Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Computational Geometry

Winter term 2014/15

Orthogonal Range Queries

or

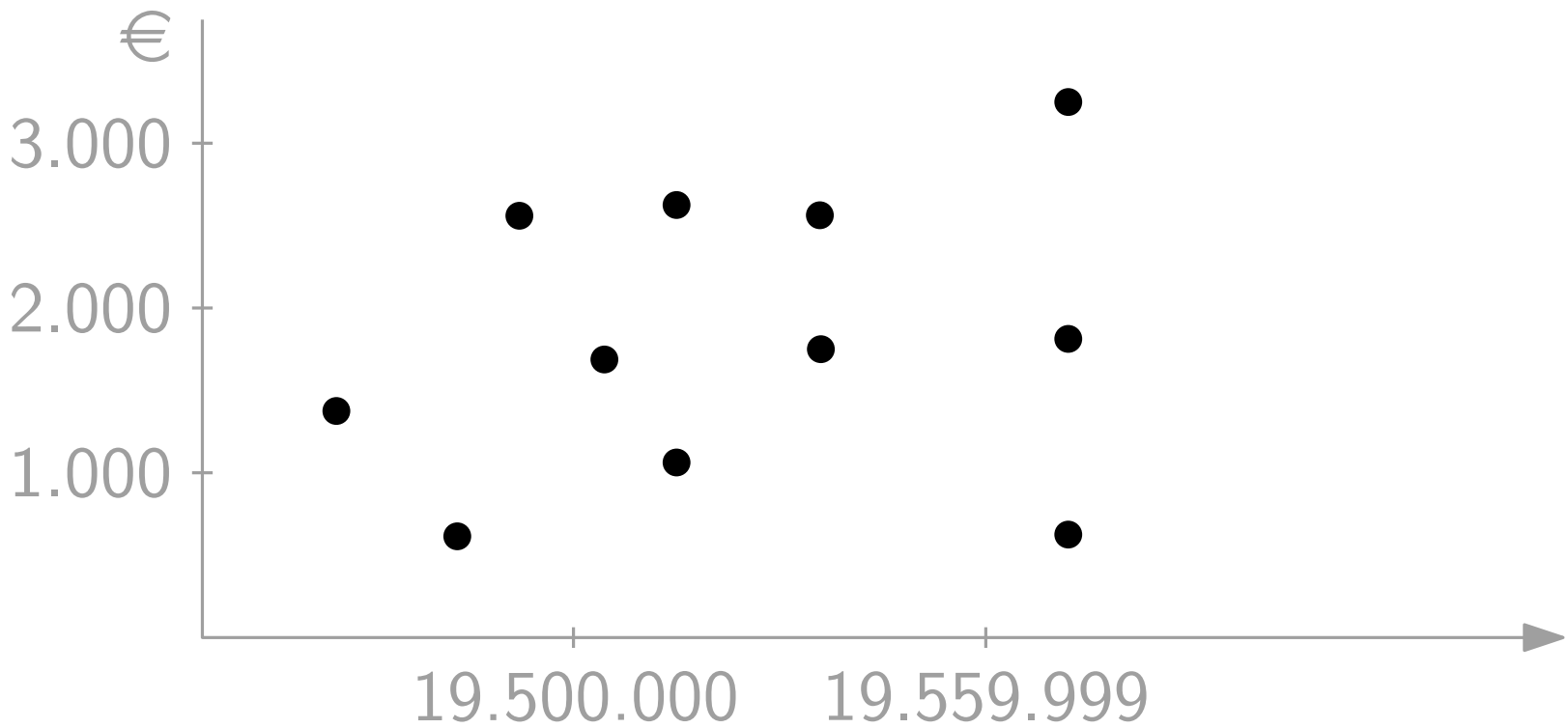
Fast Access to Data Bases

Lecture #5

Dipl.-Inf. Philipp Kindermann,
Prof. Dr. Alexander Wolff *Chair for Computer Science I*

Orthogonal Range Queries

Example: Personnel management in a company



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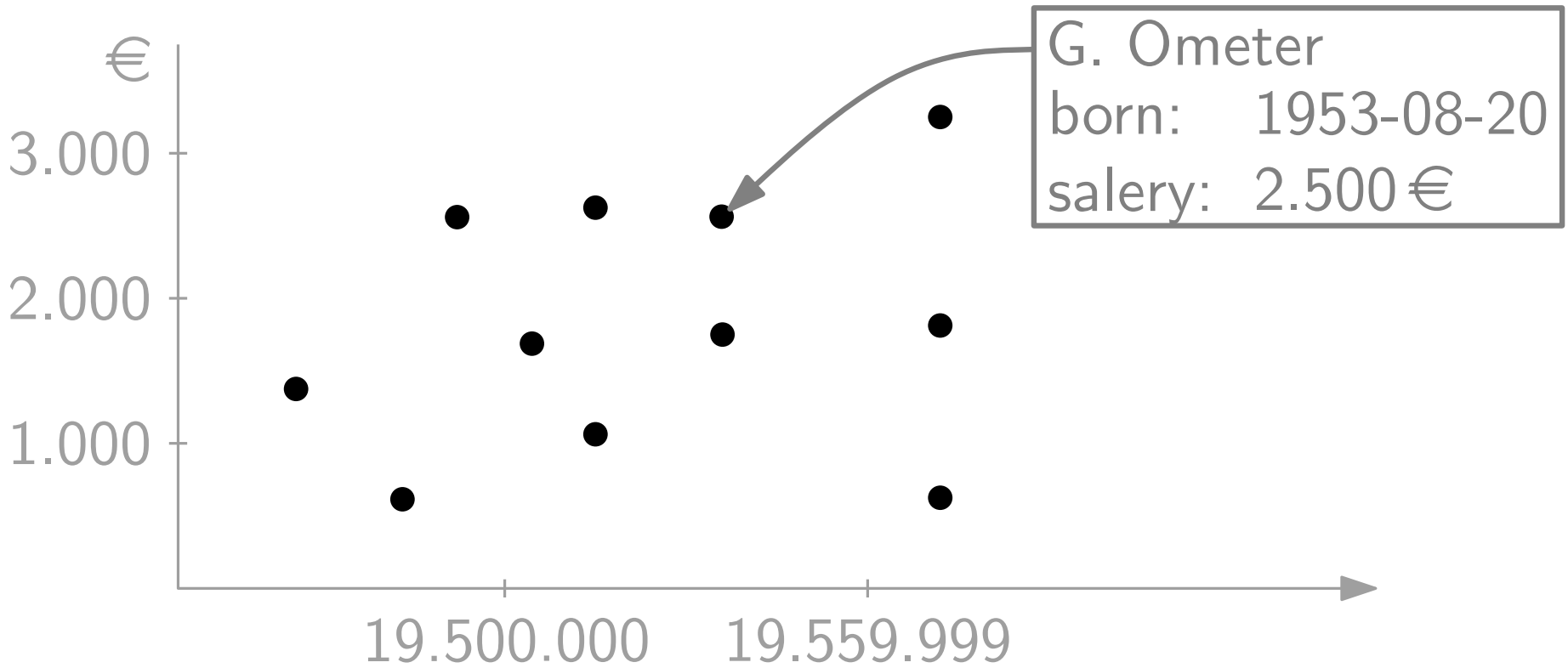

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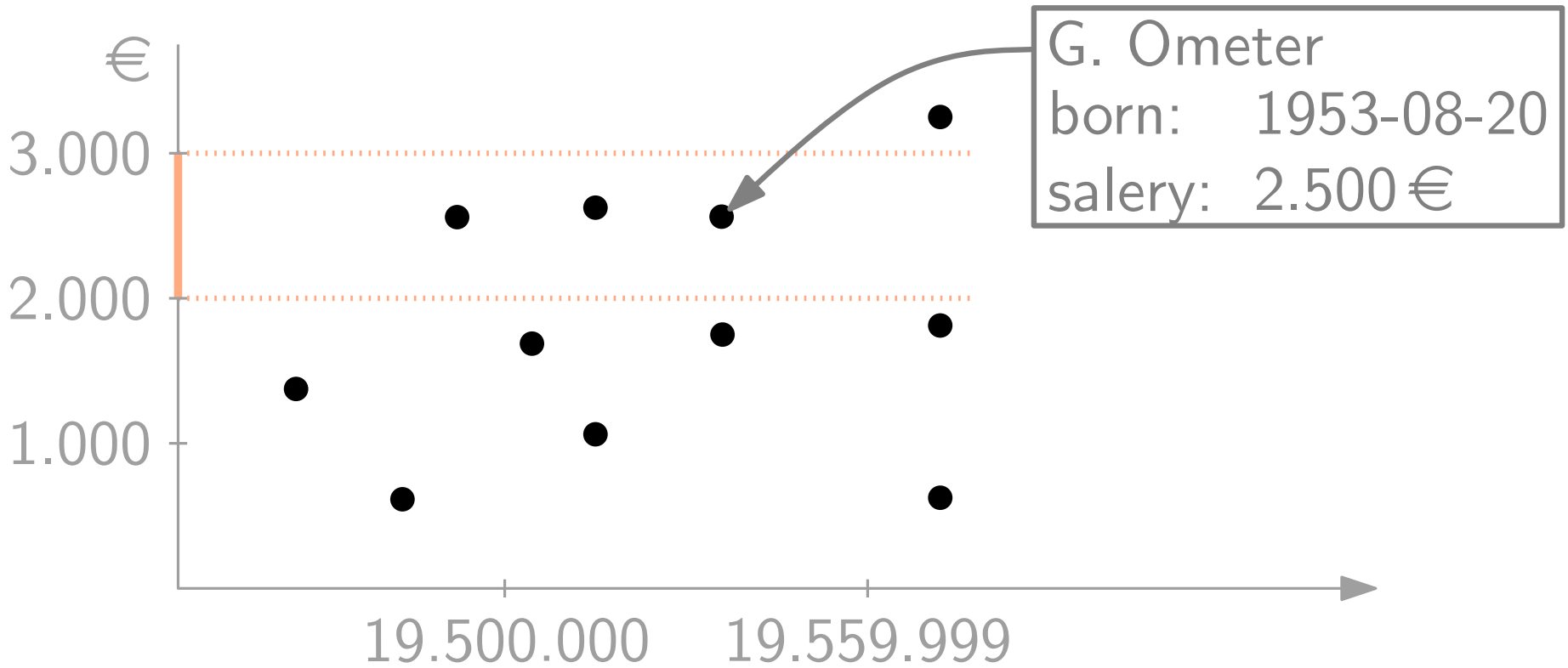

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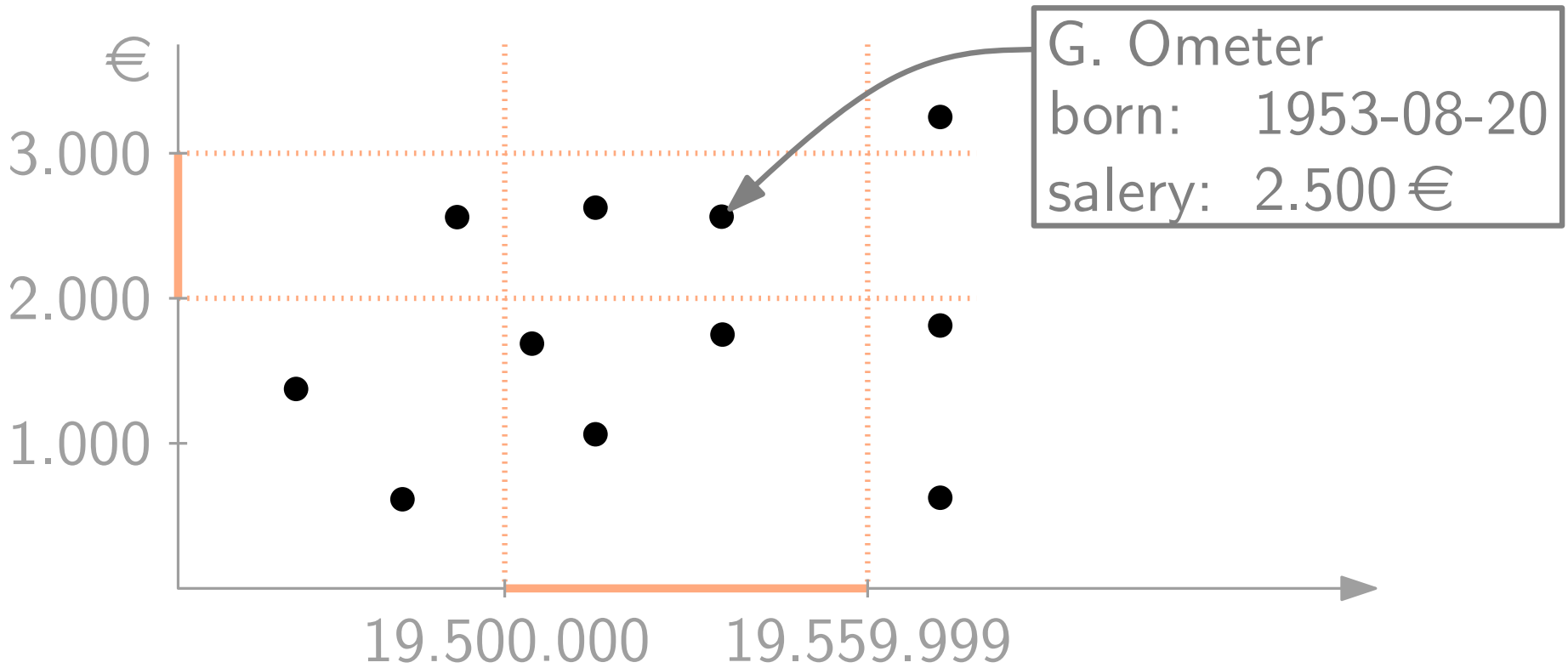

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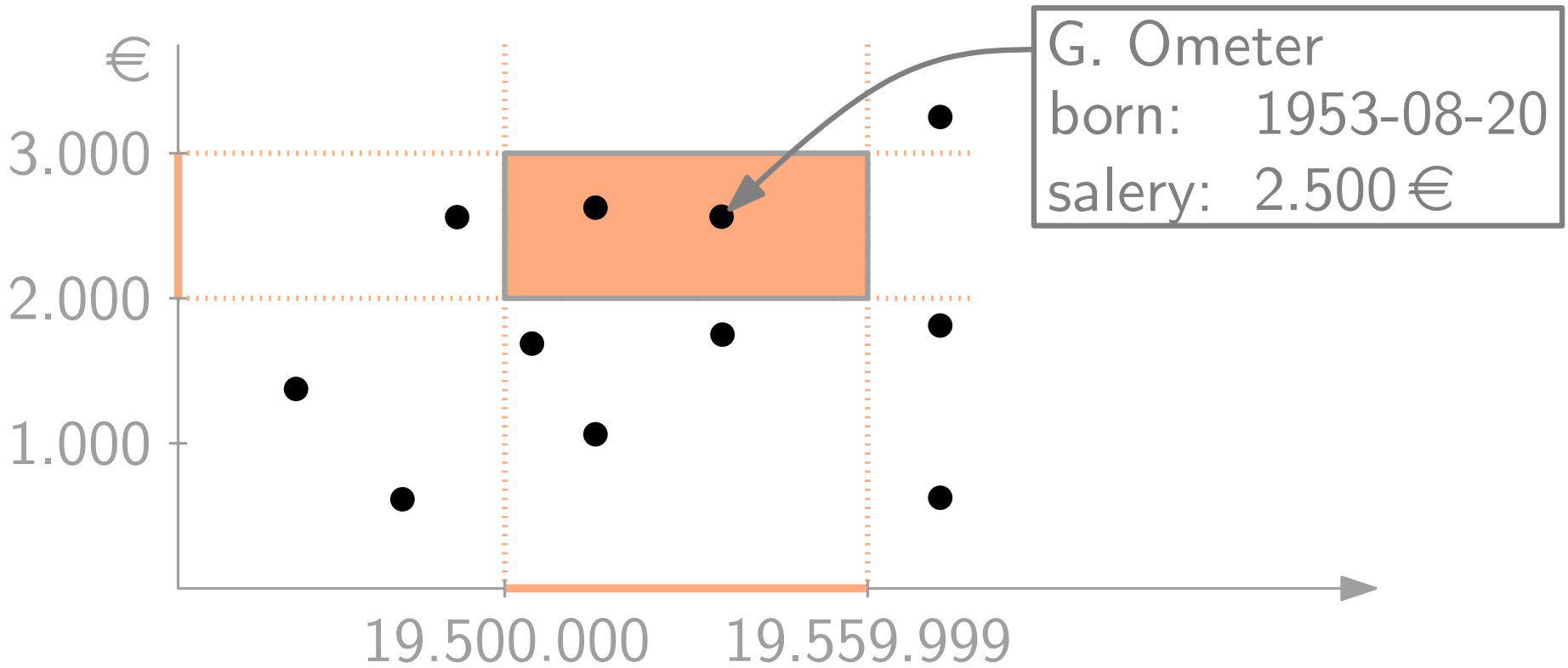

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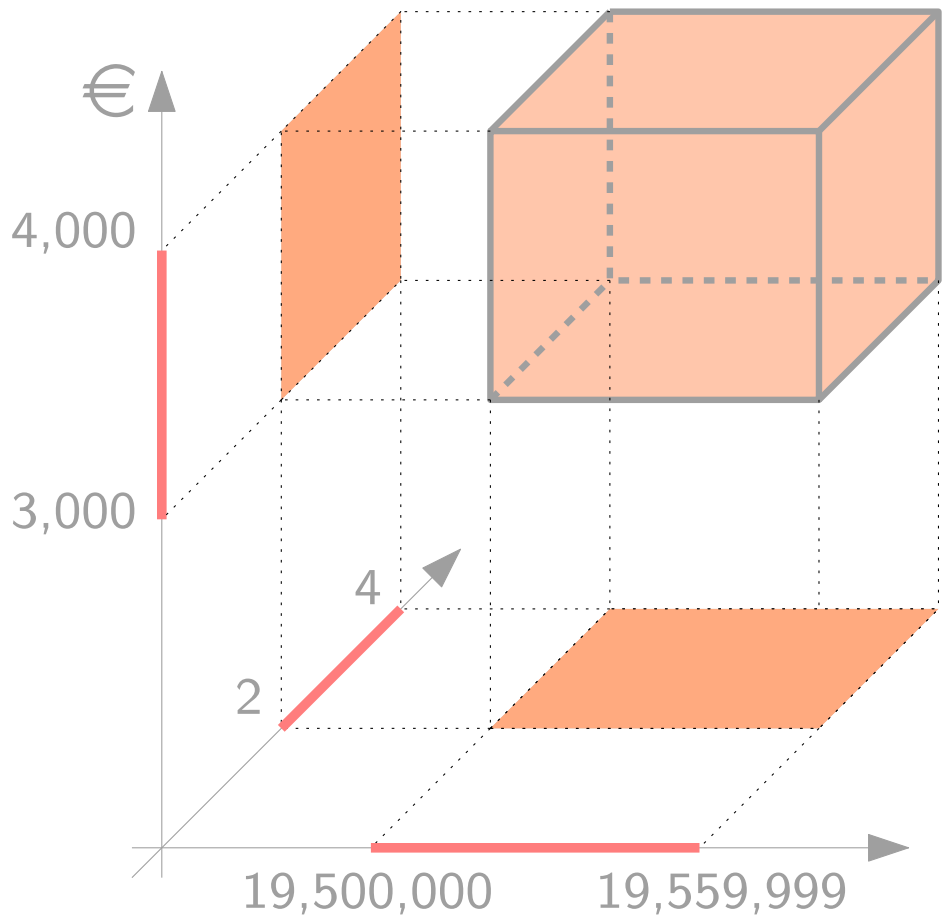

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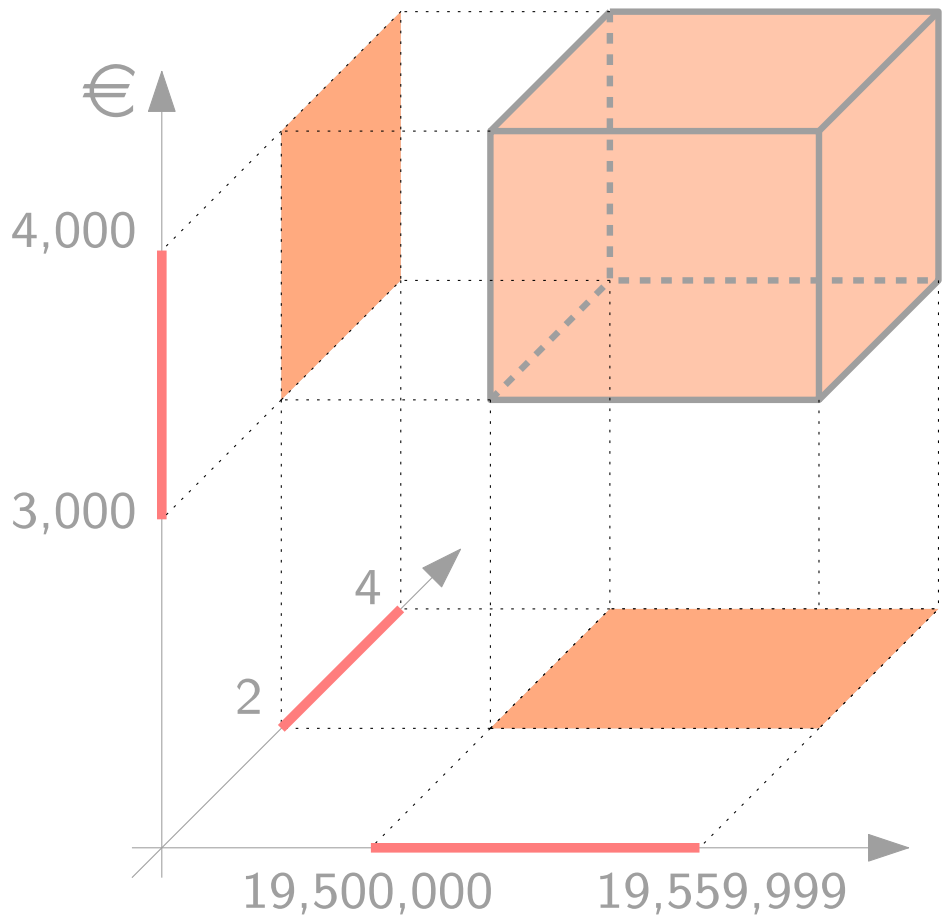

Typical queries for data bases!

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Task: Preprocess a finite set $P \subset \mathbb{R}$ such that for any interval $[x, x']$ the set $P \cap [x, x']$ can be reported quickly. [2 min]

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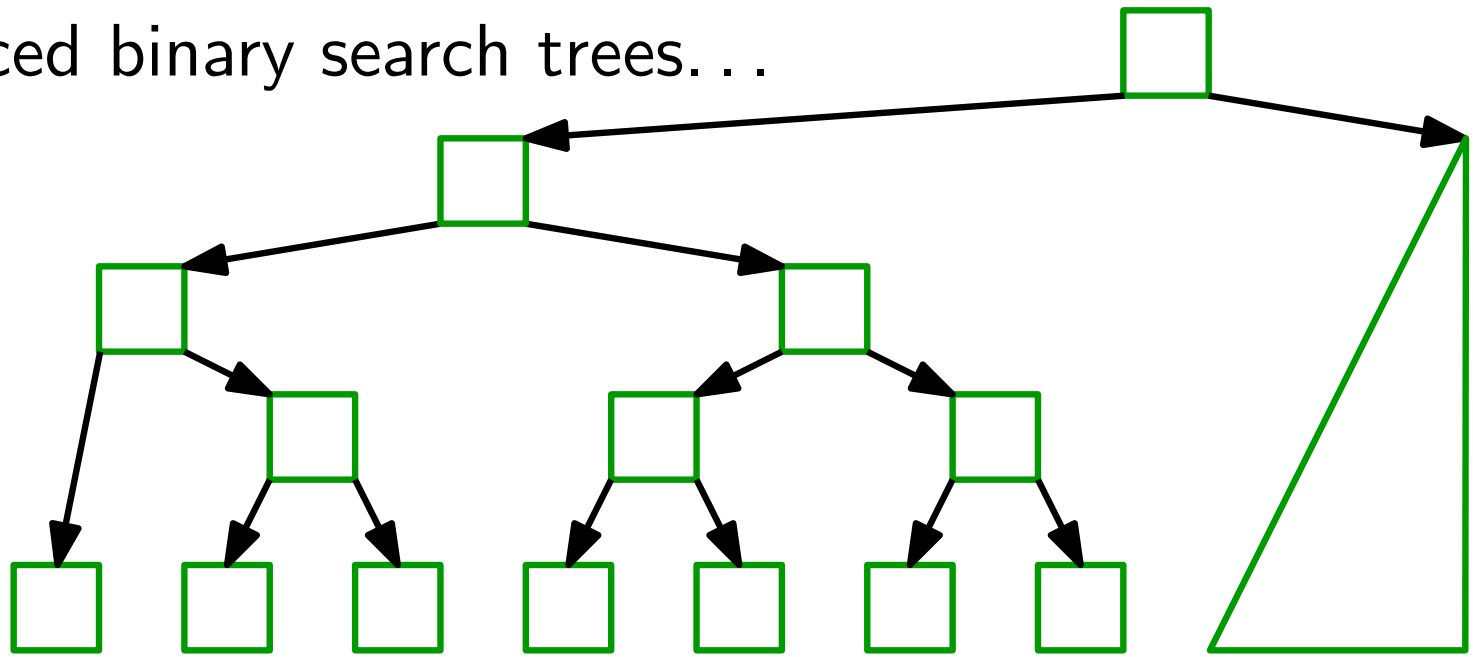

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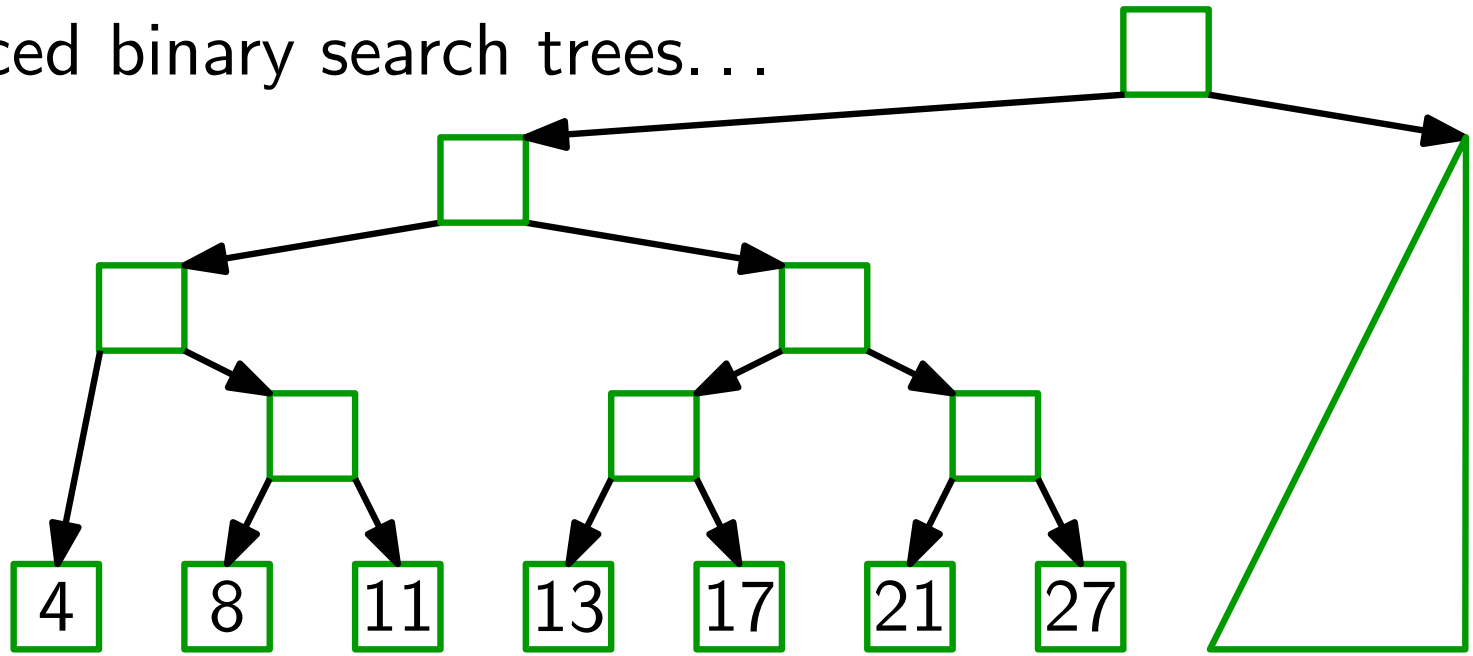

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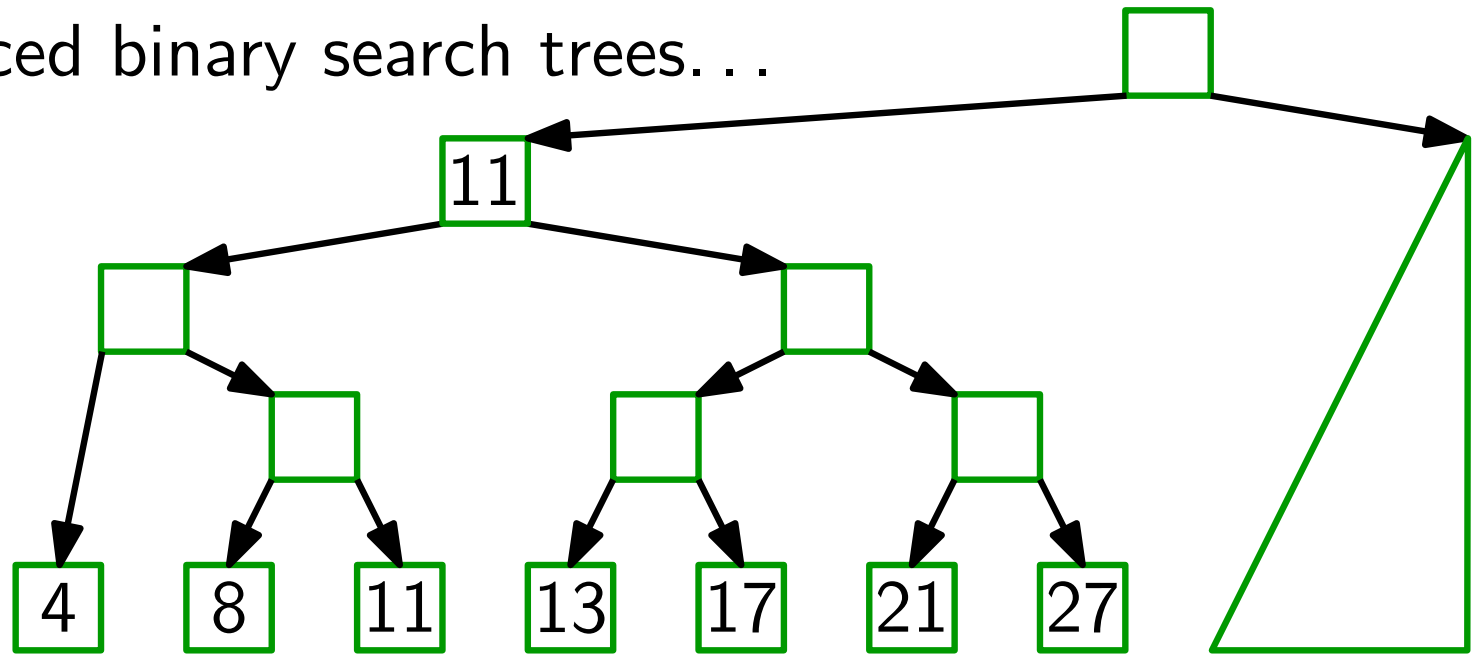

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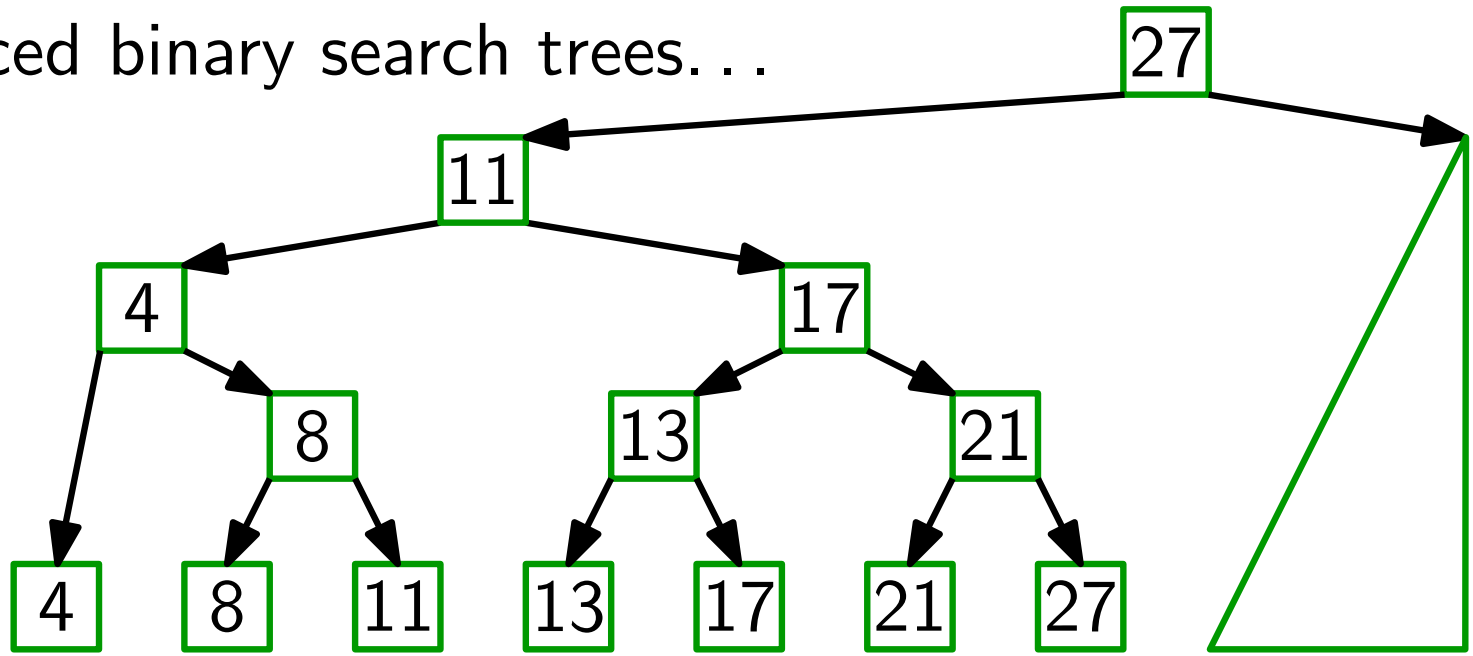

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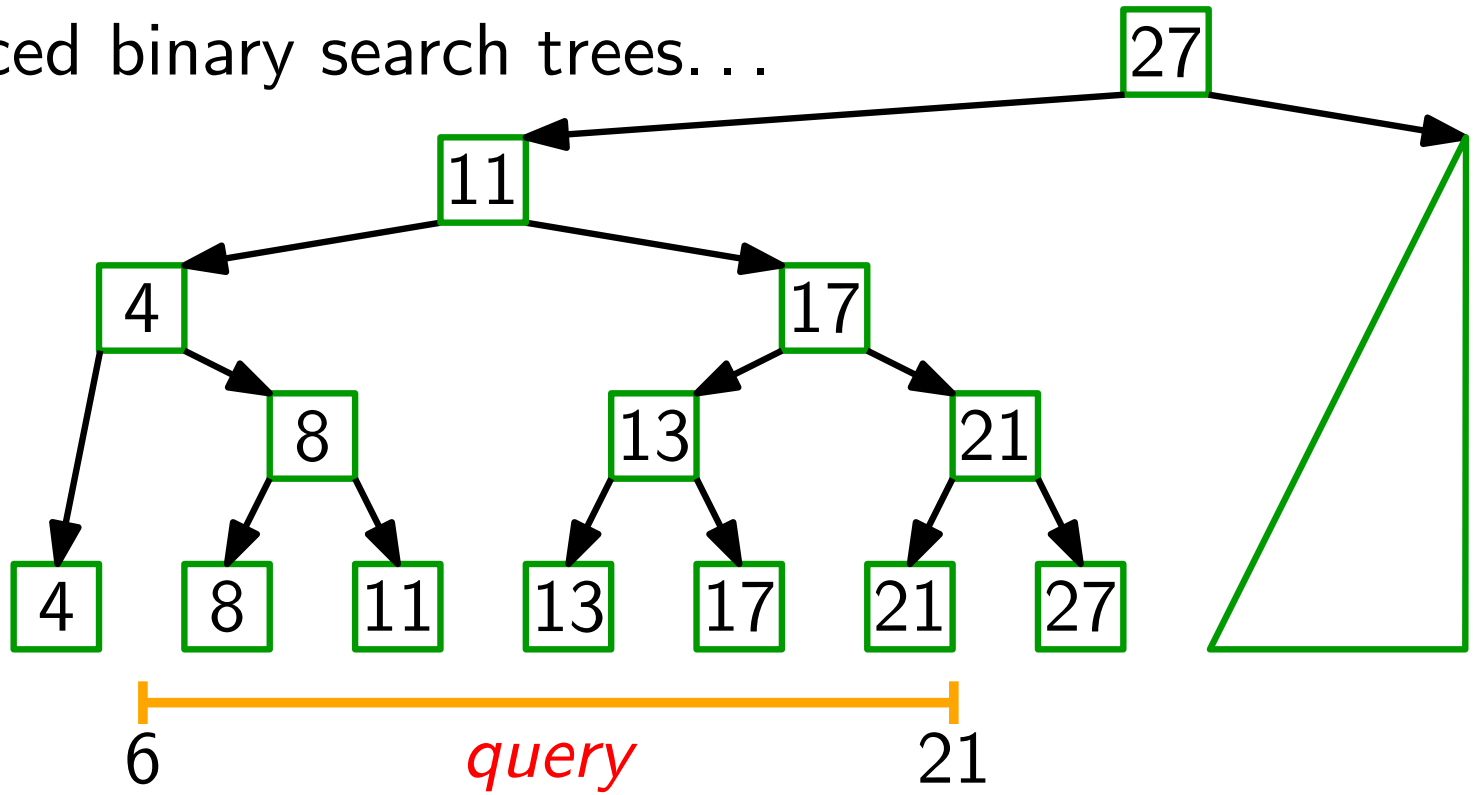

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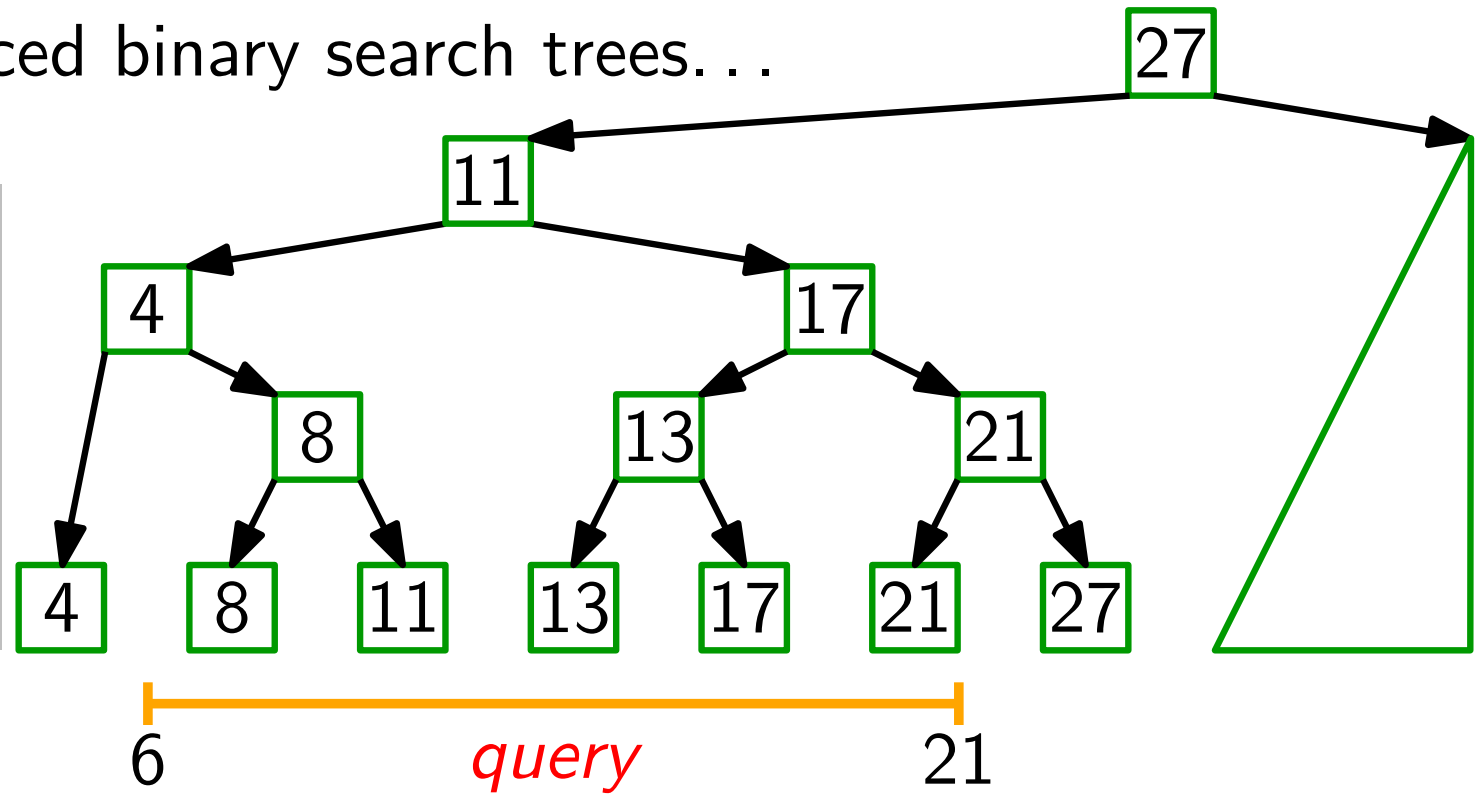

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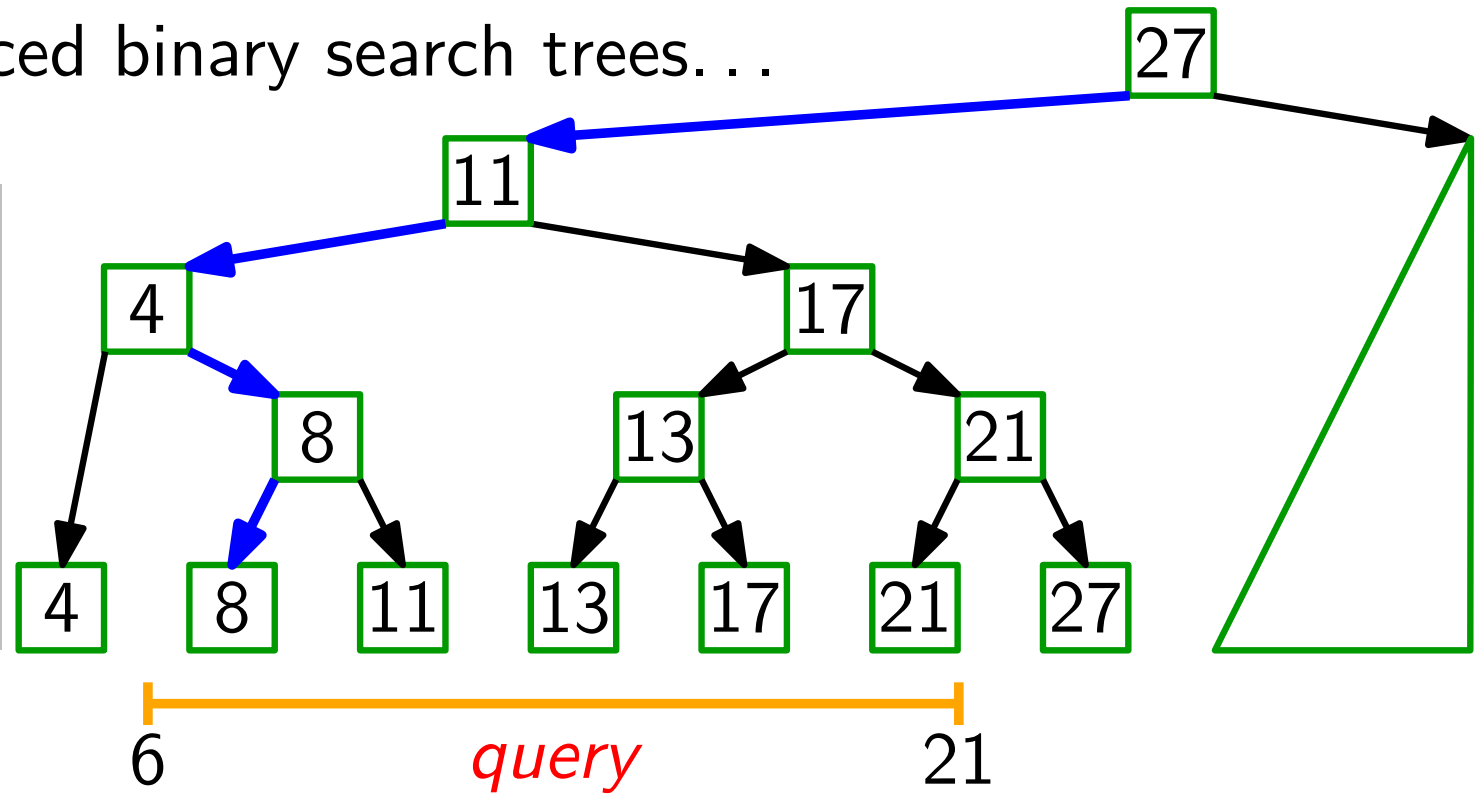

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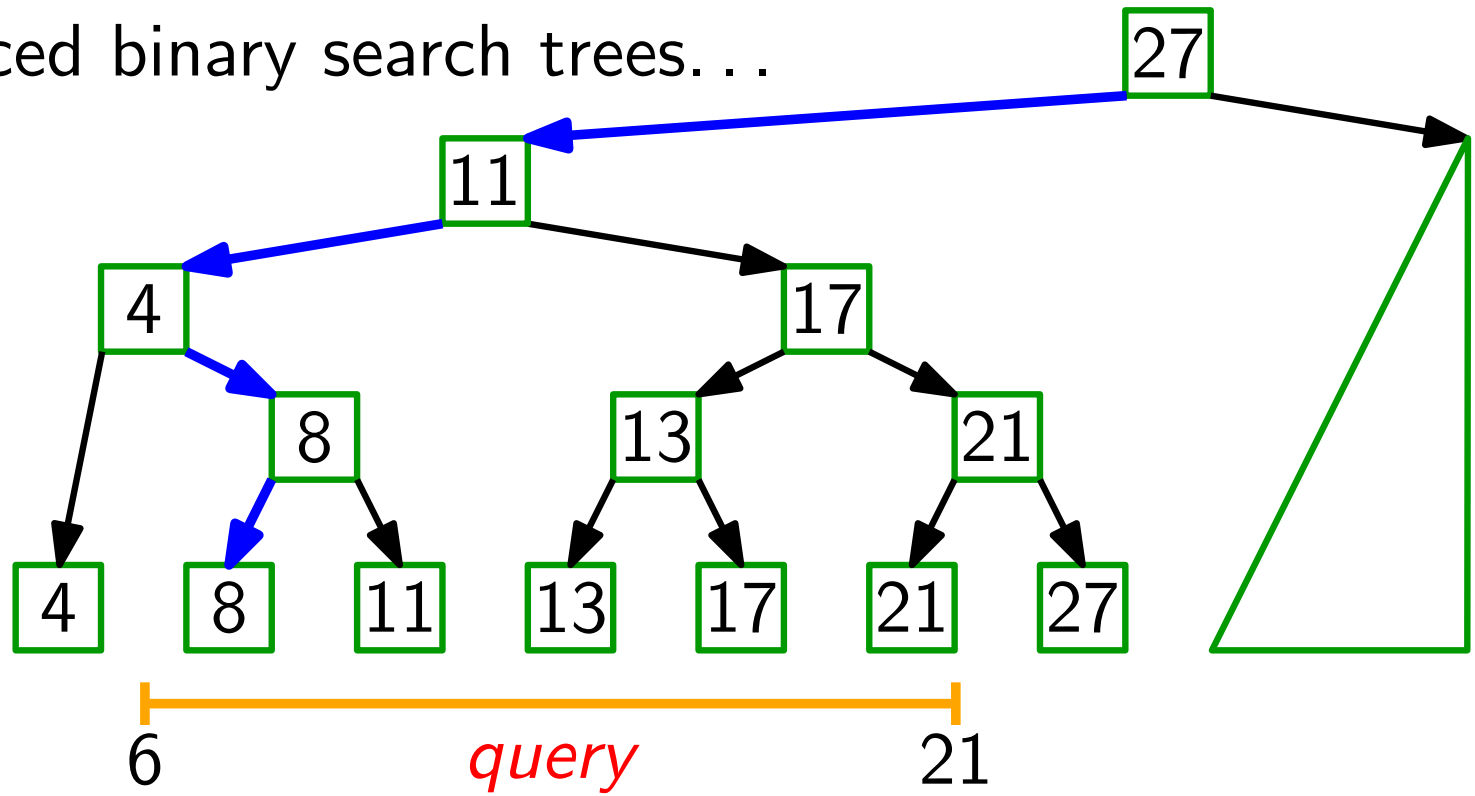

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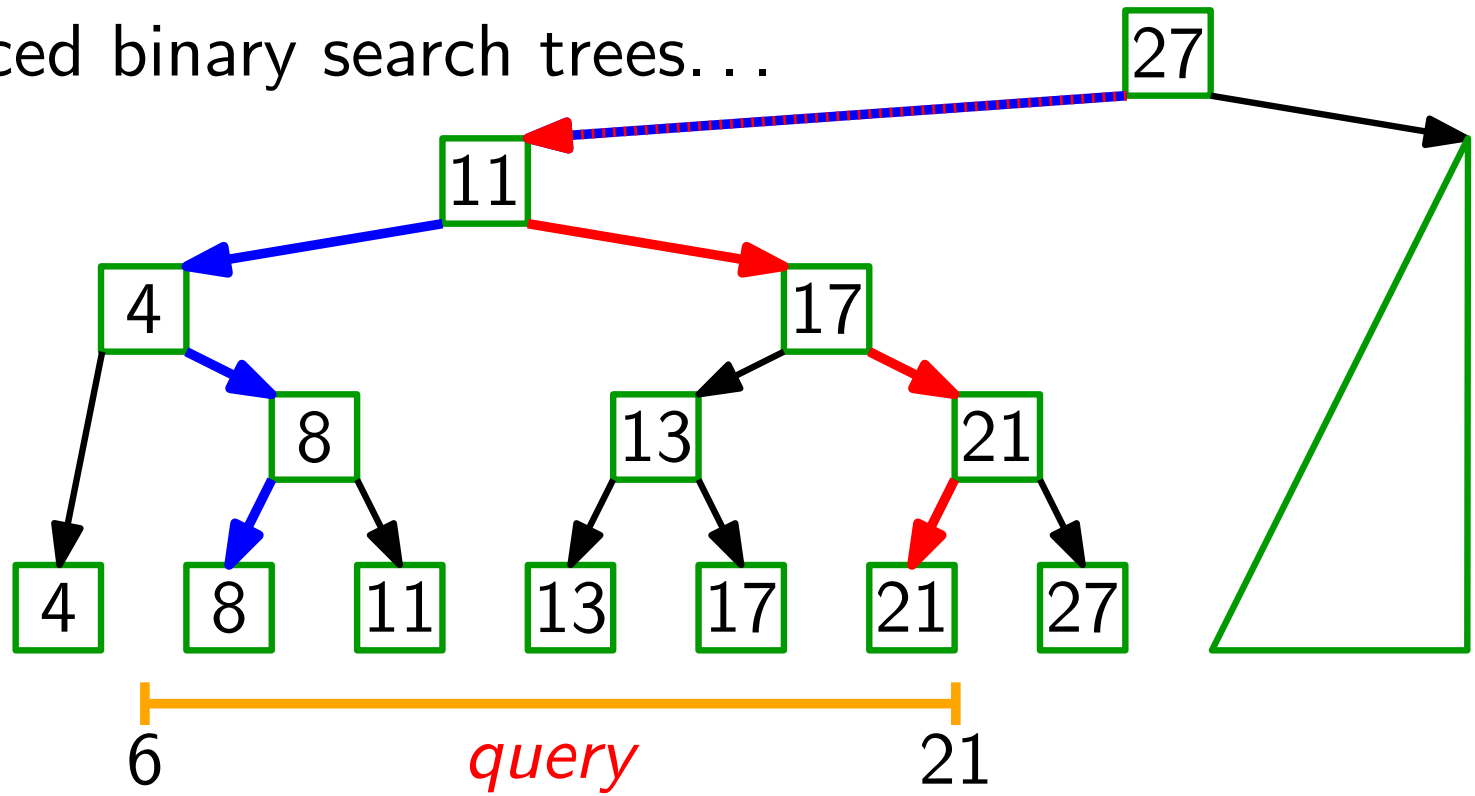

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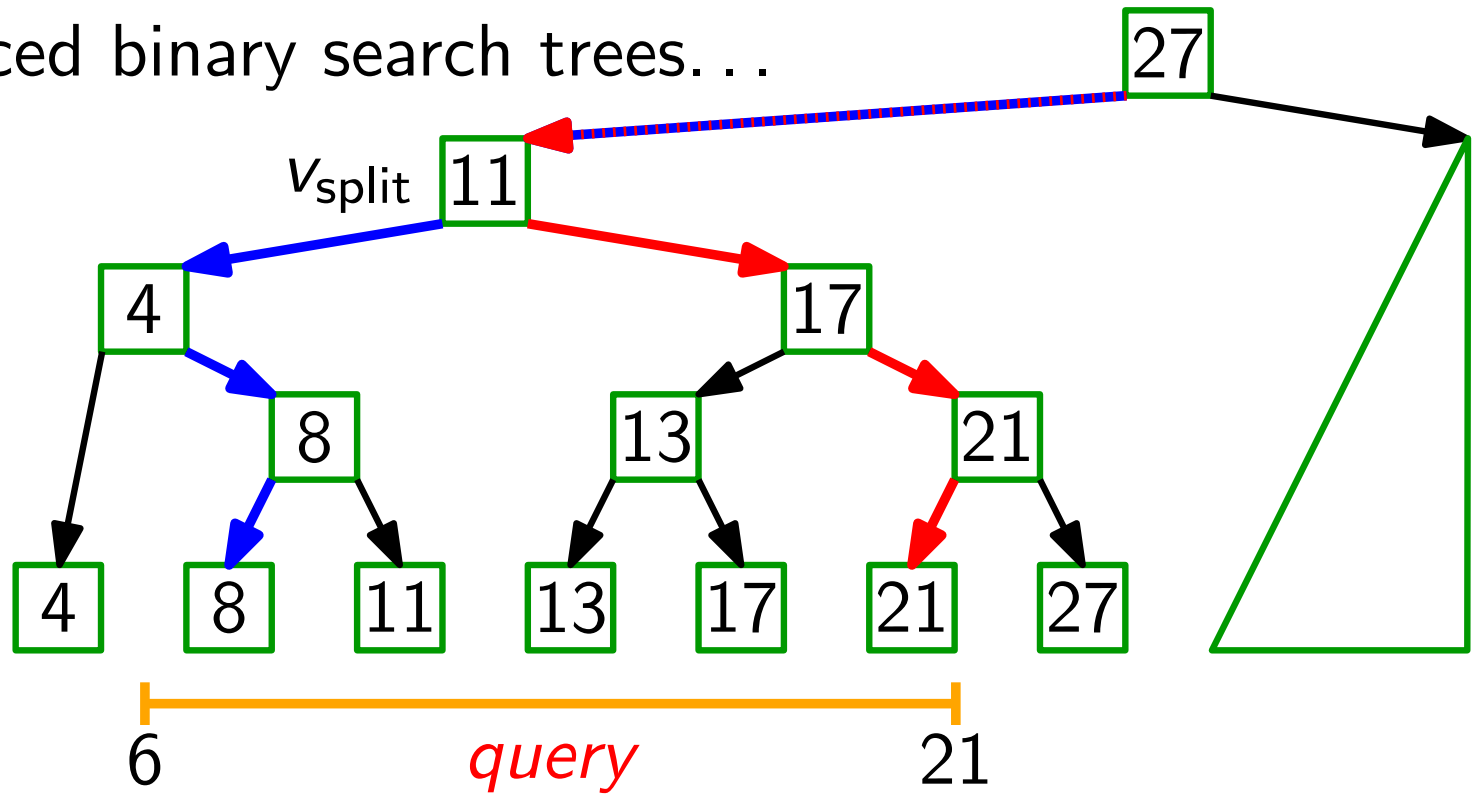

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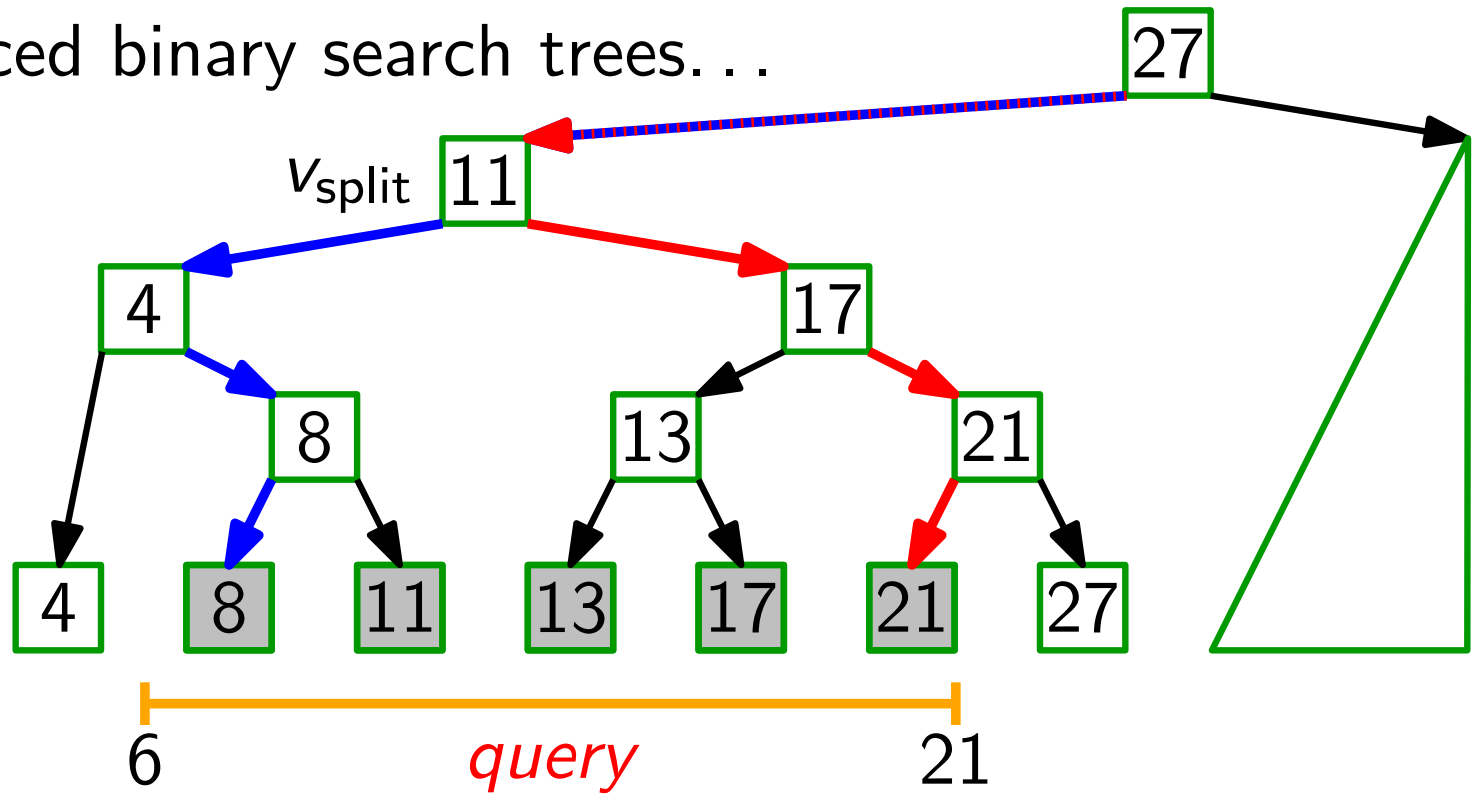

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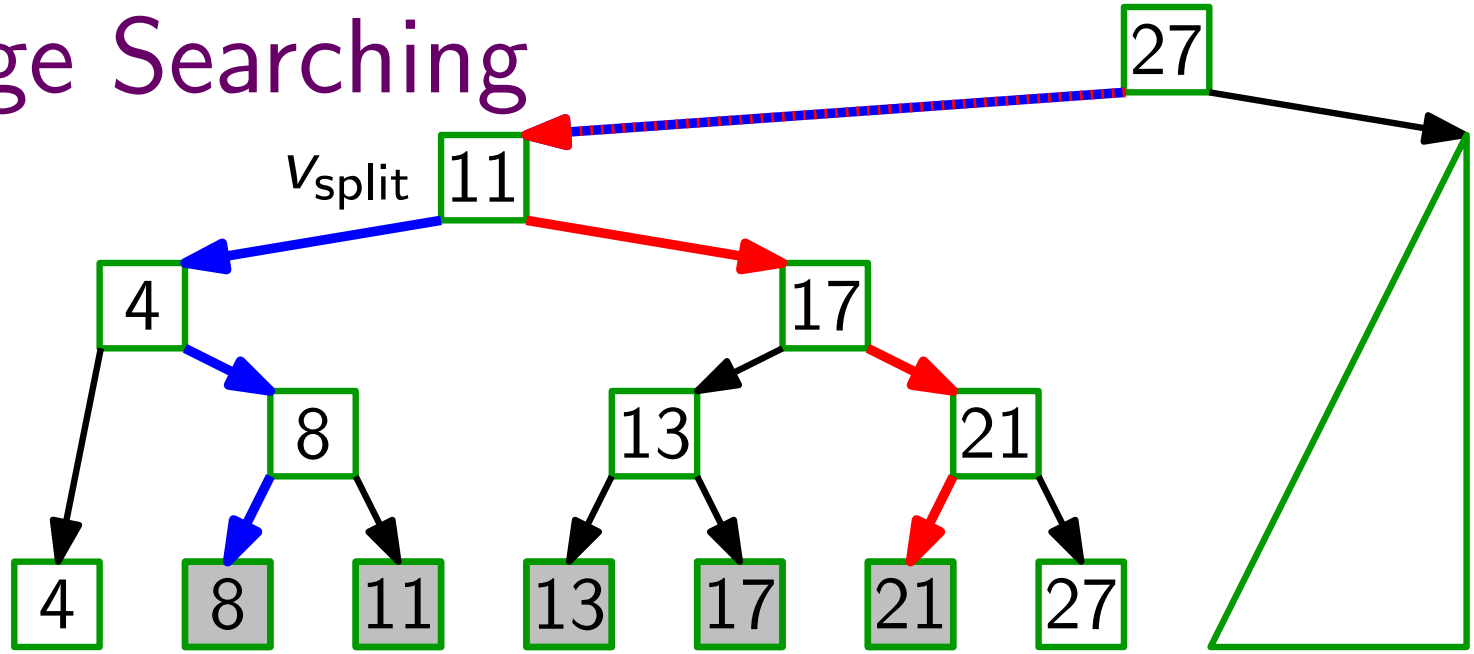

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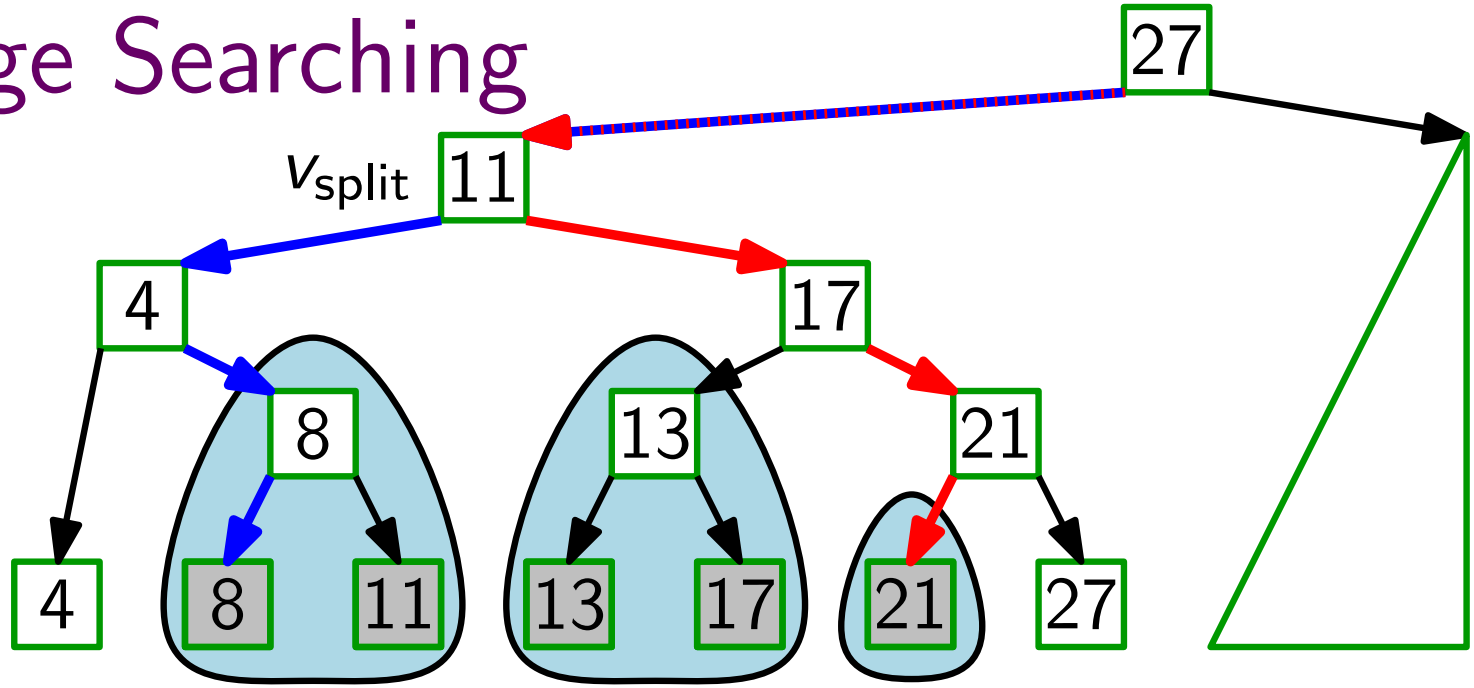

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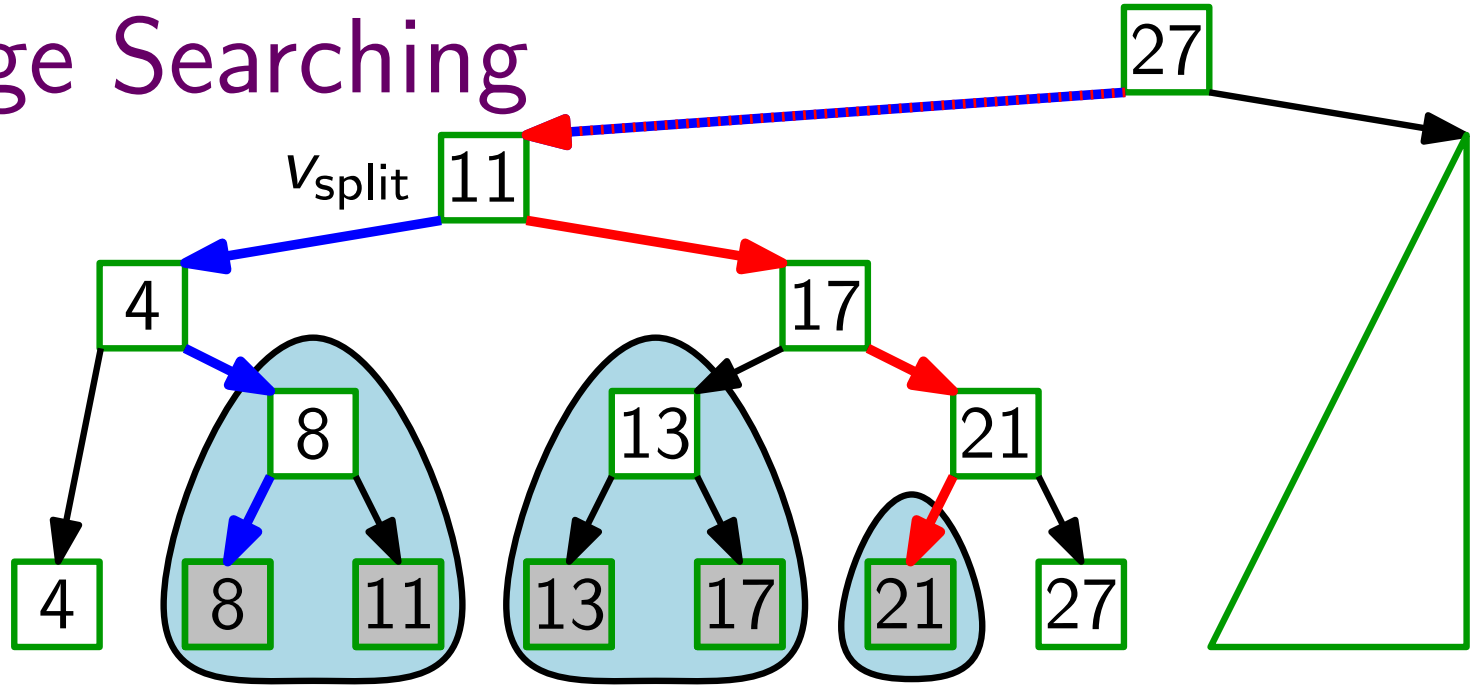

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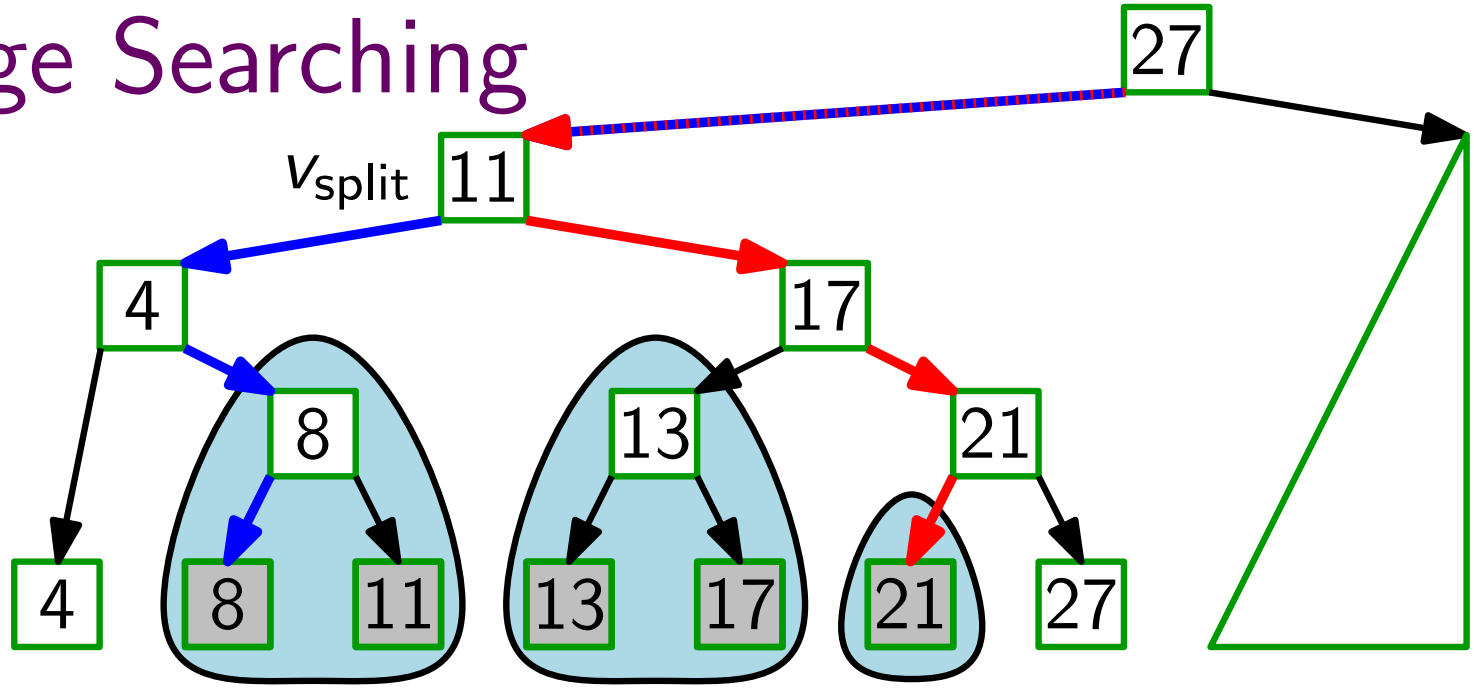

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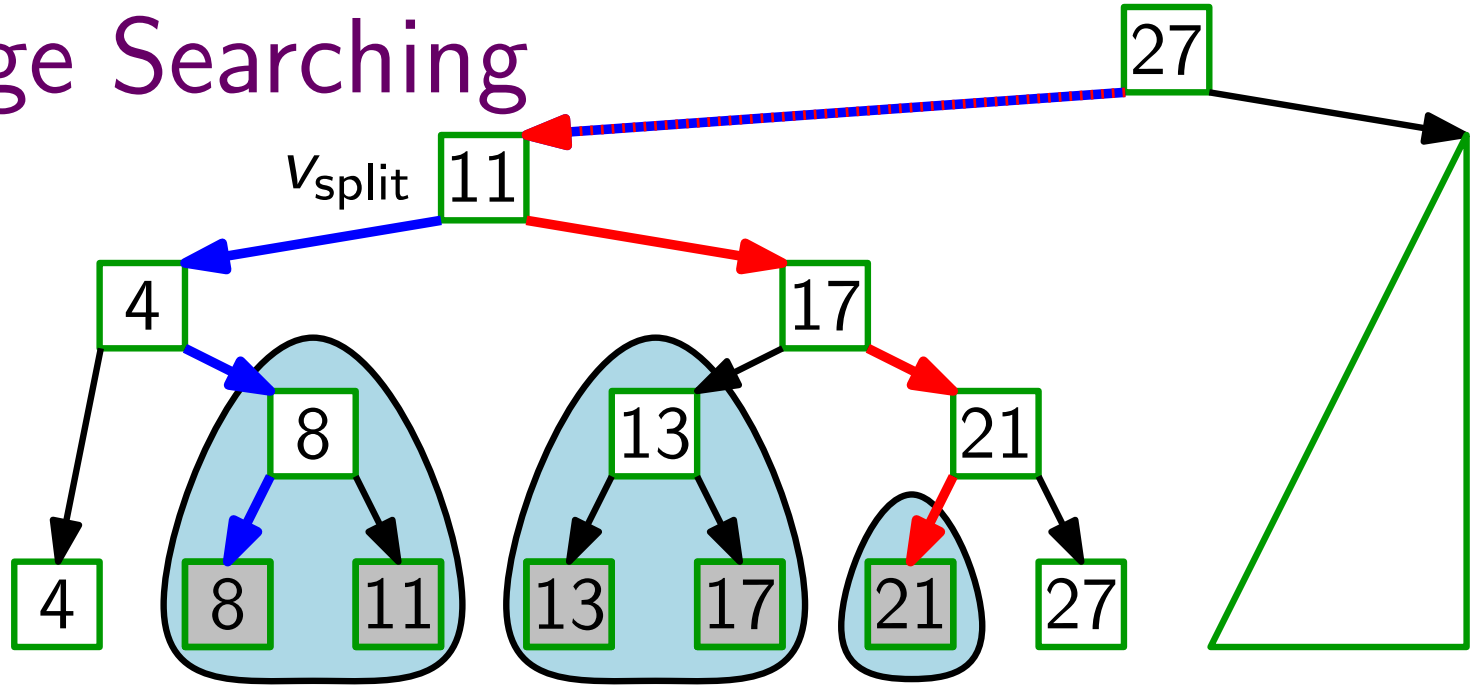

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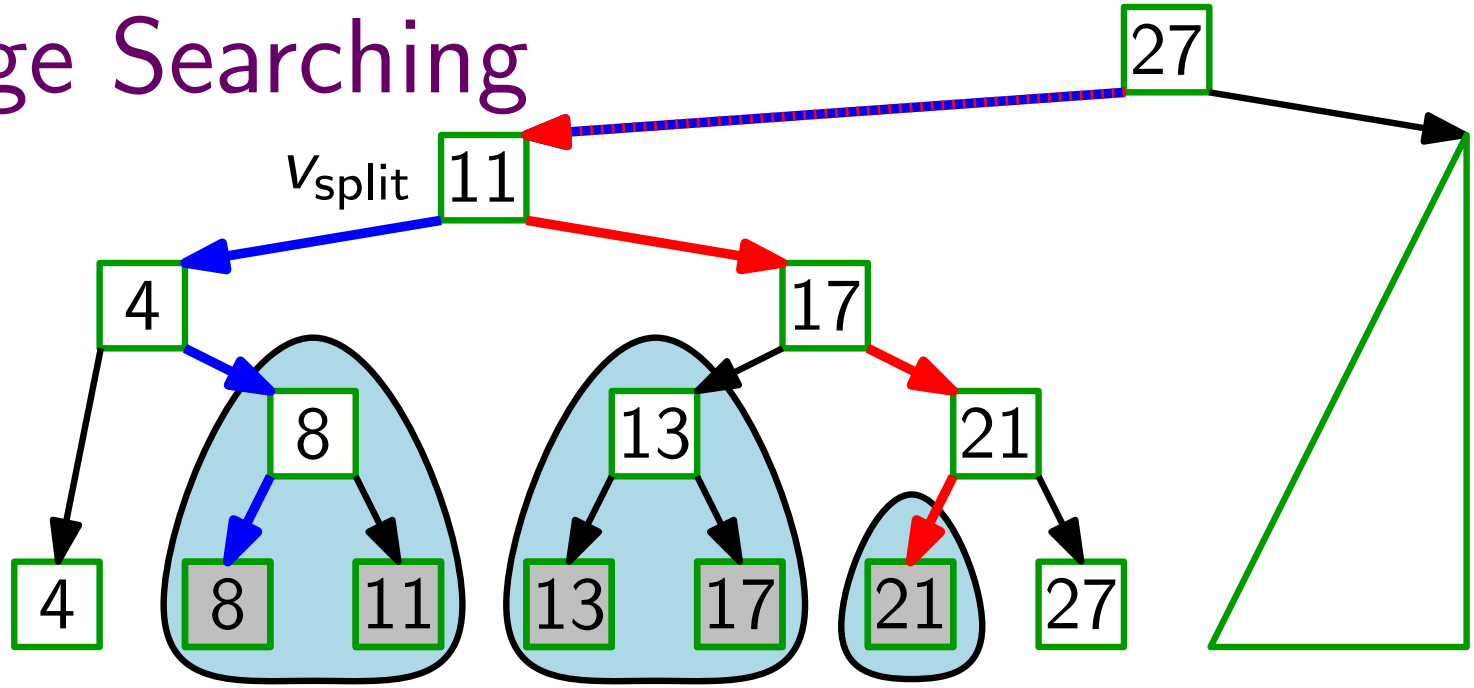

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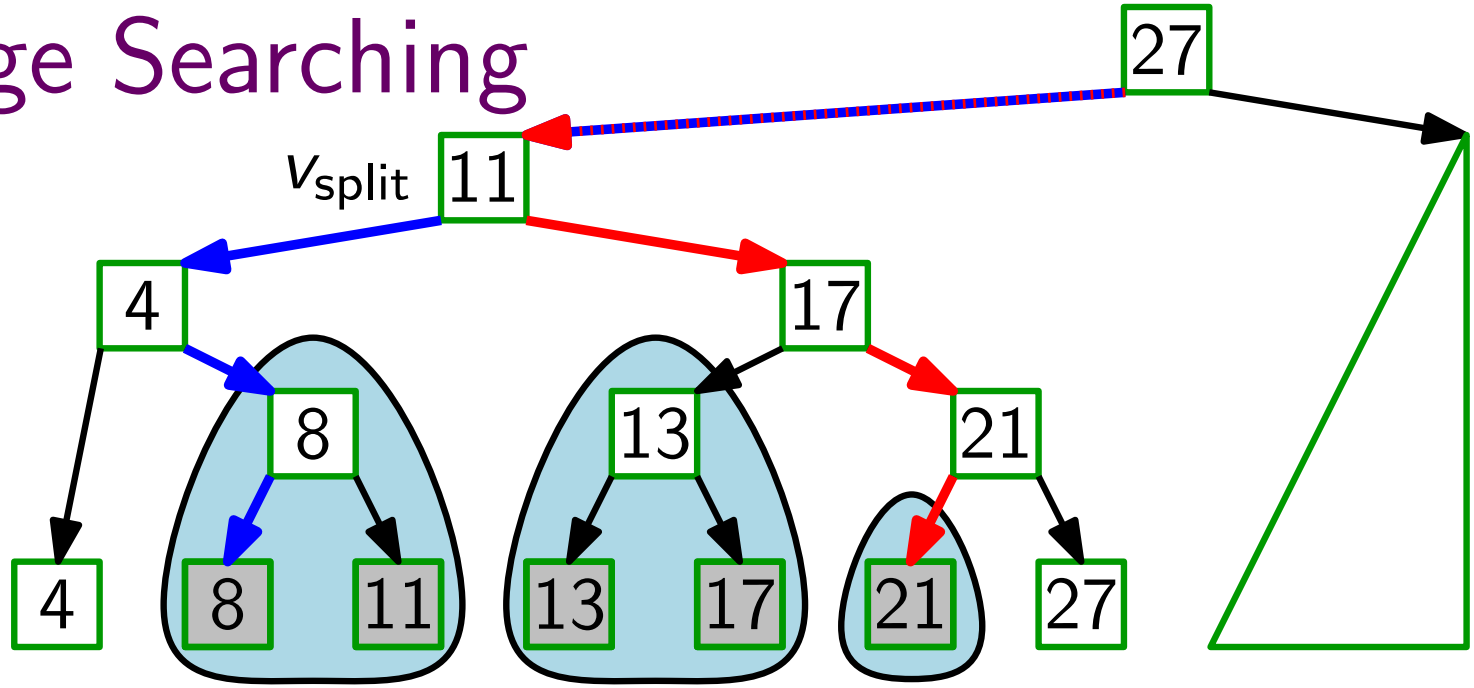

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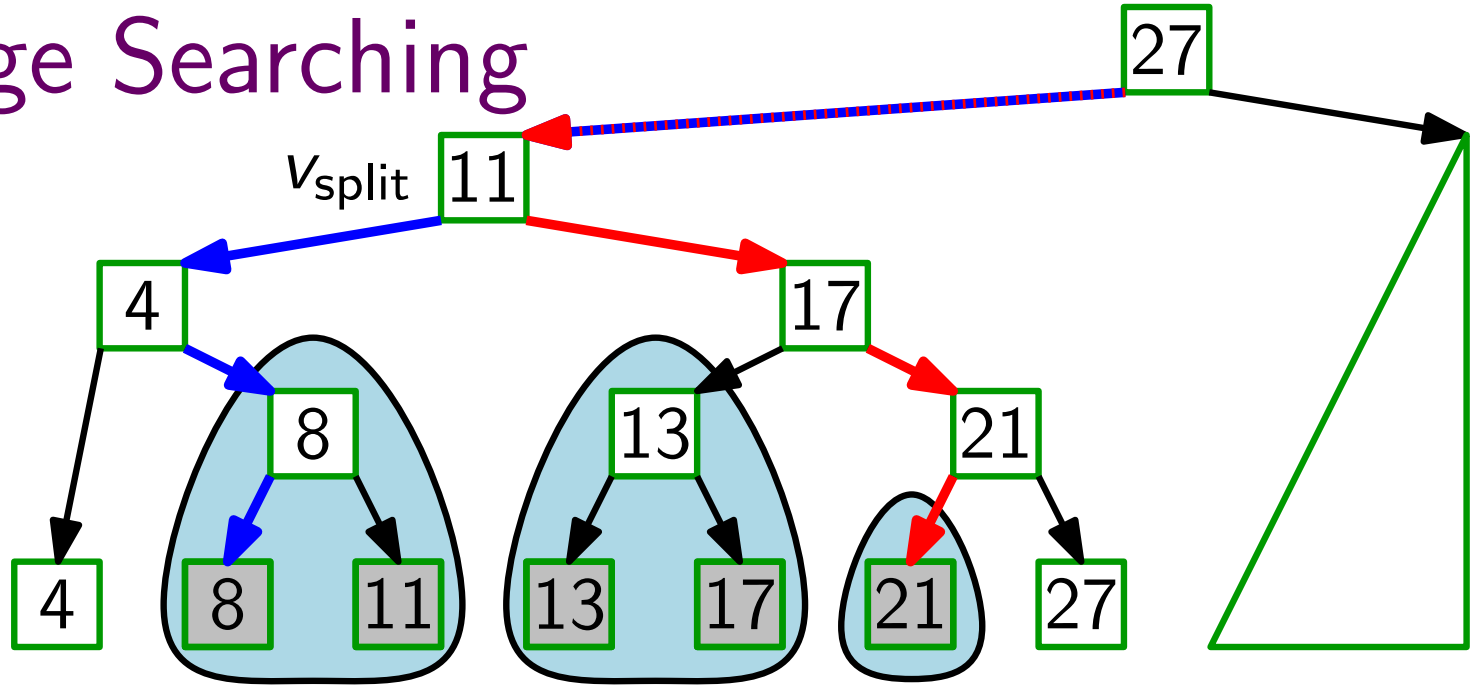

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output sensitive!

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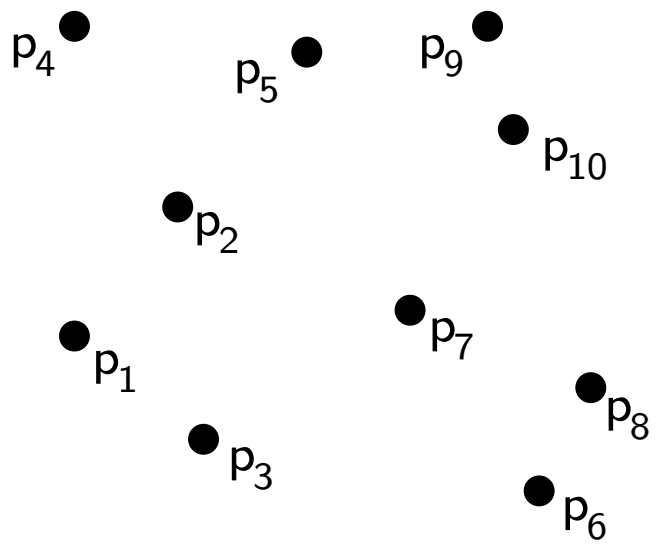
Here: no two points have the same x- or y-coordinate.

Kd-Trees: Example

[dBCvKO'08]



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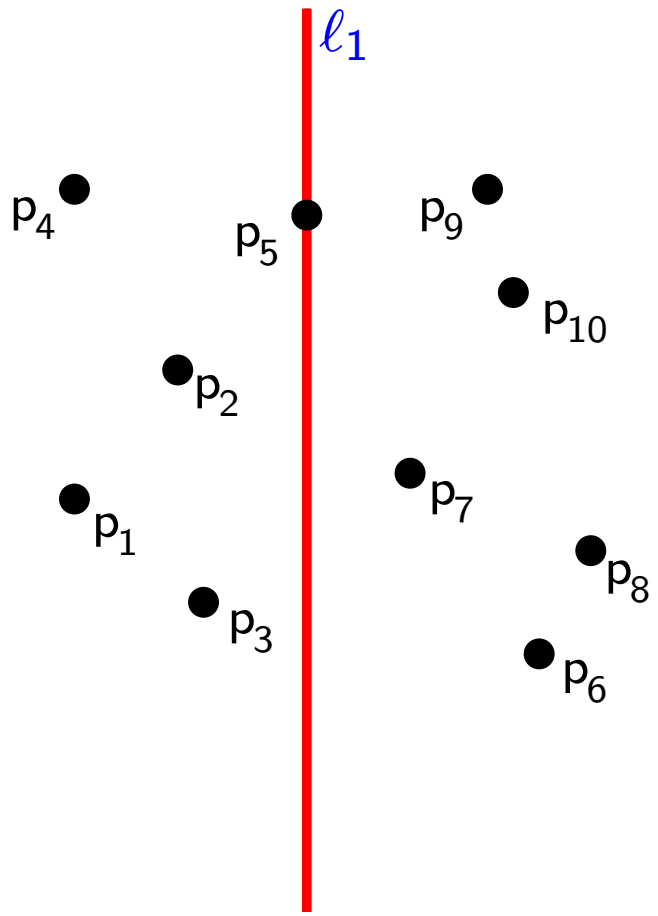

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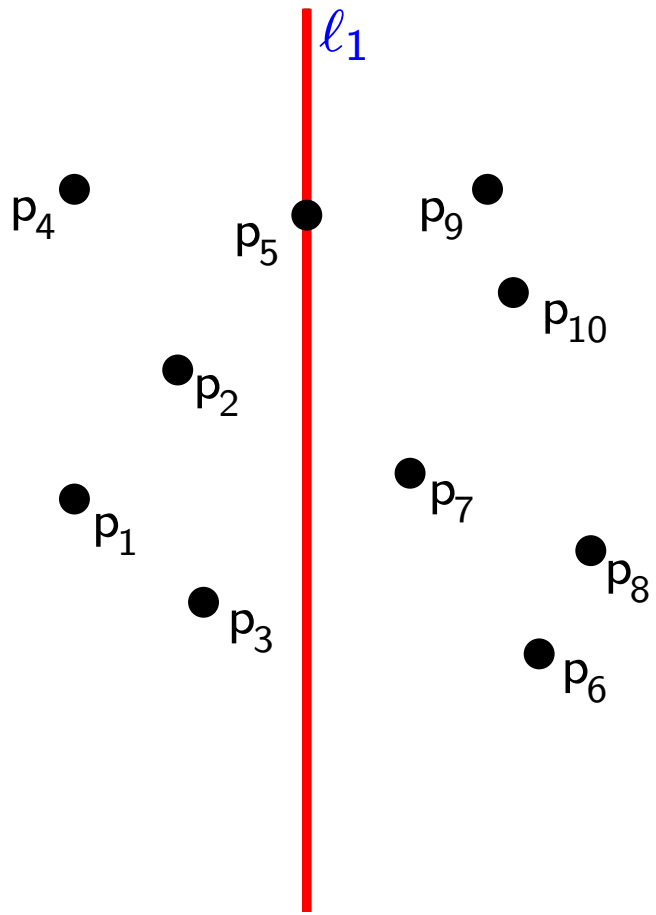

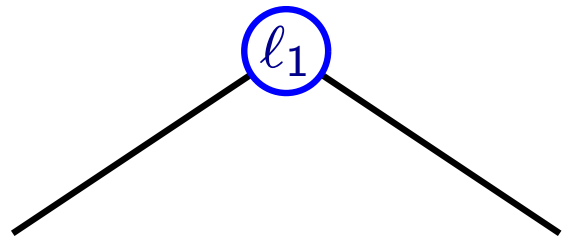

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- Split any reagion that contains more than one point.

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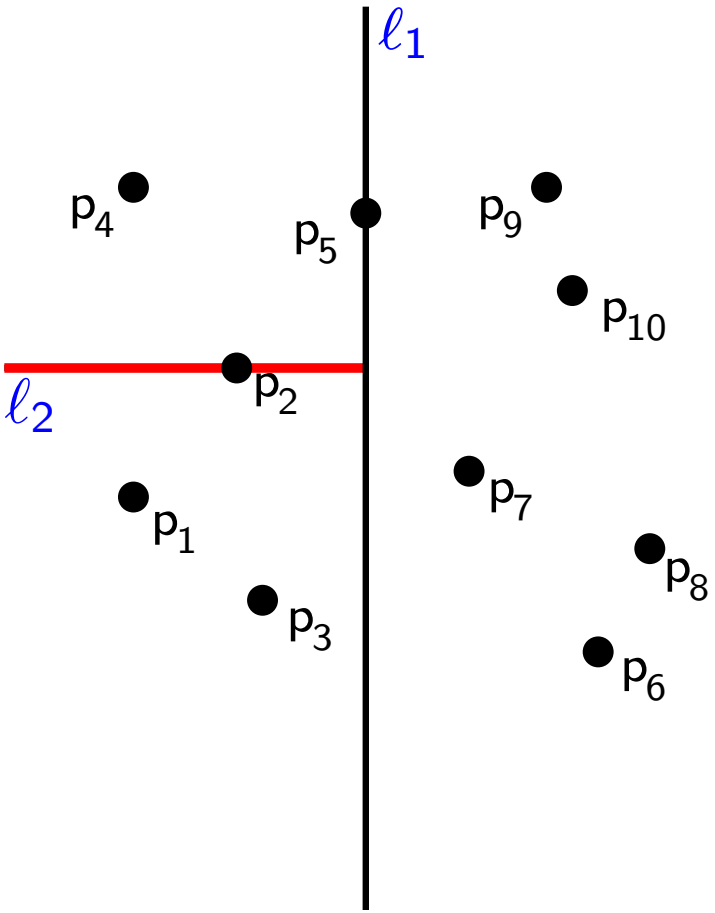

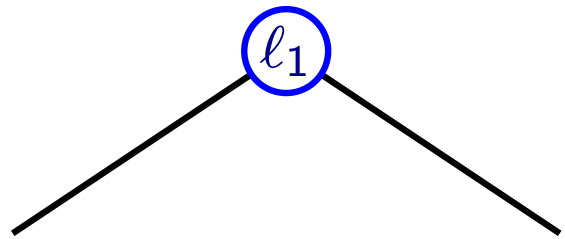

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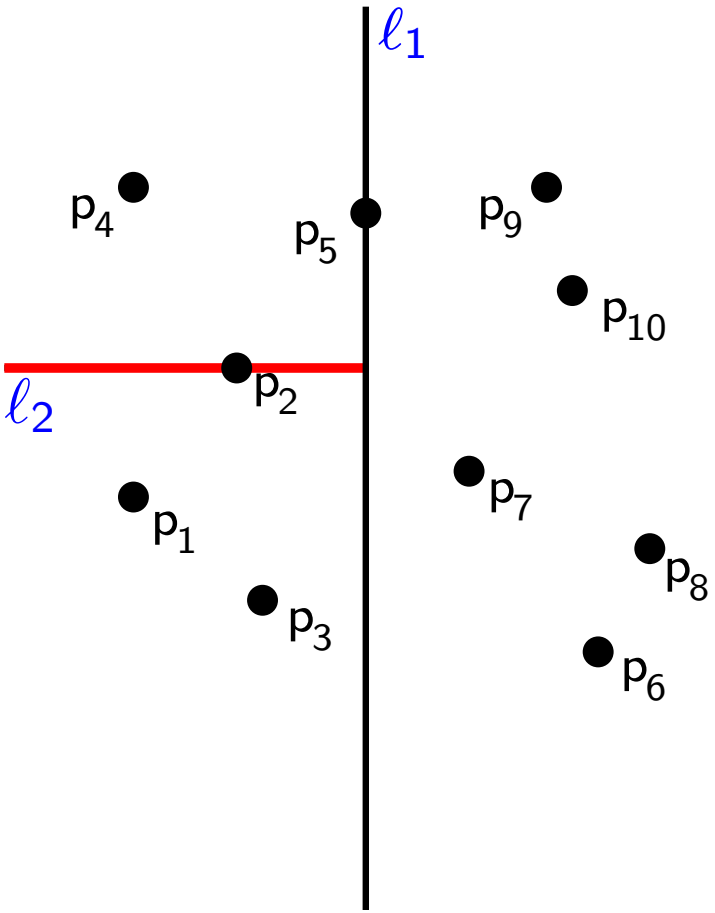

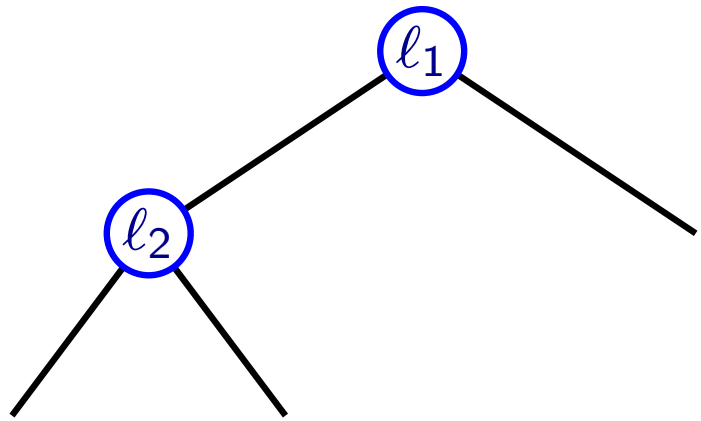

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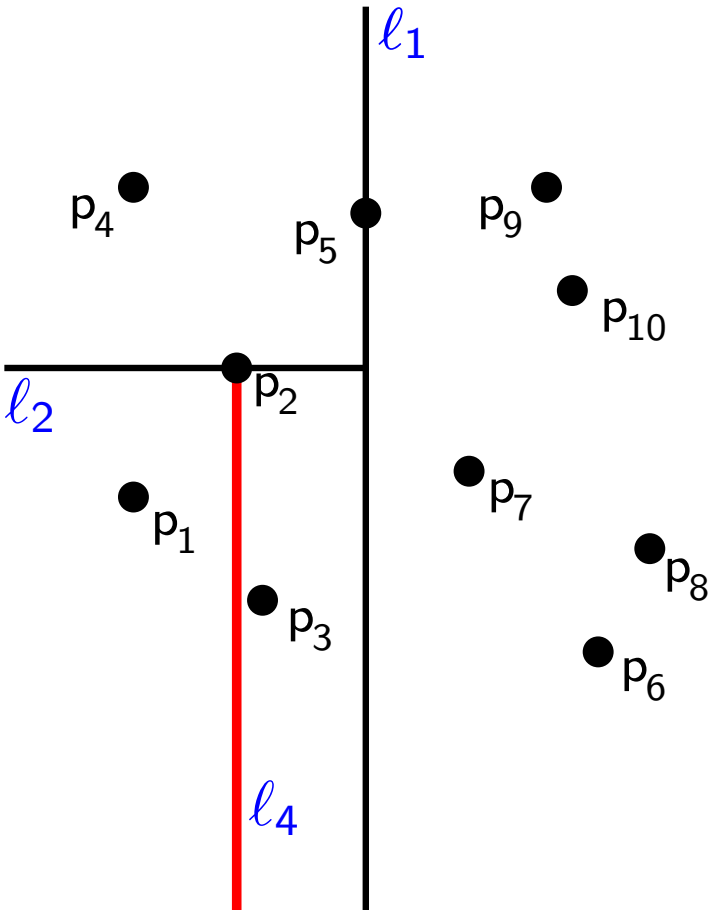

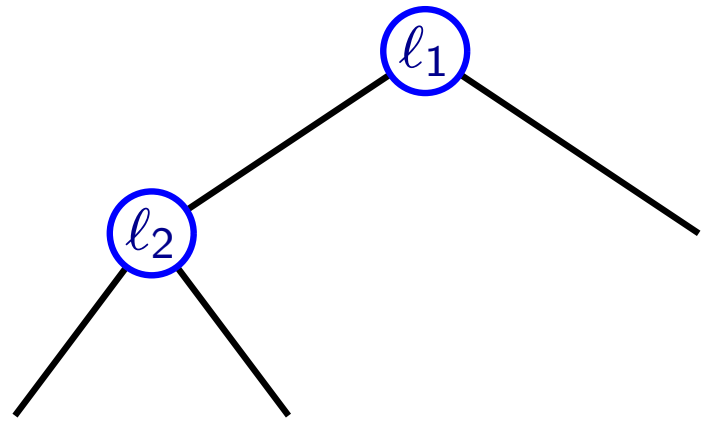

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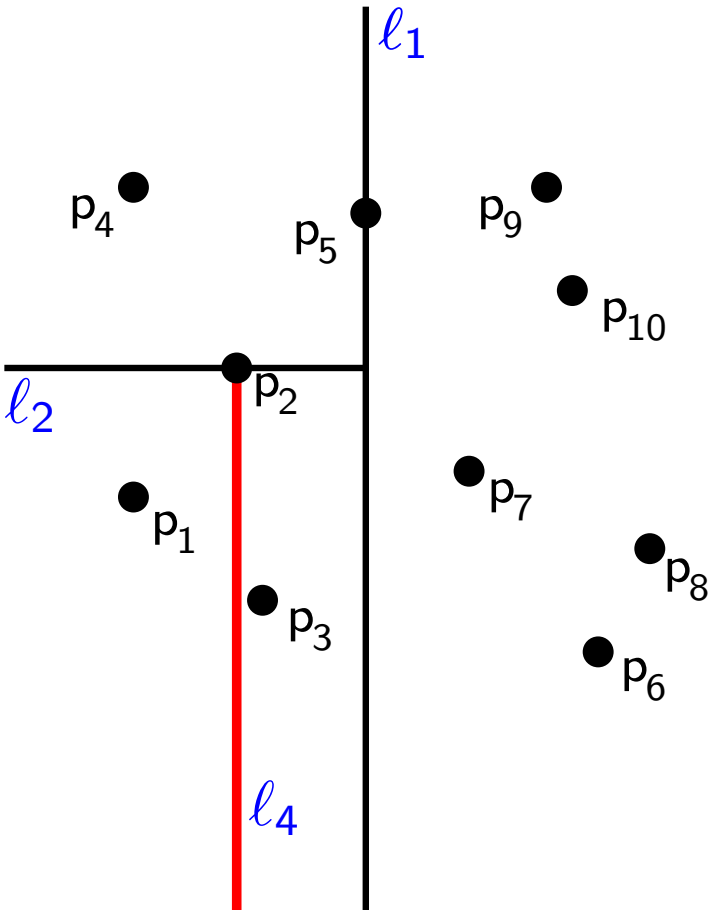

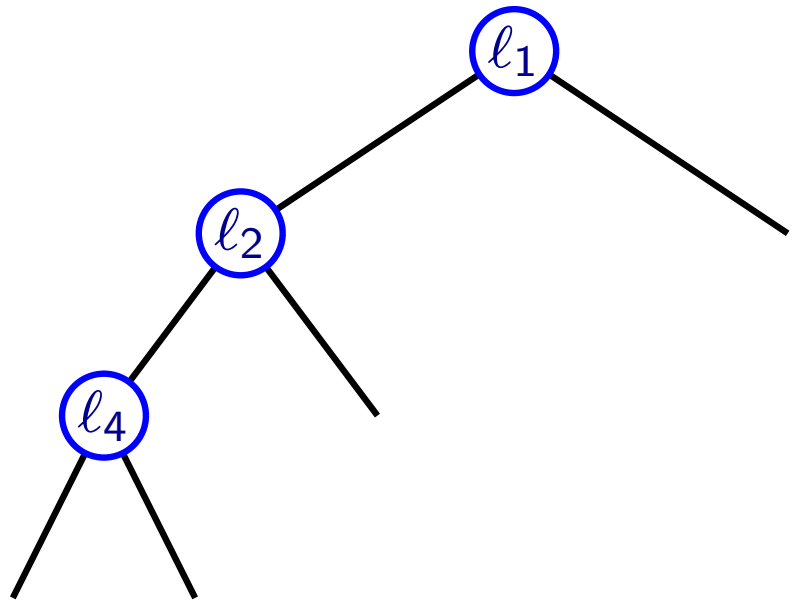

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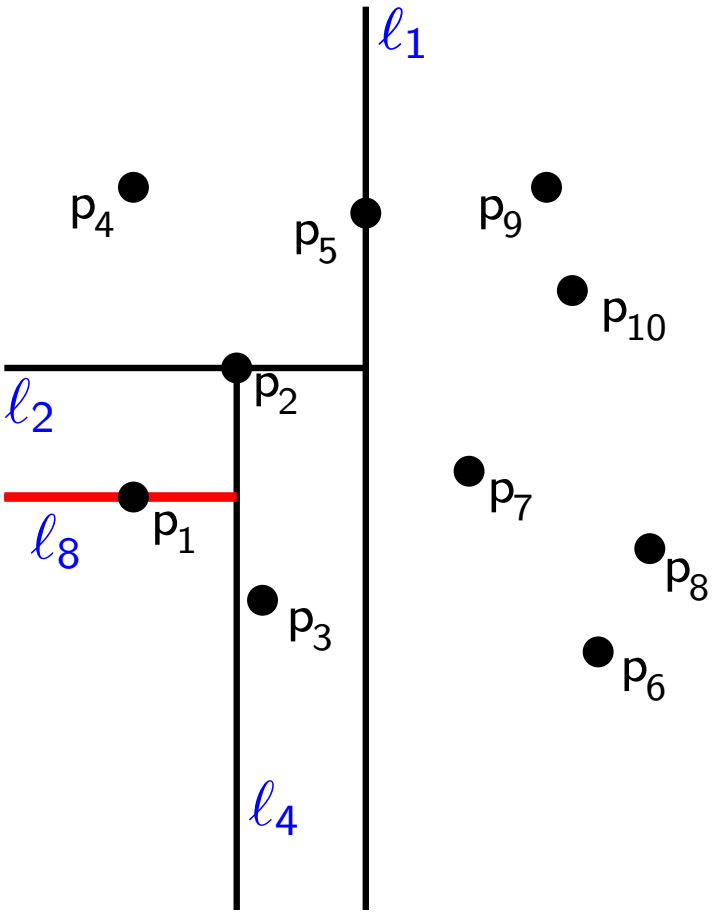
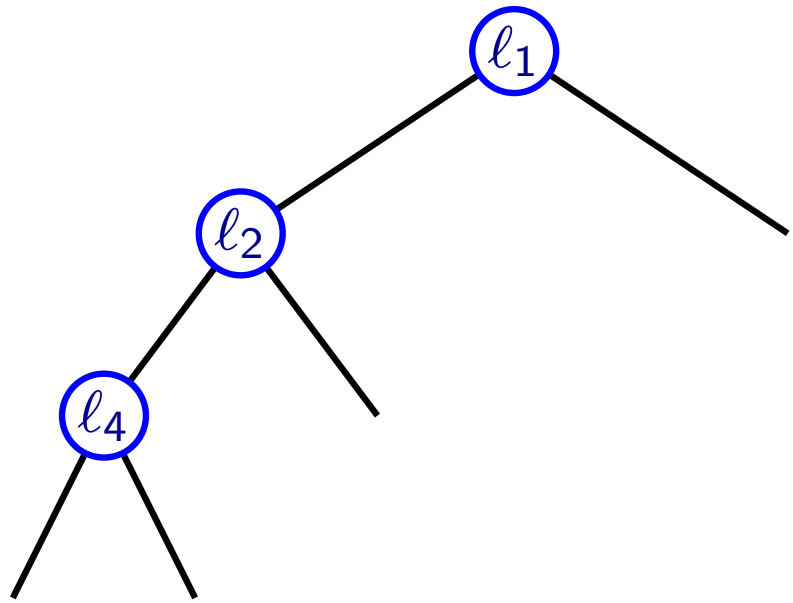
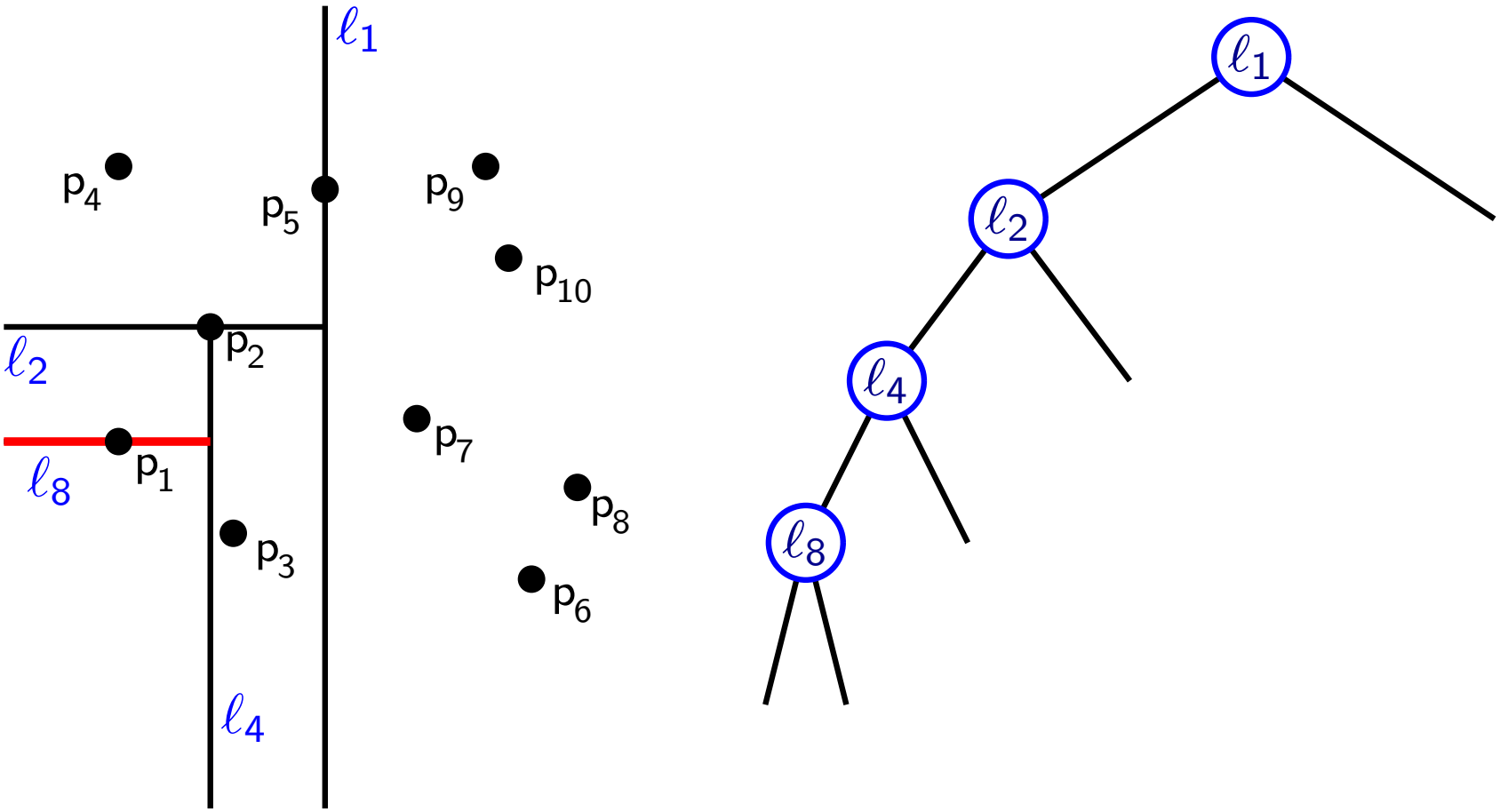




[dBCvKO'08]

- Split any reagion that contains more than one point.

Kd-Trees: Example



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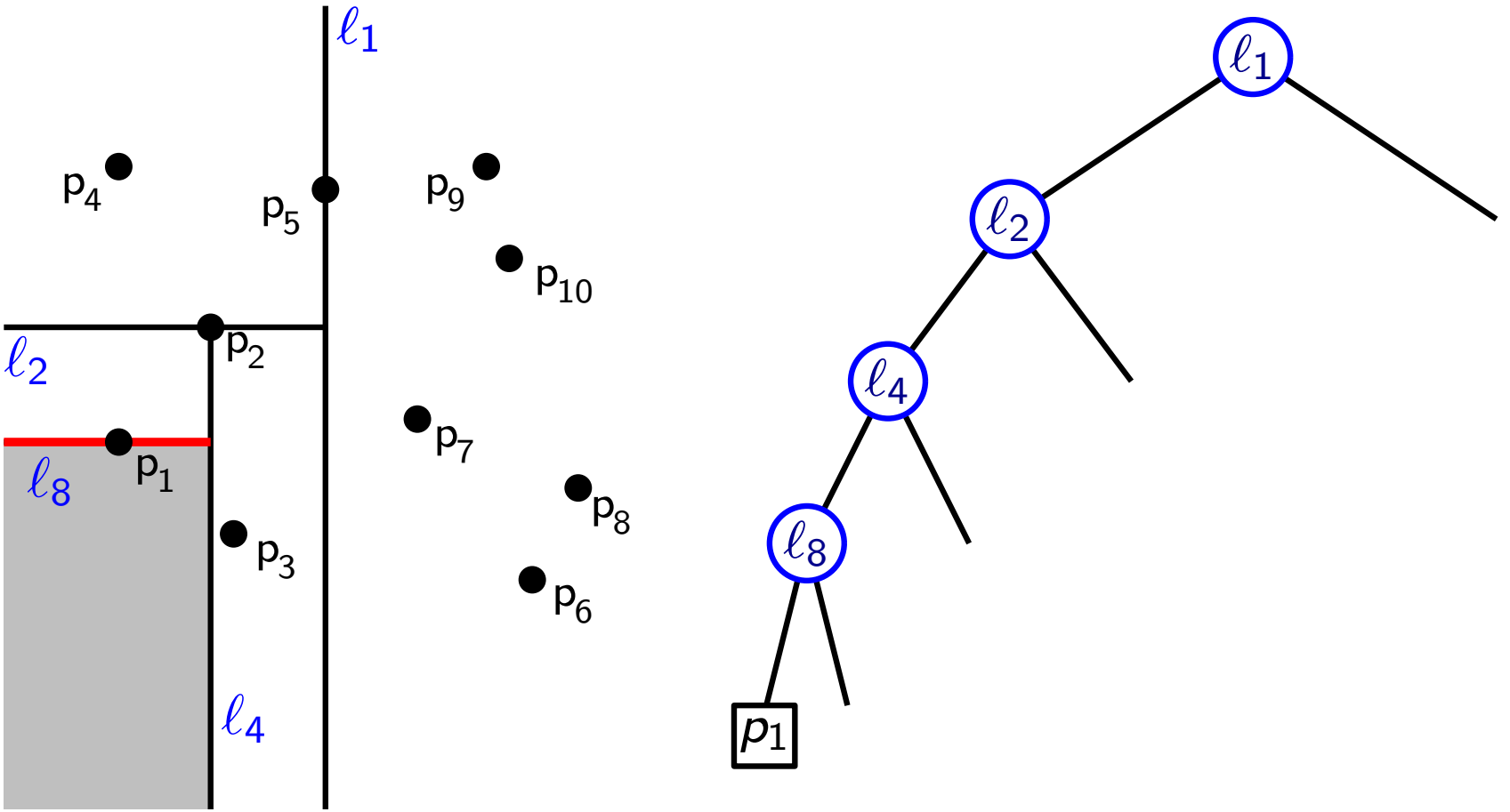

[dBCvKO'08]

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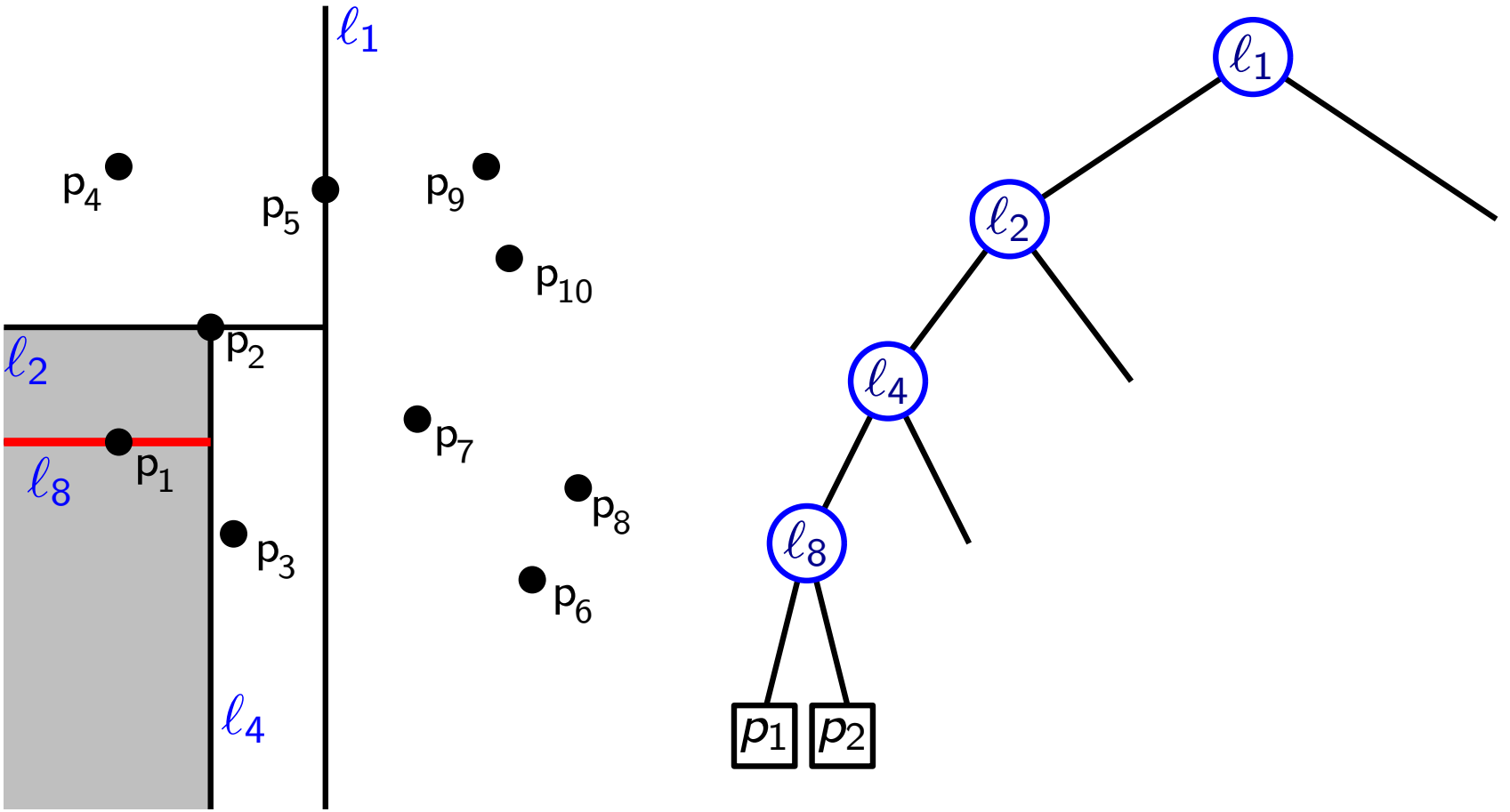
Kd-Trees: Example

[dBCvKO'08]



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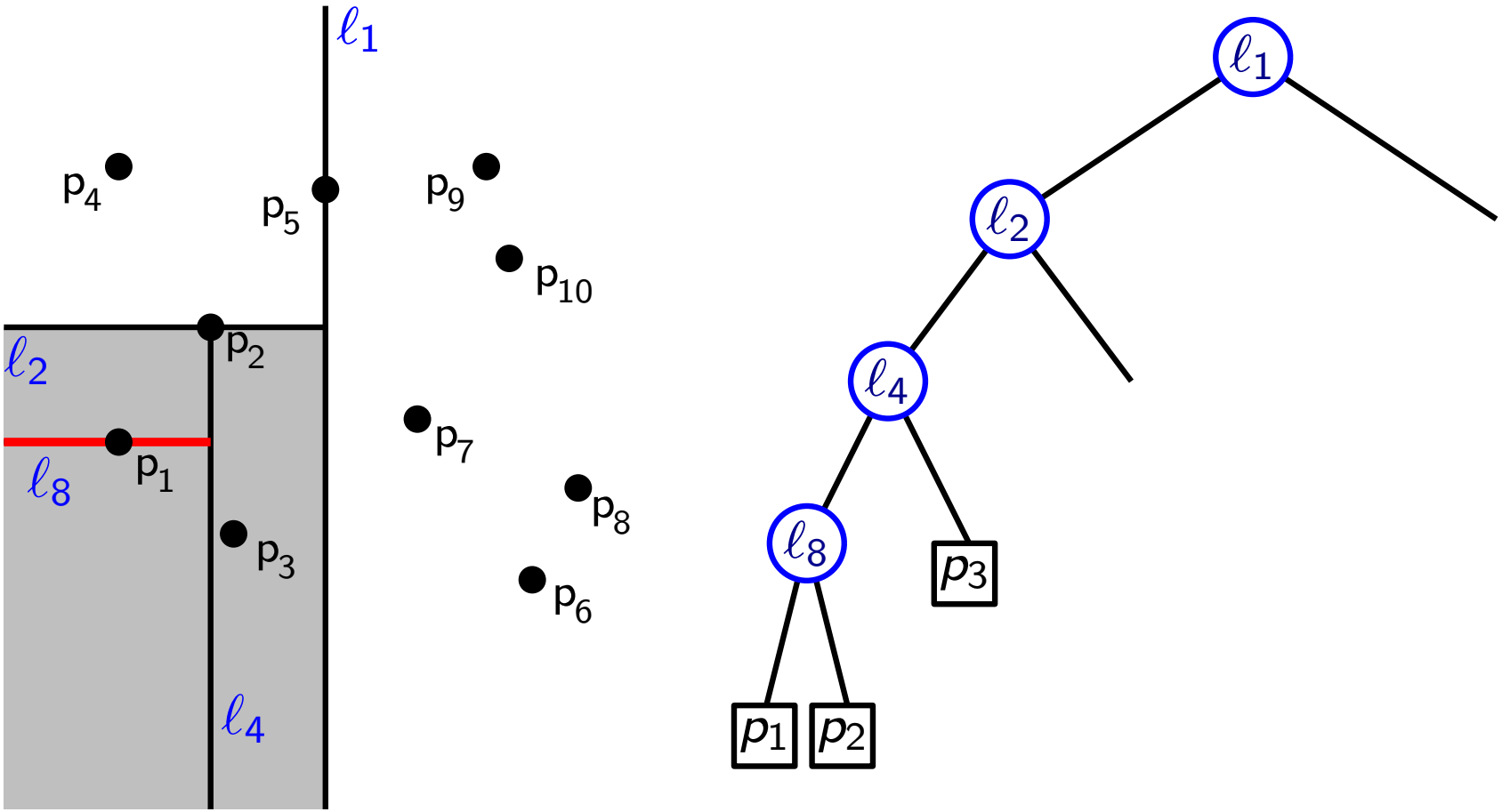
Kd-Trees: Example

[dBCvKO'08]



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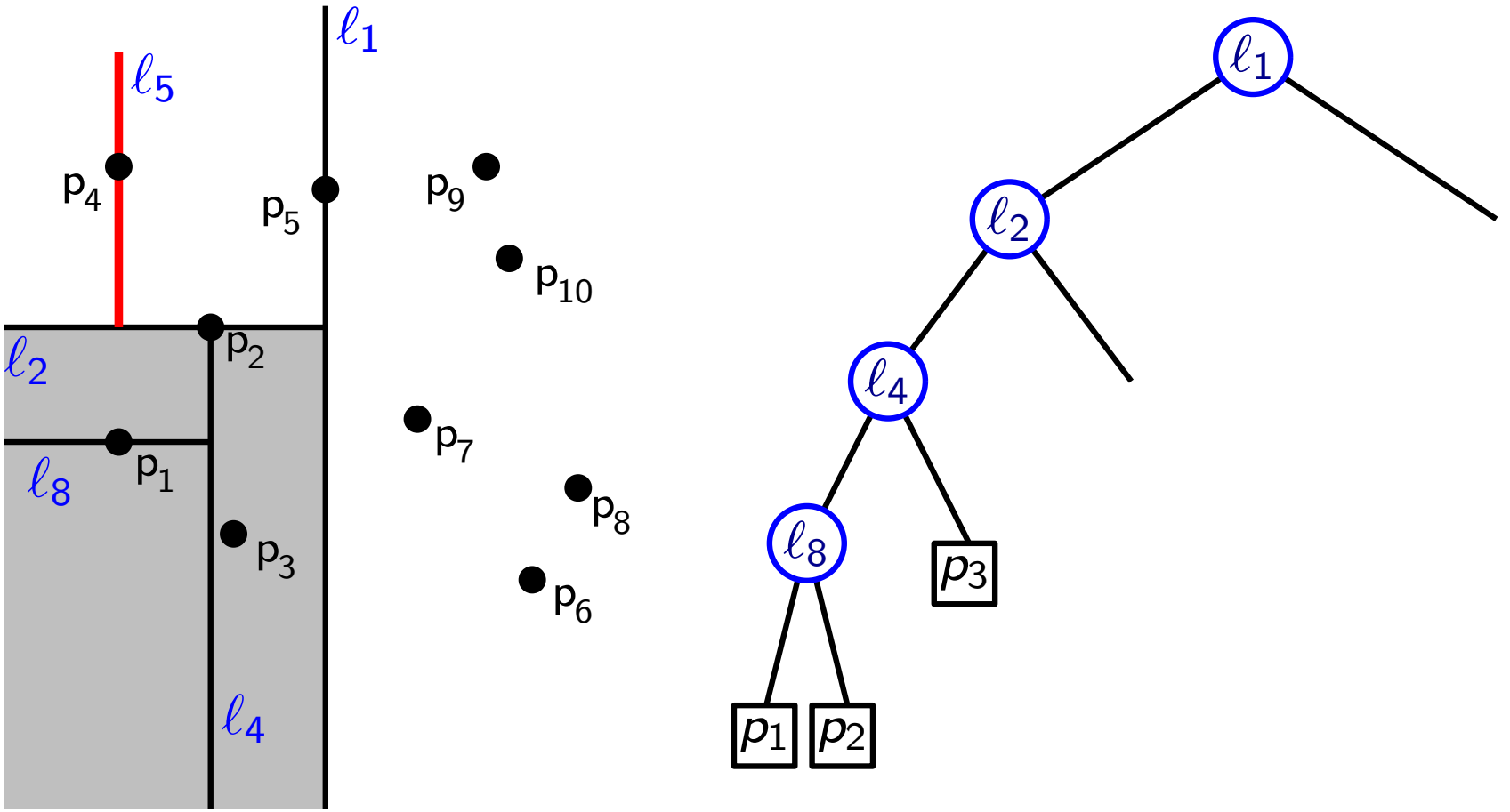

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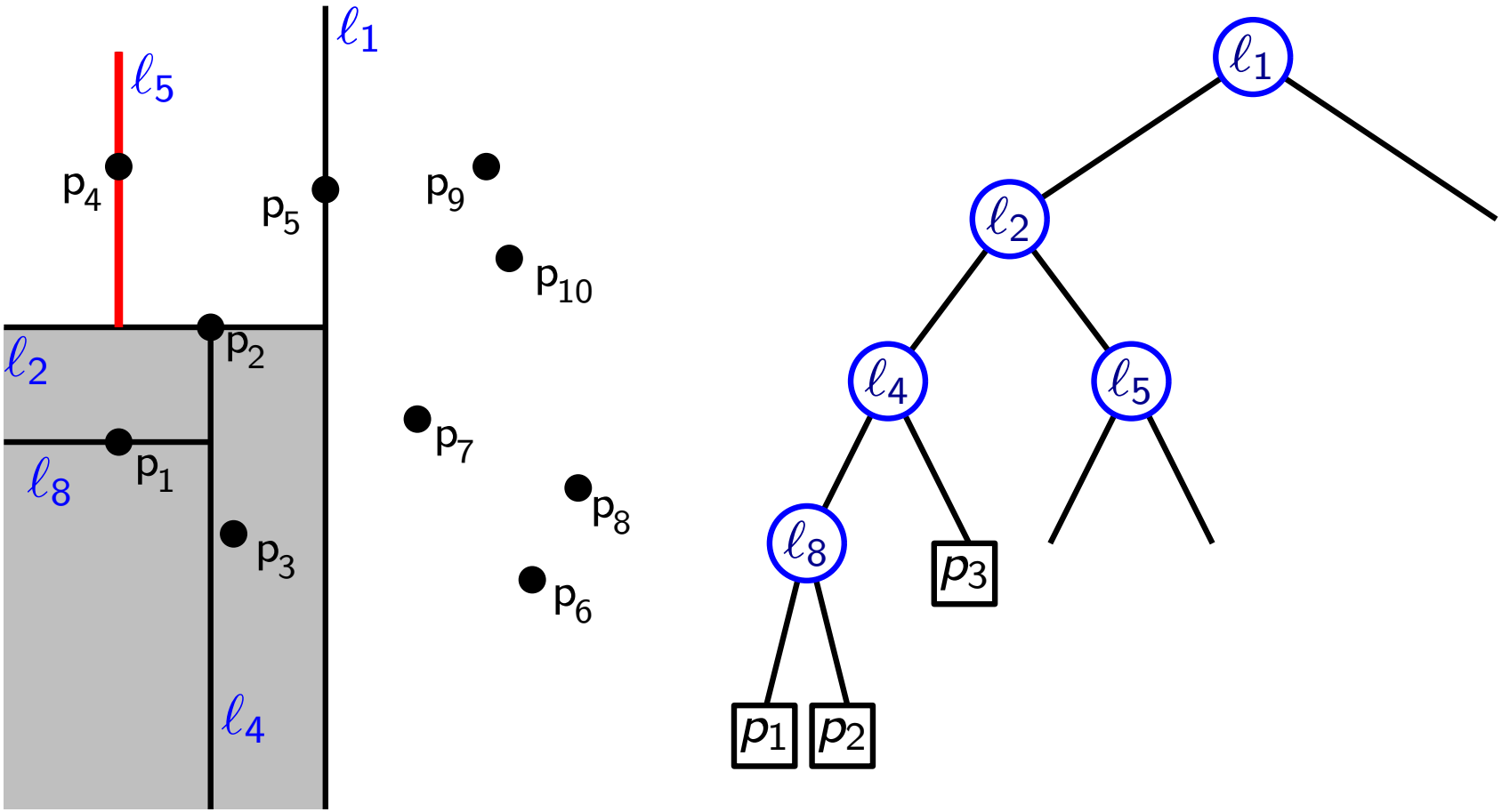

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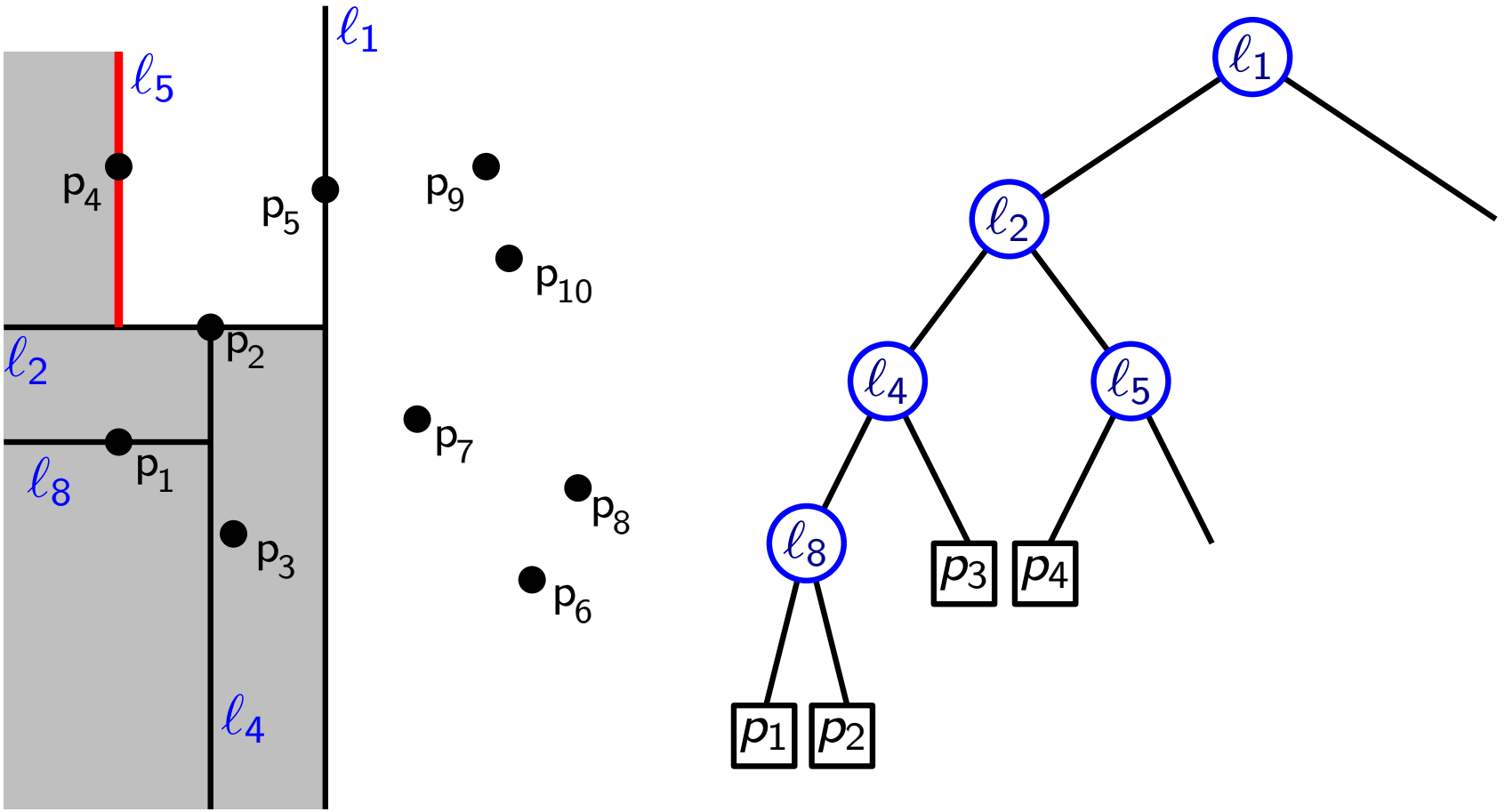

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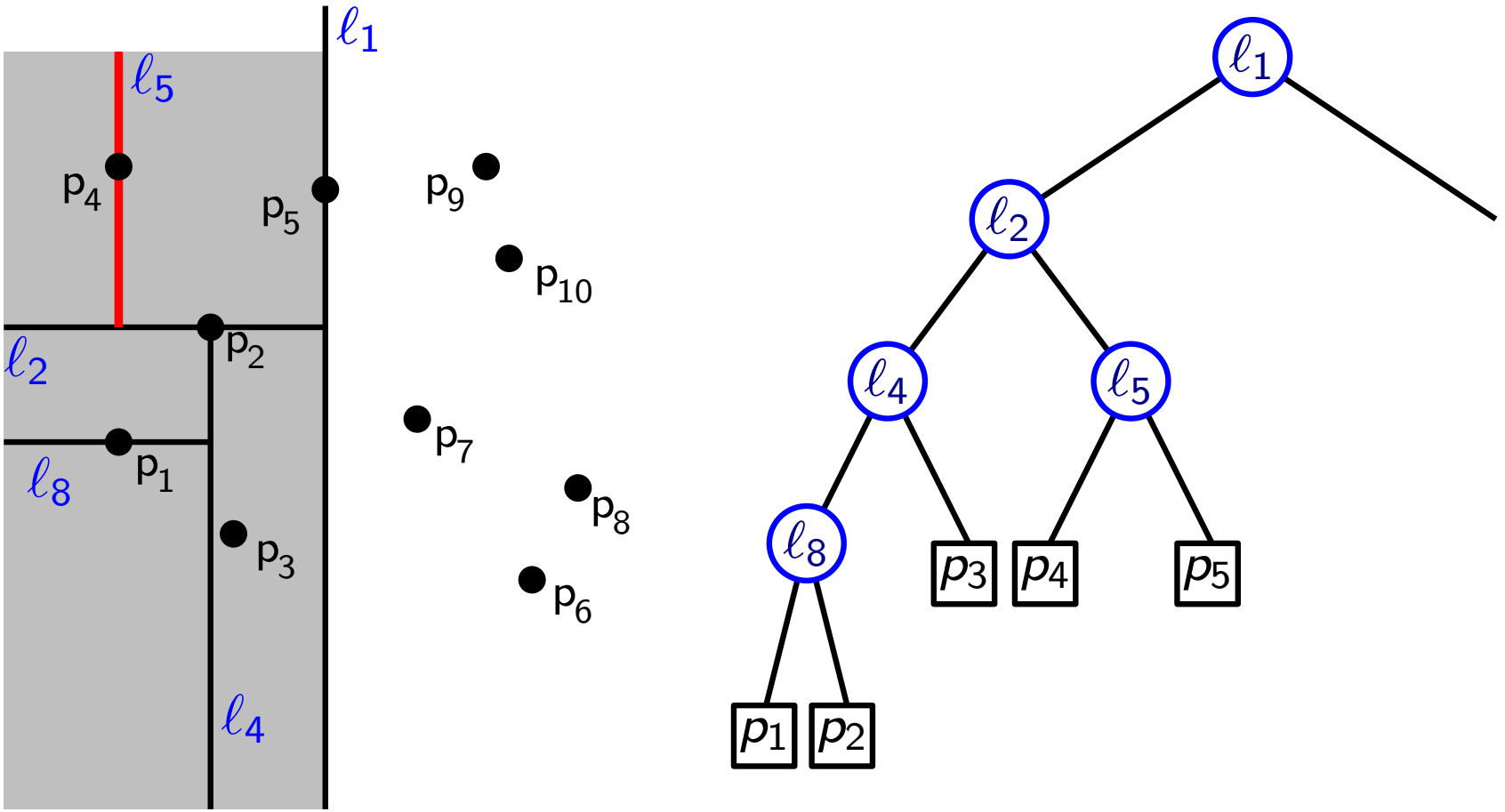

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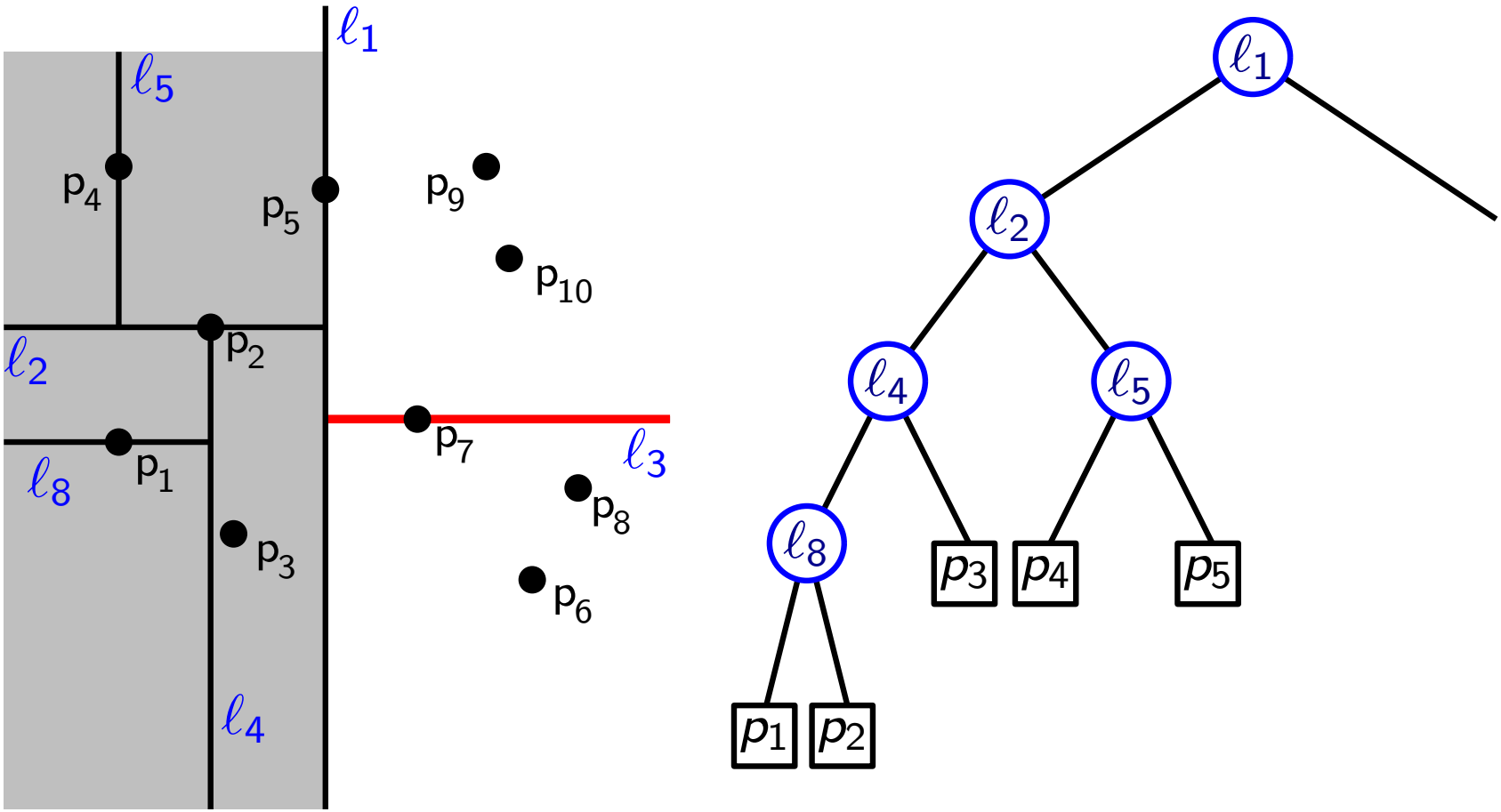

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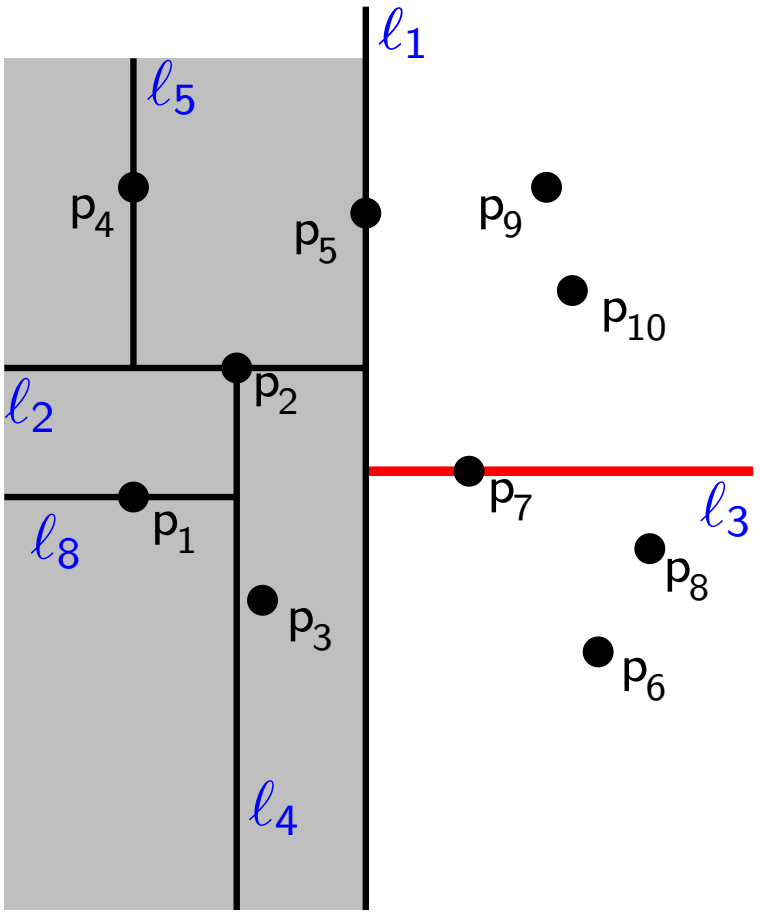
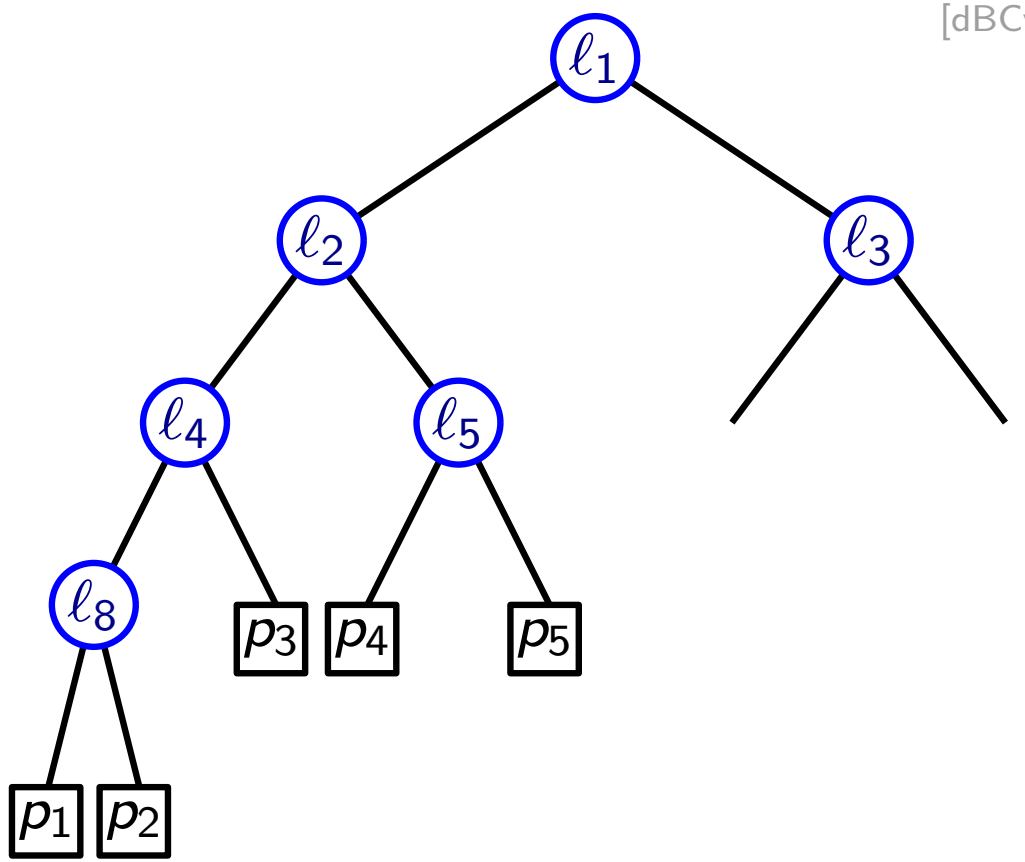


[dBCvKO'08]



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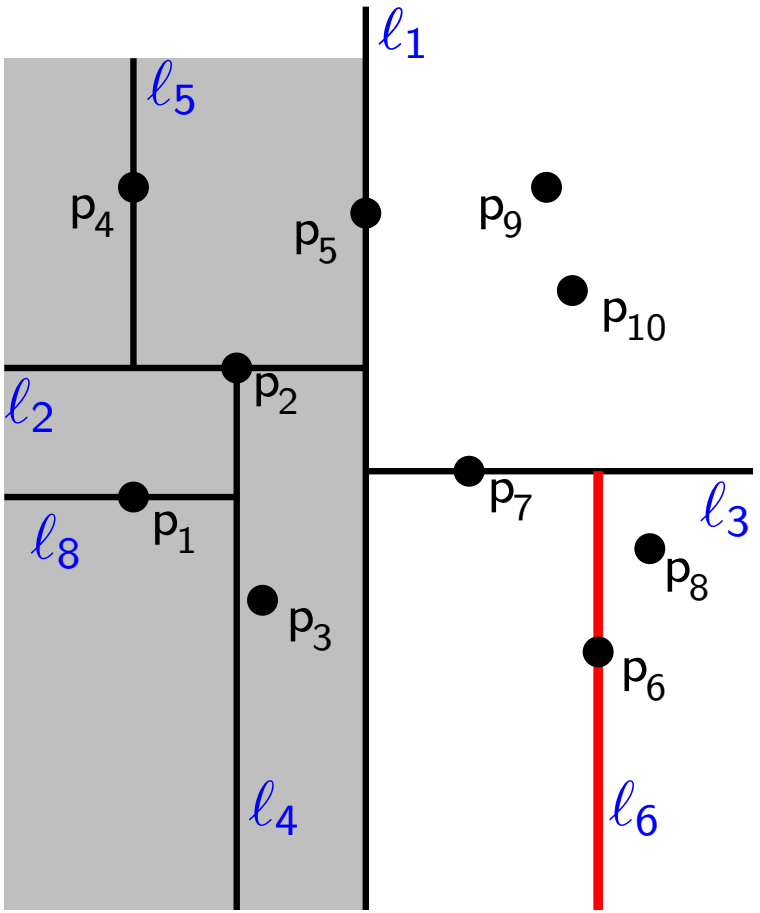

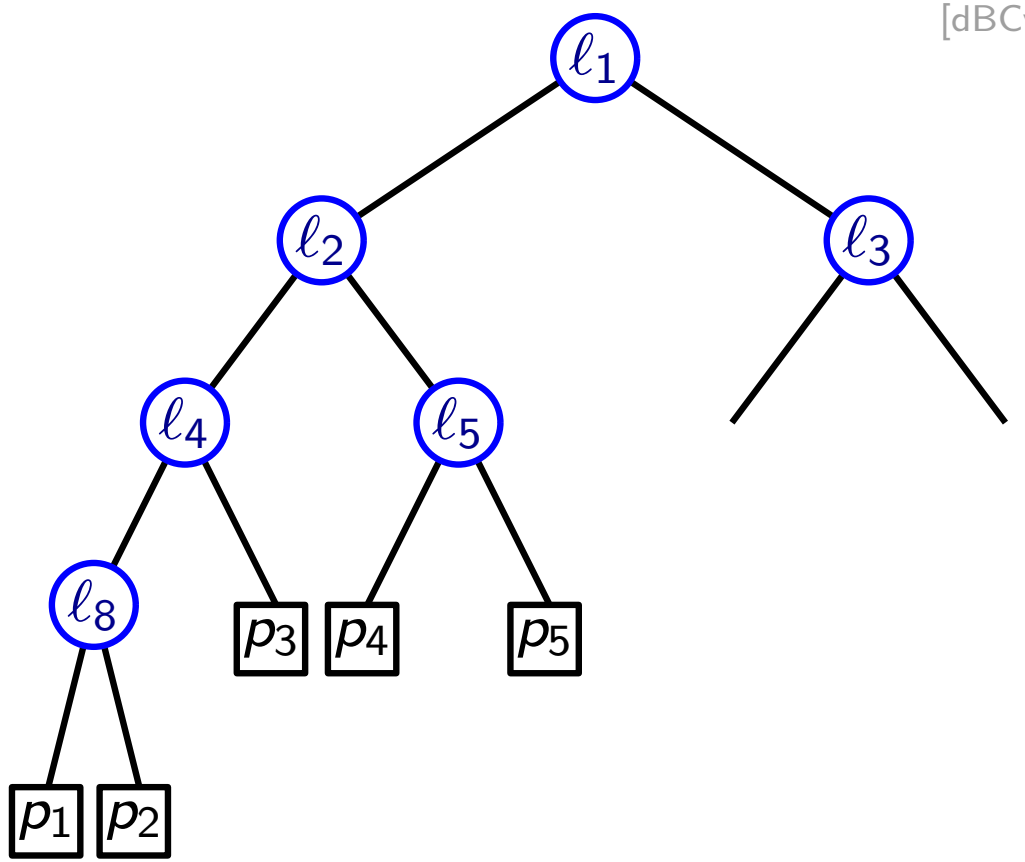

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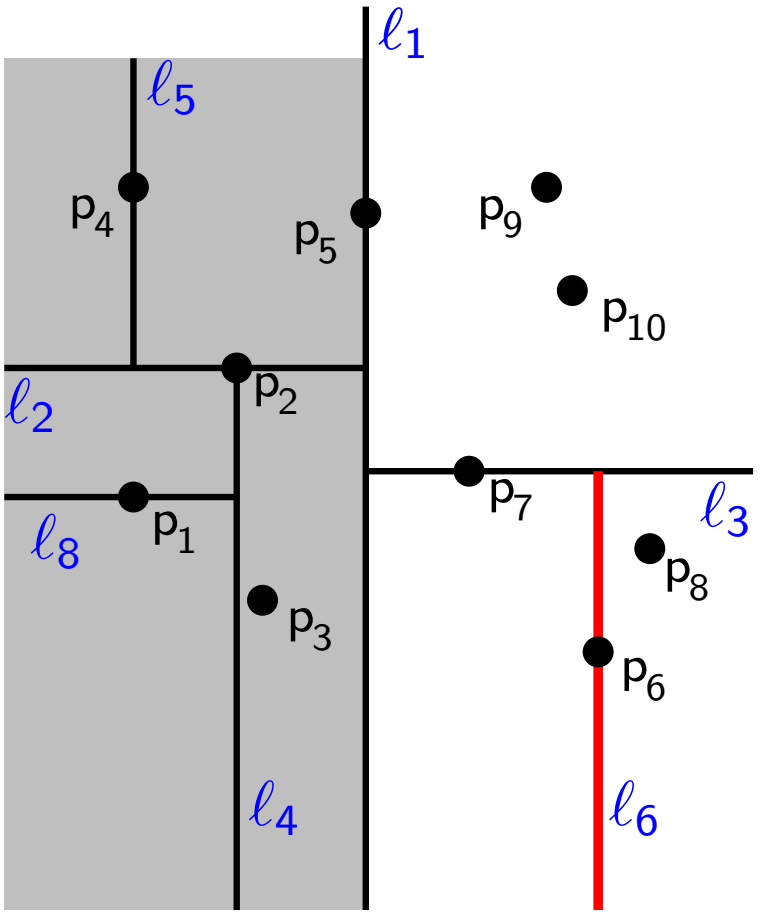

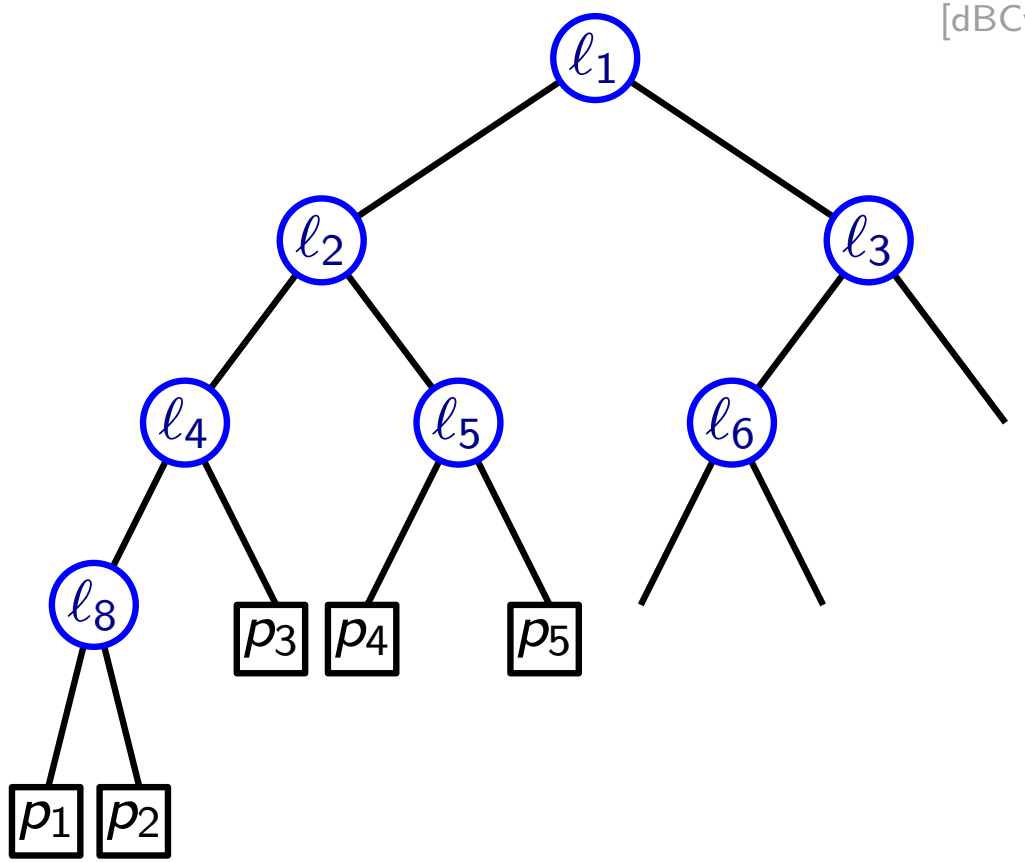

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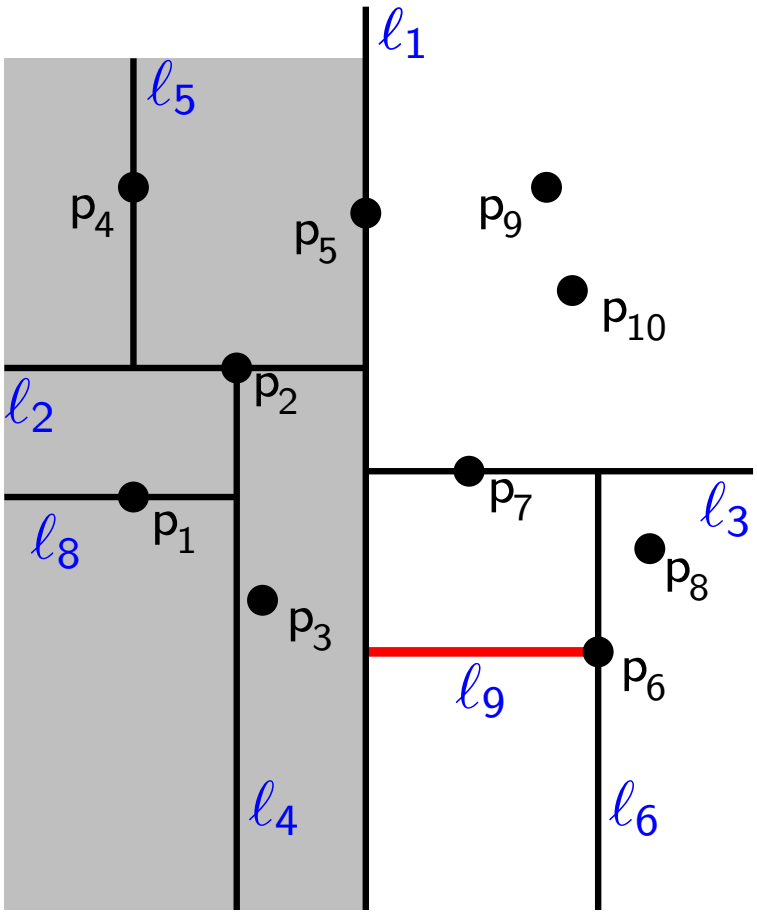

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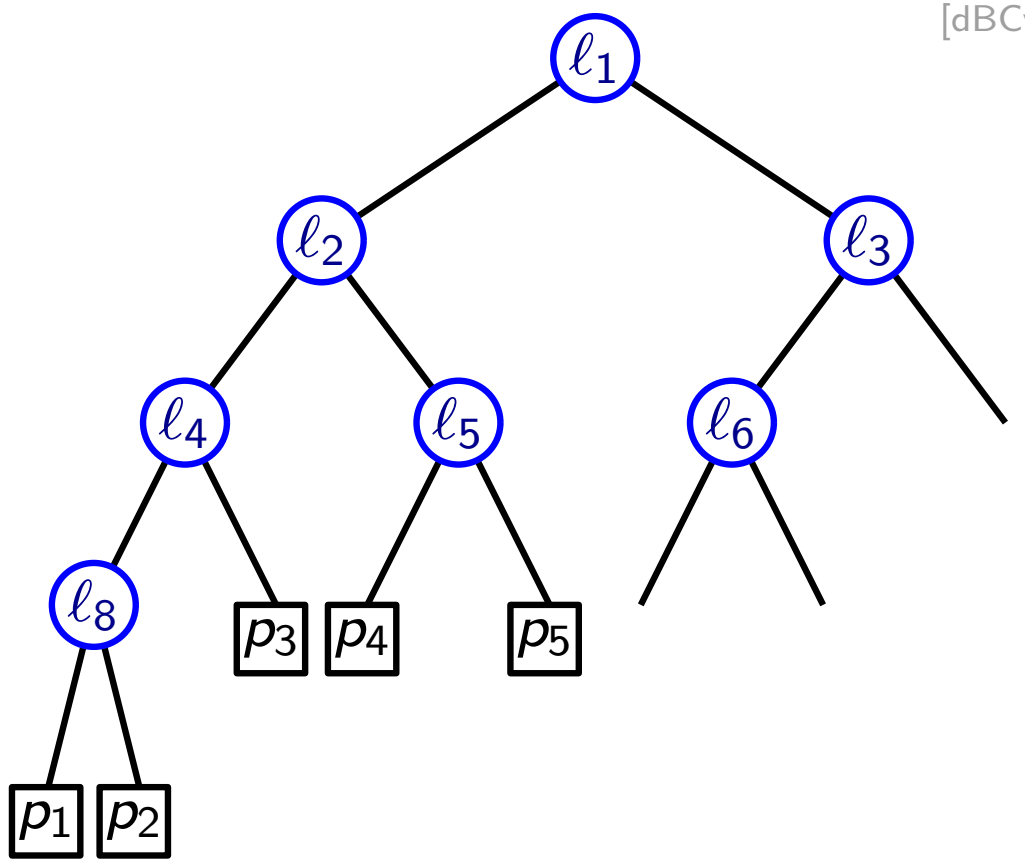

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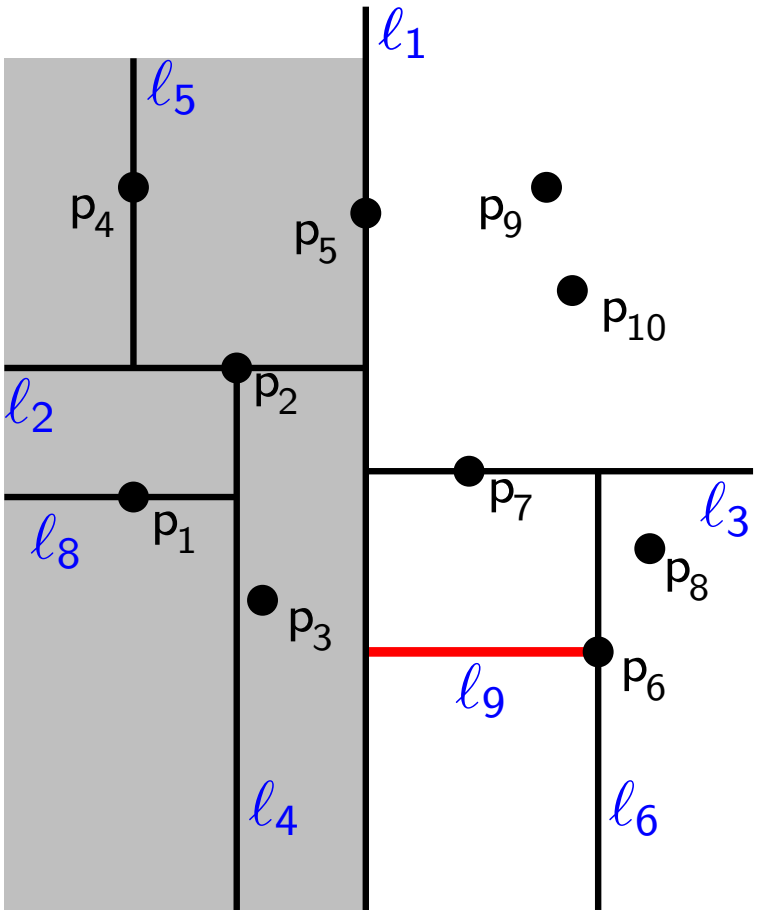

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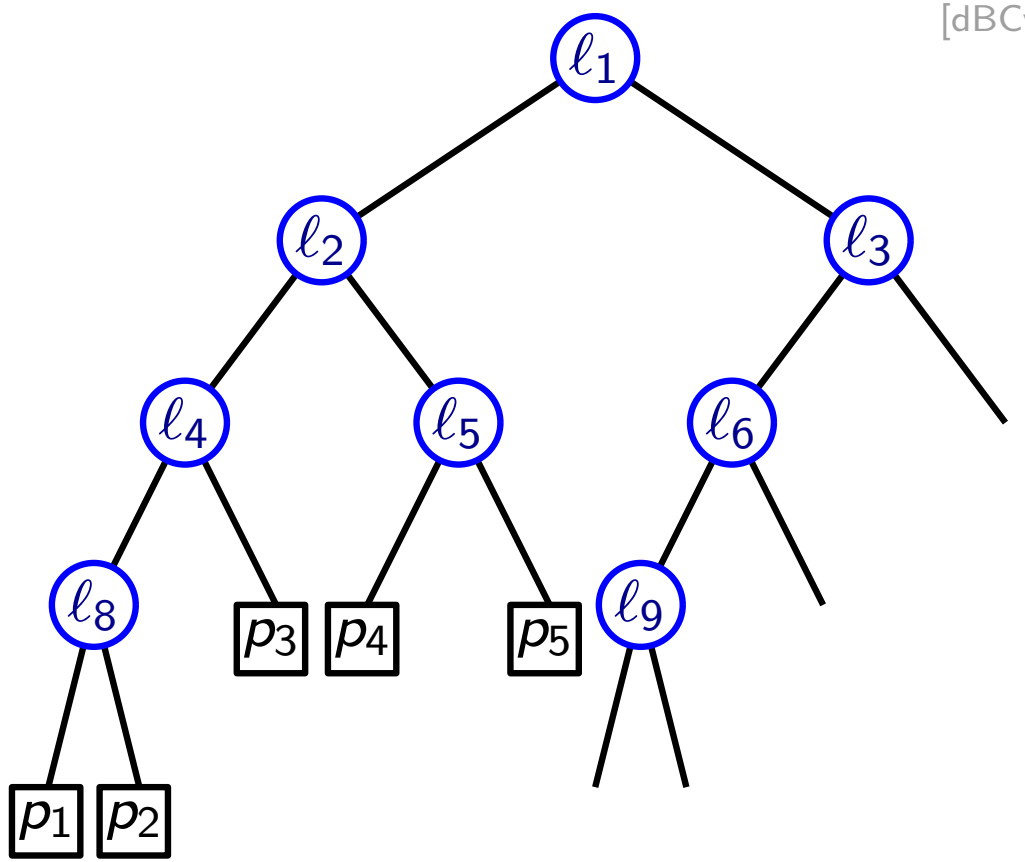

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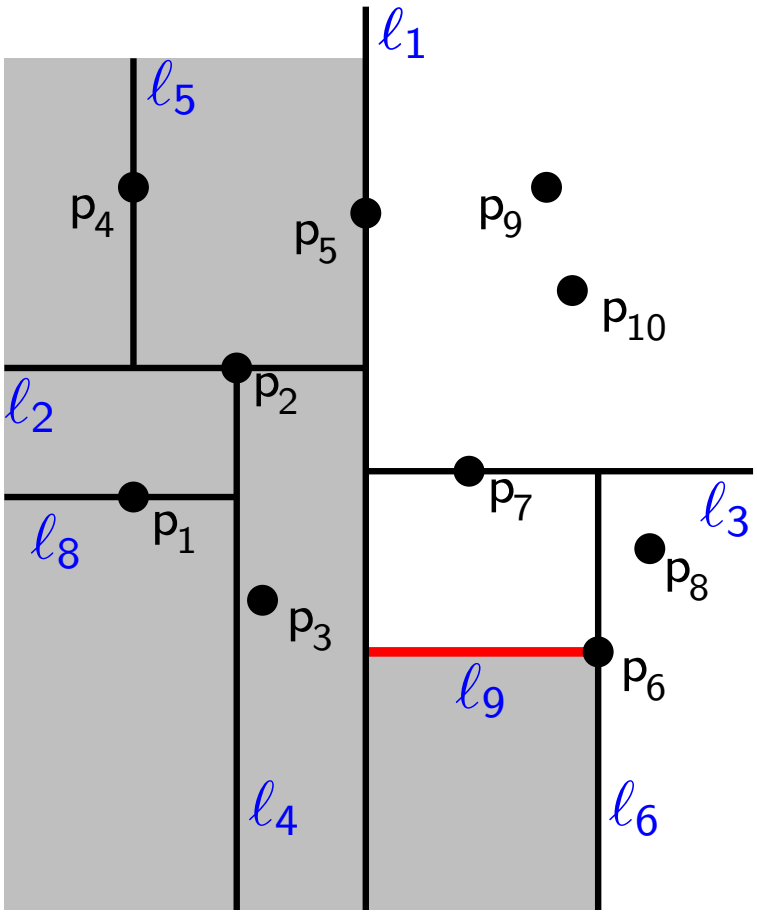
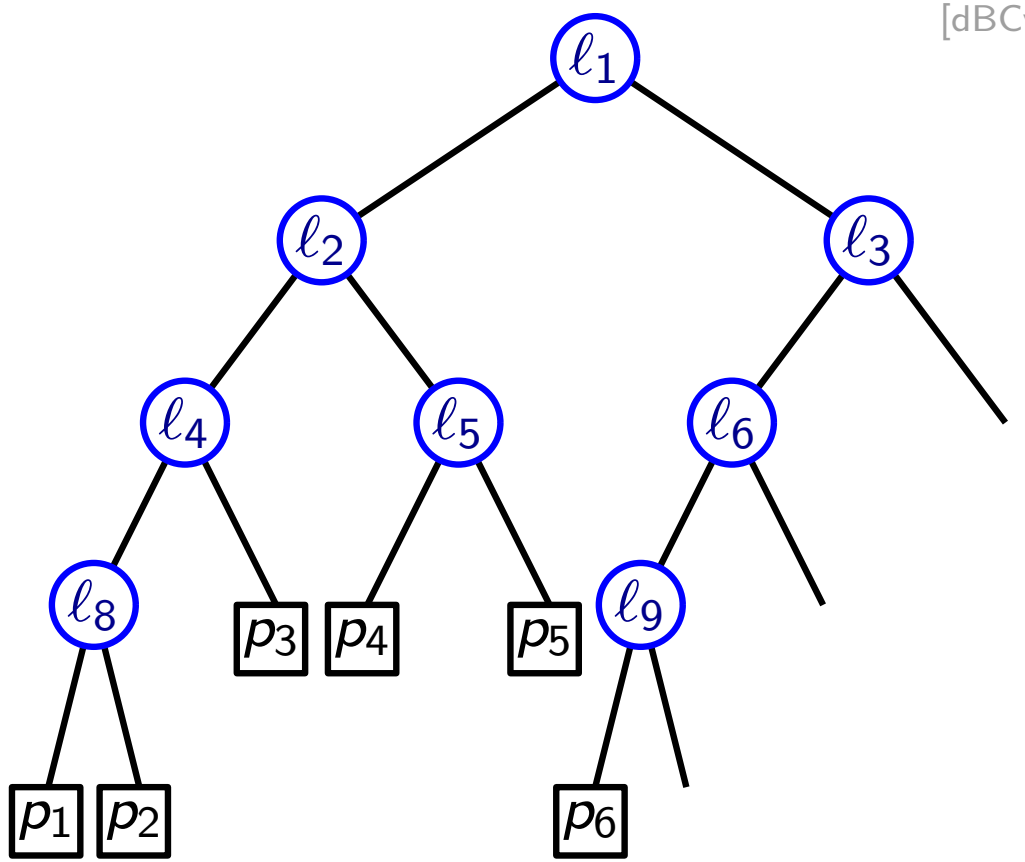


[dBCvKO'08]



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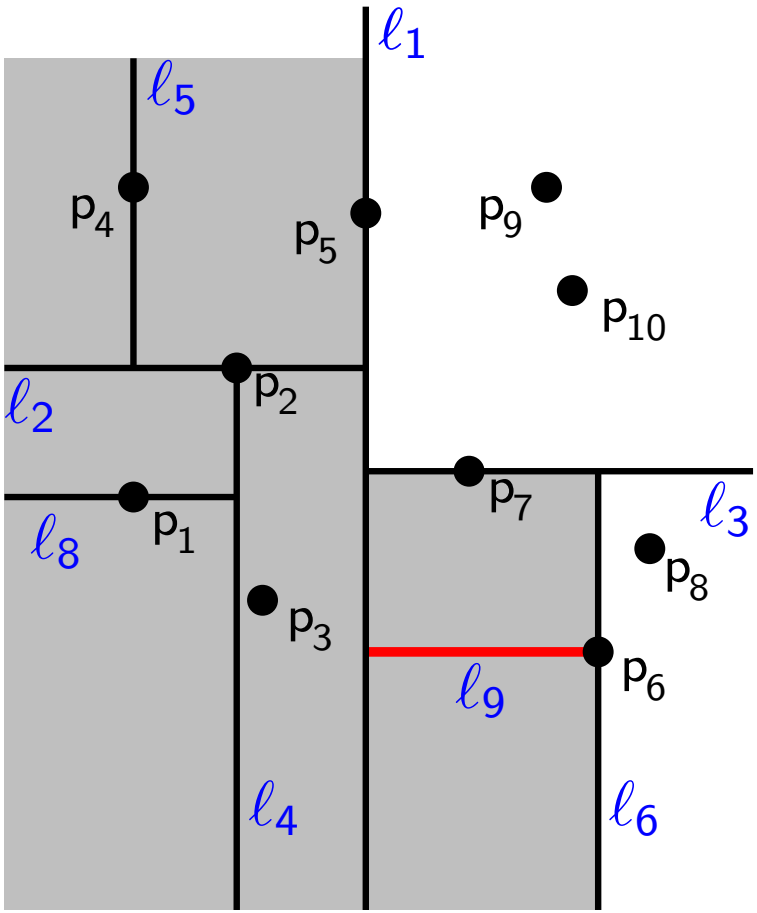

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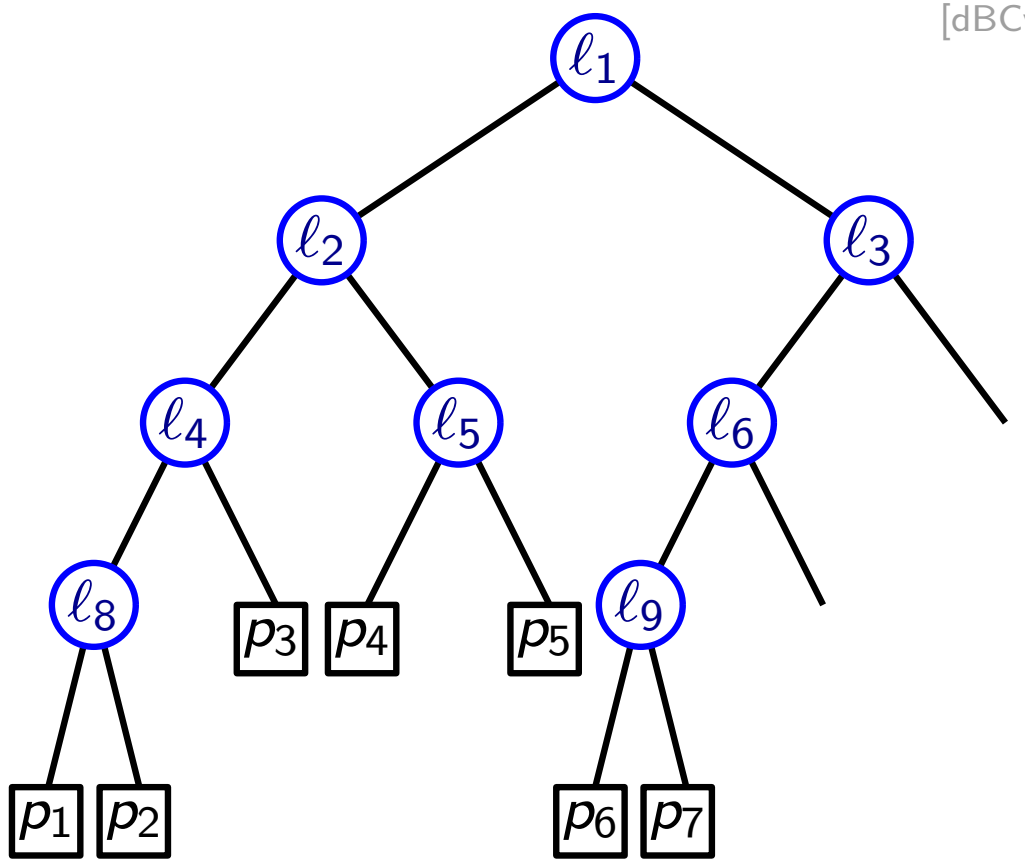

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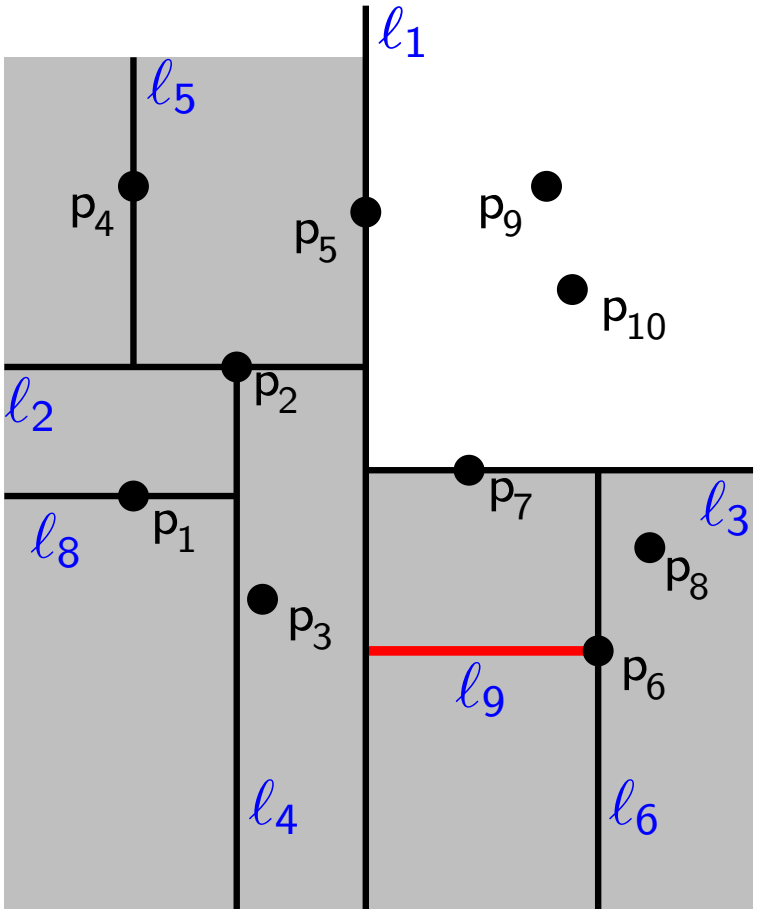

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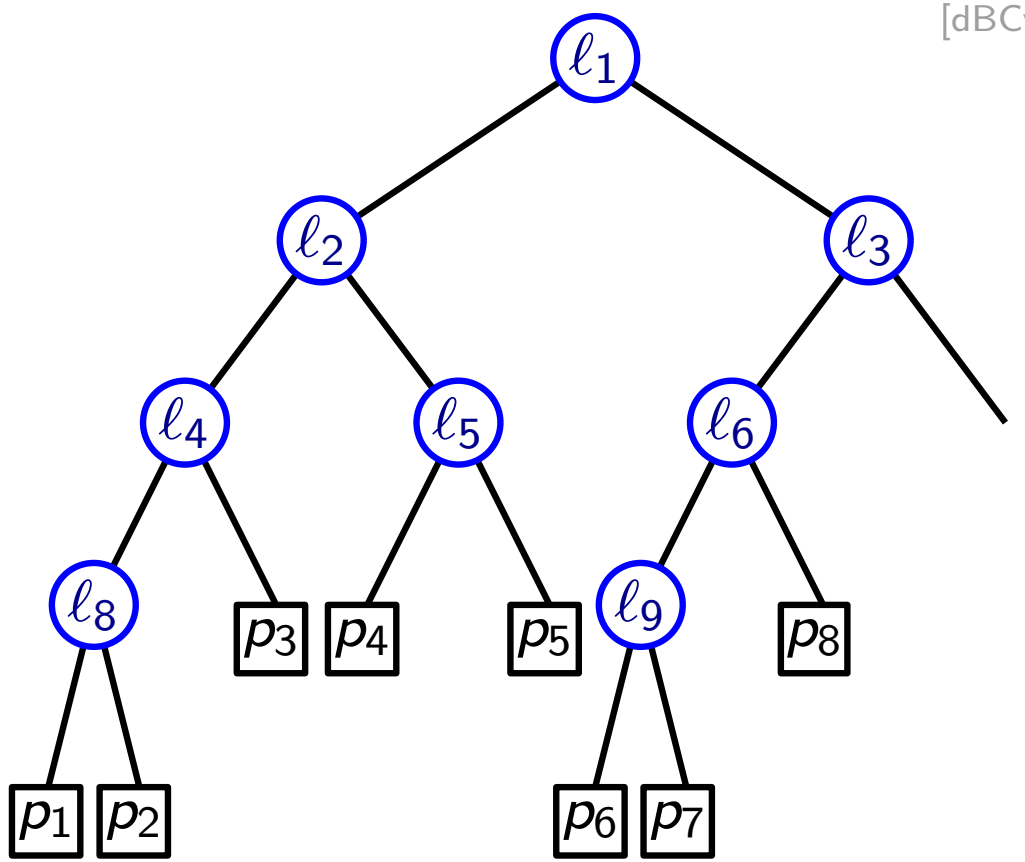

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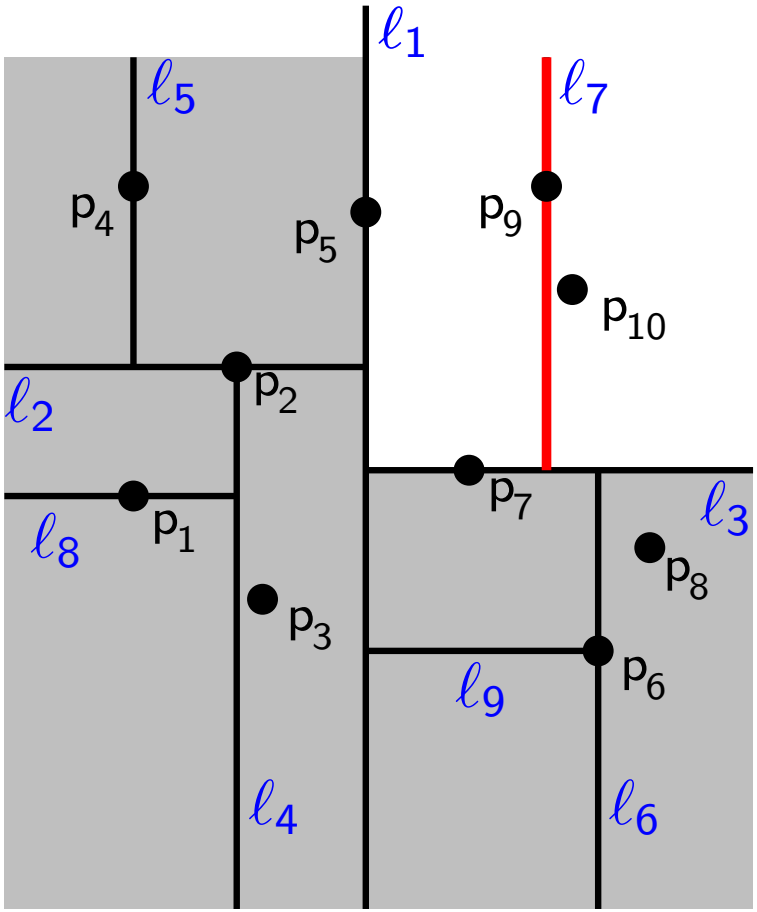

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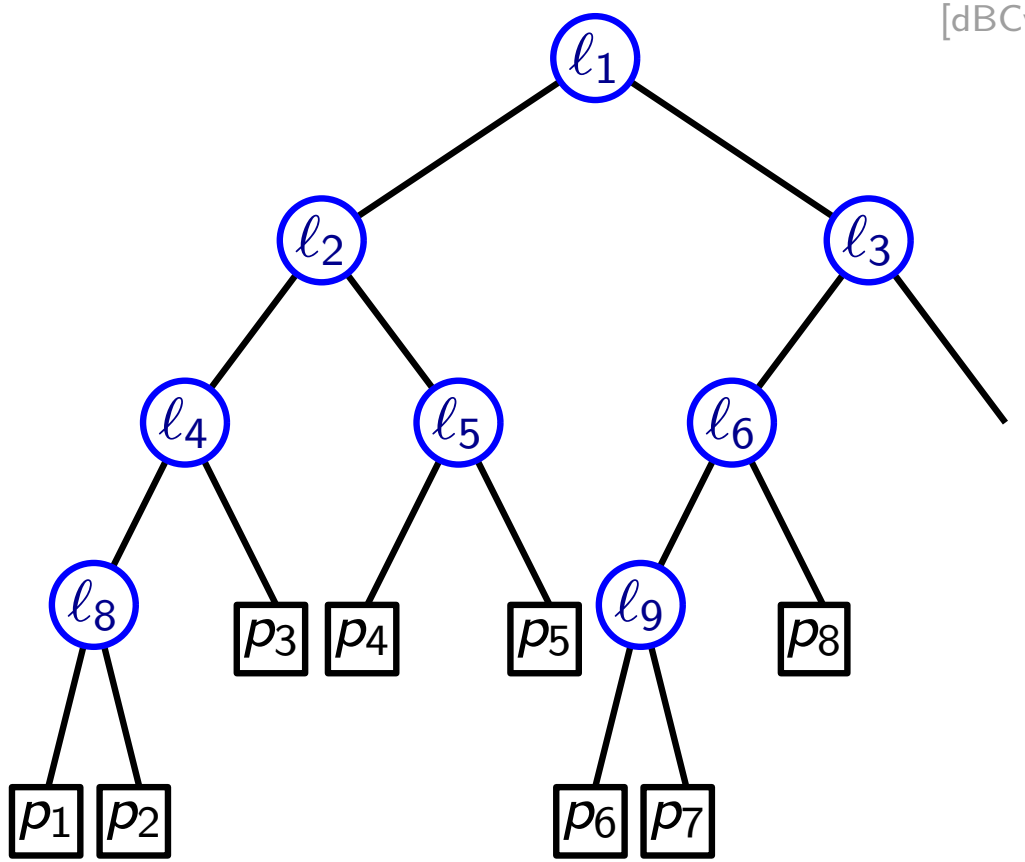

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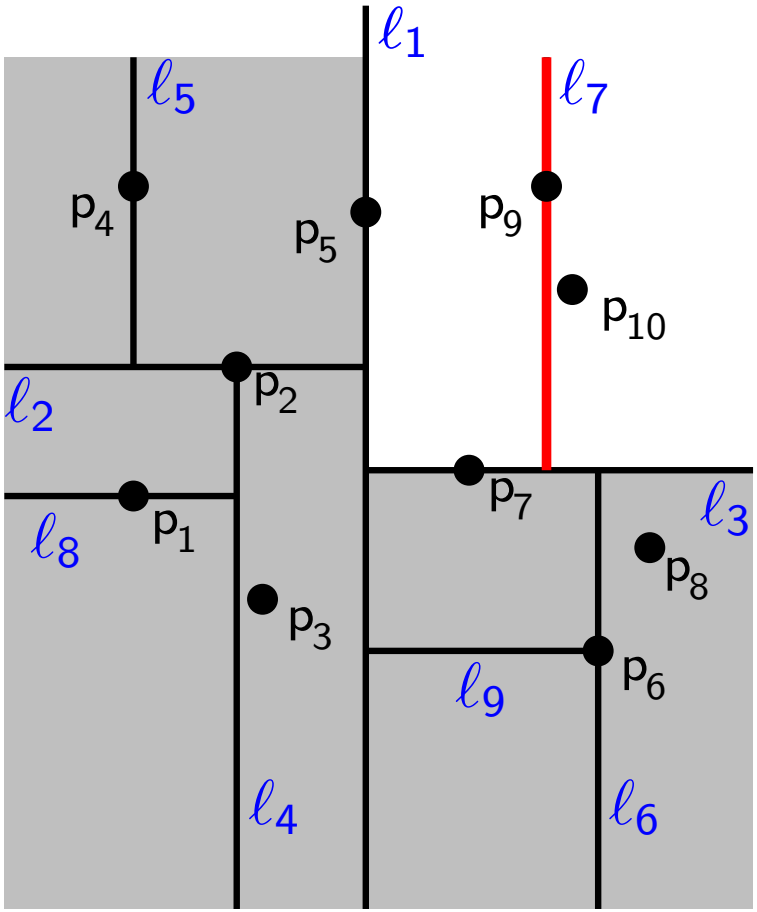

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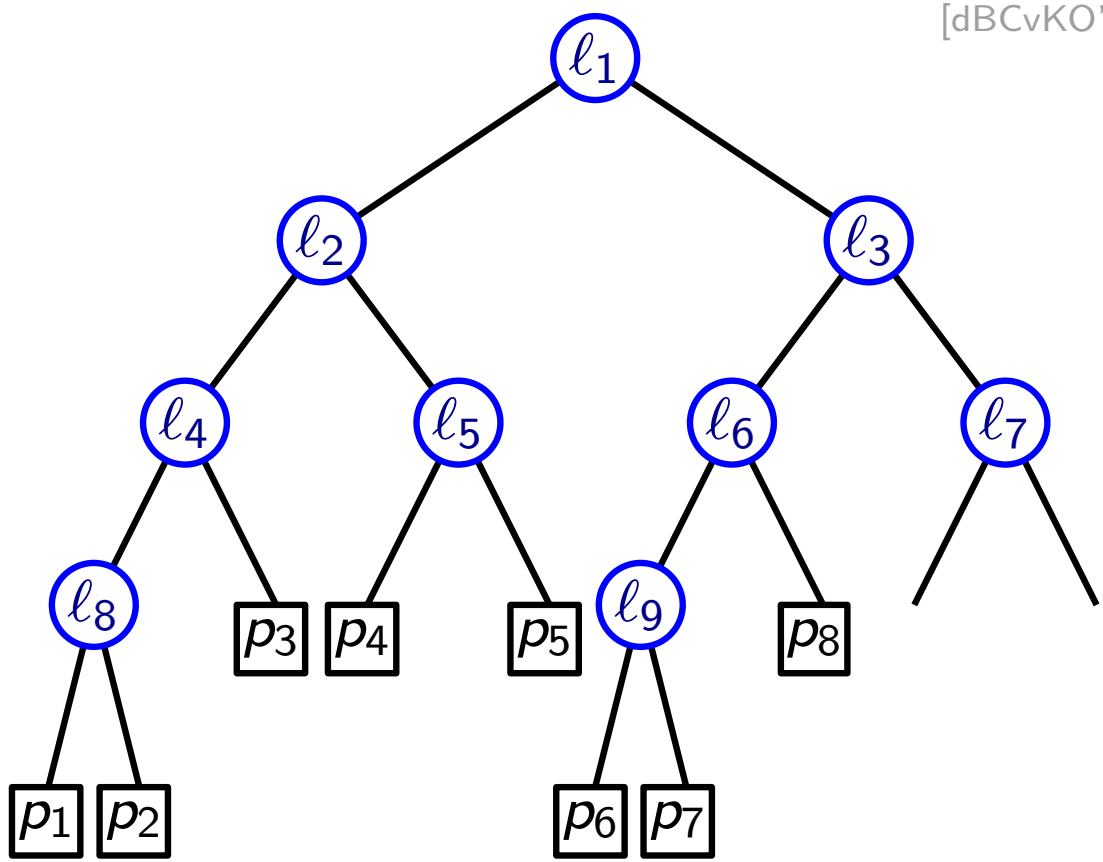

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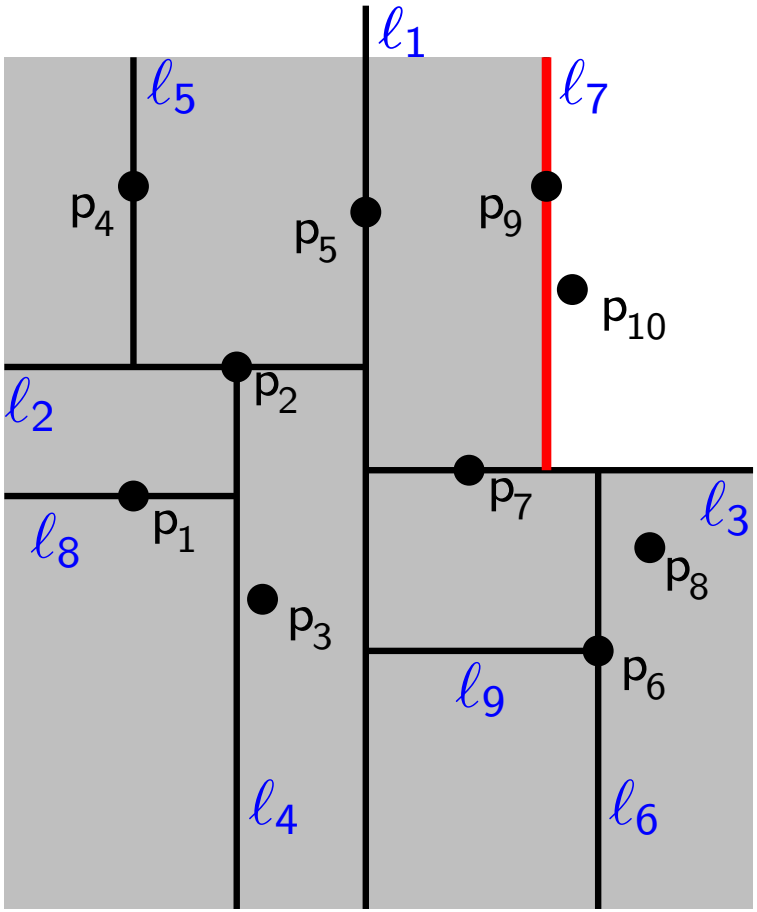

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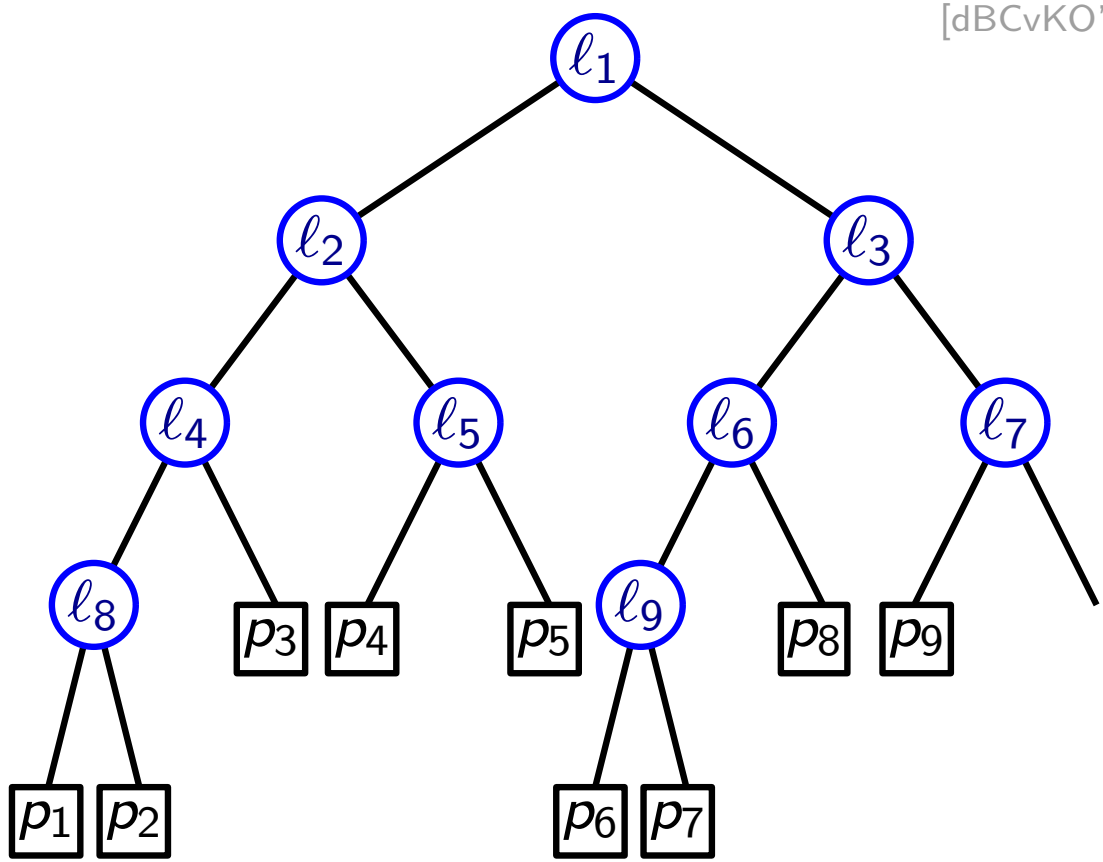

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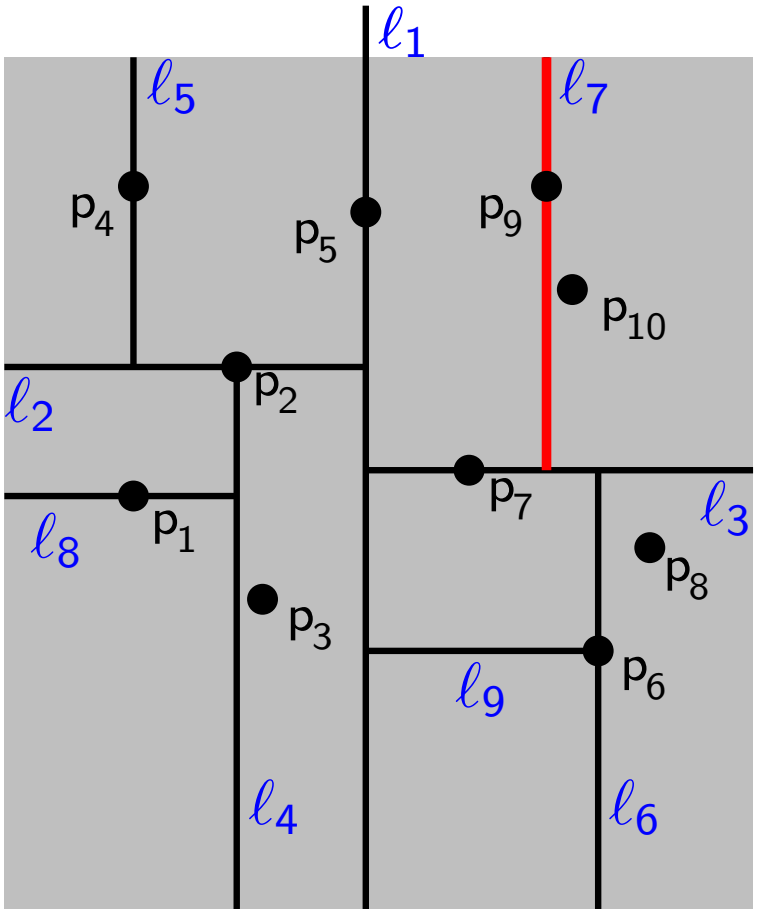

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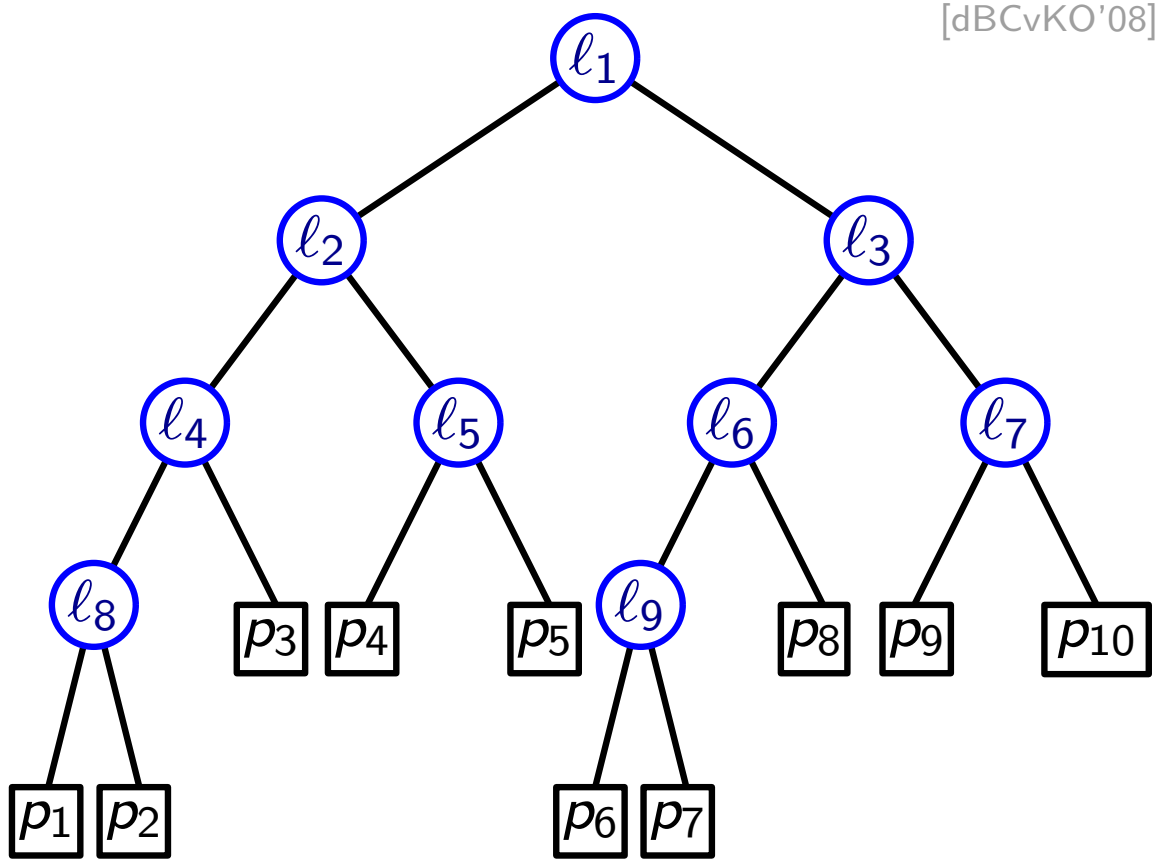

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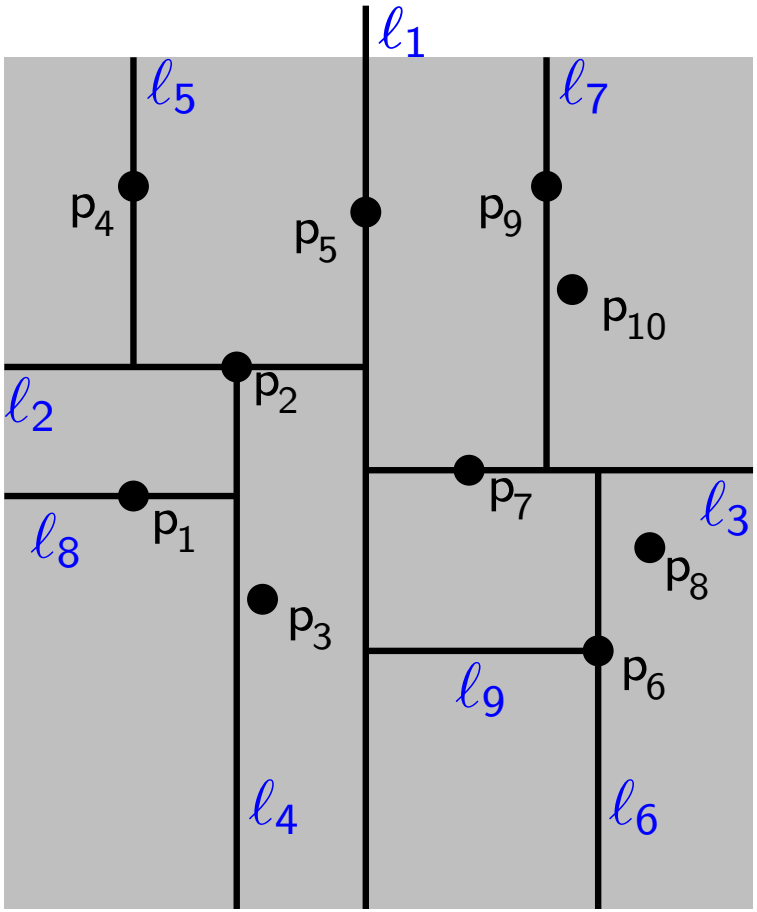

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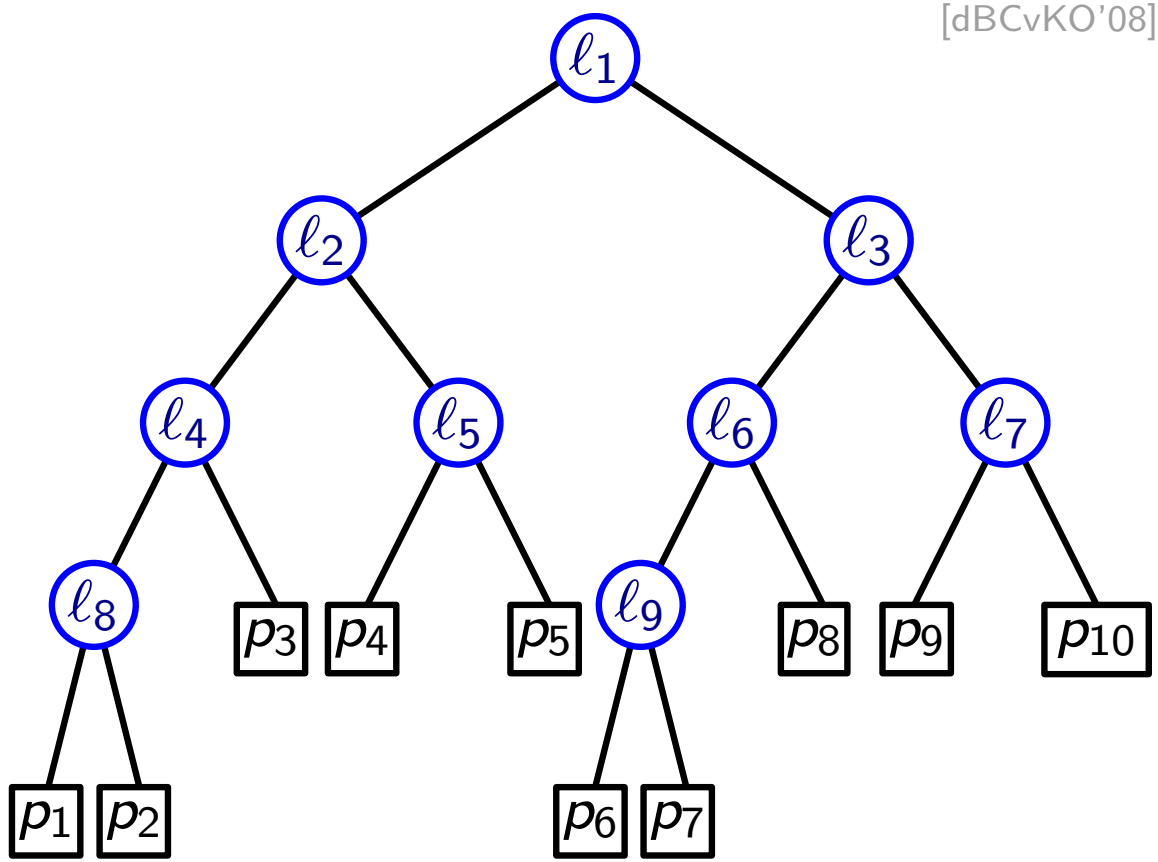

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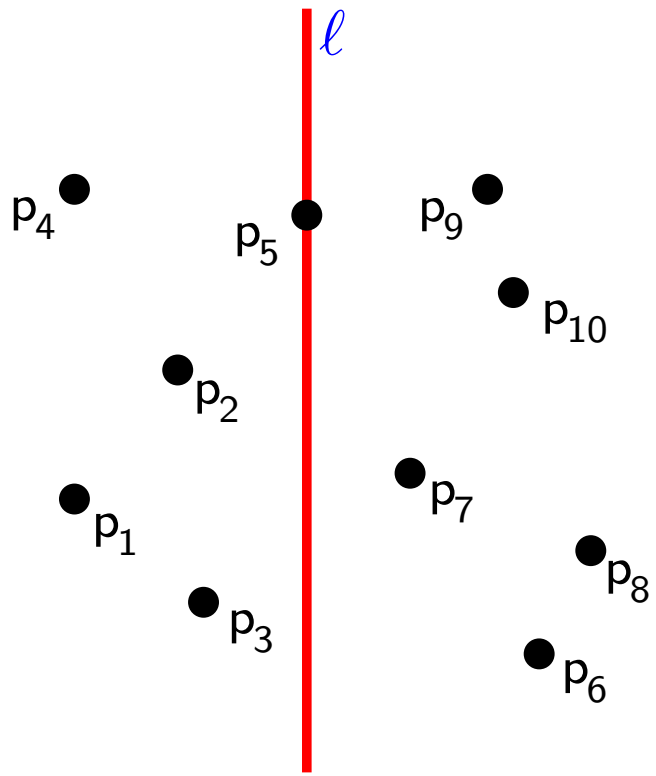

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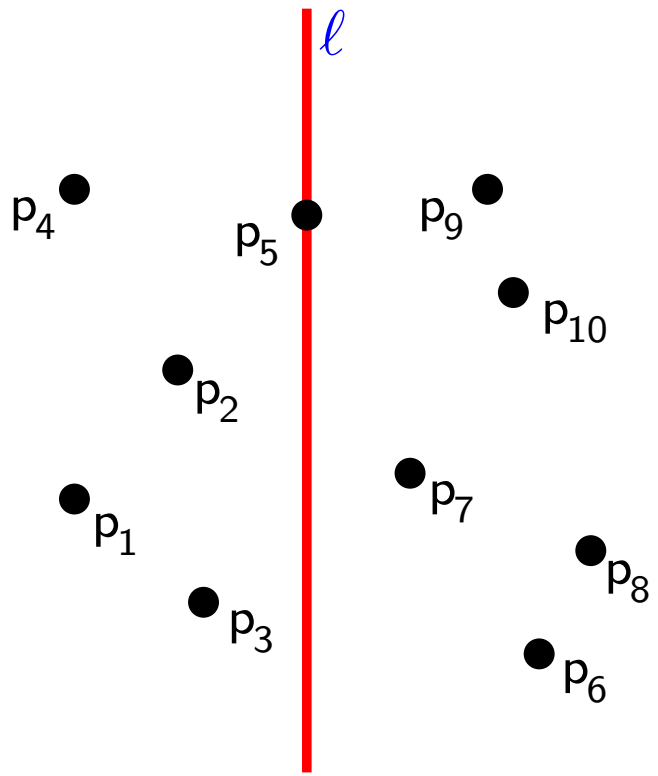

Pseudo-code:

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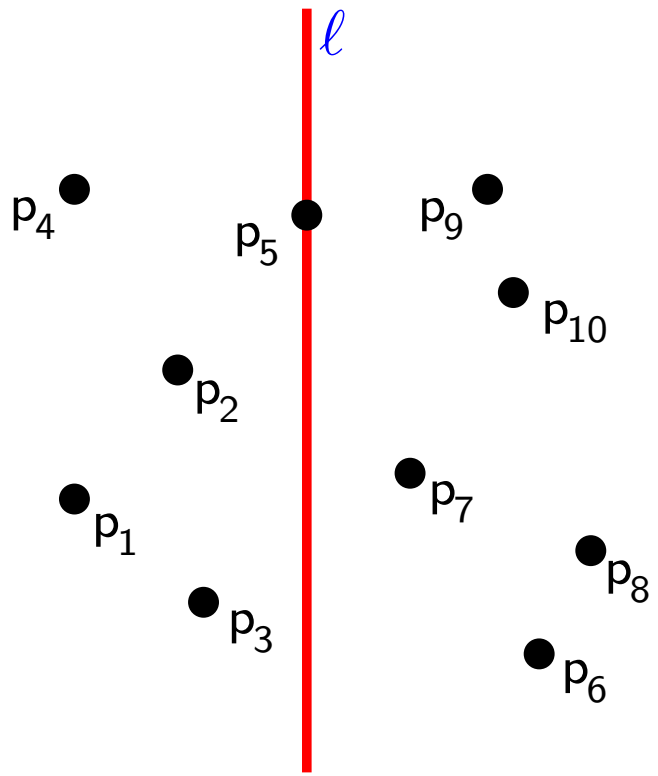
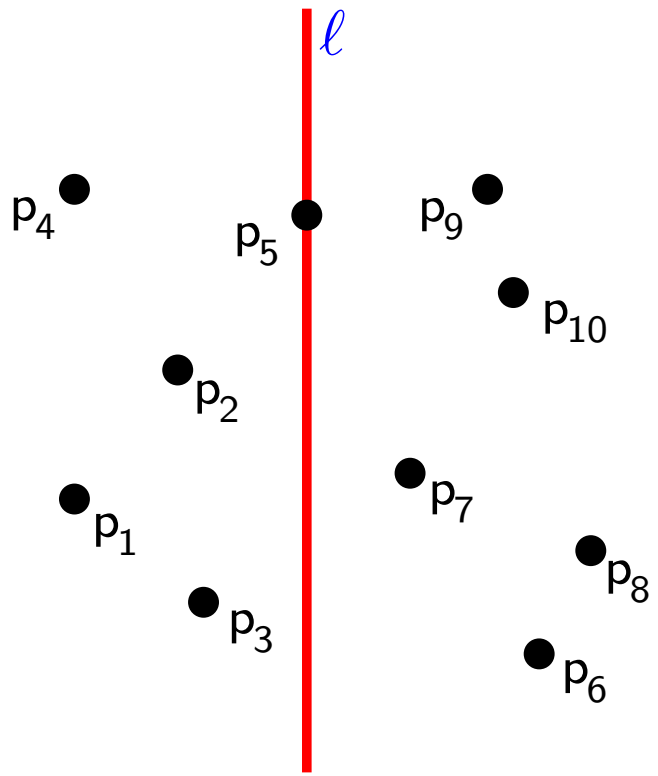


Pseudo-code:

BuildKdTree(points P, int $depth$)

Kd-Trees: Construction



Pseudo-code:

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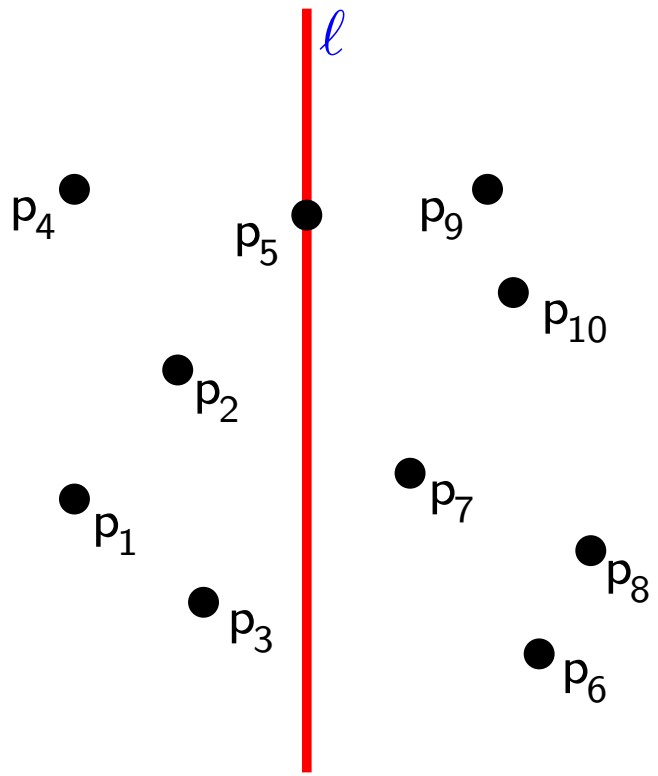
  if $|P| = 1$ **then**
    return (leaf storing the pt in P)
  else

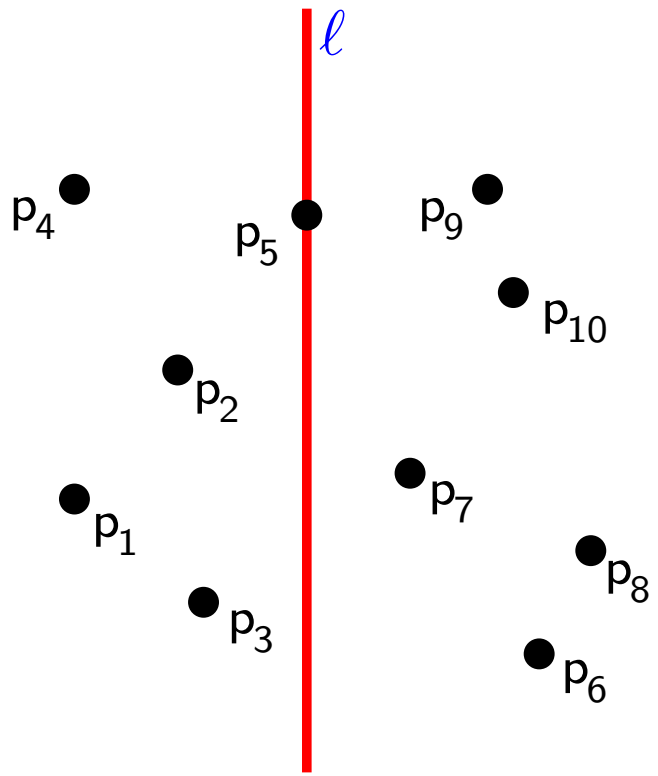
Kd-Trees: Construction



Pseudo-code:

BuildKdTree(points P, int $depth$)
  if $|P| = 1$ **then**
    return (leaf storing the pt in P)
  else
    if $depth$ is even **then**
      else

Kd-Trees: Construction



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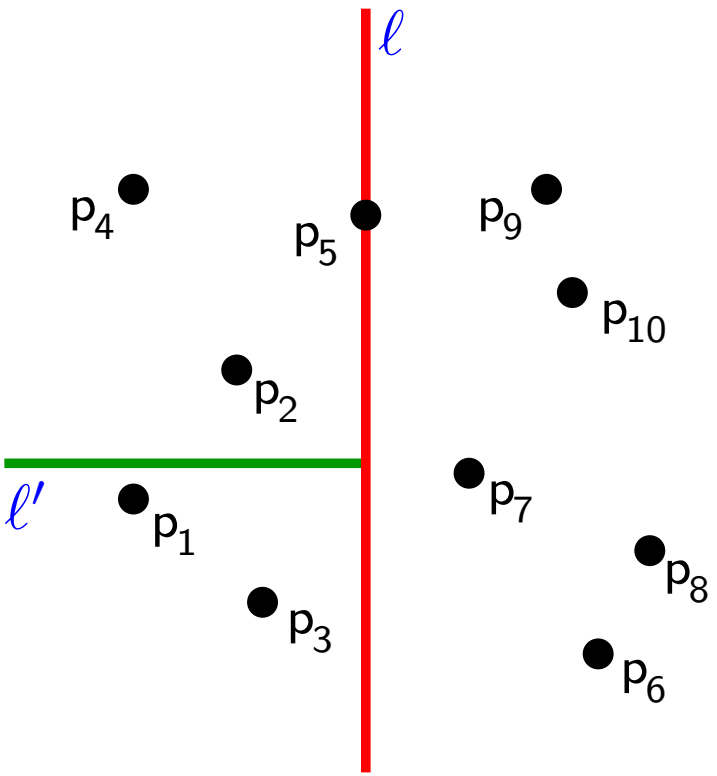
BuildKdTree(points P, int $depth$)
  if $|P| = 1$ **then**
    return (leaf storing the pt in P)
  else
    if $depth$ is even **then**
      split P with the vertical line
        ℓ: $x = x_{\text{median}(P)}$ into
        P_1 (pts left of or on ℓ) and
        $P_2 = P \setminus P_1$
    else

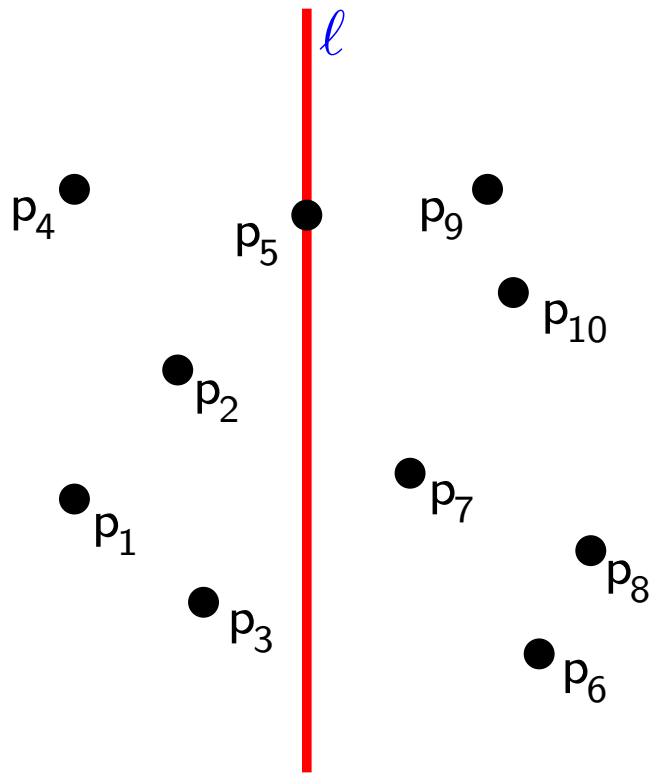
Kd-Trees: Construction



Pseudo-code:

BuildKdTree(points P, int $depth$)
- **if** $|P| = 1$ **then**
  - **return** (leaf storing the pt in P)
- **else**
  - **if** $depth$ is even **then**
    - split P with the vertical line ℓ: $x = x_{\text{median}(P)}$ into P_1 (pts left of or on ℓ) and $P_2 = P \setminus P_1$
  - **else**
    - split P horizontally...

Kd-Trees: Construction



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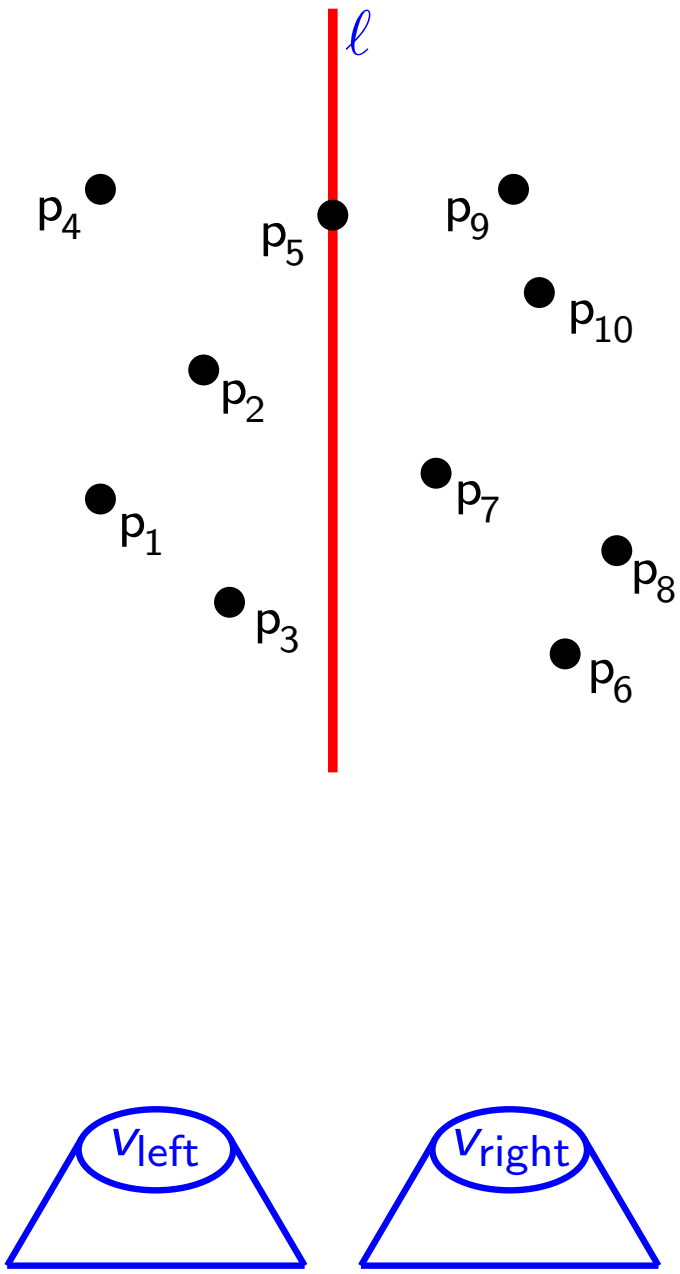
    $v_{\text{left}} \leftarrow$ BuildKdTree(P_1, $depth + 1$)
    $v_{\text{right}} \leftarrow$ BuildKdTree(P_2, $depth + 1$)

Kd-Trees: Construction



Pseudo-code:

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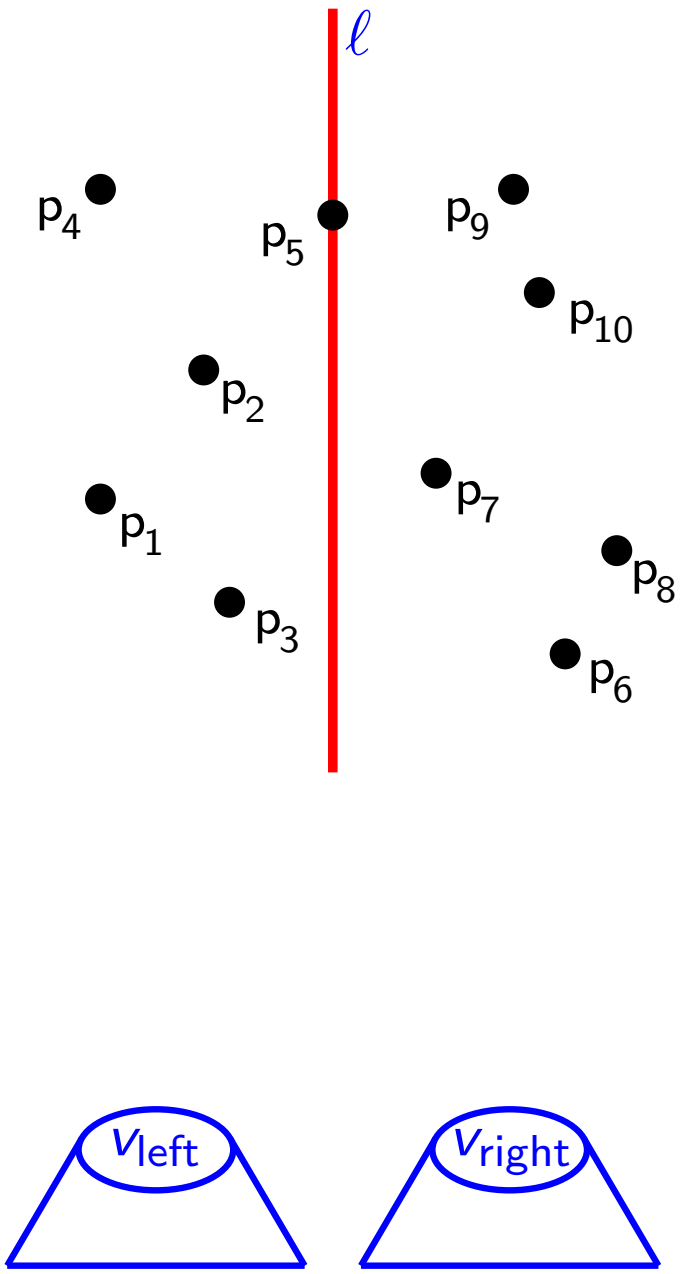
BuildKdTree(points P, int depth)
  if |P| = 1 then
    return (leaf storing the pt in P)
  else
    if depth is even then
      split P with the vertical line
        ℓ: x = x_median(P) into
        P1 (pts left of or on ℓ) and
        P2 = P \ P1
    else
      split P horizontally...
    v_left ← BuildKdTree(P1, depth + 1)
    v_right ← BuildKdTree(P2, depth + 1)
```

Kd-Trees: Construction



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    $v_{\text{left}} \leftarrow$ BuildKdTree(P_1, $depth + 1$)
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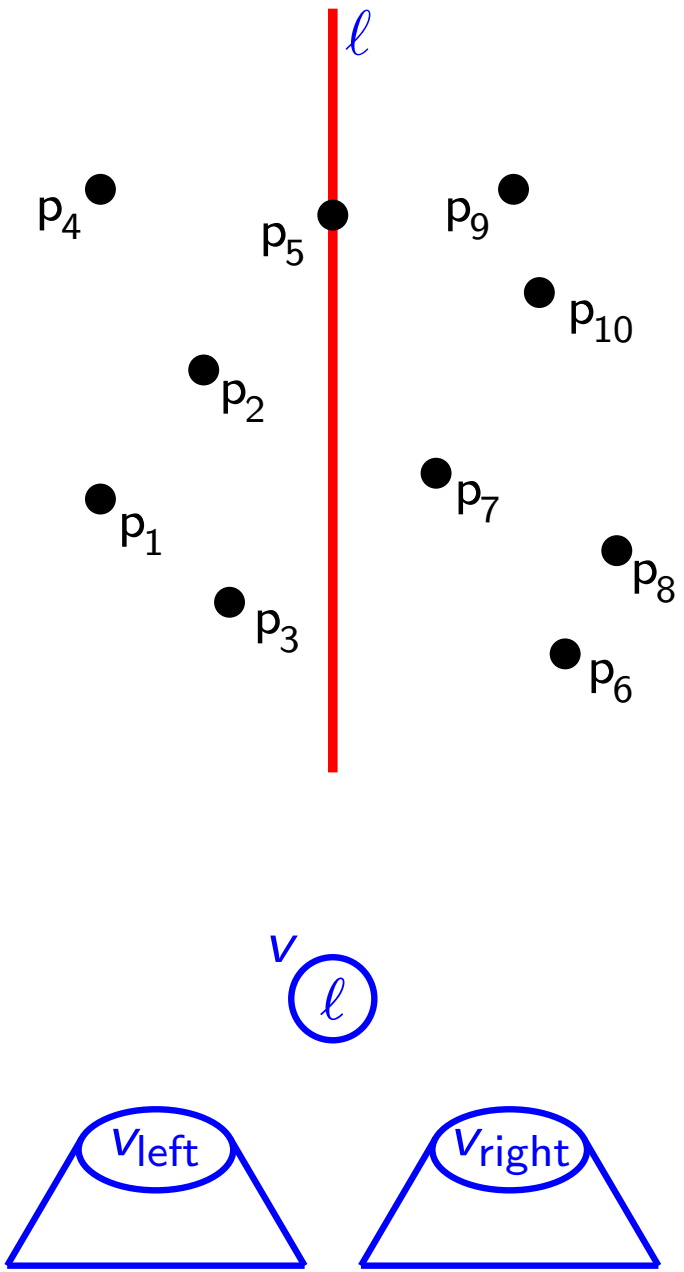
Kd-Trees: Construction



Pseudo-code:

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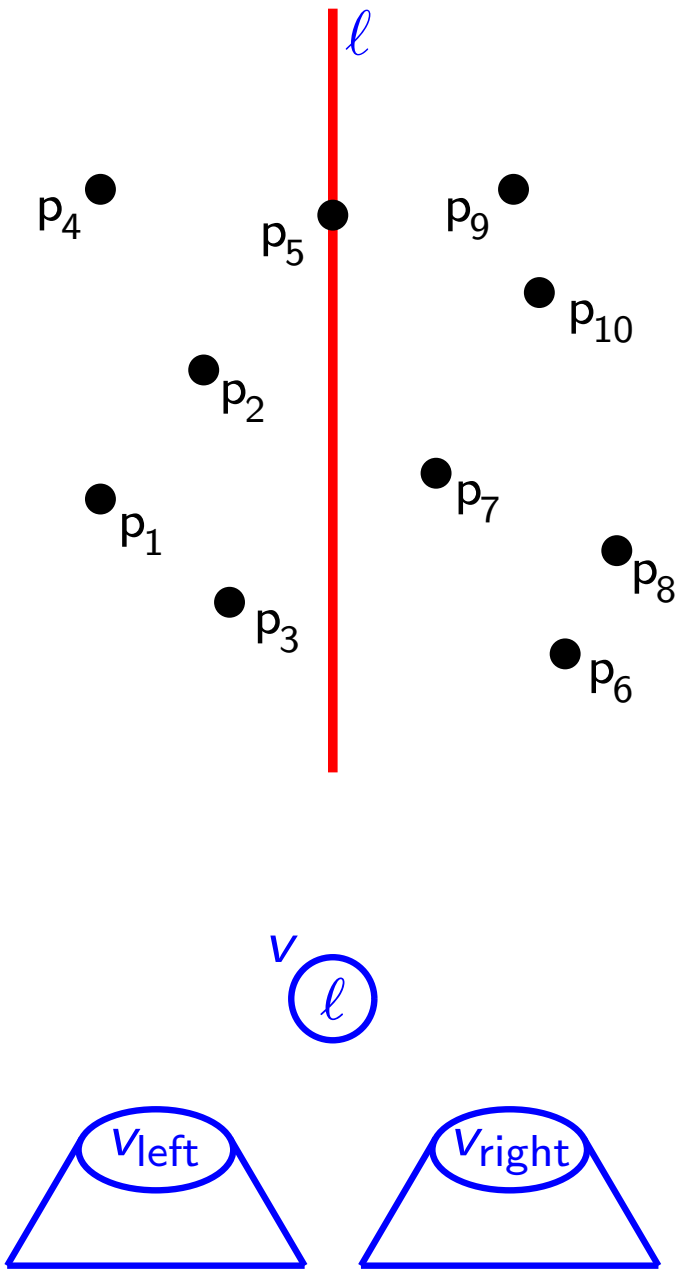

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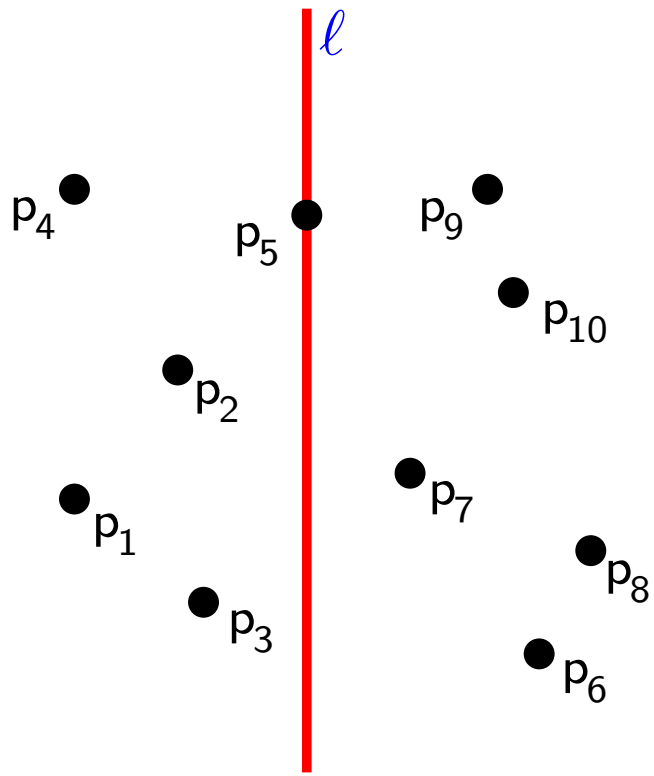

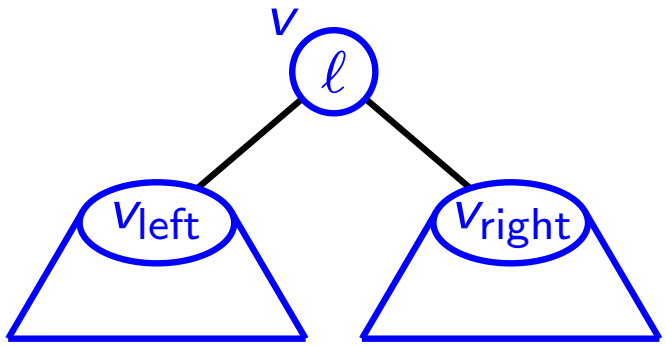

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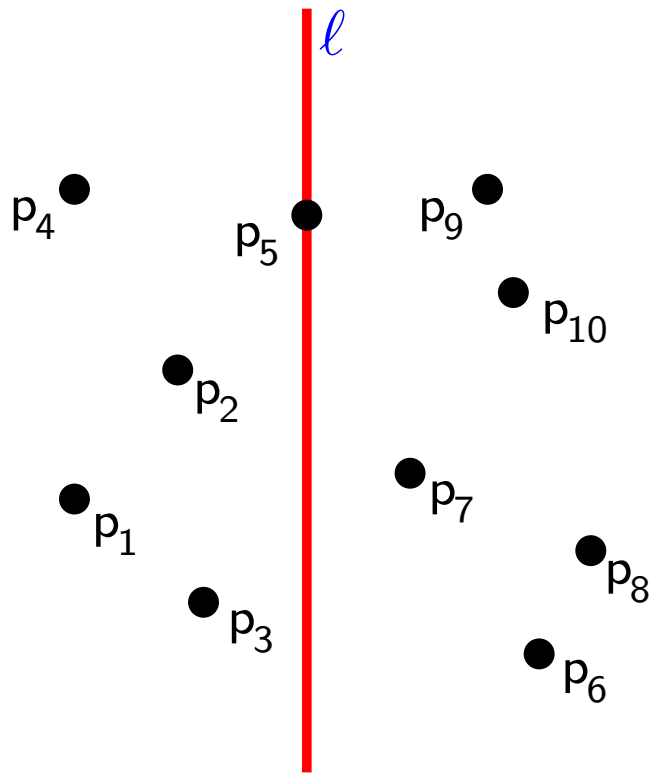

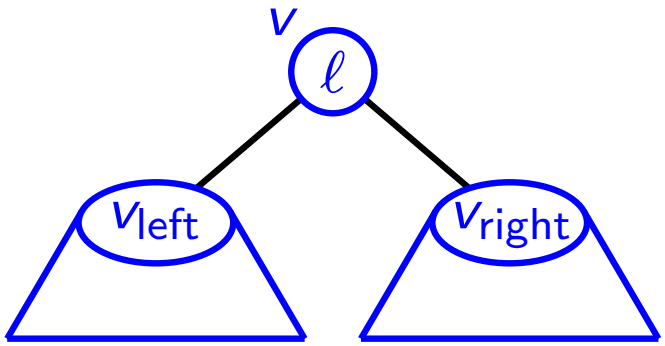

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Kd-Trees: Analysis

Construction time?

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$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ O(n) + 2T(\lceil n/2 \rceil) & \text{else.} \end{cases}$$

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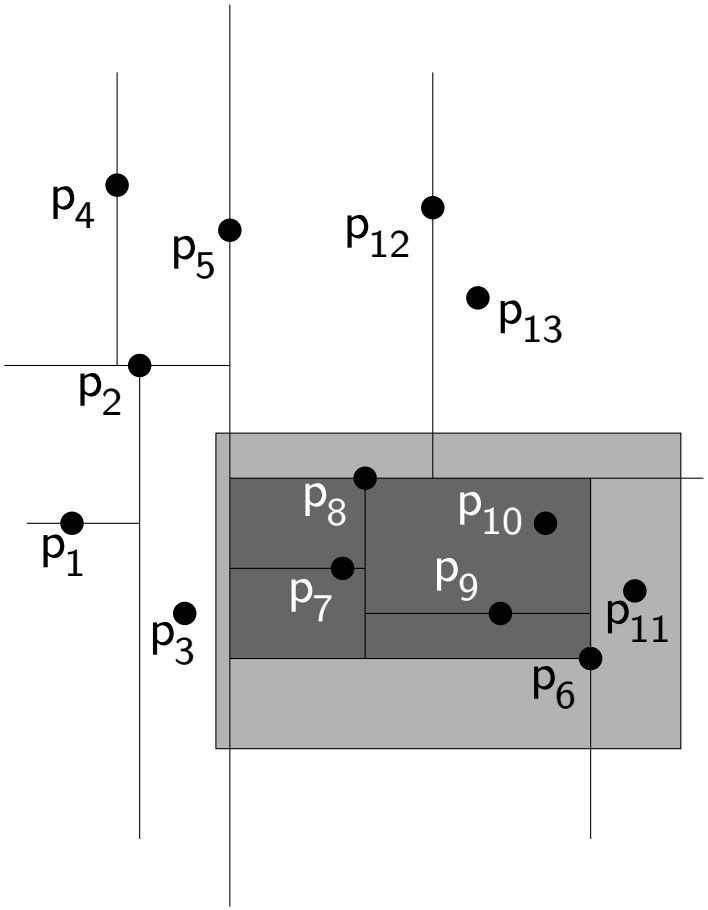

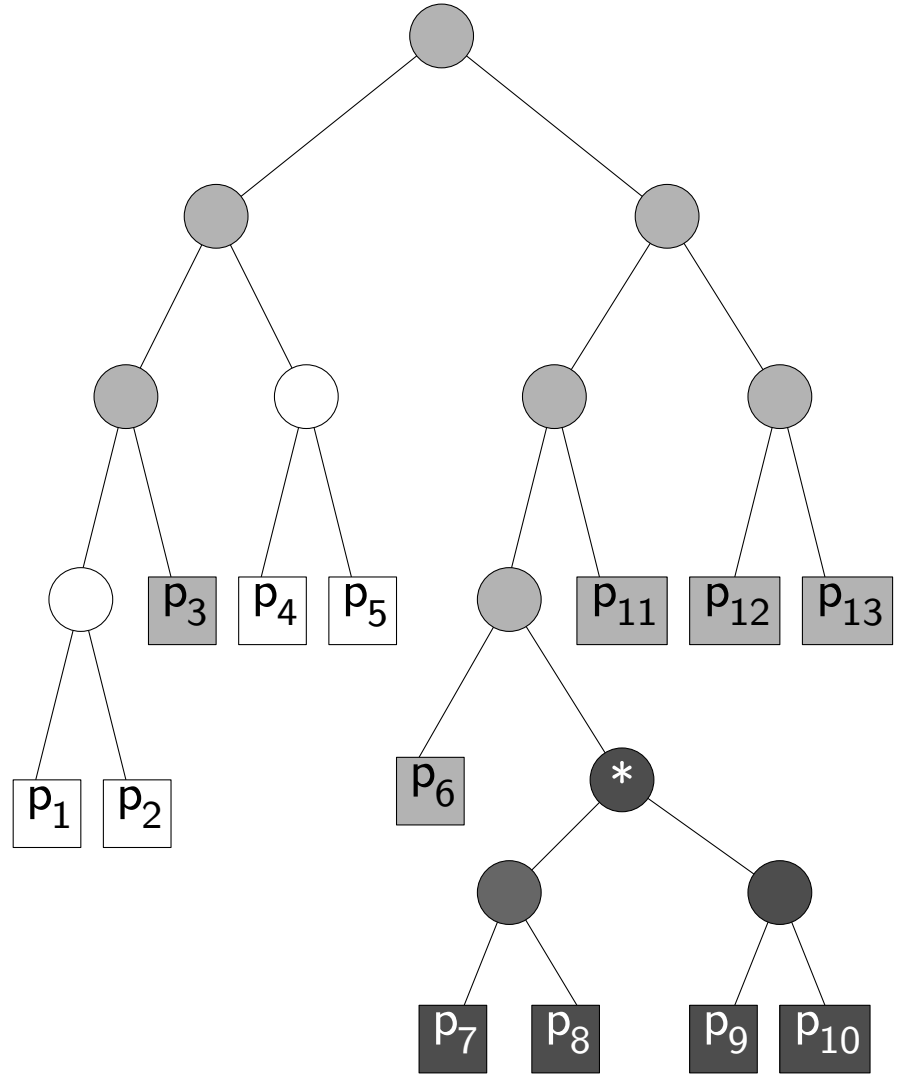

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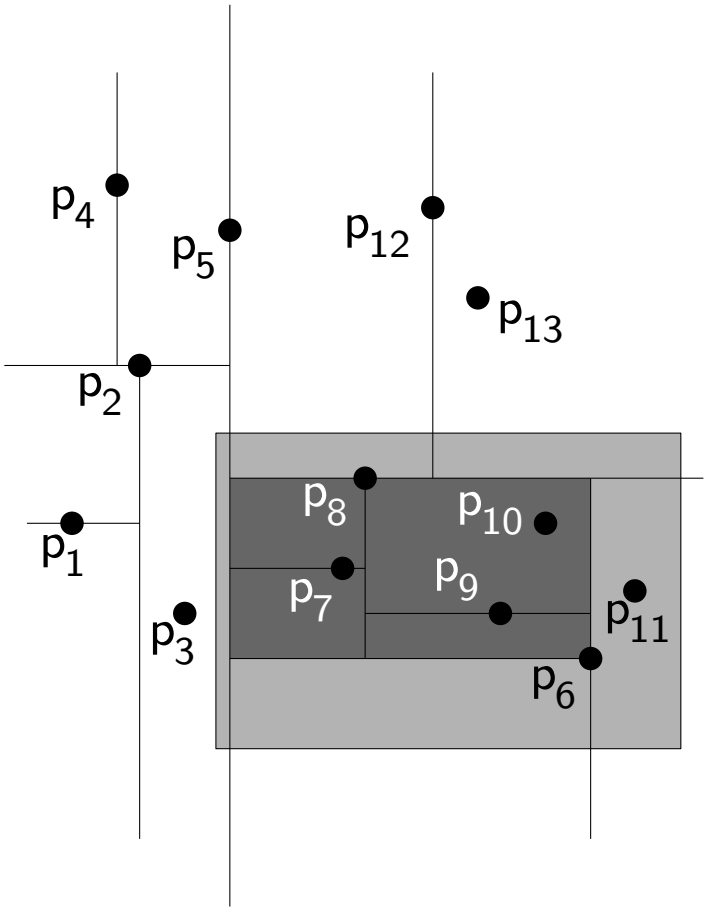
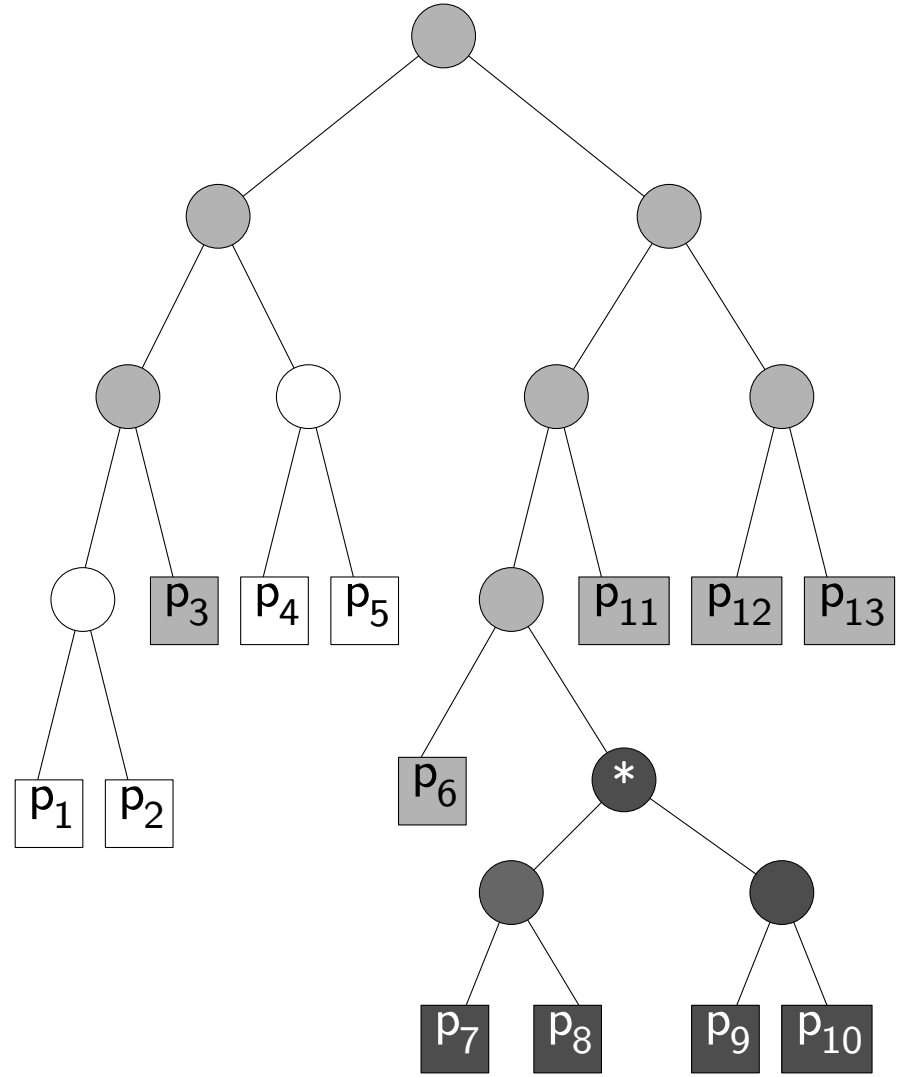
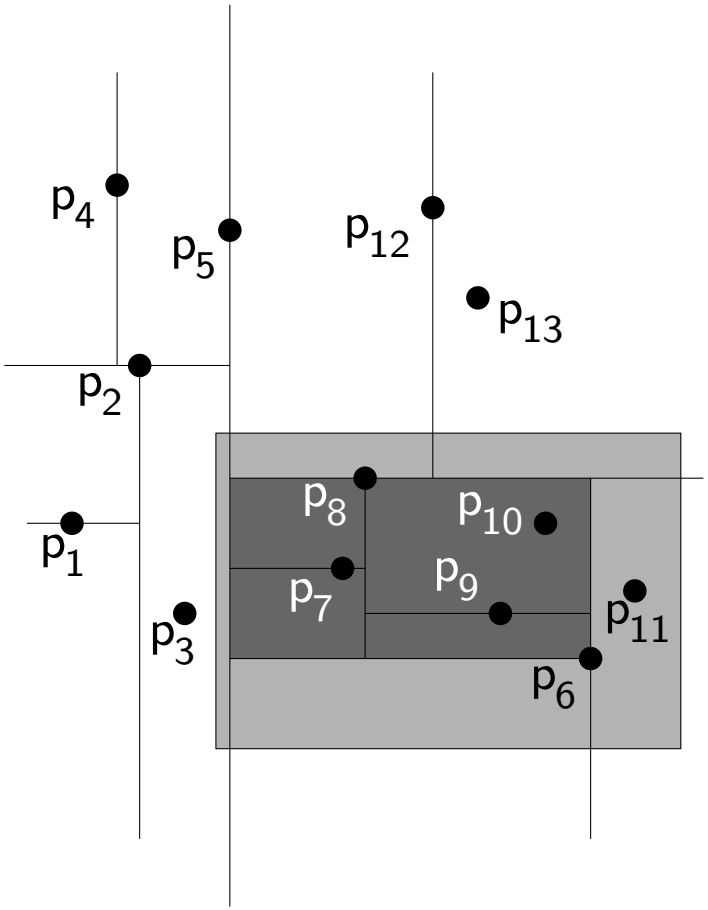
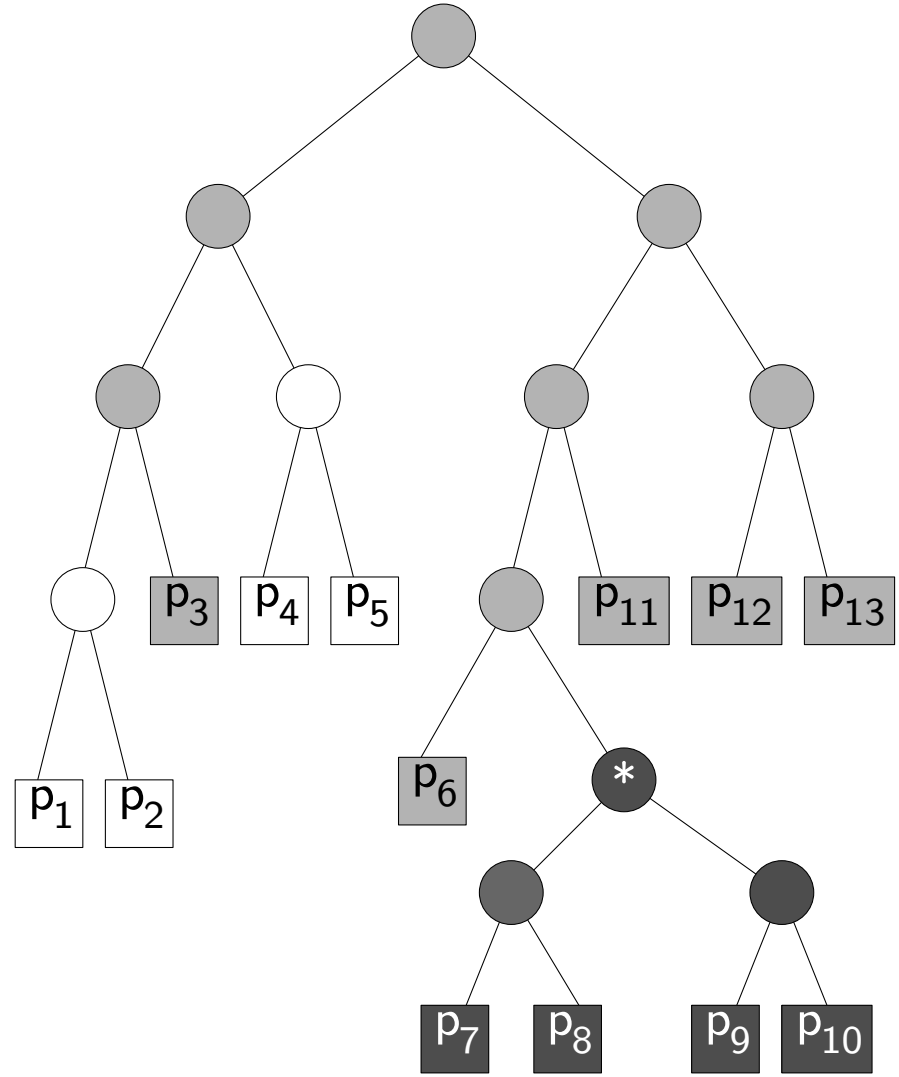




Lemma. Querying a kd-tree for n pts in the plane with an axis-parallel rectangle R takes $O(k + \sqrt{n})$ time, where $k = |\text{output}|$.

Kd-Trees: Querying





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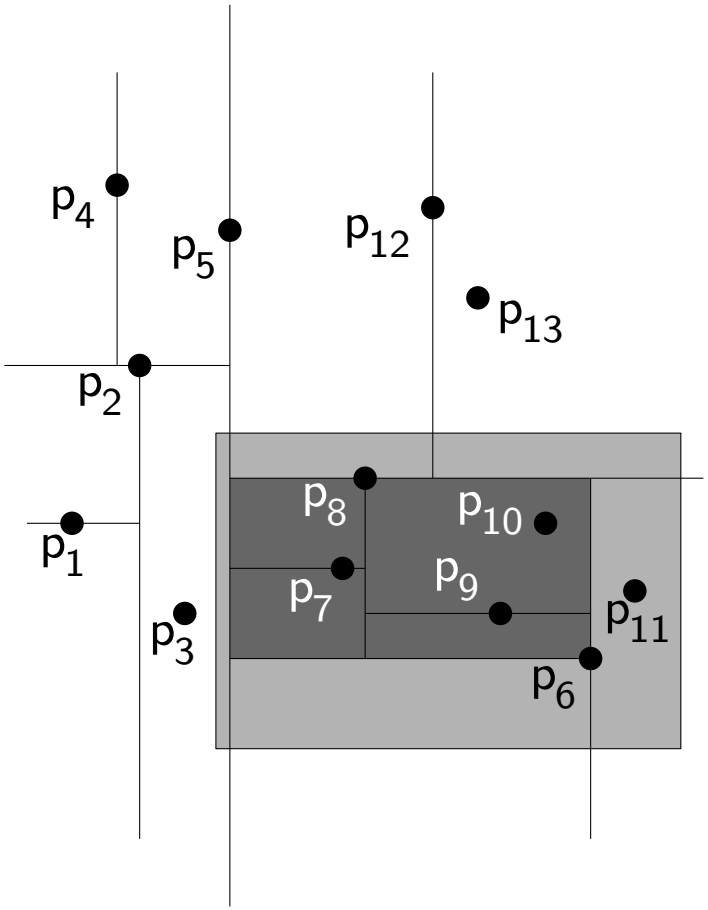

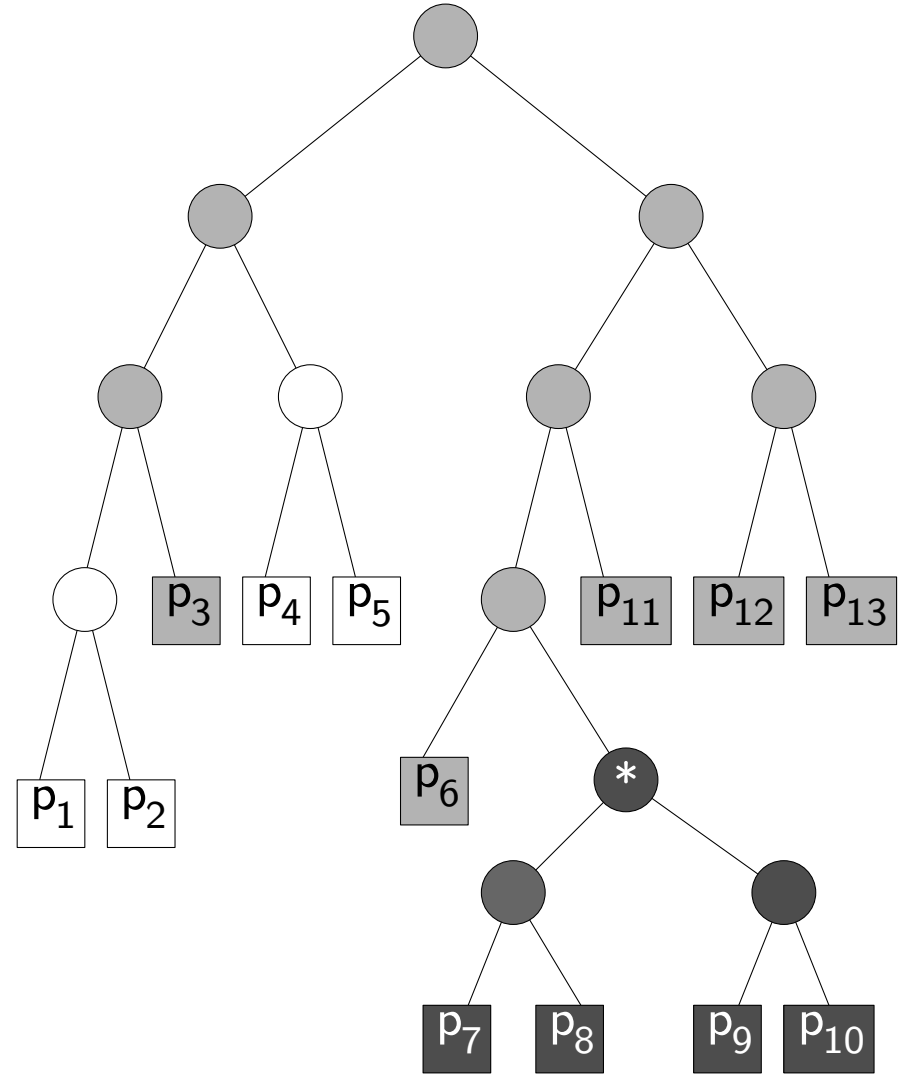

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query time: $O(k + n^{1-1/d})$

Range Trees: Query Algorithm

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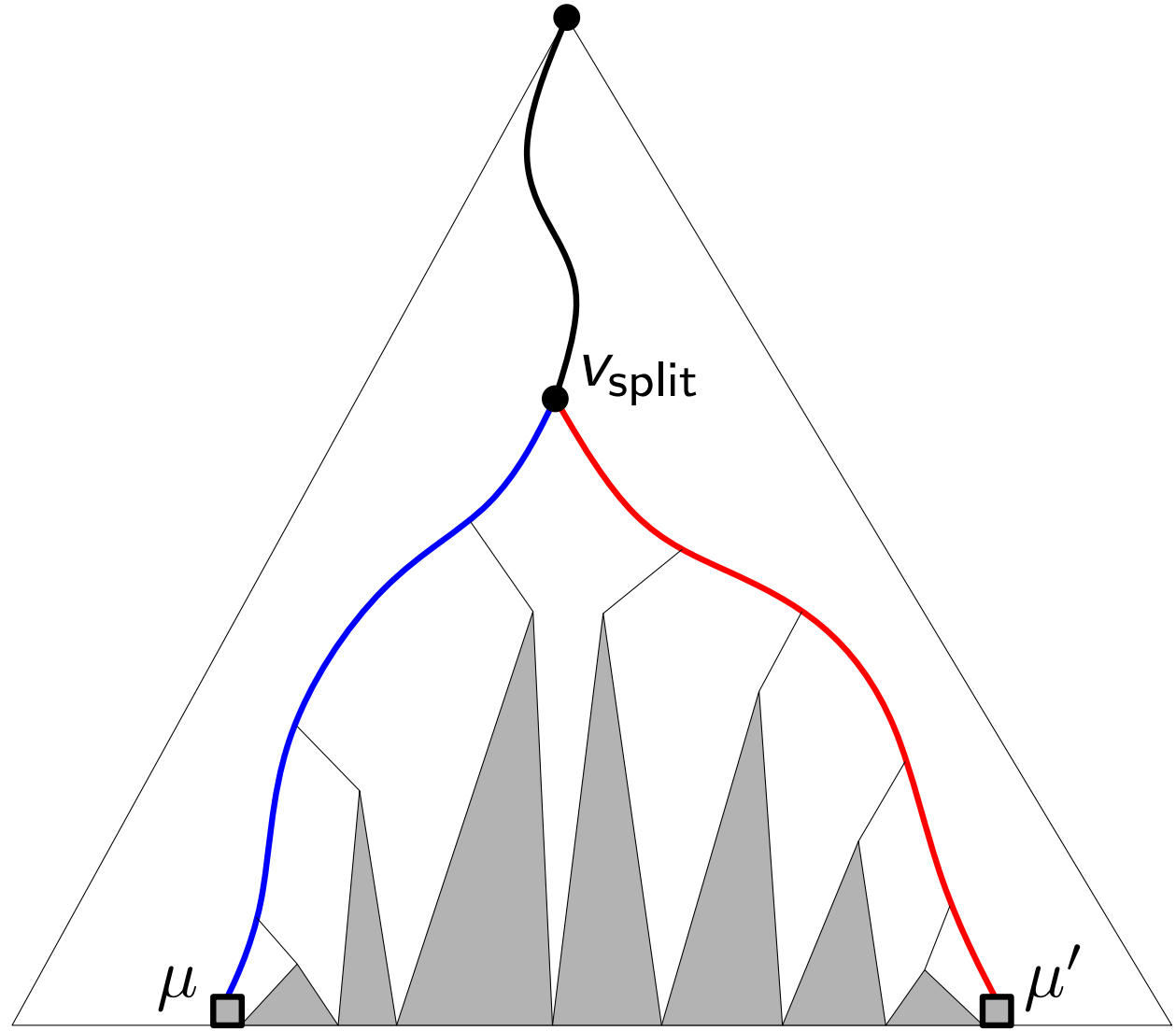

Range Trees: Query Algorithm

1. Search in main tree for x-coordinate
2. For each node u
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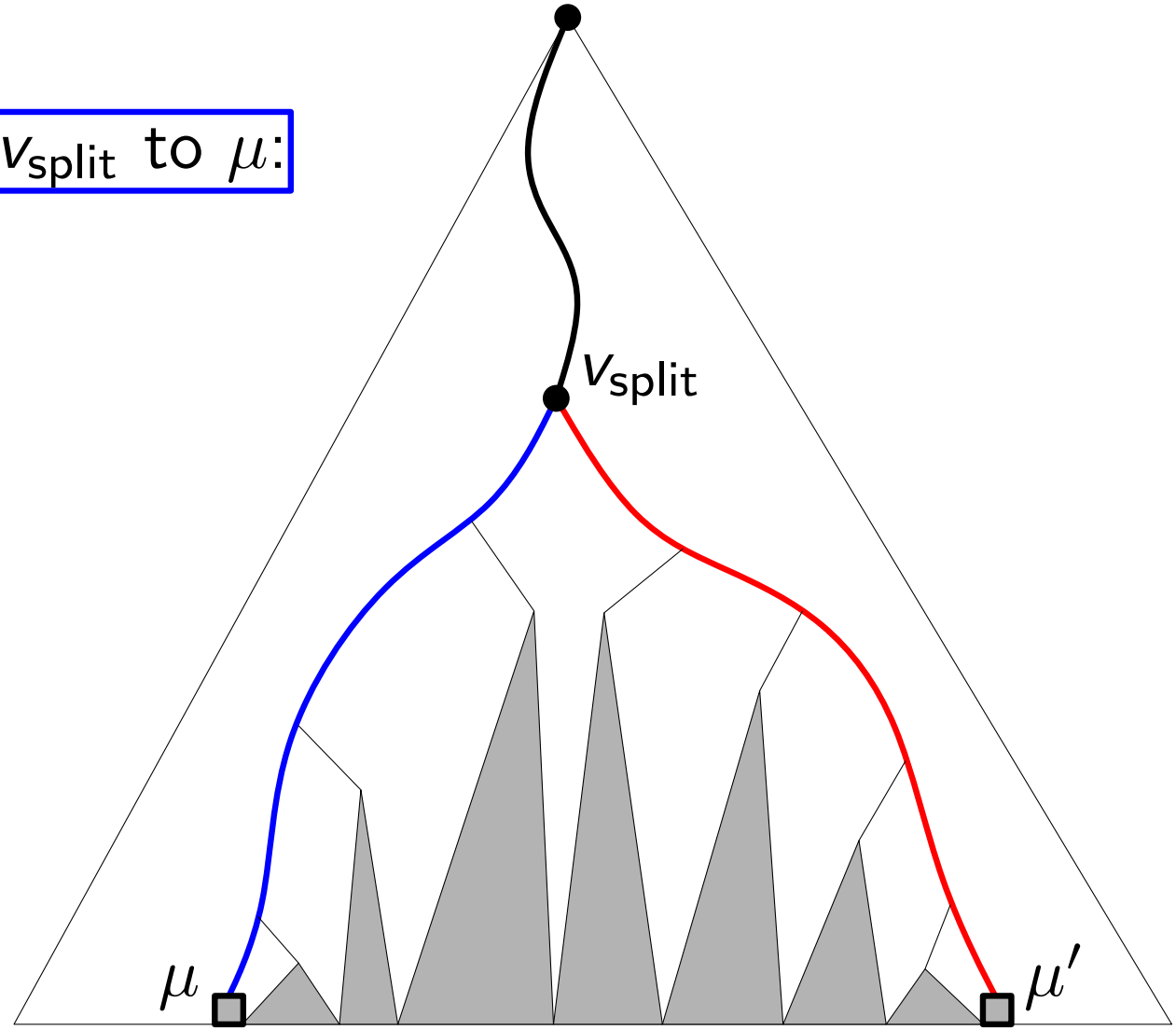

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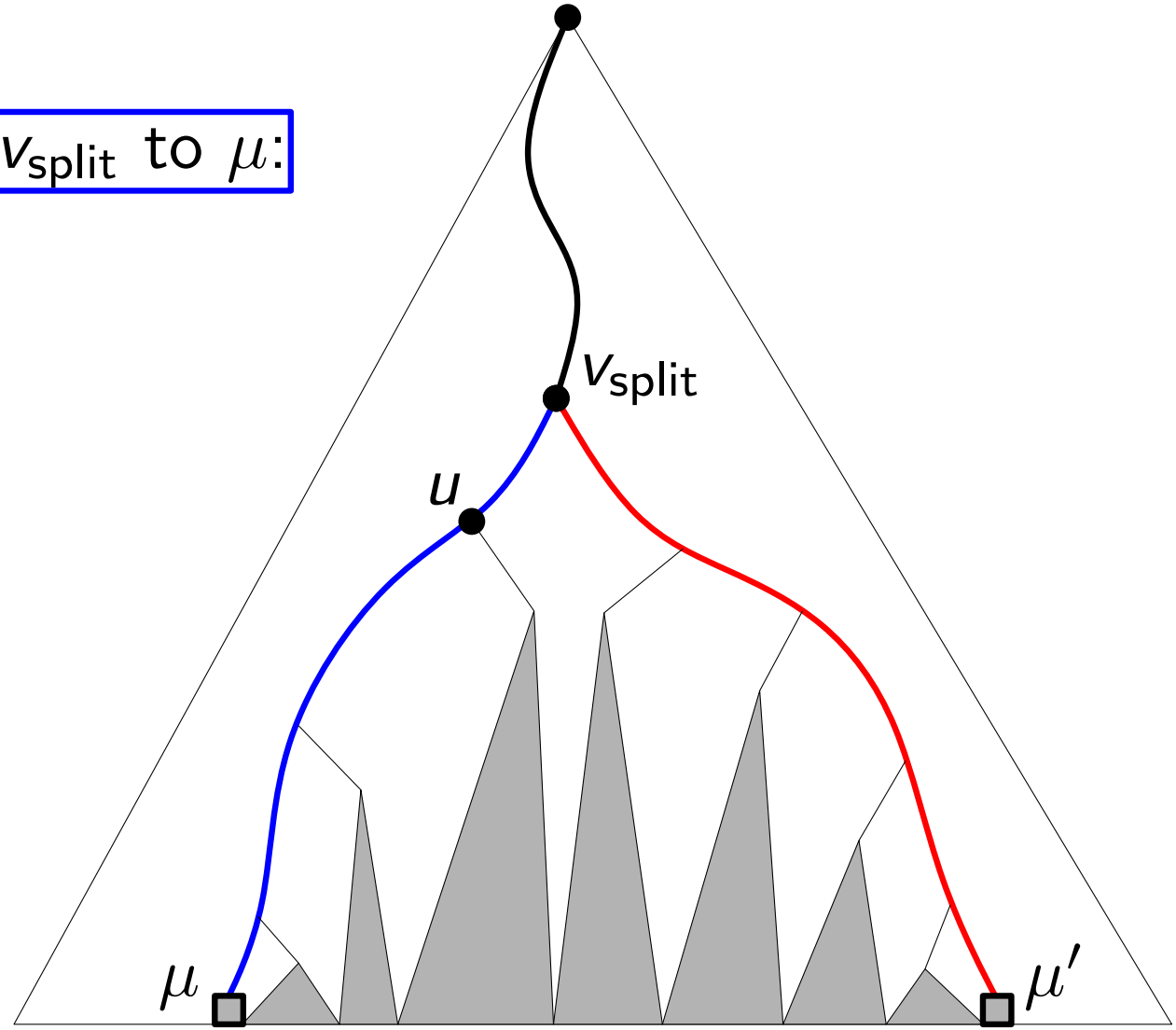

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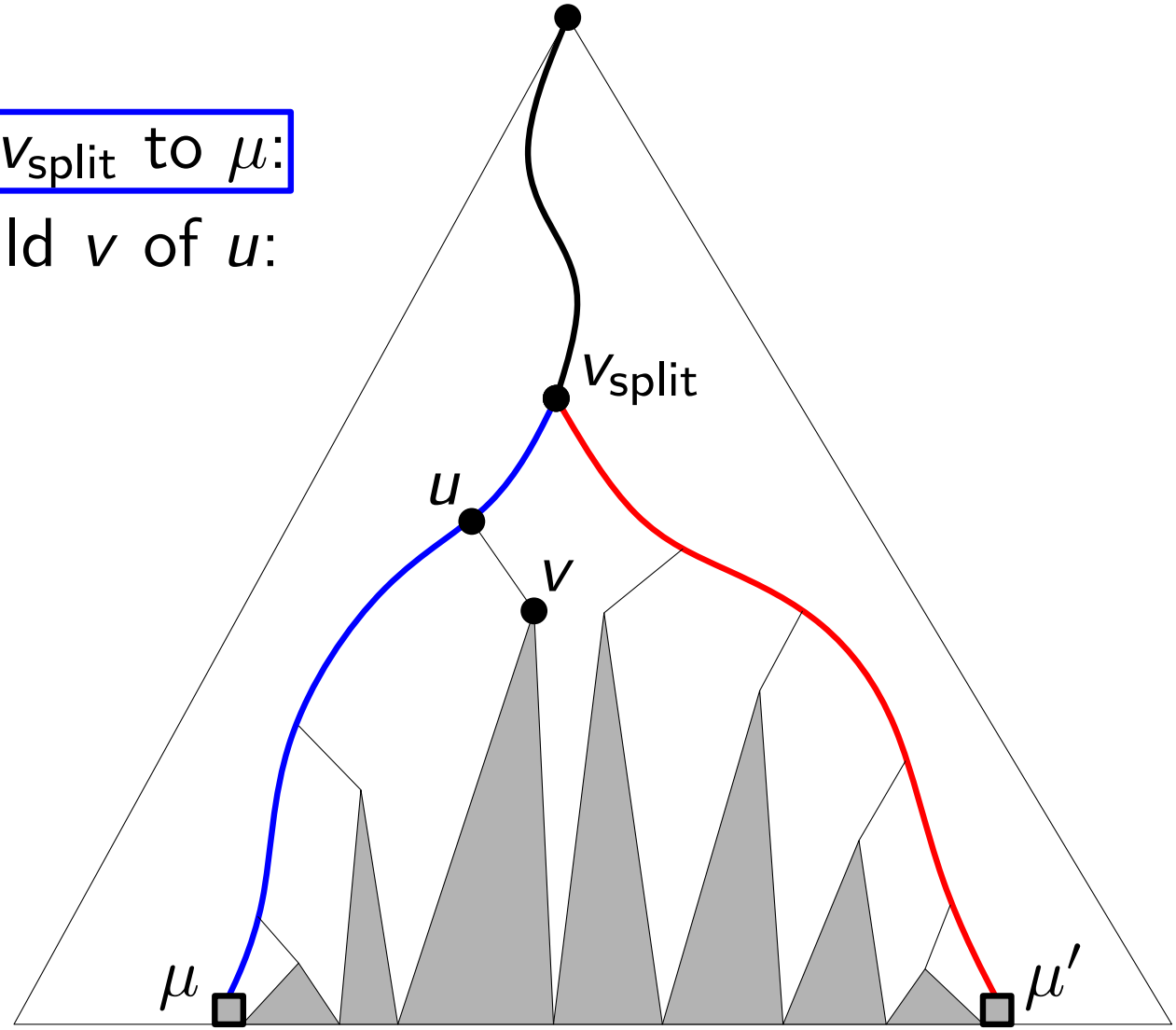

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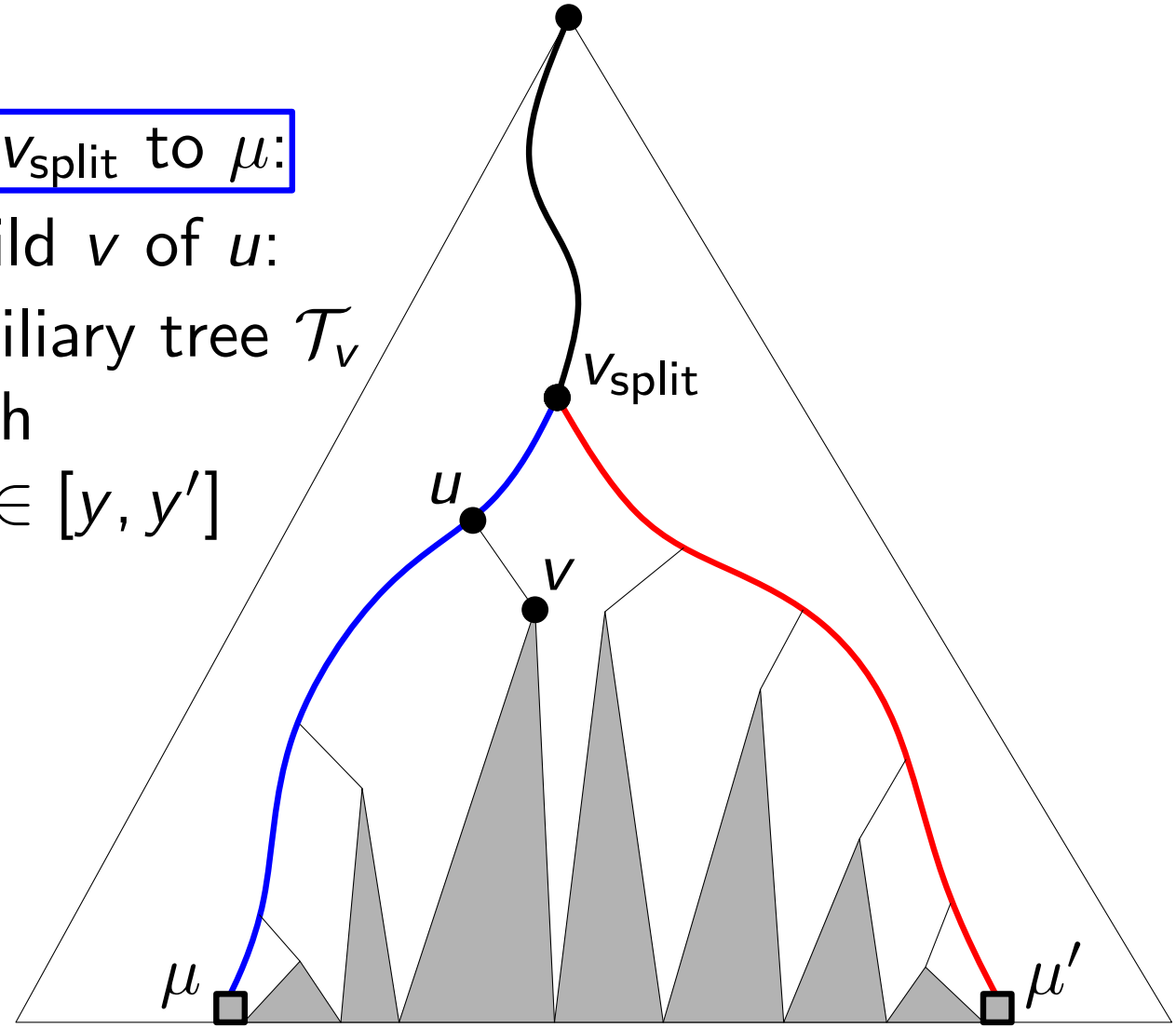

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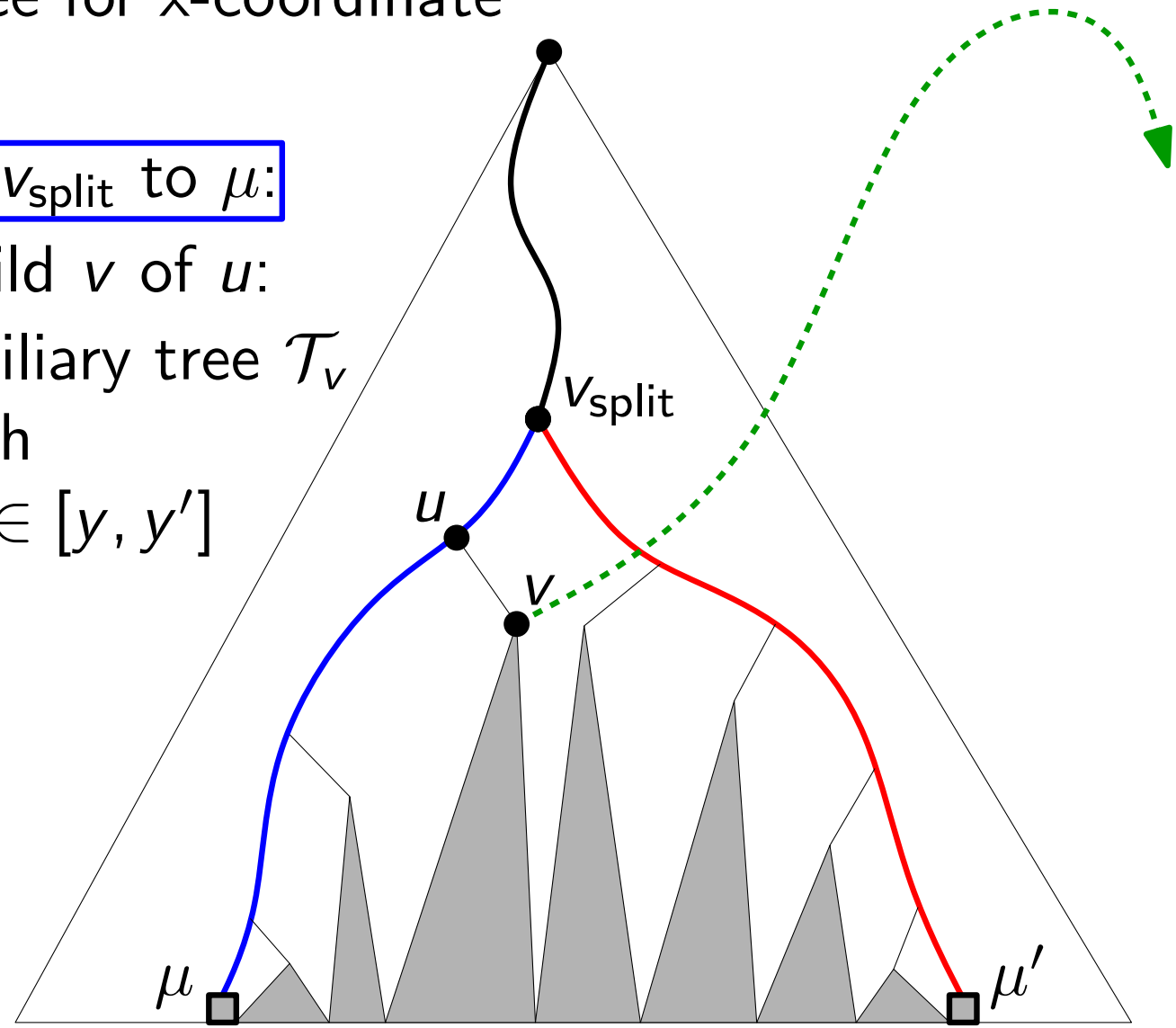

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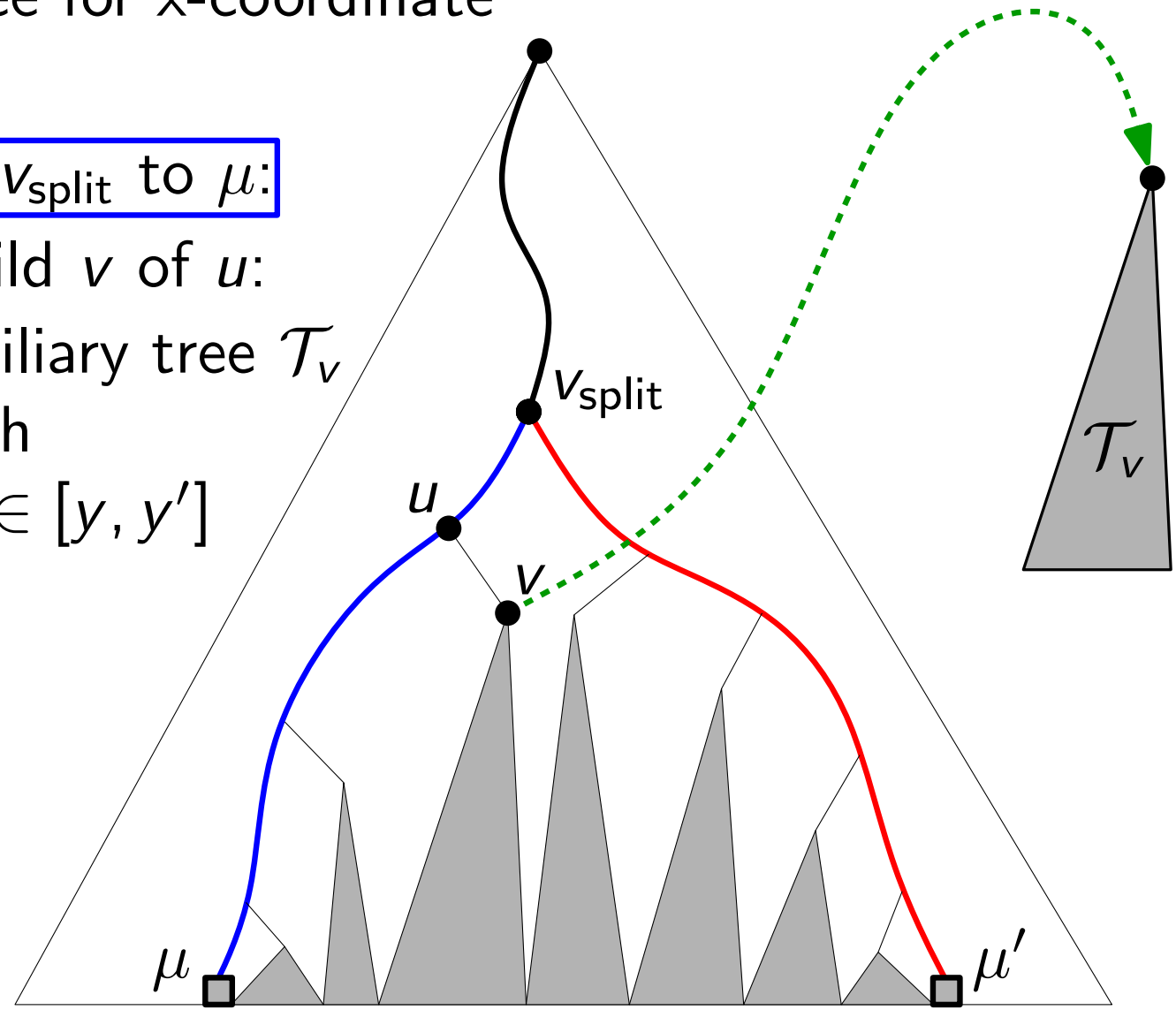

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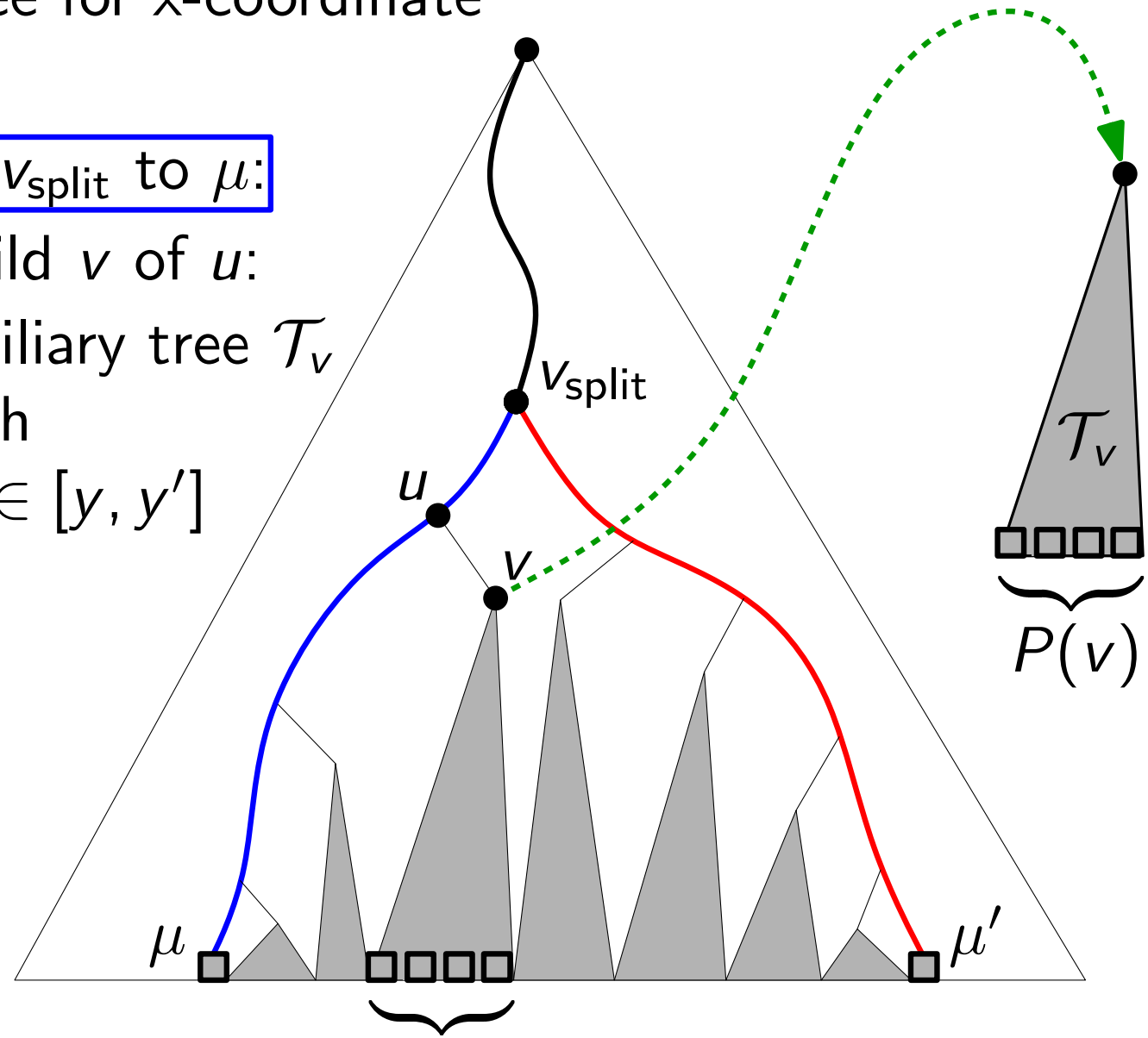

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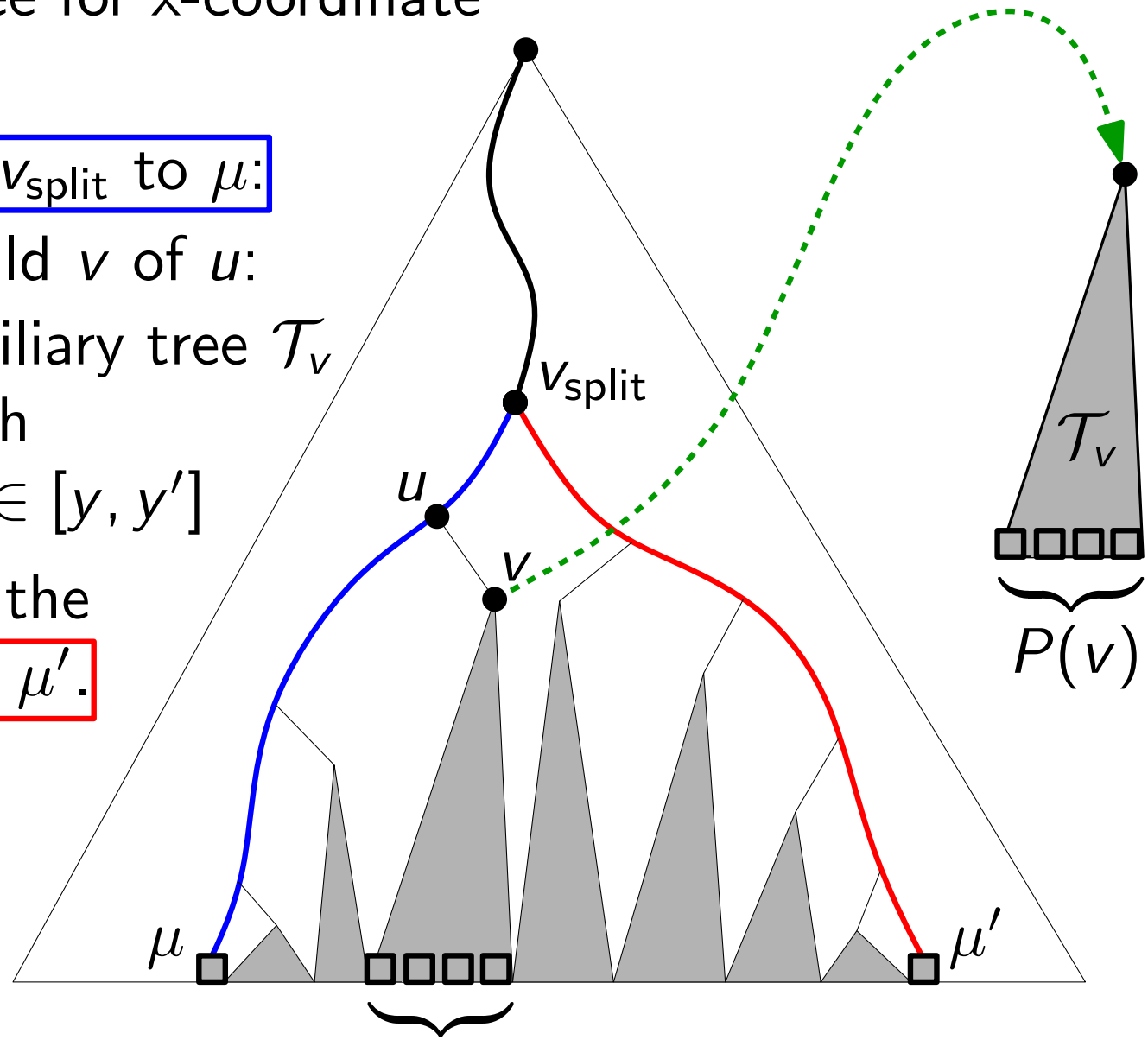

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Build2DRangeTree(point[] P)

construct 2nd-level tree $\mathcal{T}_P$ on P (y-order)

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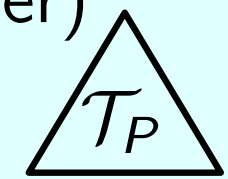

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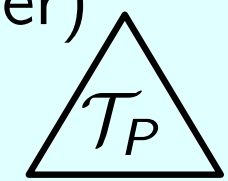

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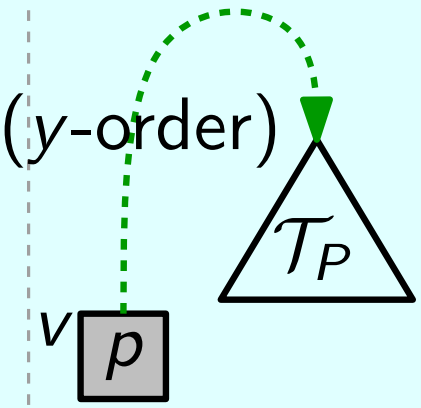

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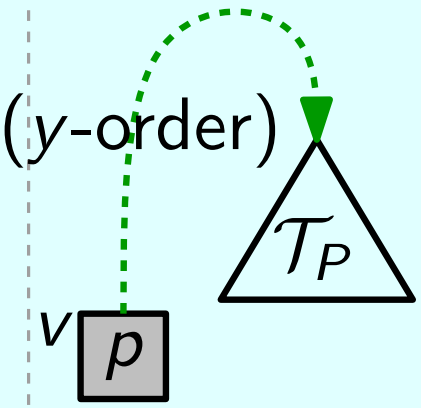

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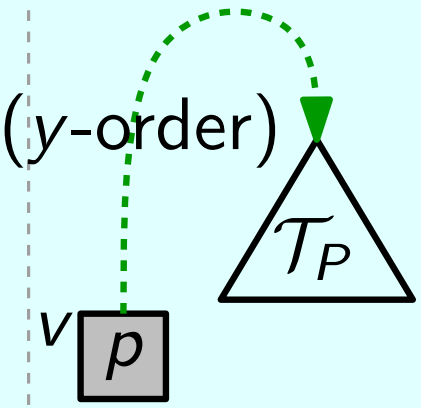

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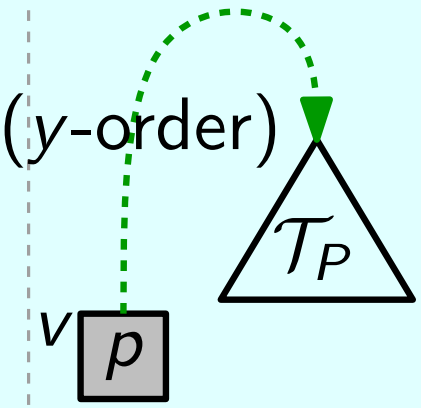

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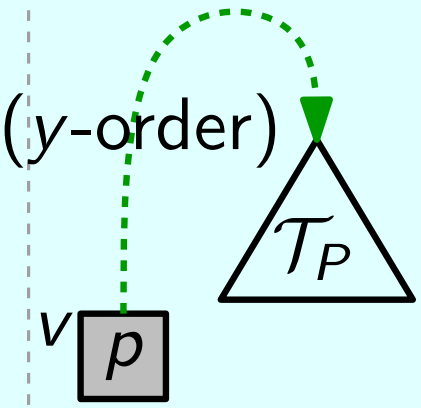

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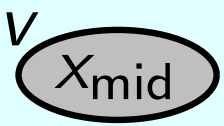

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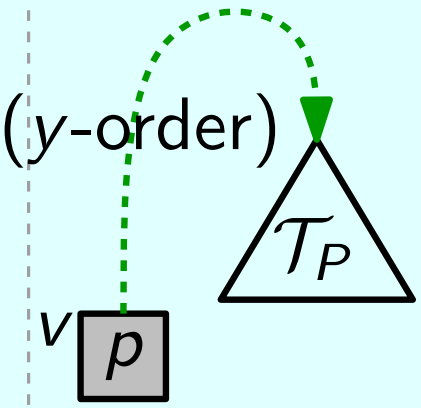

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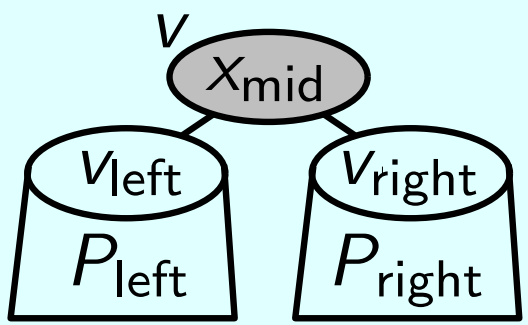

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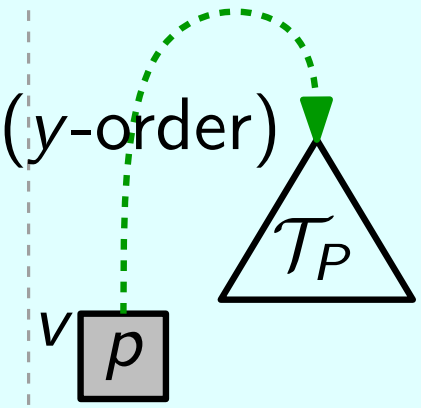

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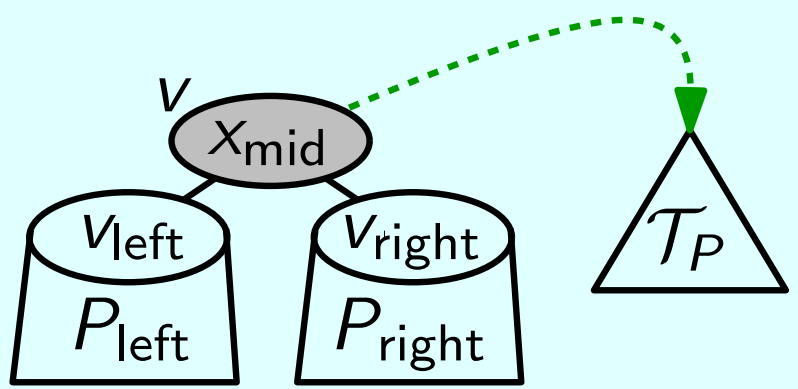

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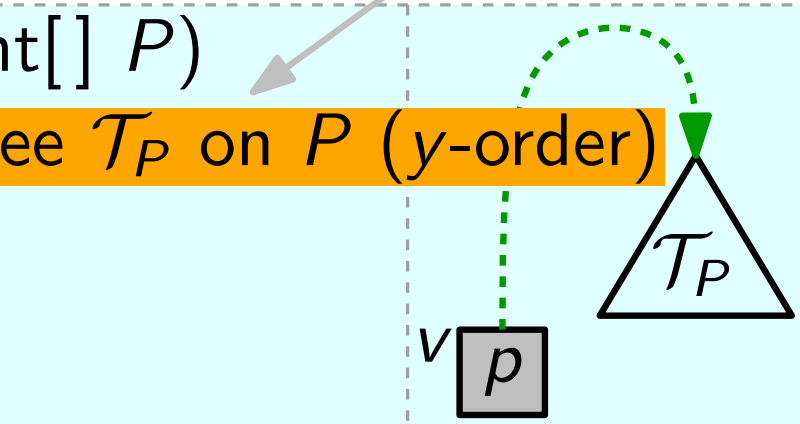

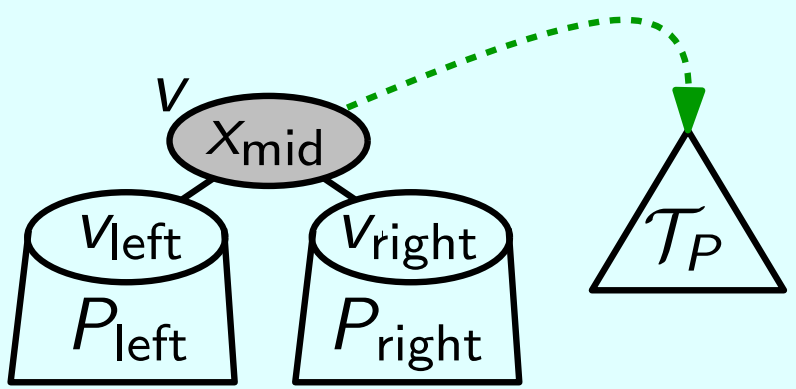

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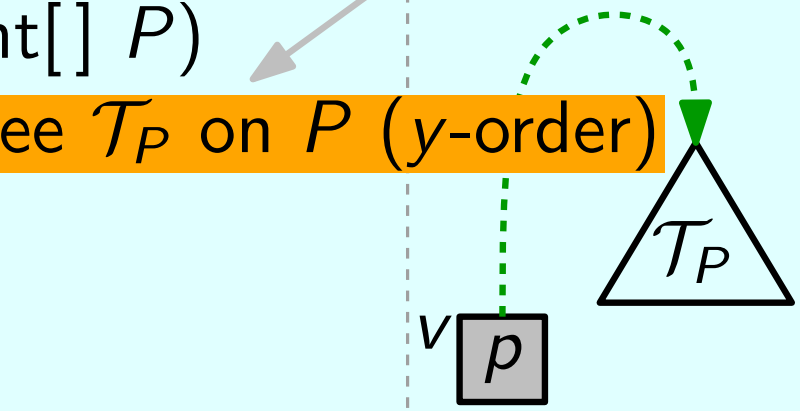

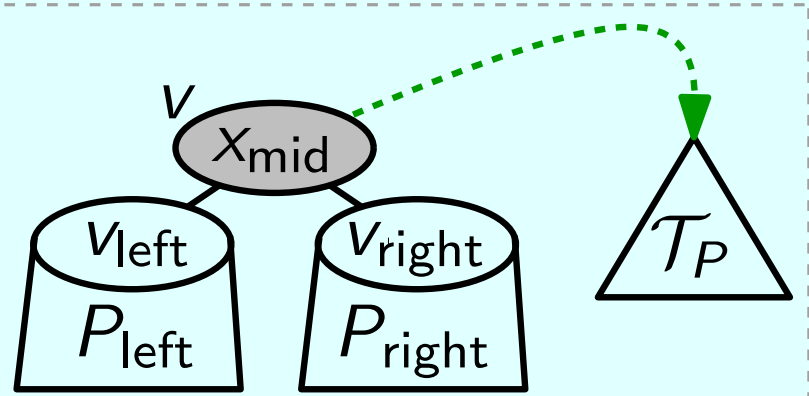

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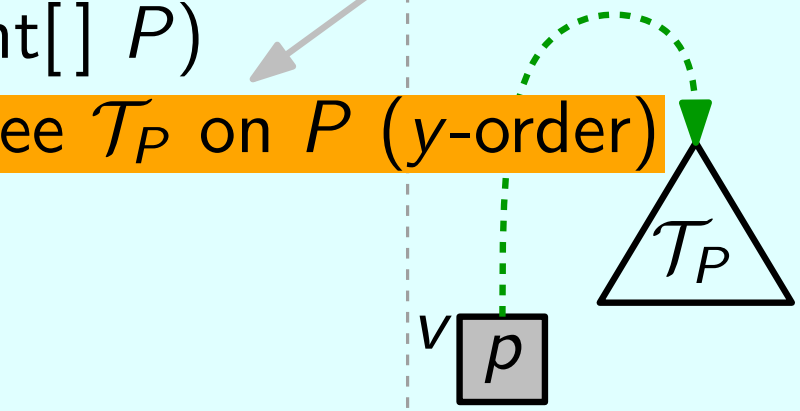

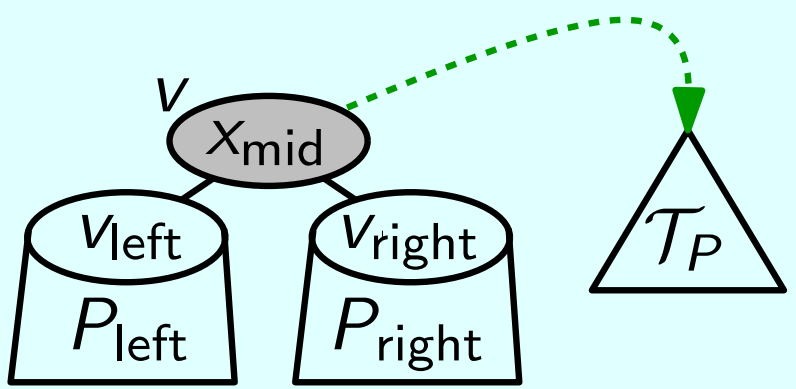

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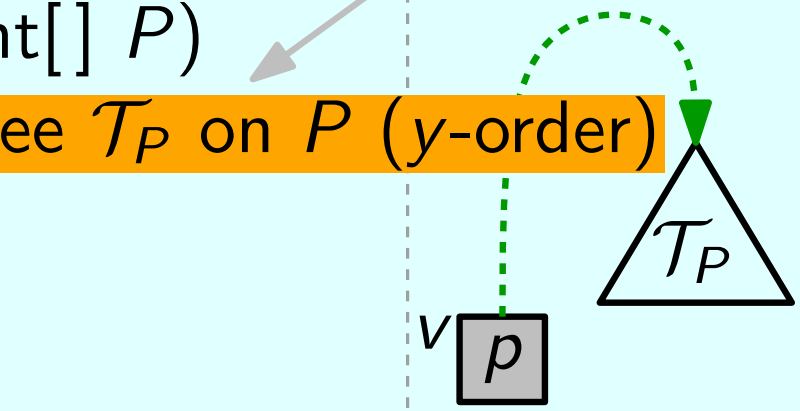

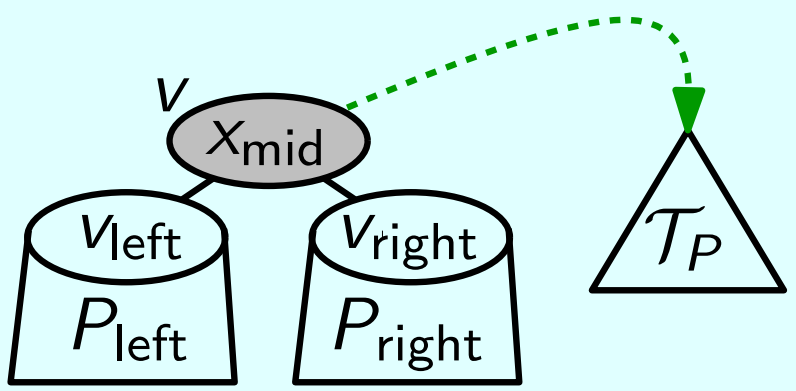

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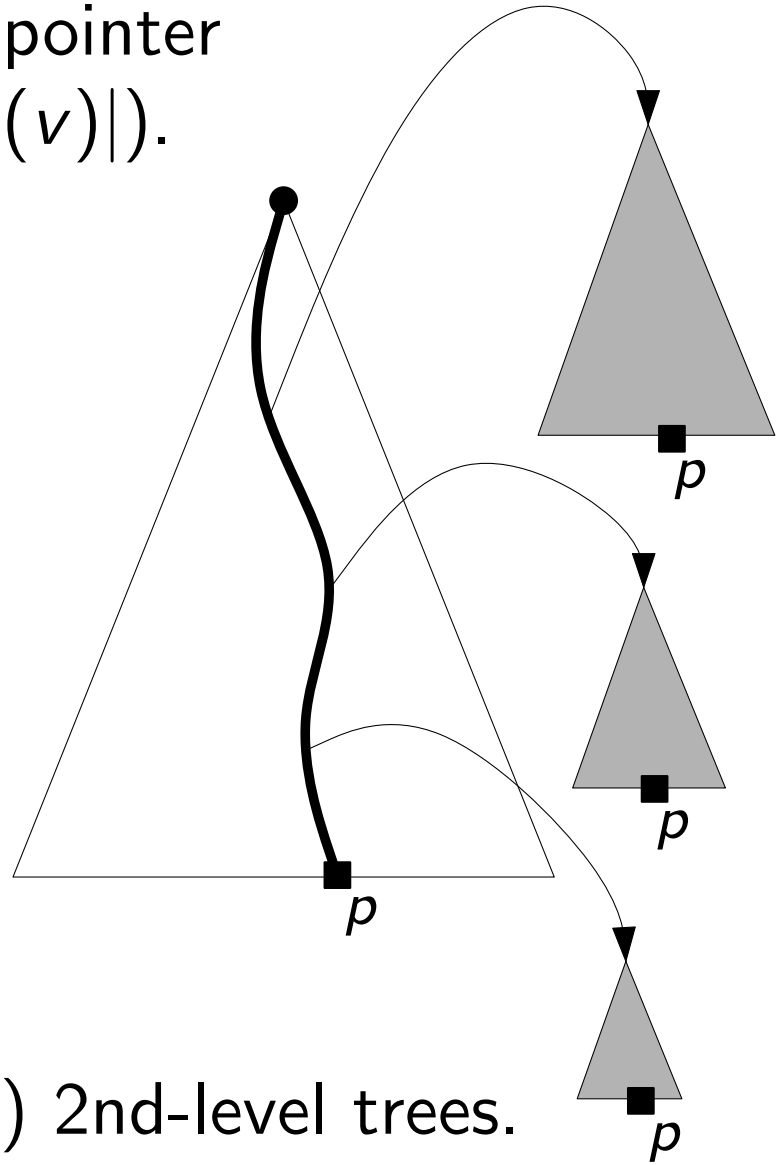

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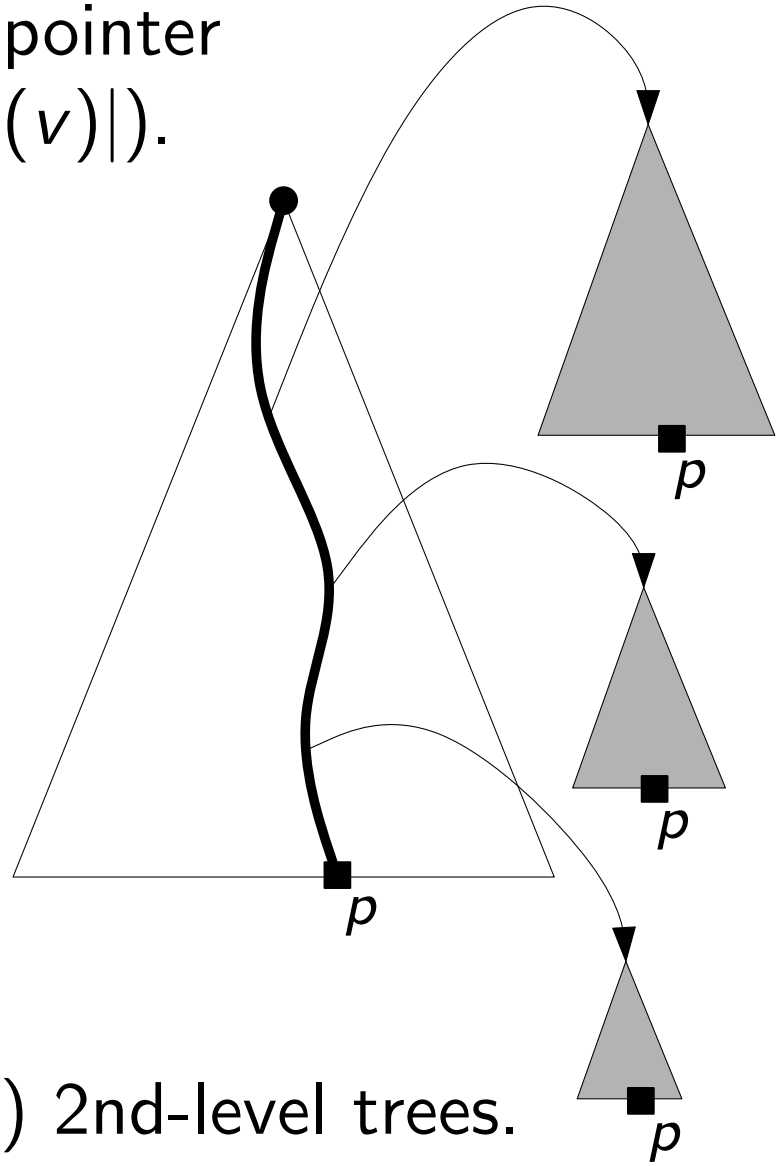

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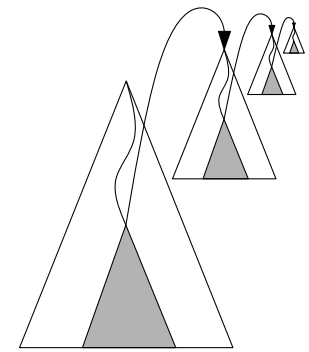

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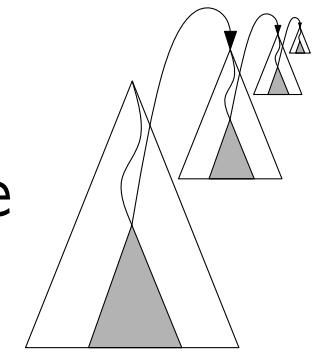

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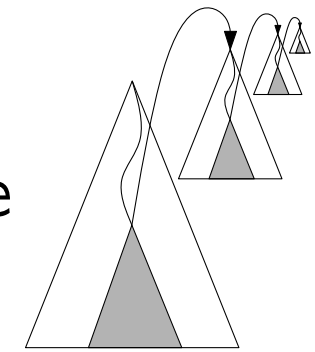

See Chapter 5.4 in the book
Computational Geometry: Algorithms and Applications
by de Berg et al., Springer-Verlag, 3rd edition 2008.

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Note: *trade-off* between space and query time

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[dBCvKO'08]

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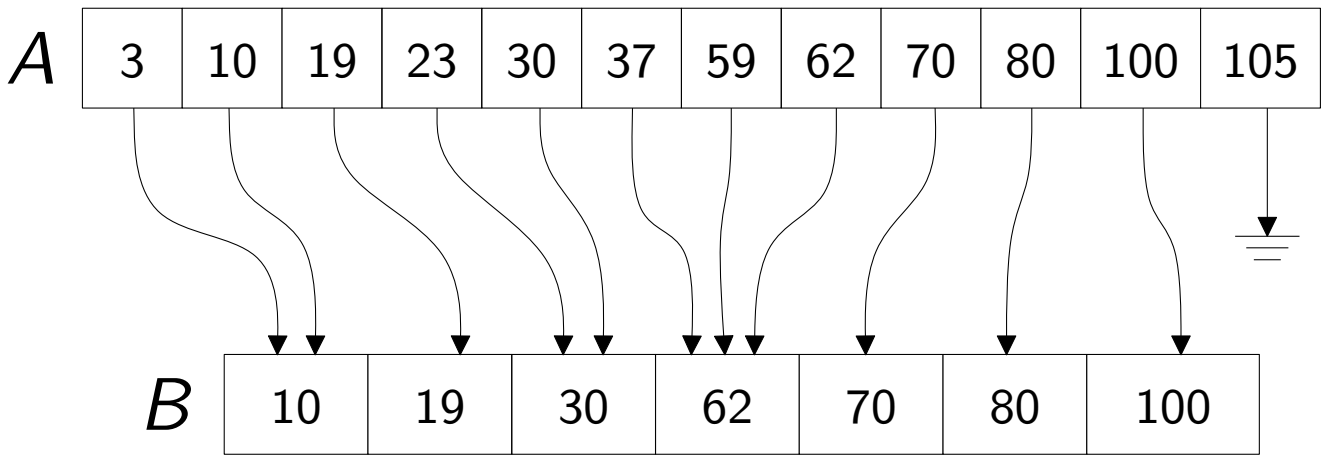

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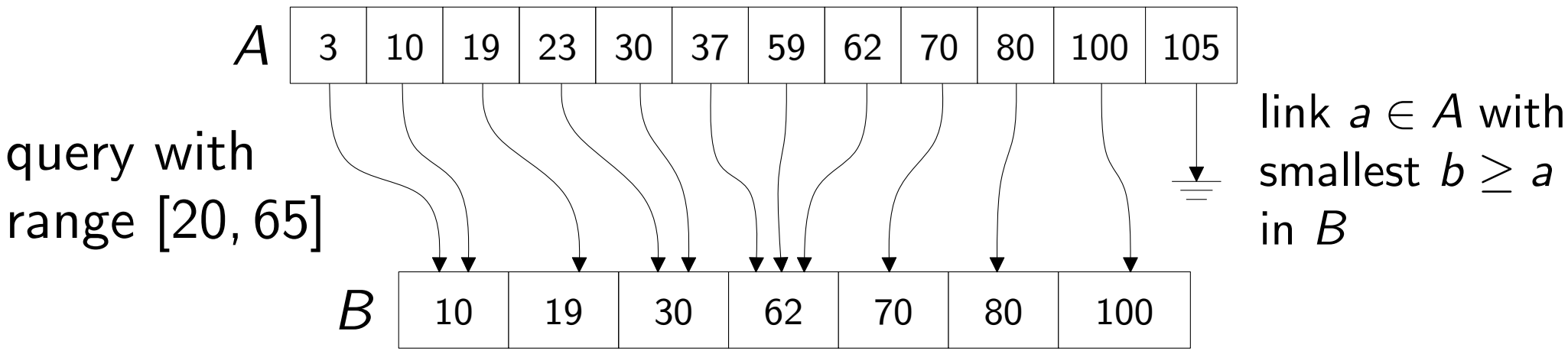

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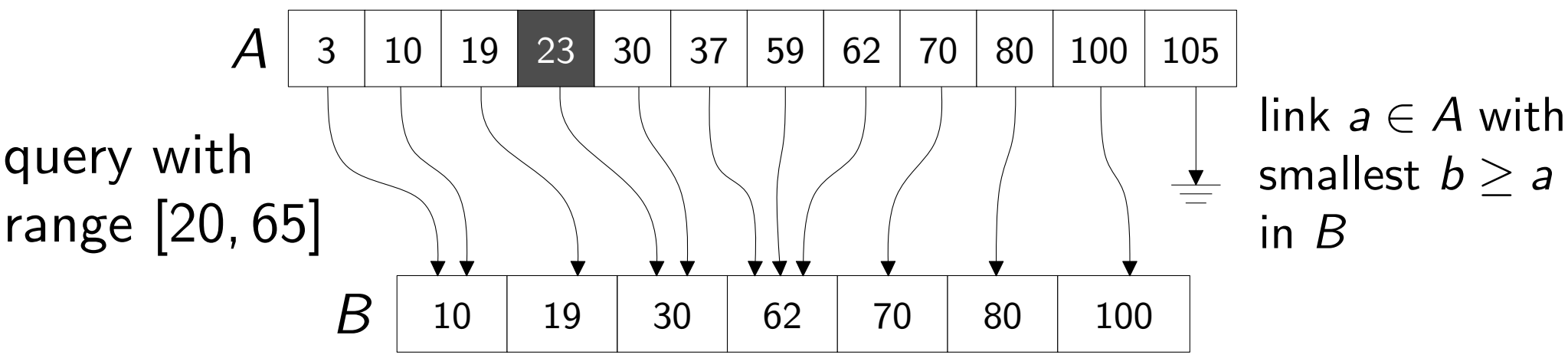

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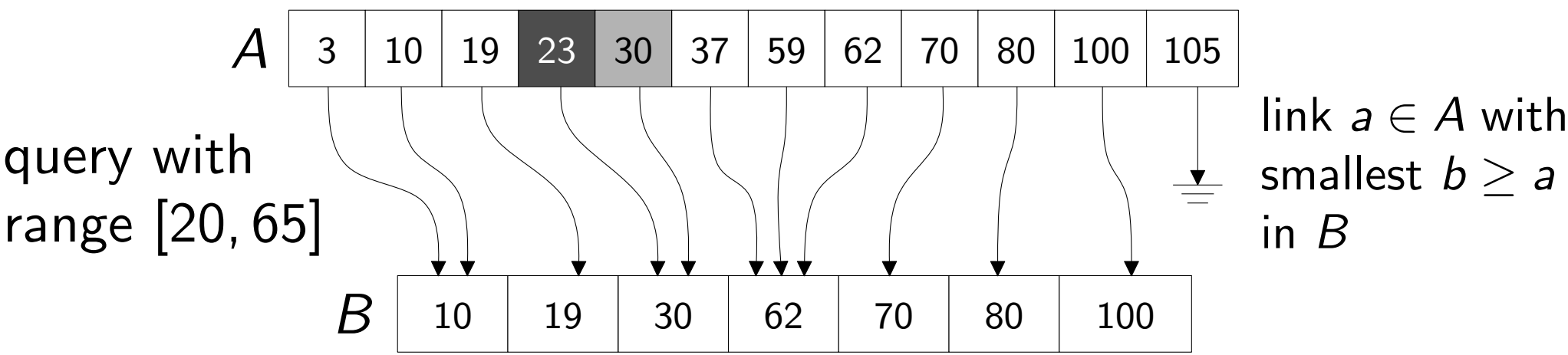

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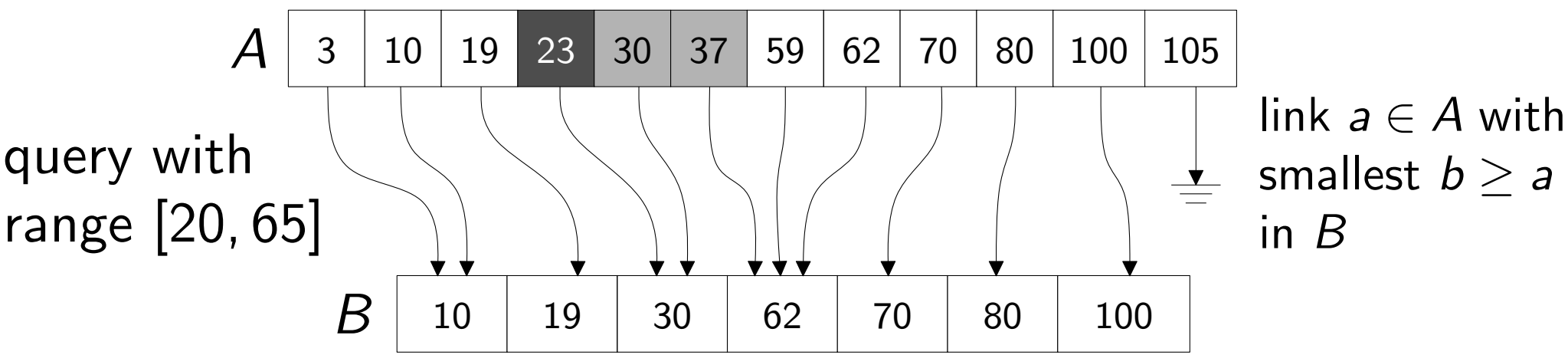

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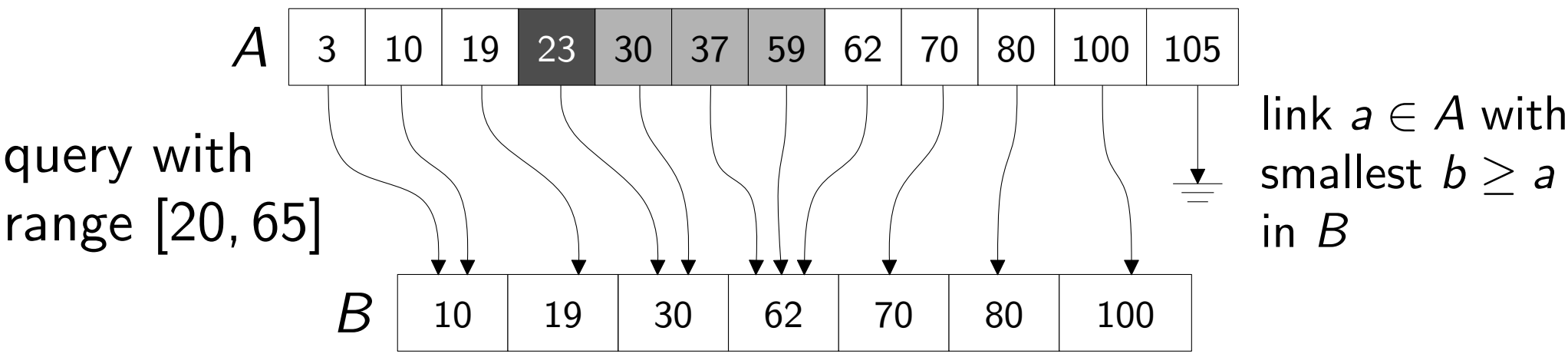

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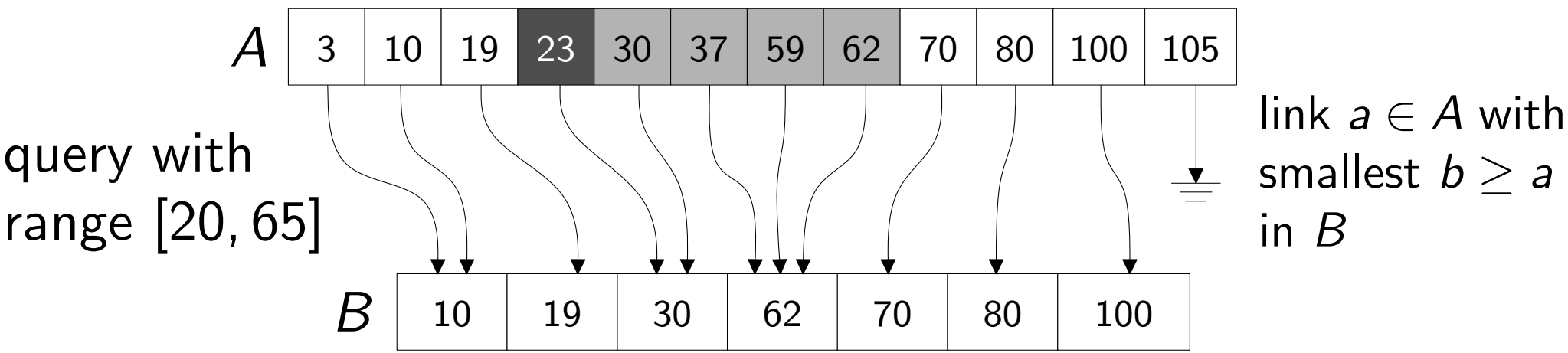

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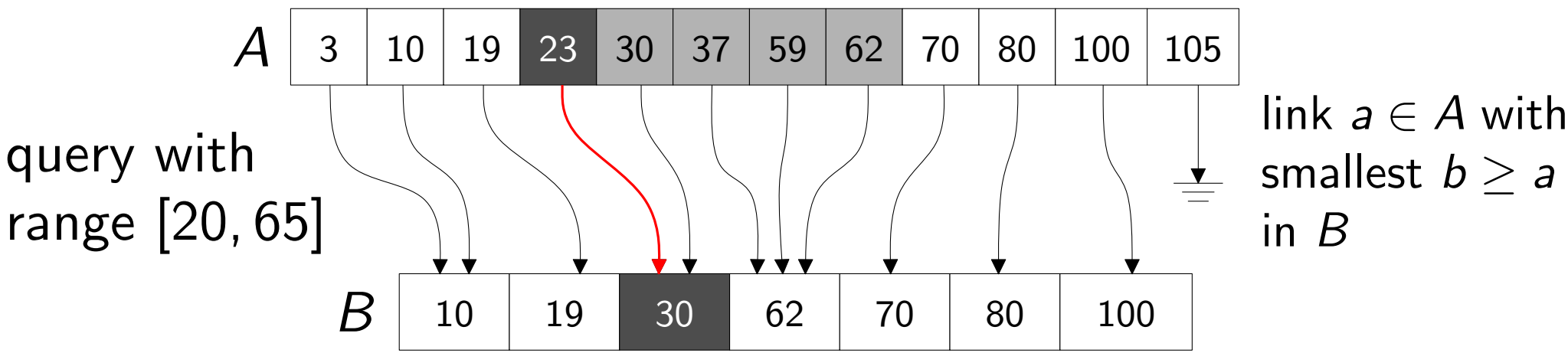


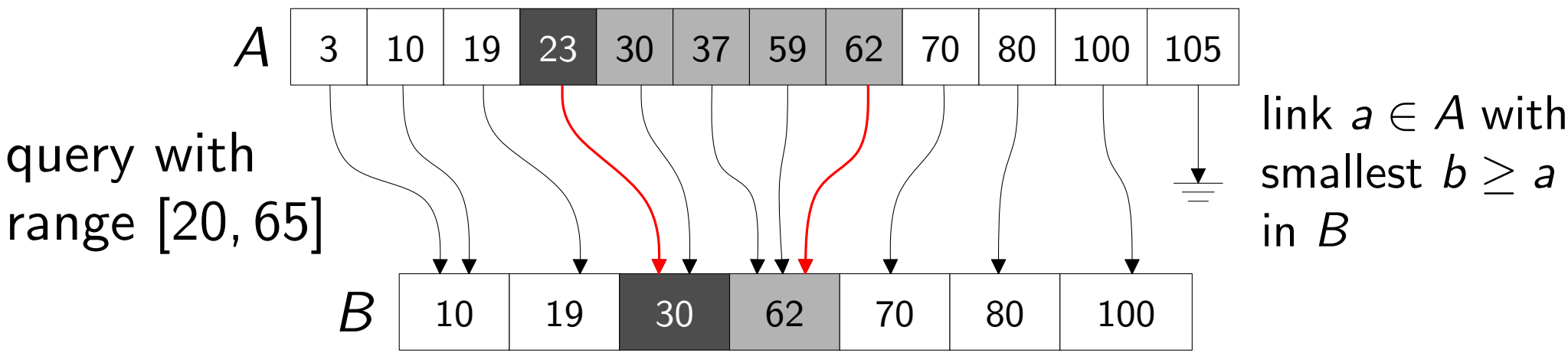
[dBCvKO'08]

Fractional Cascading

Task 1: Given sets $B \subset A \subset \mathbb{N}$ stored in sorted order in arrays $A[1..n]$ and $B[1..m]$, support 1d range queries in the multiset $A \cup B$ in $k + 1 \cdot \log n$ time!

We allow $n \log m$ bits extra space.

Task 2: Assuming that task 1 can be solved, speed up 2d range queries: $O(k + \log^2 n) \rightarrow O(k + \log n)$ time!



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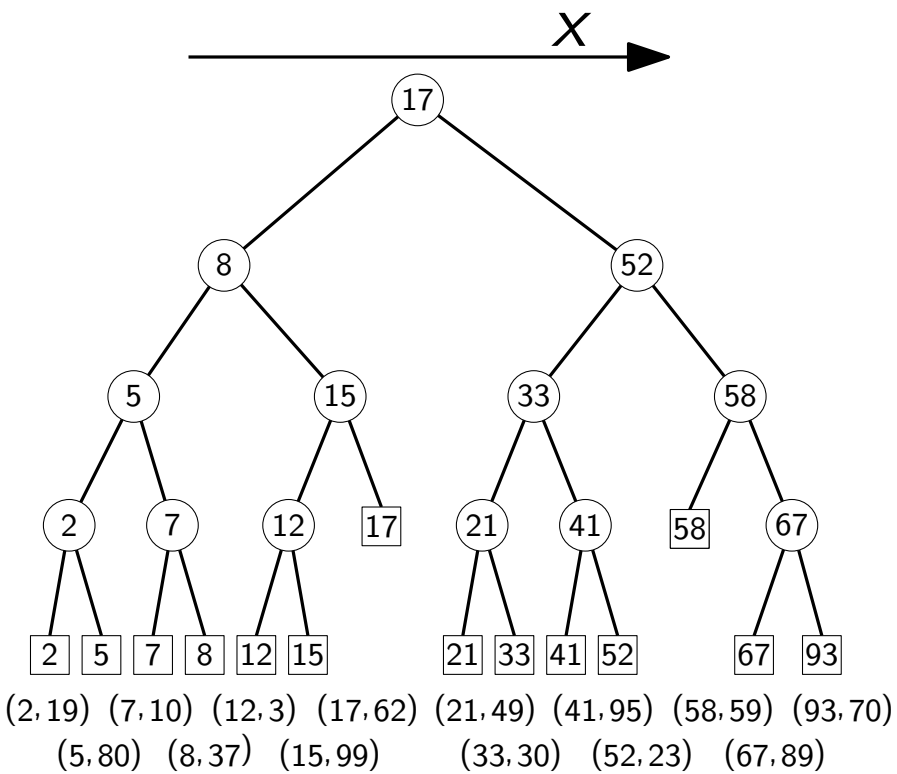
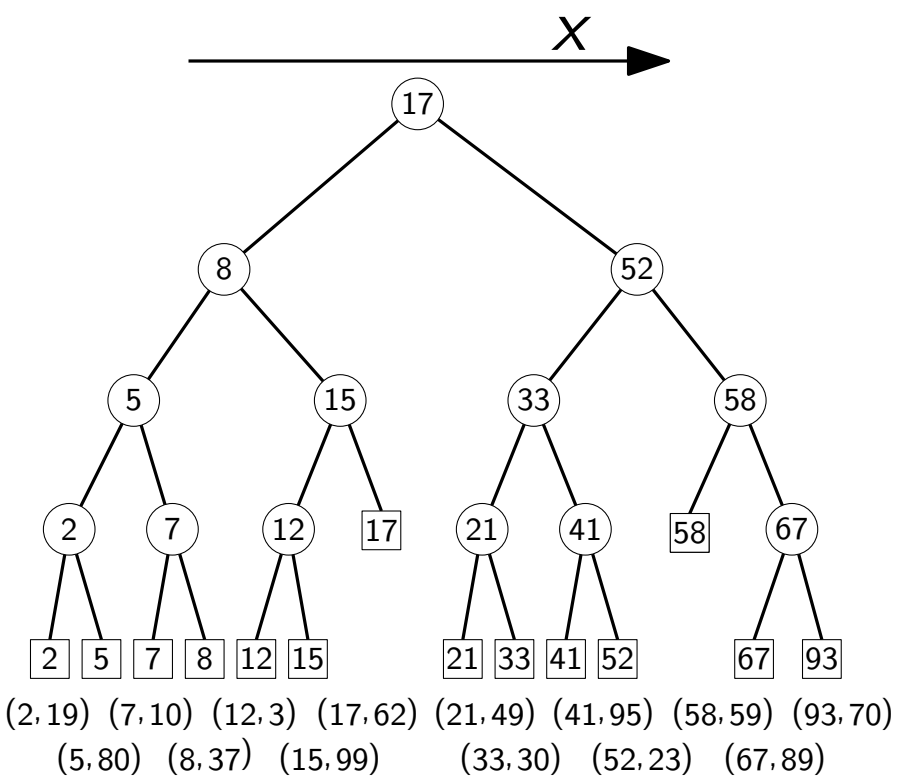
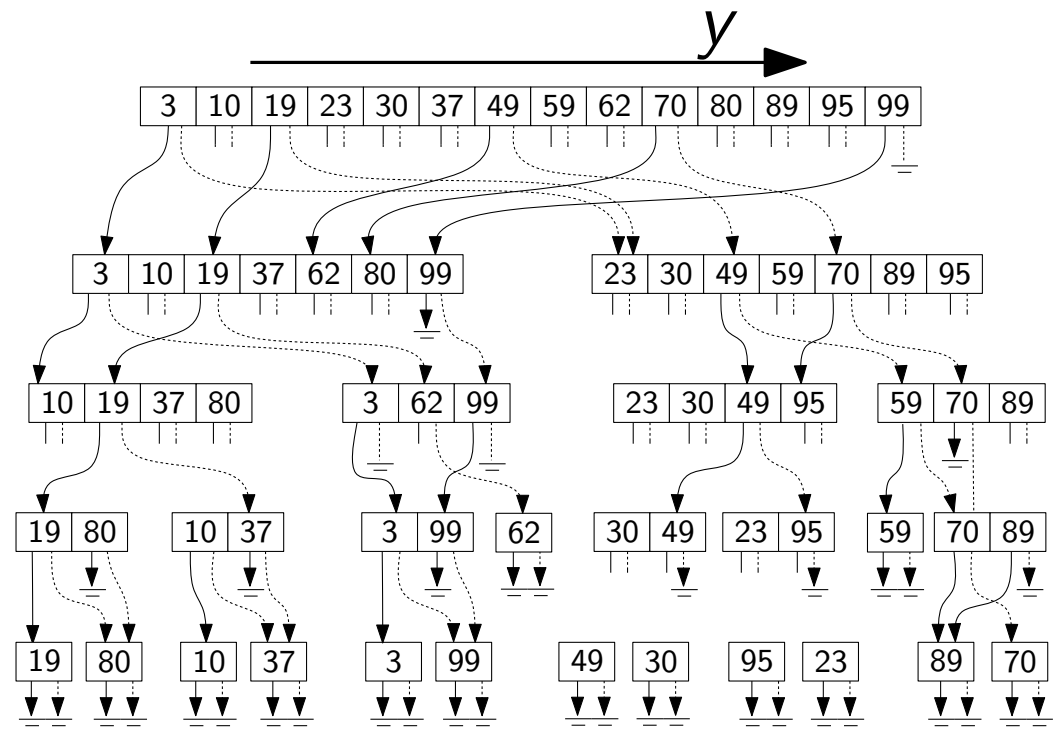
Layered Range Trees

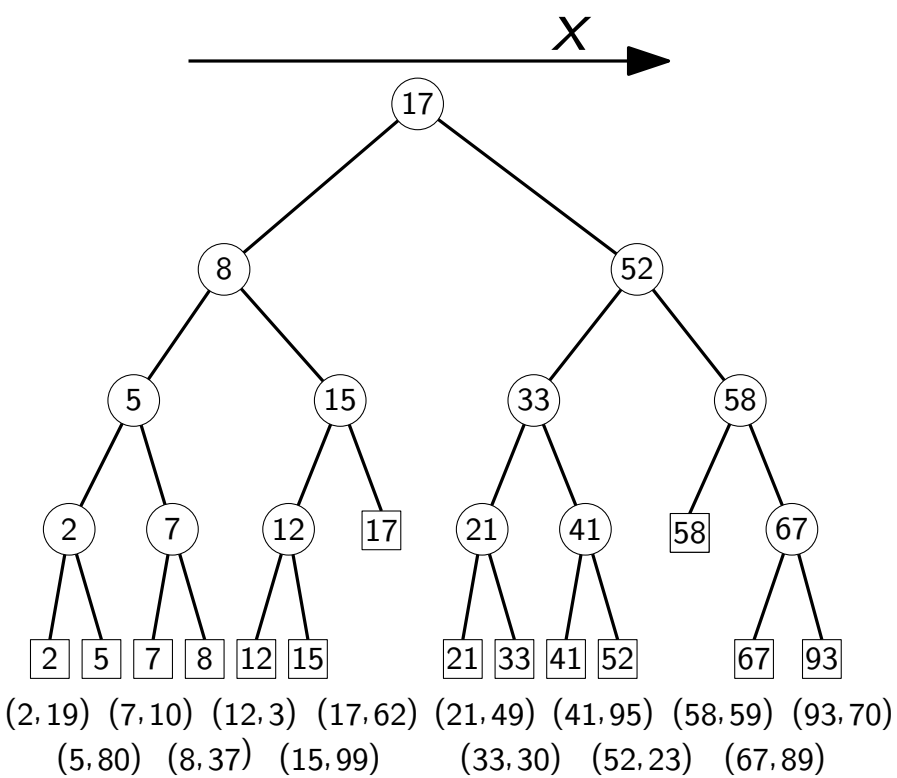


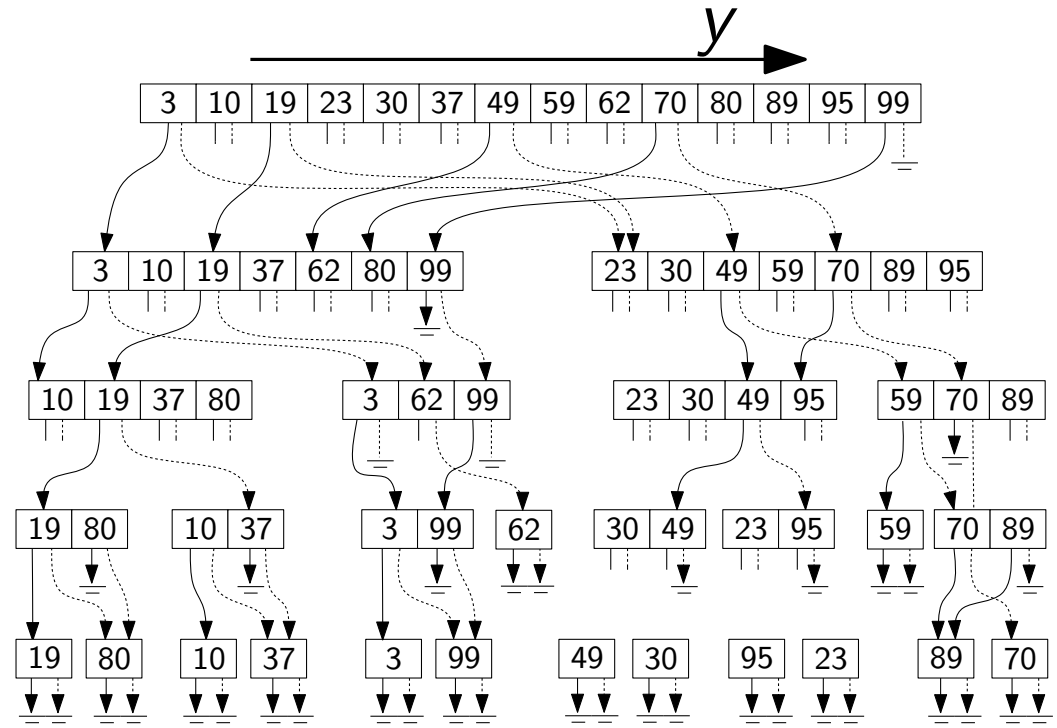
Layered Range Trees





Layered Range Trees

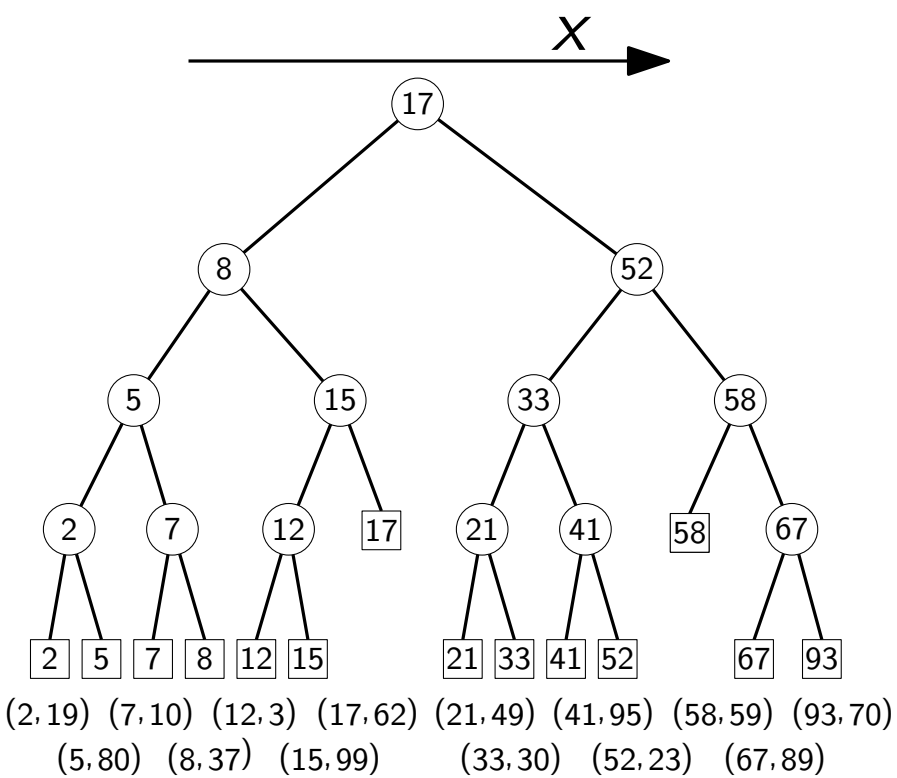


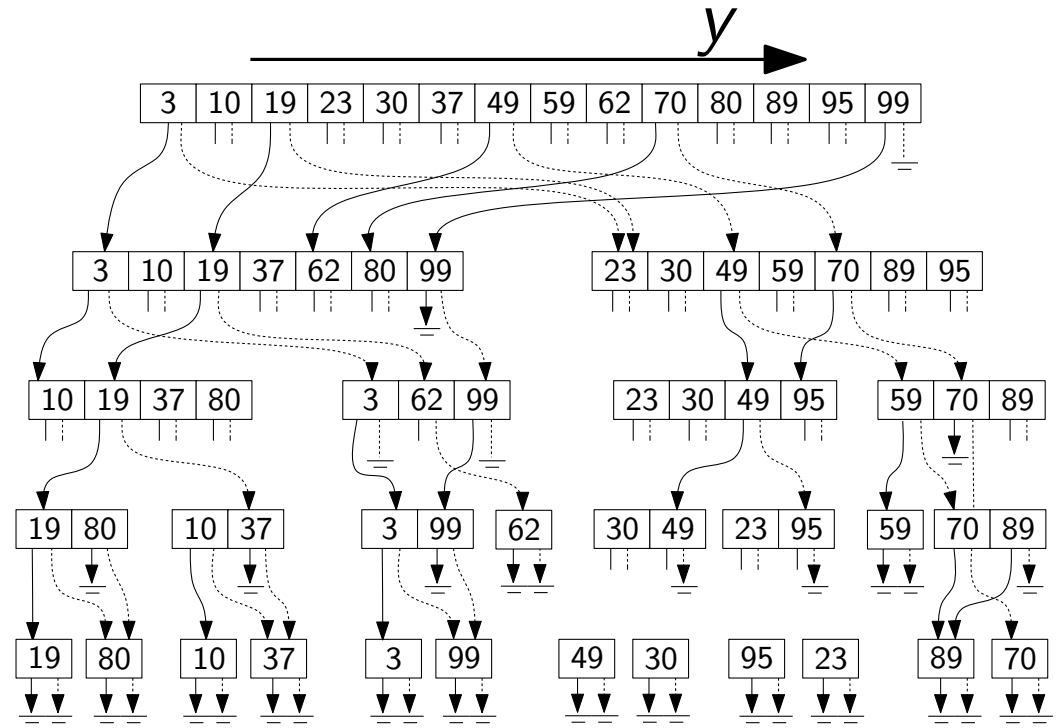


[dBCvKO'08]

Theorem: Let $d \geq 2$ and let P be a set of n pts in $\mathbb{R}^d$. Given $O(n \log^{d-1} n)$ preprocessing time & storage, d-dim range queries on P can be answered in $O(k + \log^{d-1} n)$ time.

Layered Range Trees





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