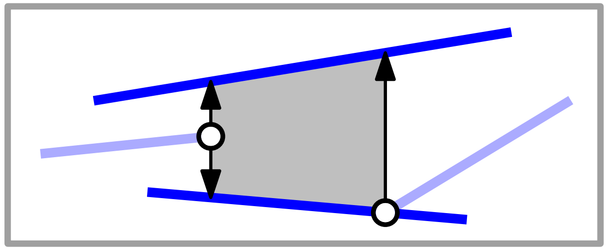

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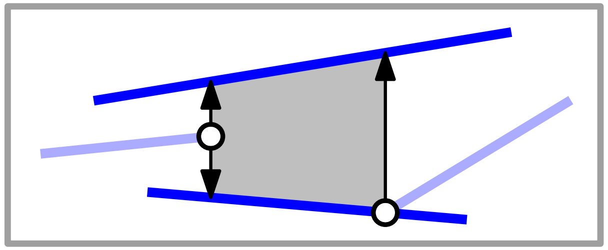

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algorithm-design paradigm!

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Given a set S of n real numbers...

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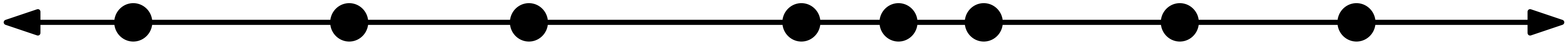

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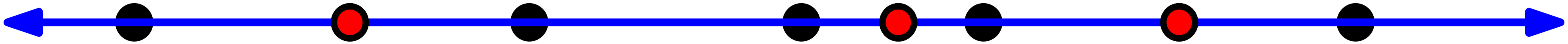

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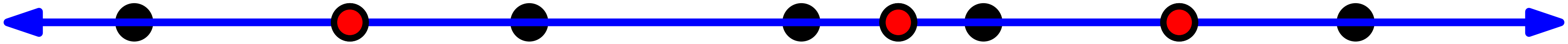

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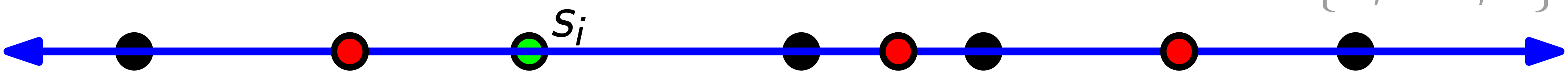

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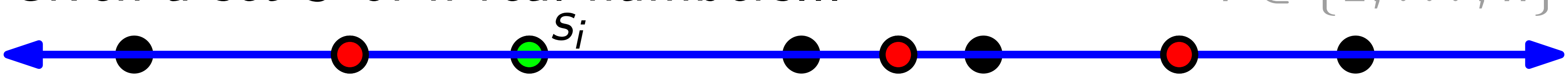

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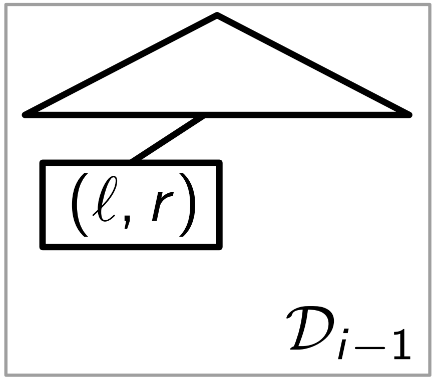

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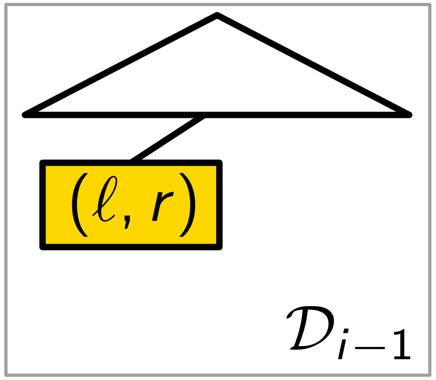

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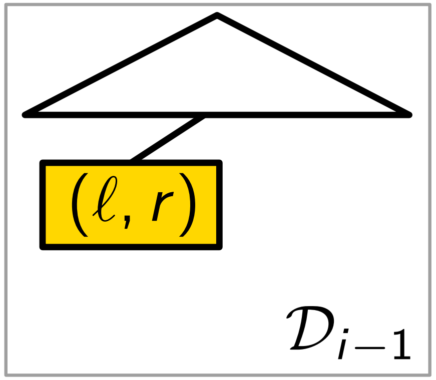

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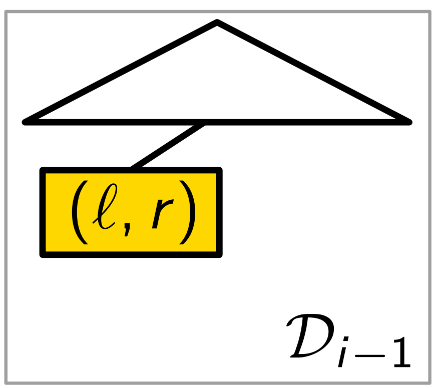

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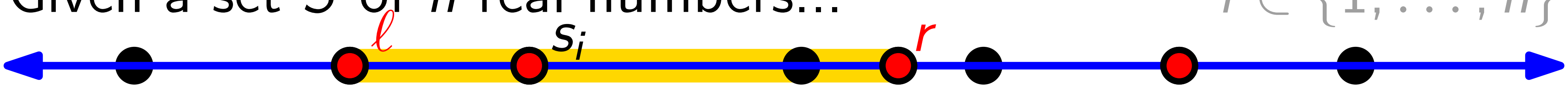

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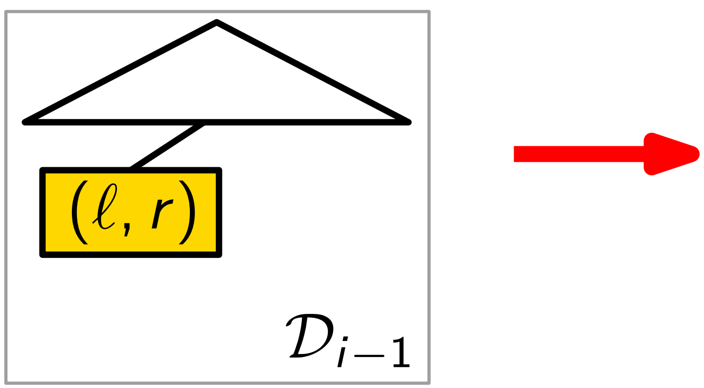

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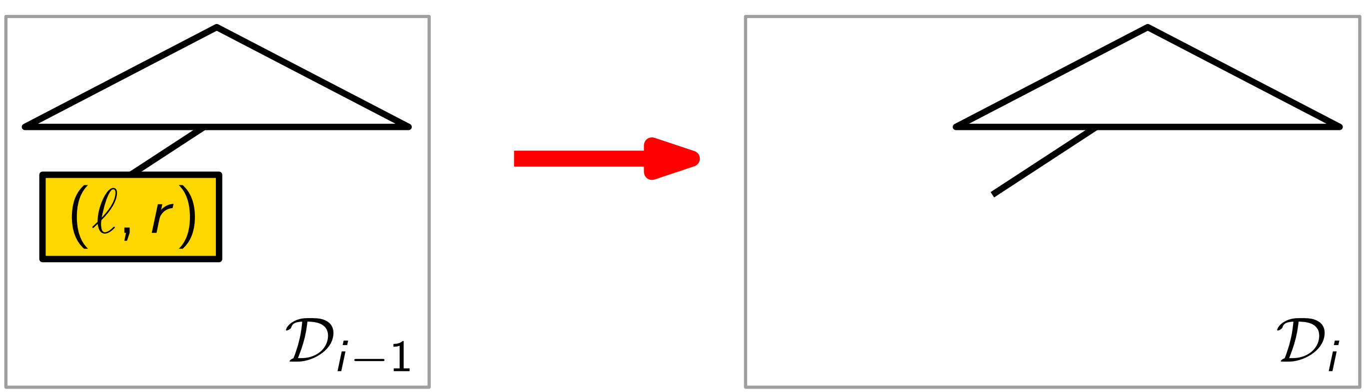

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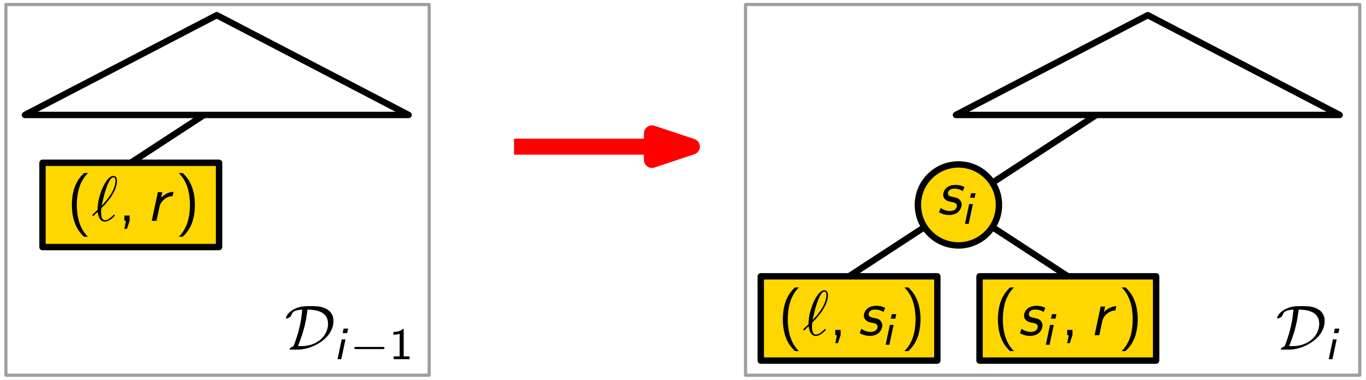

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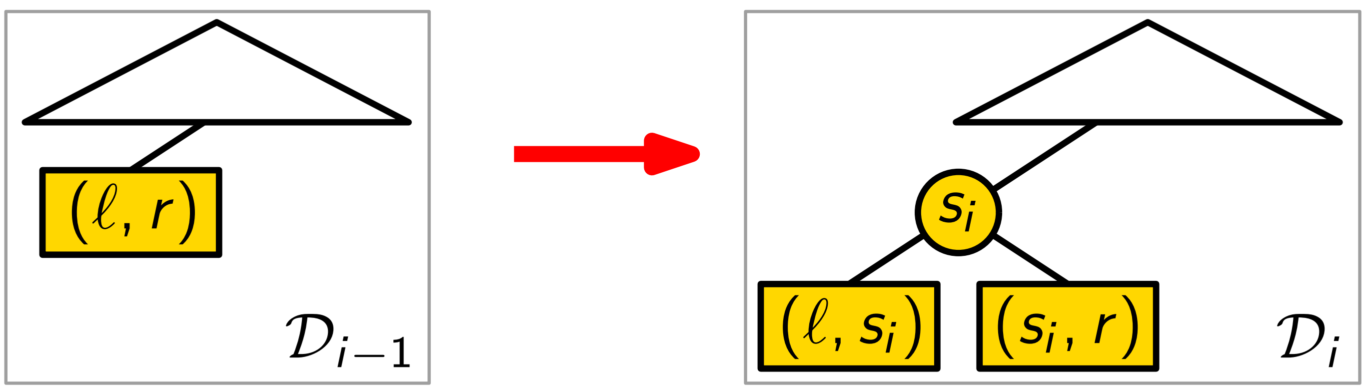

Problem:

The 1d-Problem

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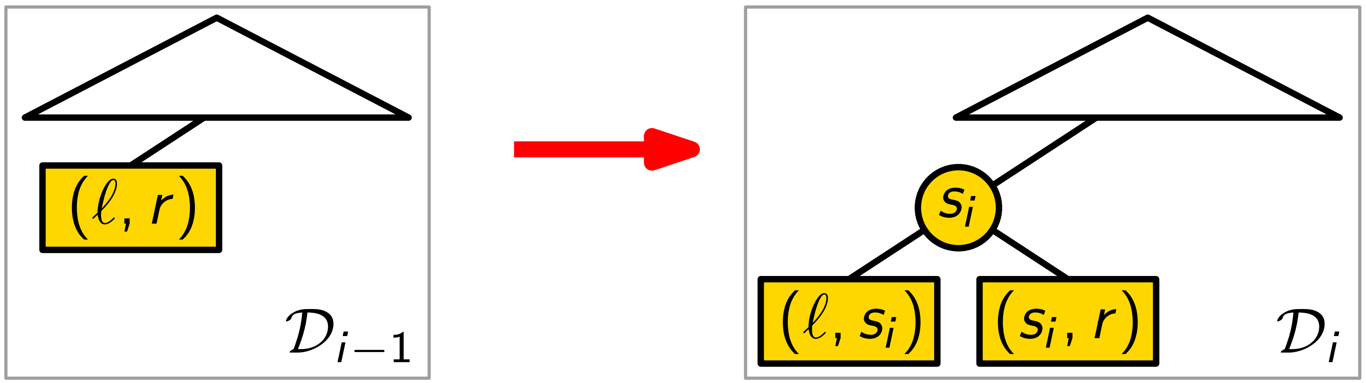

Problem: *looong* search paths!

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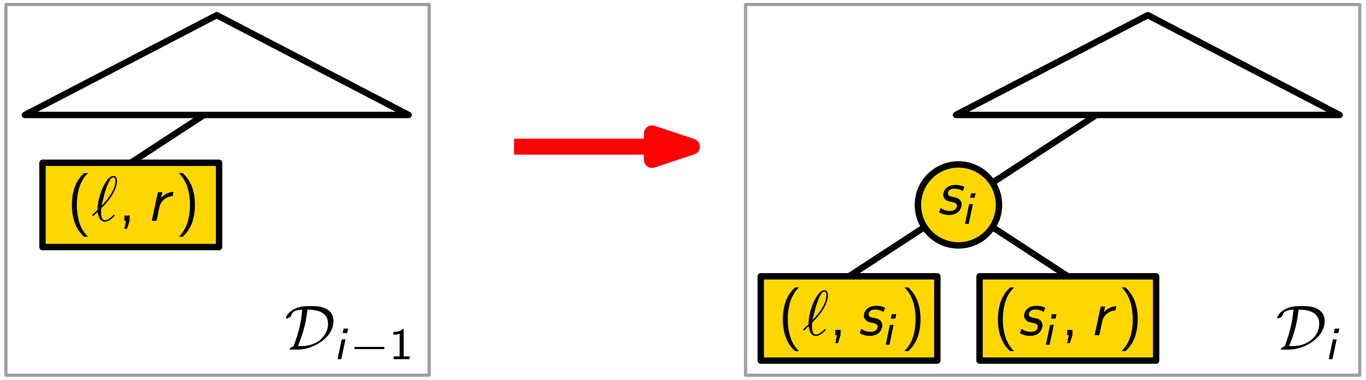

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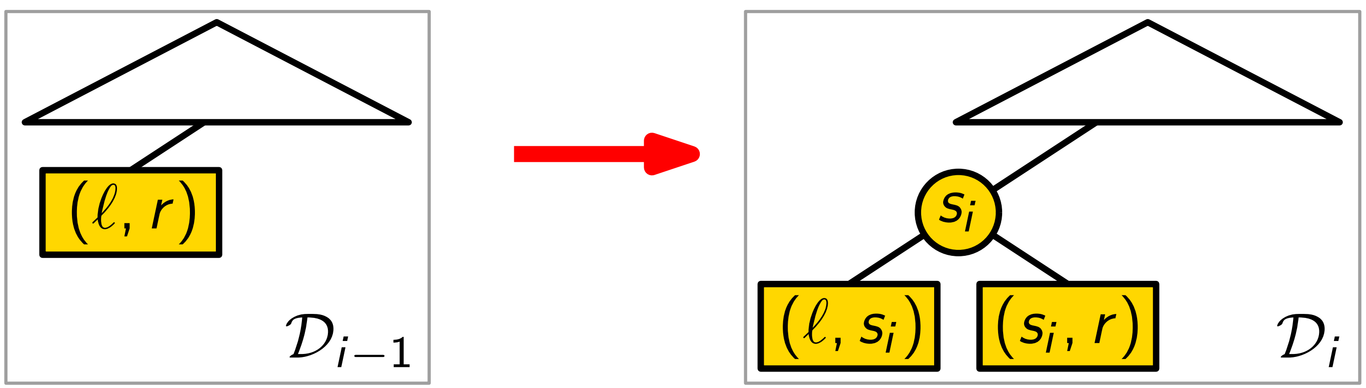

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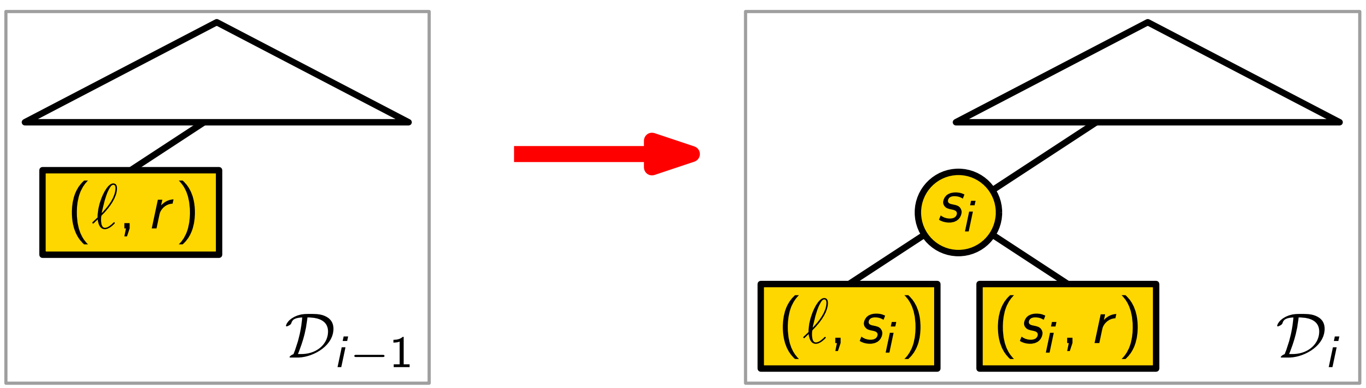

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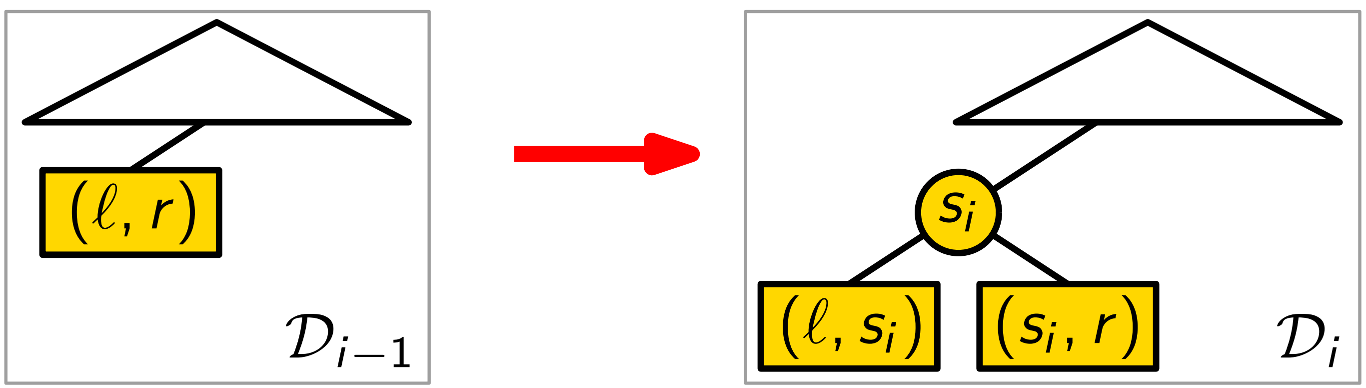

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Expected Query Time of $\mathcal{D}_n$

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The 1d-Result

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The 2d-Problem

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

The 2d-Problem

trapezoidal map → $\mathcal{T}$

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

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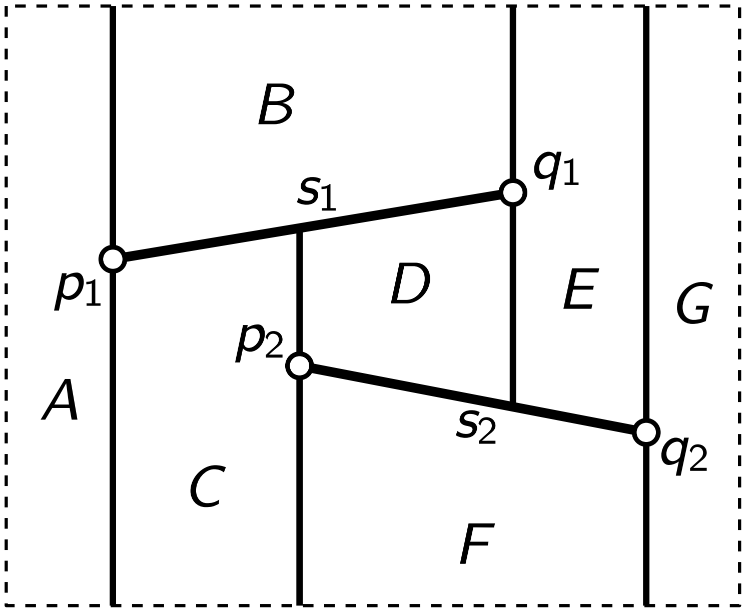
$\mathcal{T}$

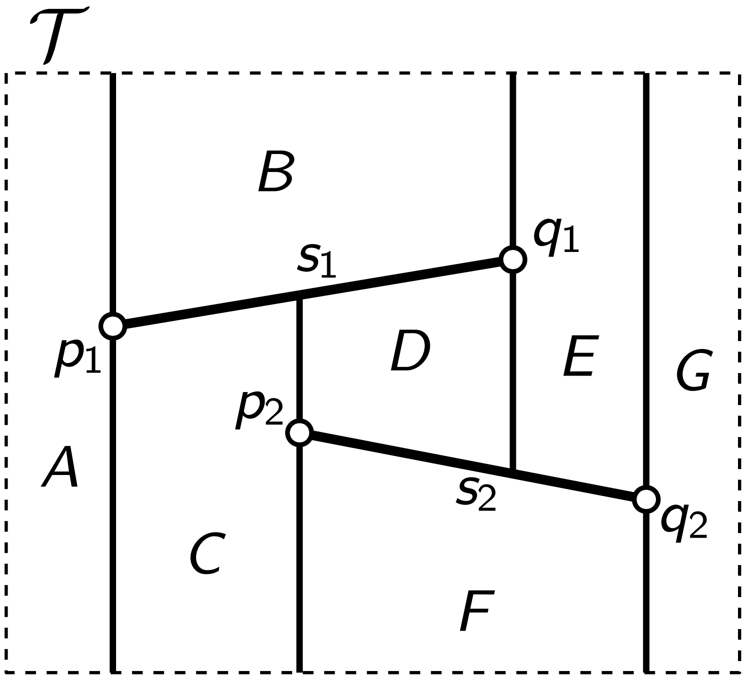


The 2d-Problem

point-location data structure (DAG)

trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$



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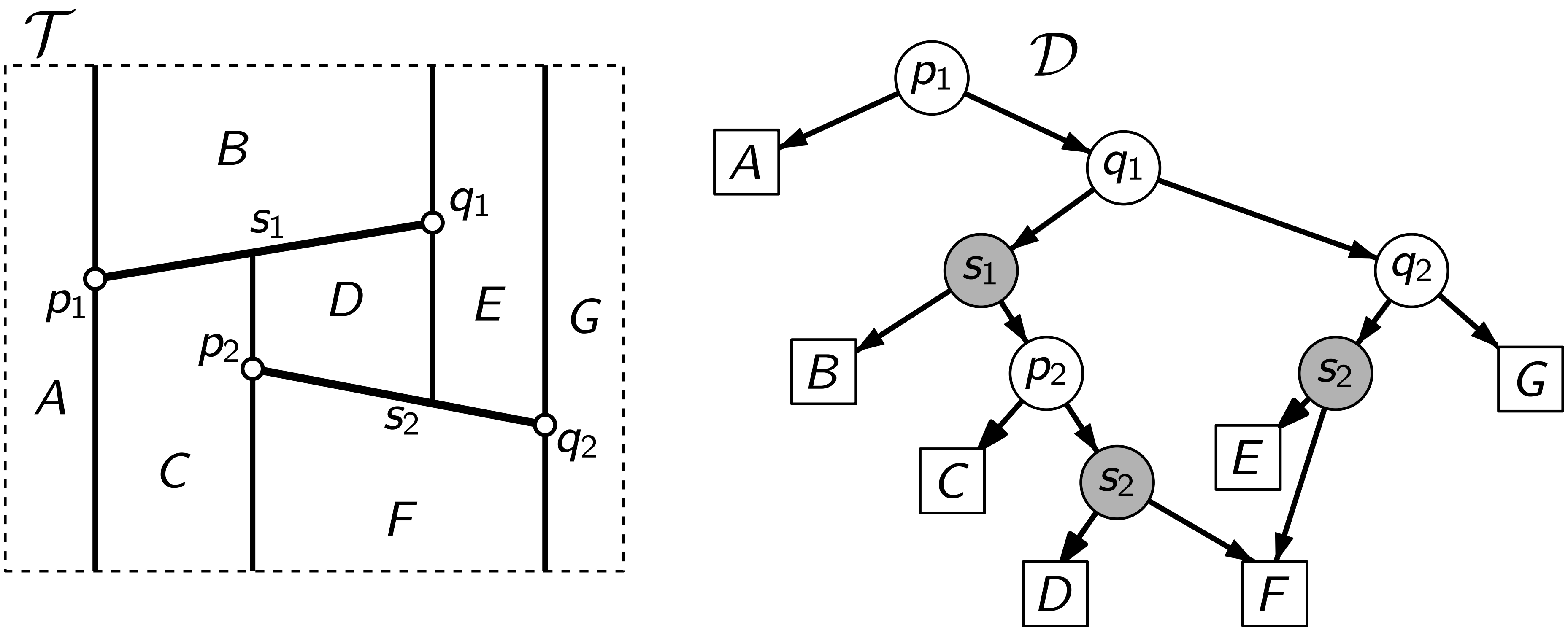

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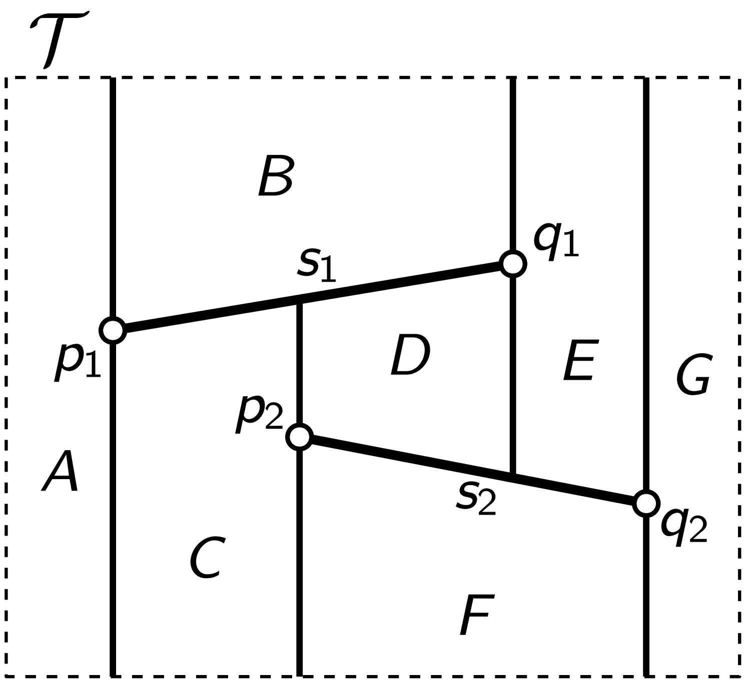
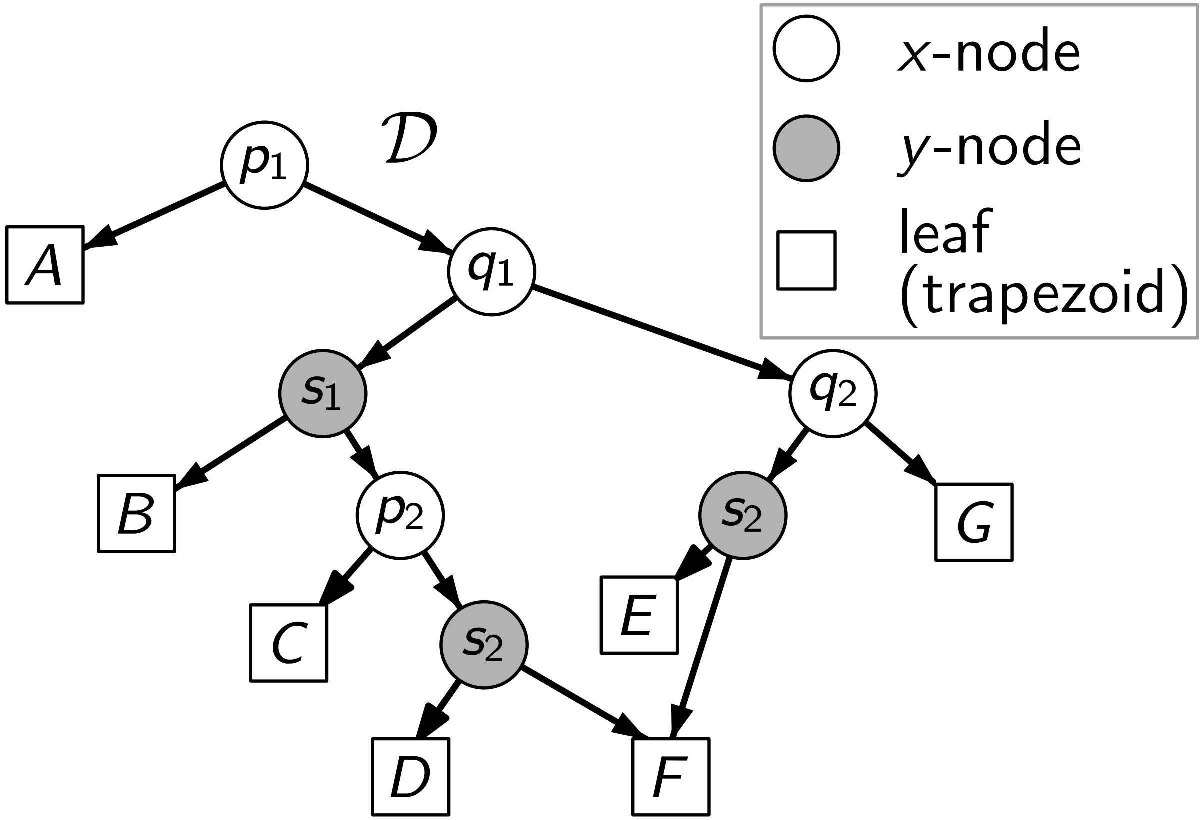




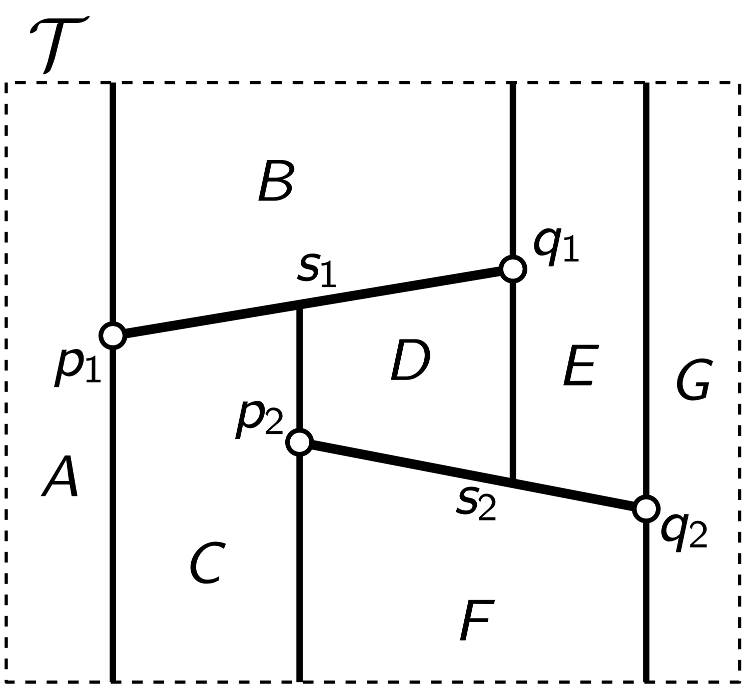
The 2d-Problem

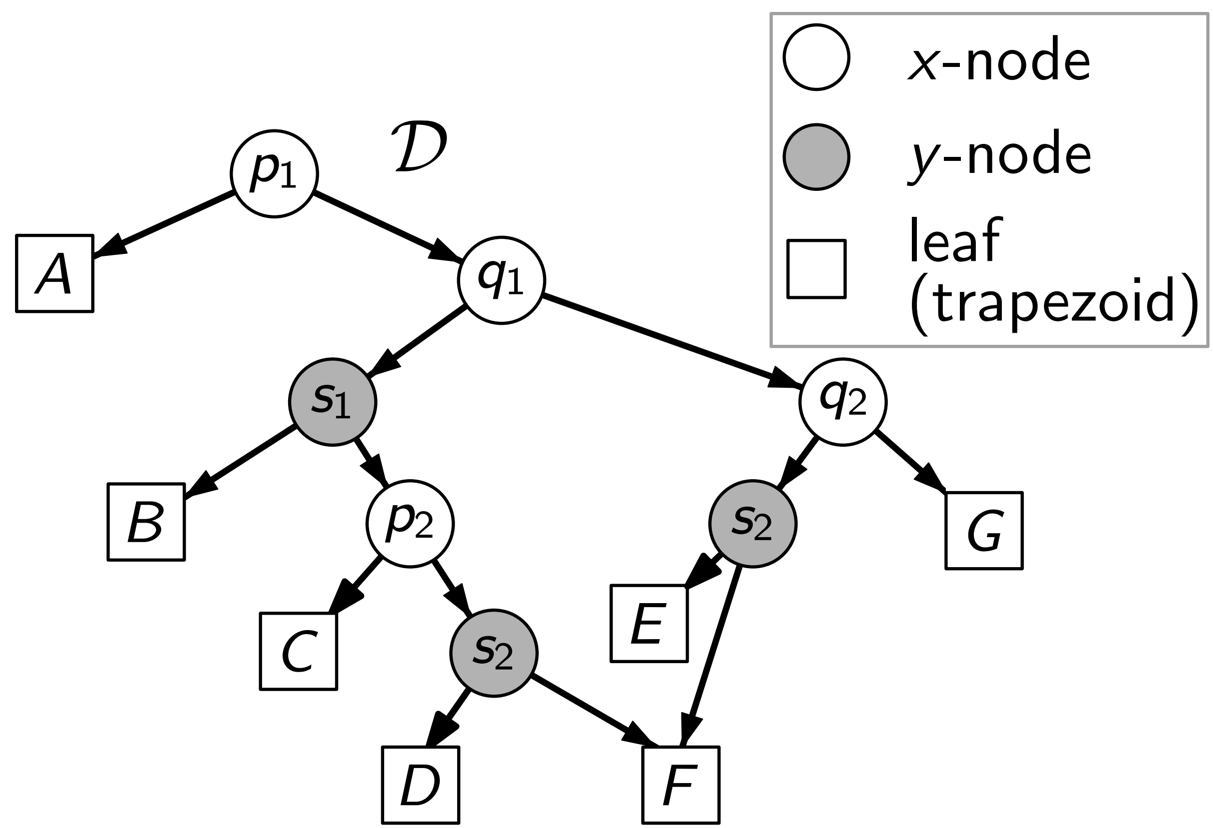
point-location data structure (DAG)

trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

– use $\mathcal{D}$ to locate left endpoint of next segment s





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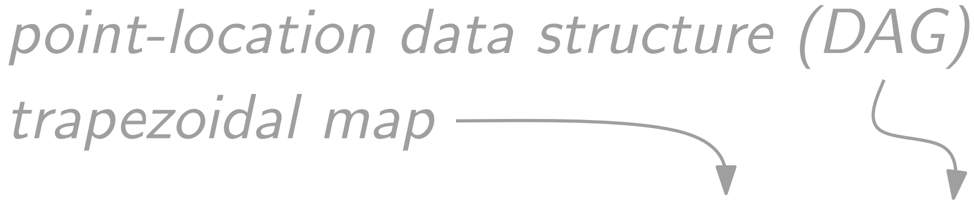
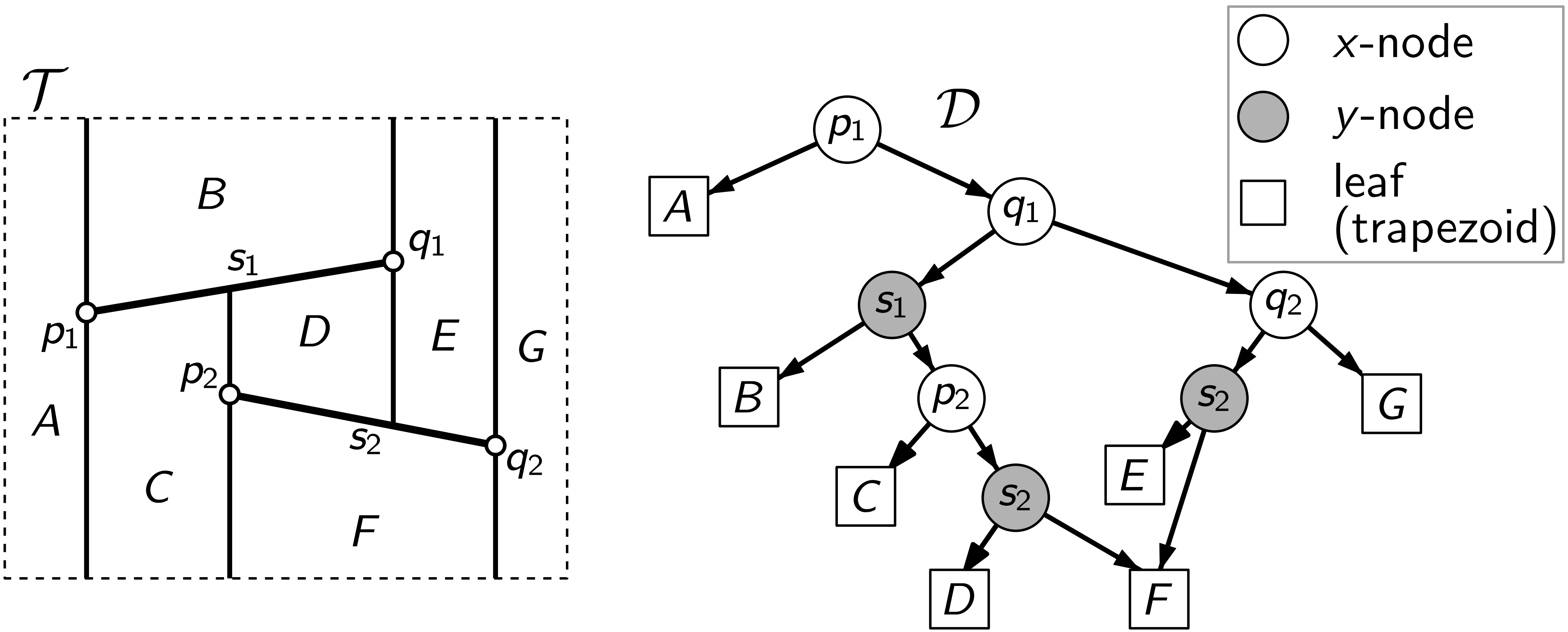


Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

– use $\mathcal{D}$ to locate left endpoint of next segment s

– "walk" along s through $\mathcal{T}$



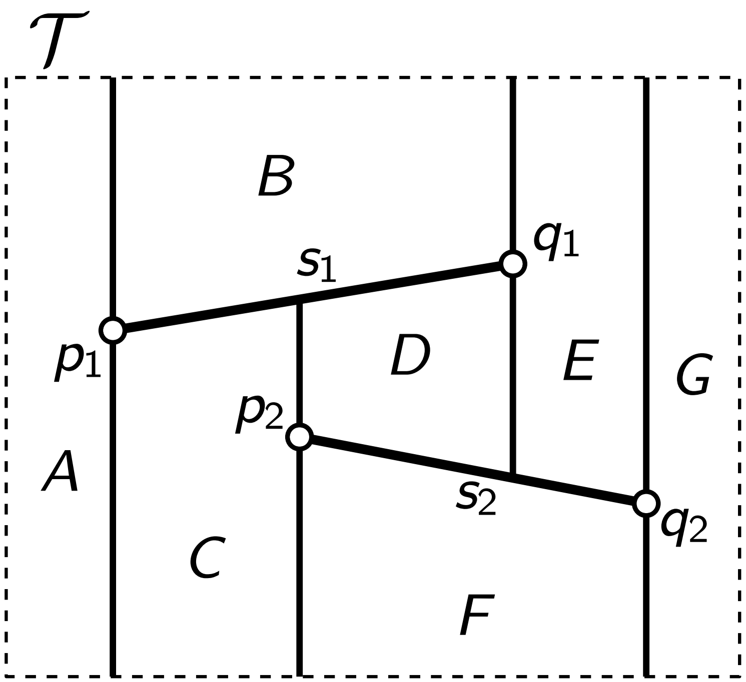
The 2d-Problem

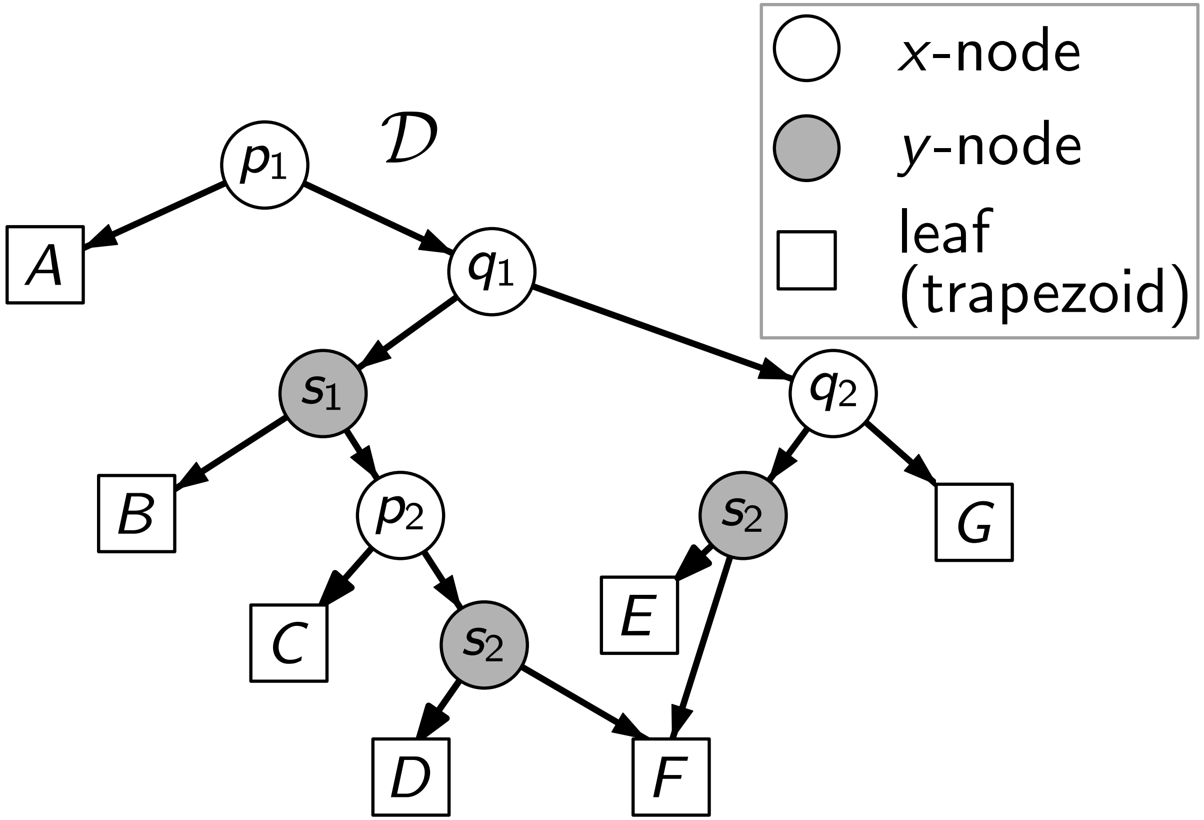
point-location data structure (DAG)

trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

- use $\mathcal{D}$ to locate left endpoint of next segment s
- "walk" along s through $\mathcal{T}$
- destroy all trapezoids of $\mathcal{T}$ intersecting s





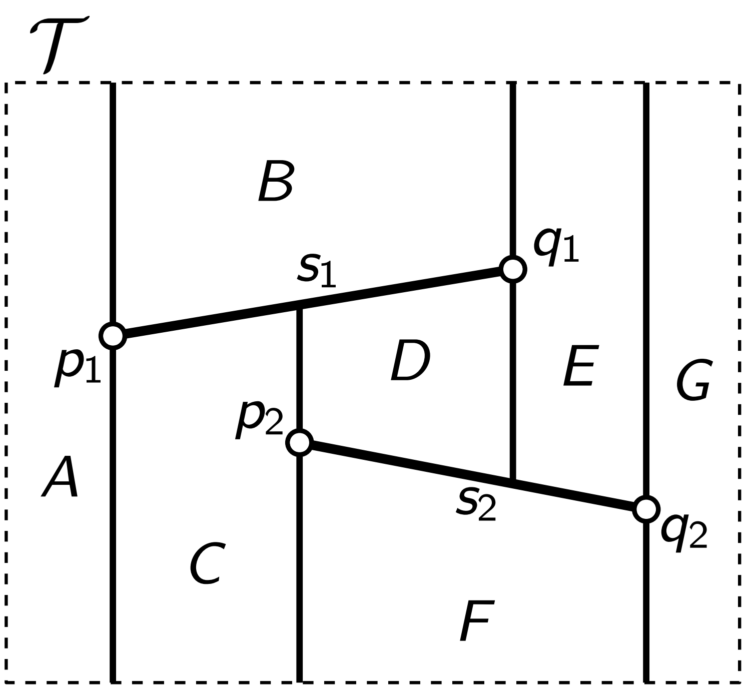
The 2d-Problem

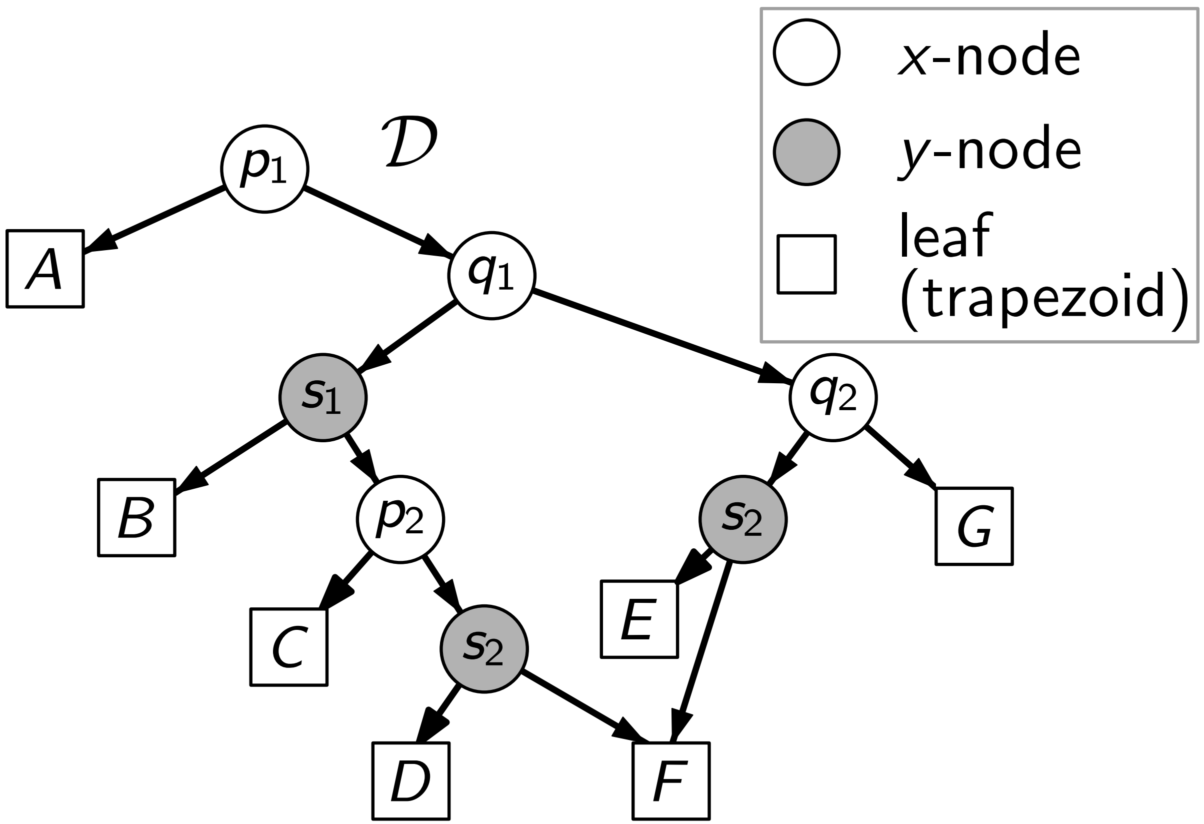
point-location data structure (DAG)

trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

- use $\mathcal{D}$ to locate left endpoint of next segment s
- "walk" along s through $\mathcal{T}$
- destroy all trapezoids of $\mathcal{T}$ intersecting s
- construct new trapezoids of $\mathcal{T}$ (adjacent to s)





The 2d-Problem

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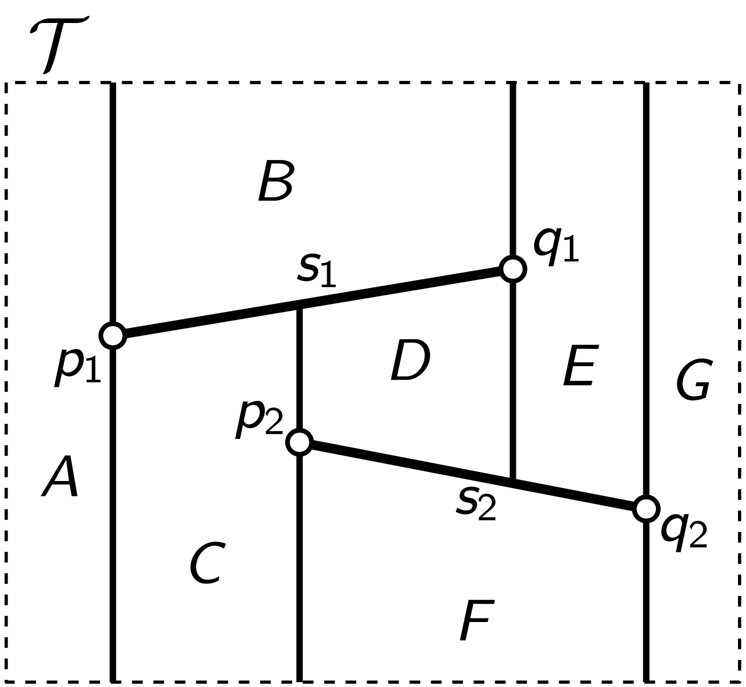
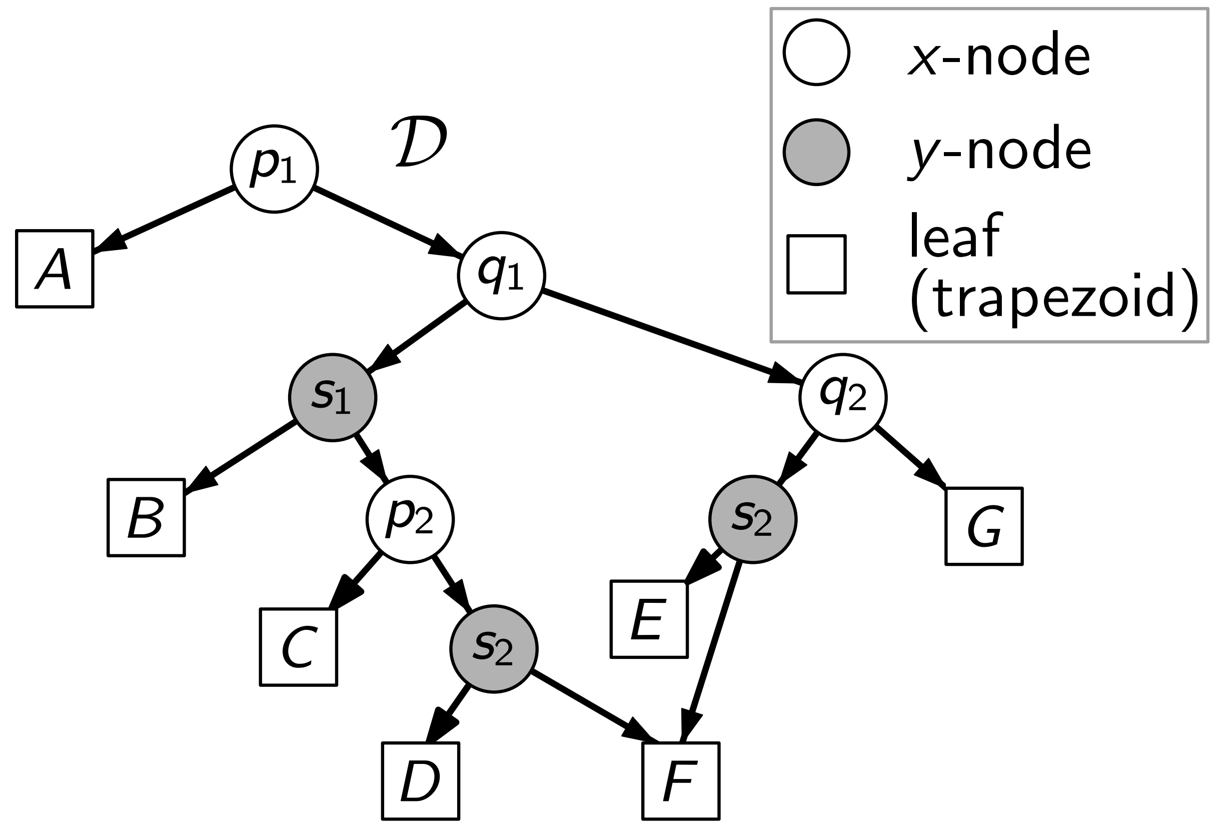
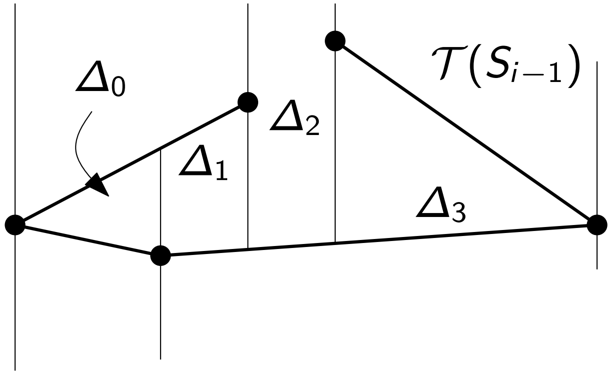
– update $\mathcal{D}$





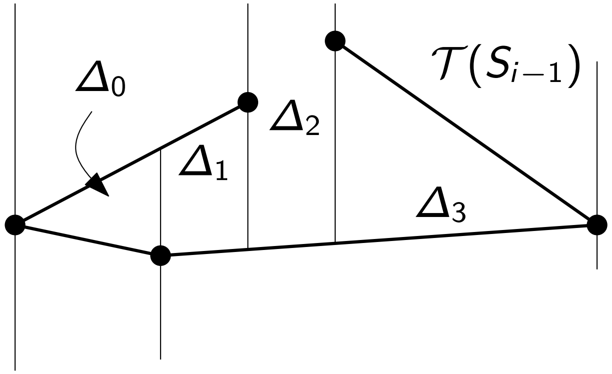
Walking through $\mathcal{T}$ and Updating $\mathcal{D}$



TrapezoidalMap(set S of n non-crossing segments)
$R = \text{BBox}(S)$; $\mathcal{T}$.init(); $\mathcal{D}$.init()
$(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$
for $i = 1$ **to** n **do**

)

Walking through $\mathcal{T}$ and Updating $\mathcal{D}$



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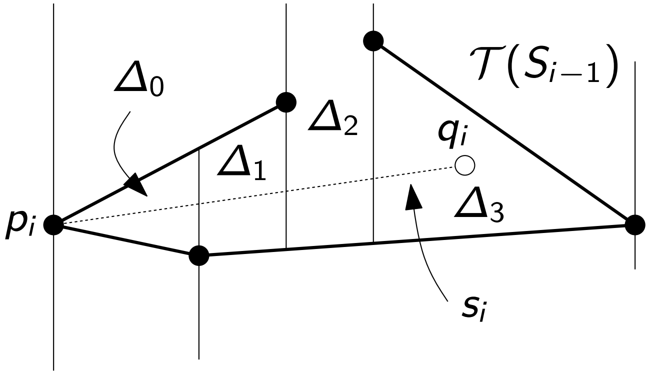

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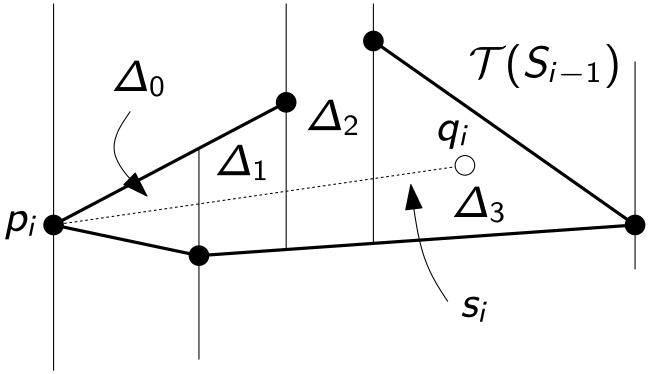

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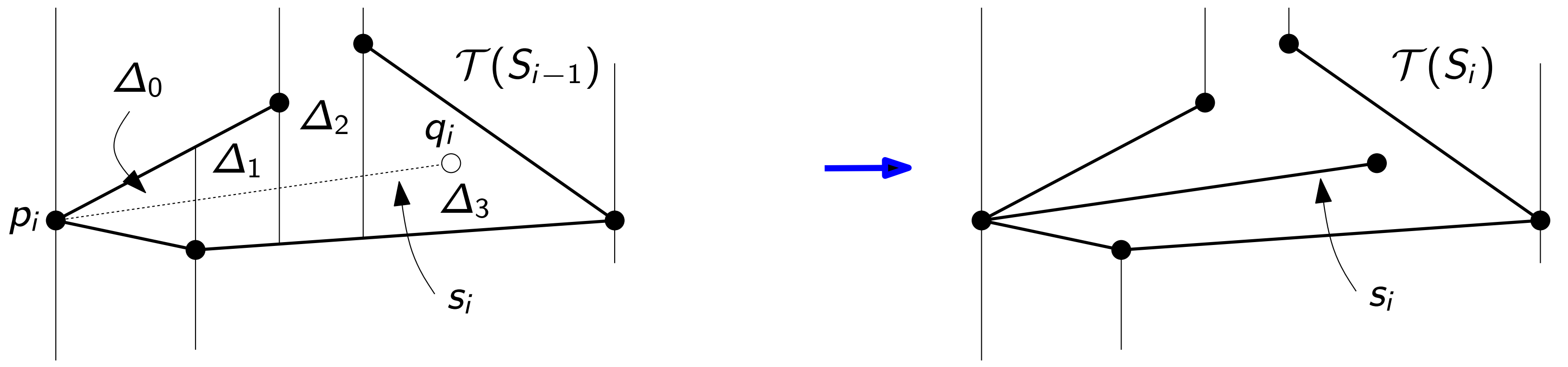
$\mathcal{T}$.remove$(\Delta_0, \ldots, \Delta_k)$

)

Walking through $\mathcal{T}$ and Updating $\mathcal{D}$



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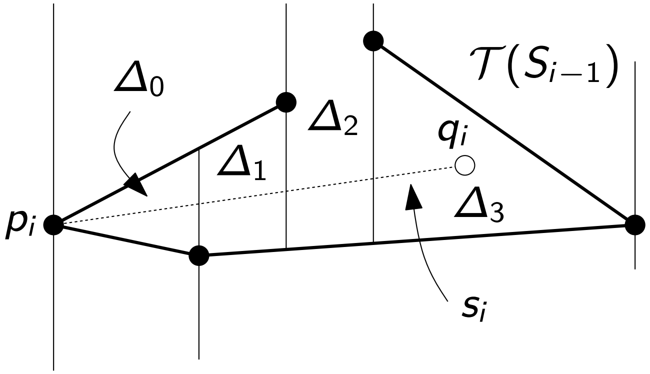

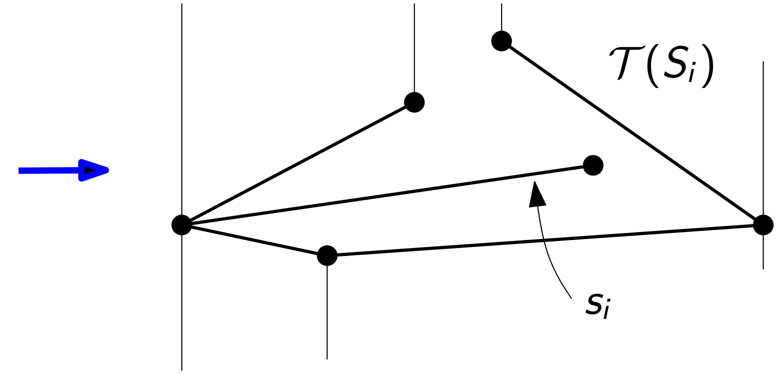

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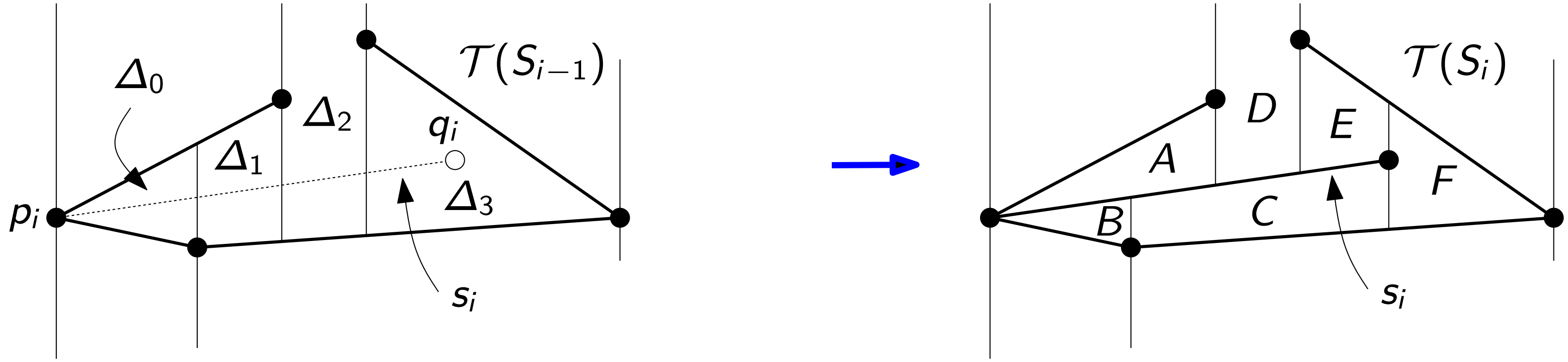

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R = BBox(S); $\mathcal{T}$.init(); $\mathcal{D}$.init()

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for $i = 1$ **to** n **do**

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)

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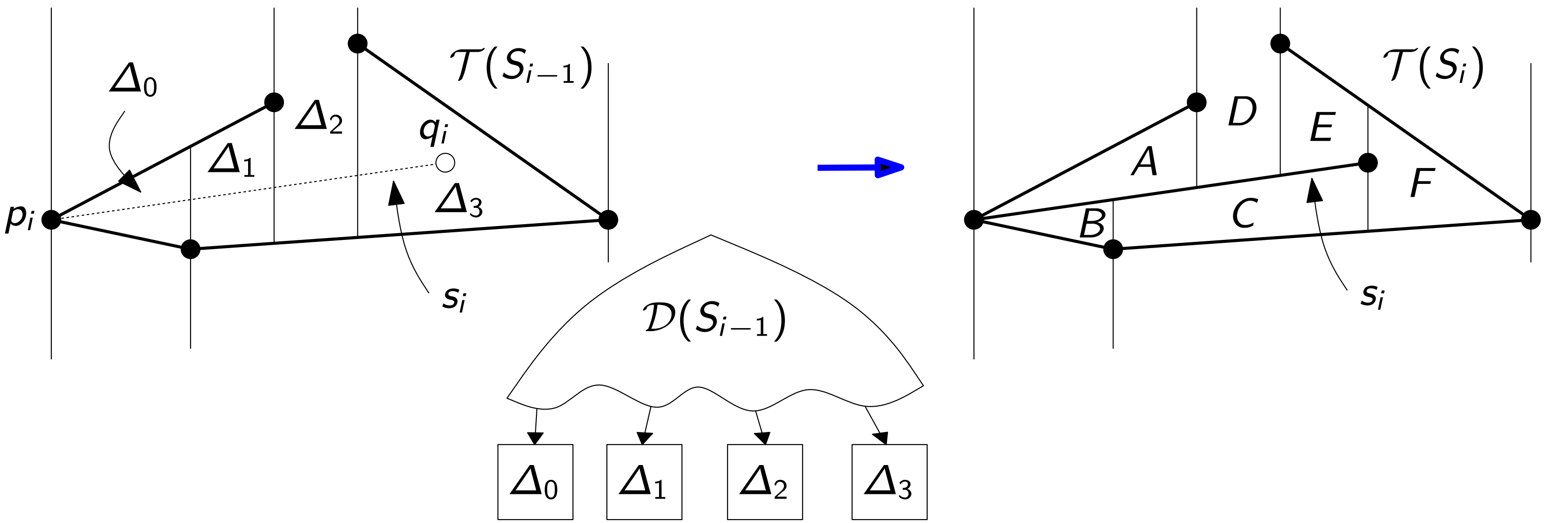

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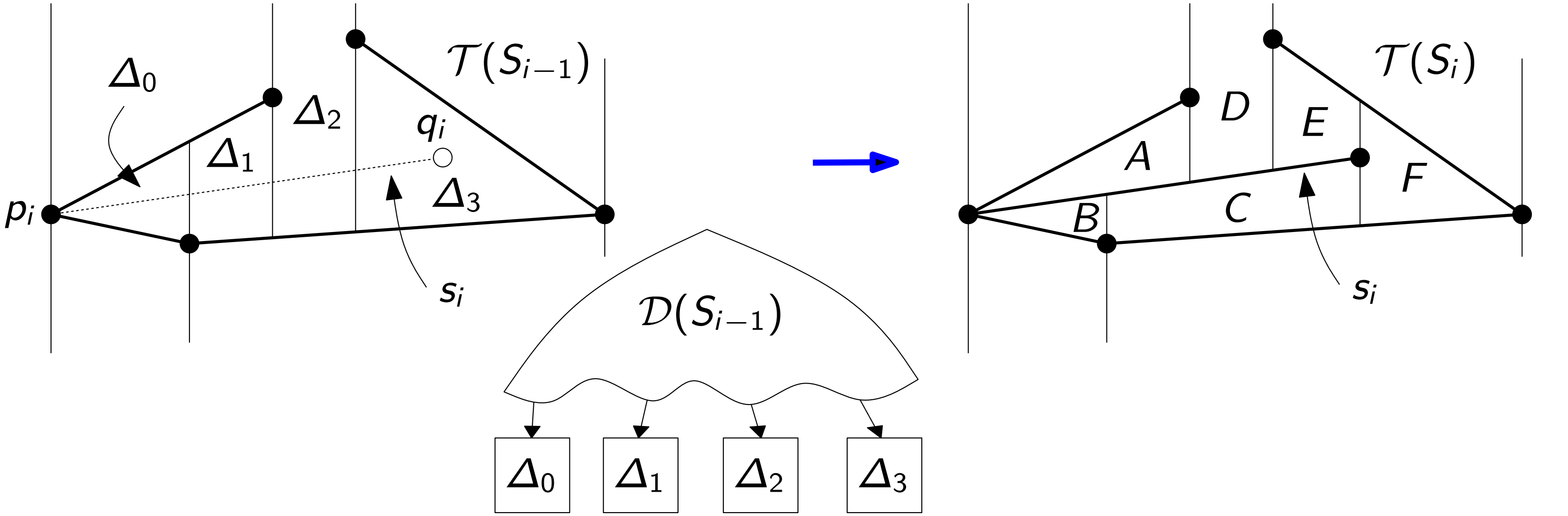

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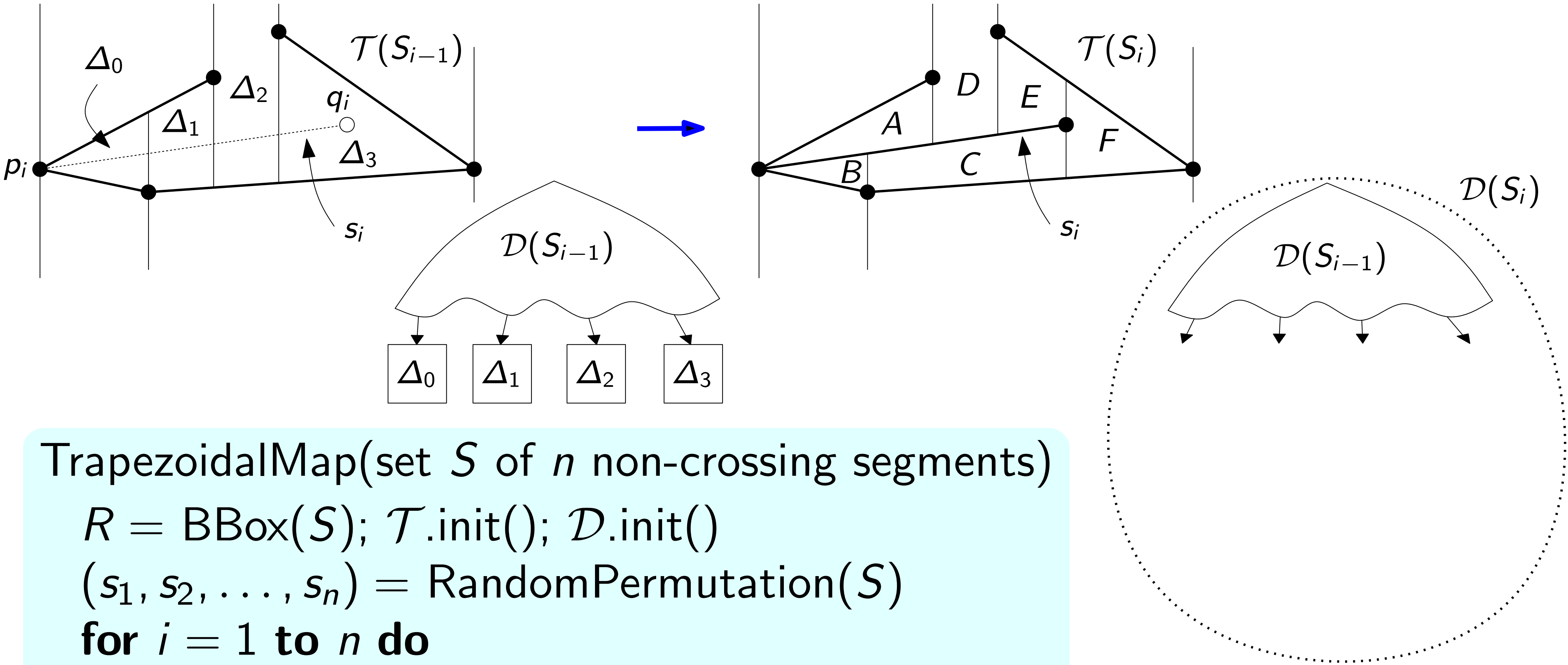
Walking through $\mathcal{T}$ and Updating $\mathcal{D}$



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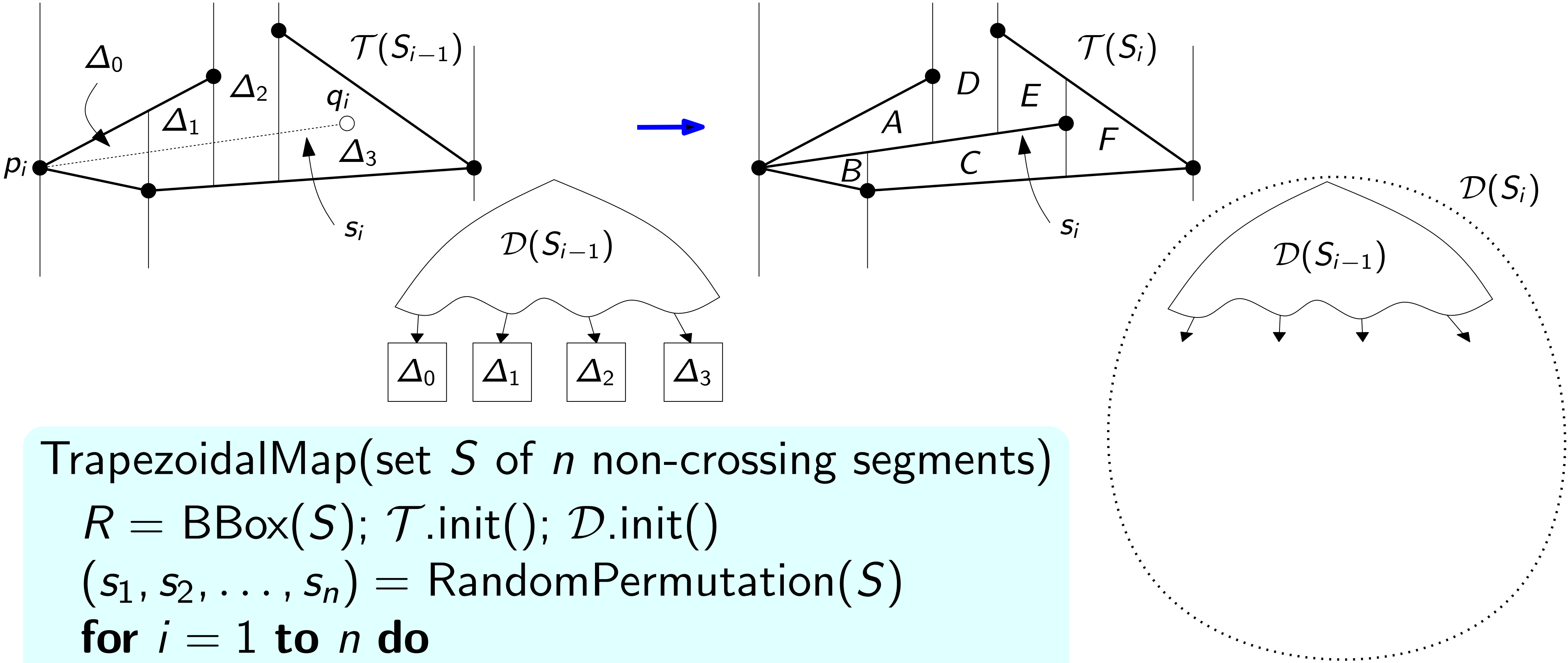

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$\mathcal{T}$.add(new trapezoids incident to s_i)
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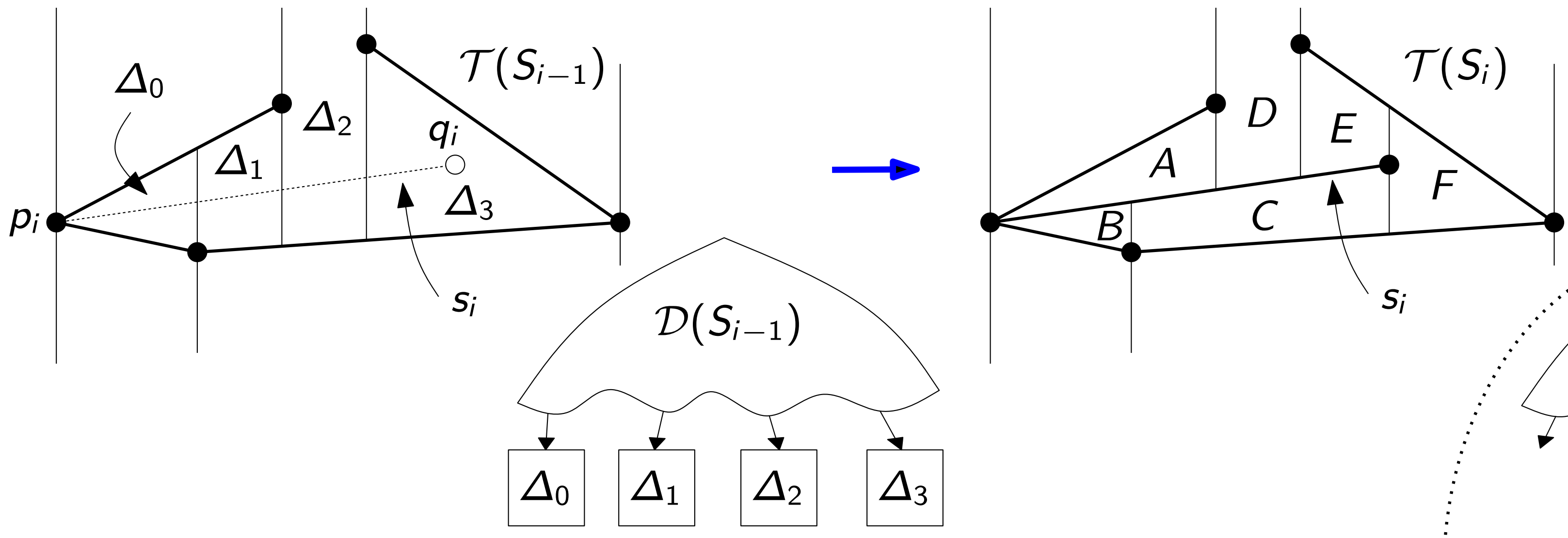
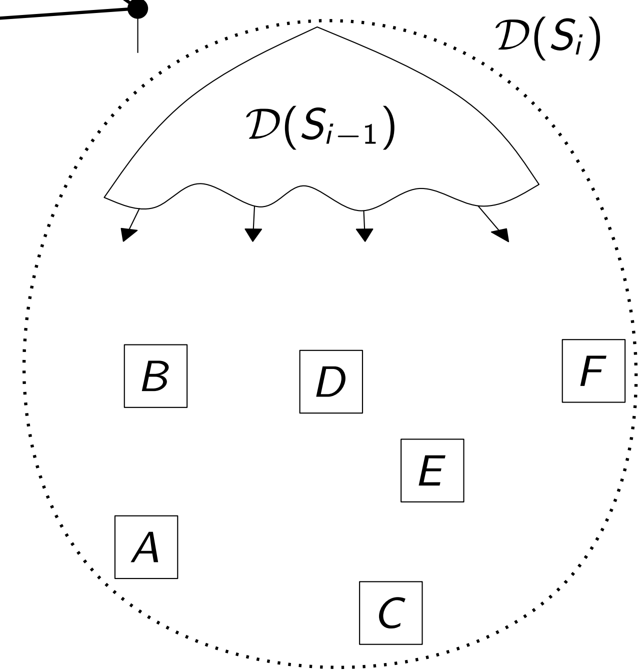
$\mathcal{D}$.add_leaves(new trapezoids incident to s_i)

Walking through $\mathcal{T}$ and Updating $\mathcal{D}$





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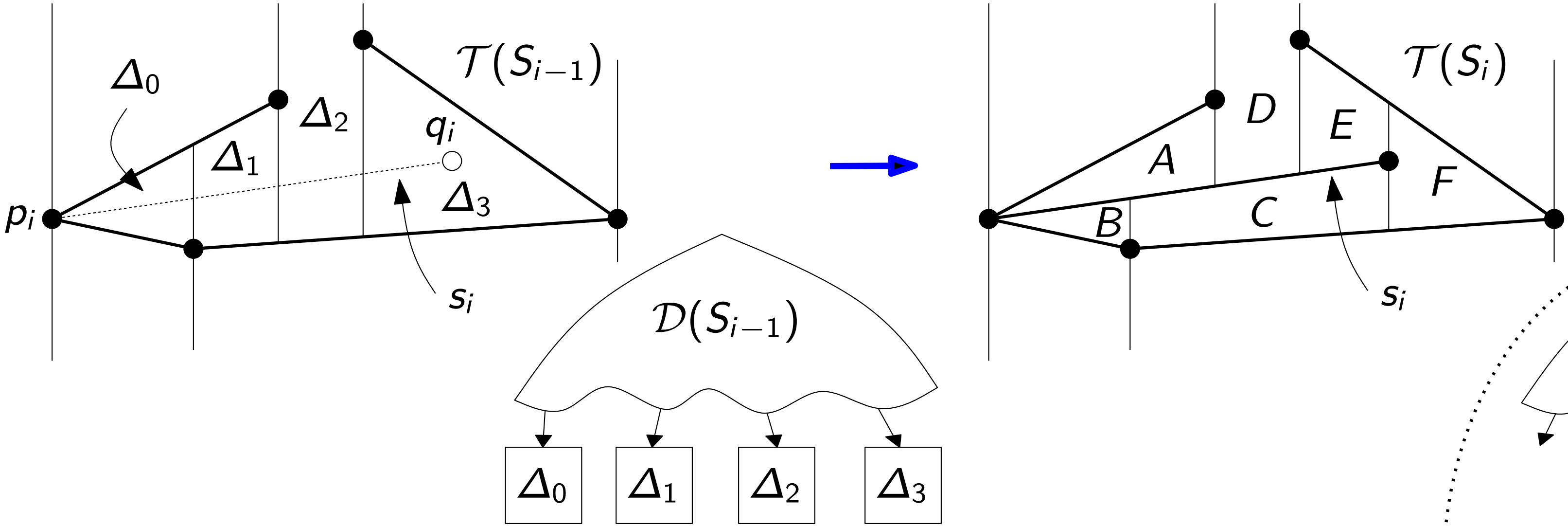

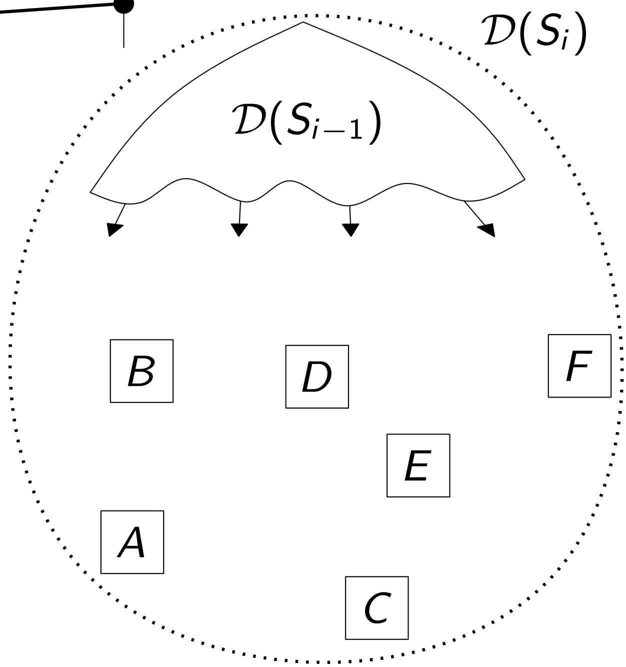

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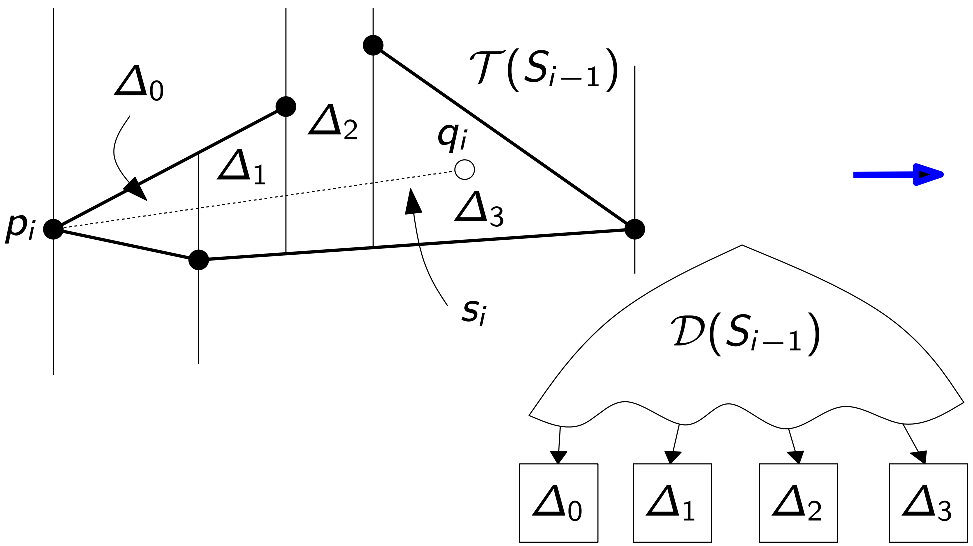

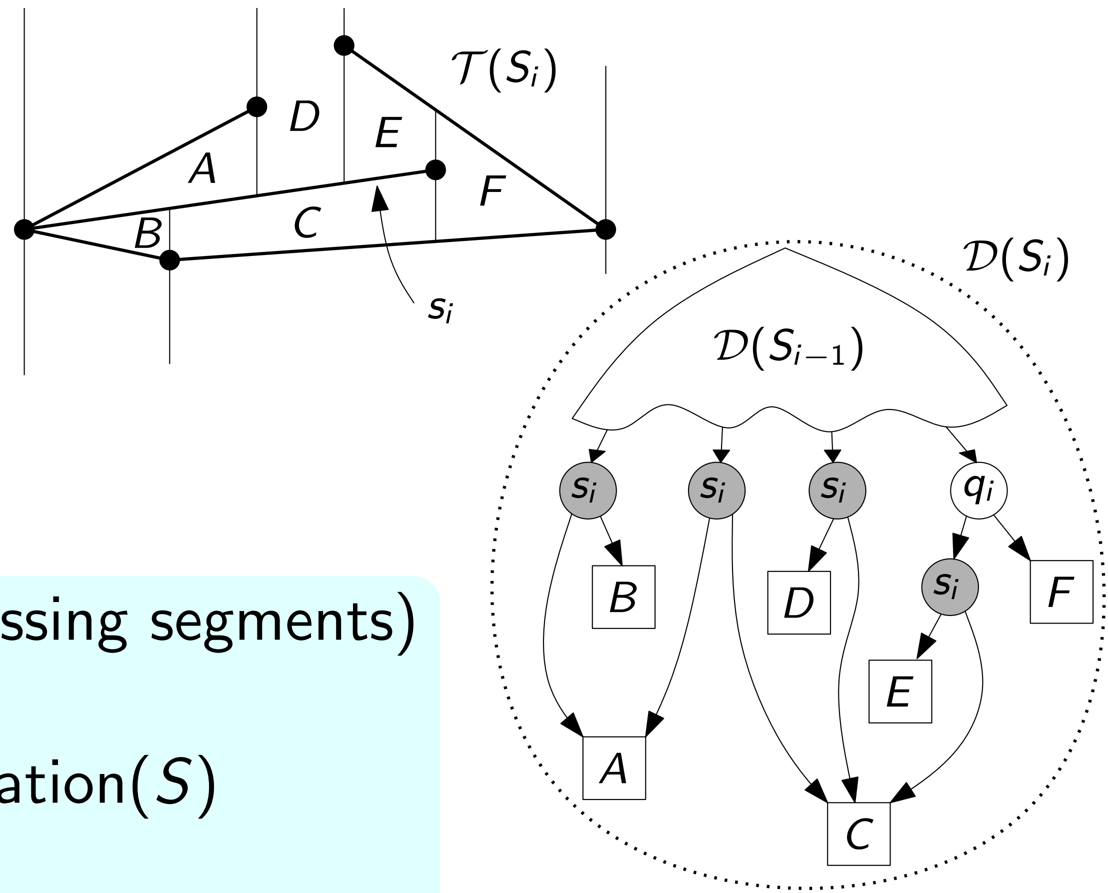

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The 2d-Result

Theorem. trapezoidalMap(S) computes $\mathcal{T}(S)$ for a set of n line segments in general position and a search structure $\mathcal{D}$ for $\mathcal{T}(S)$ in $O(n \log n)$ expected time.

The 2d-Result

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The 2d-Result

Theorem. trapezoidalMap(S) computes $\mathcal{T}(S)$ for a set of n line segments in general position and a search structure $\mathcal{D}$ for $\mathcal{T}(S)$ in $O(n \log n)$ expected time. The expected size of $\mathcal{D}$ is $O(n)$ and the expected query time is $O(\log n)$.

Invariant: Before step i, $\mathcal{T}$ is a trapezoidal map for S_{i-1} and $\mathcal{D}$ is a valid search structure for $\mathcal{T}$.

Proof.
– Correctness by loop invariant.
– Query time similar to 1d analysis.
$\Rightarrow$ construction time