

Computational Geometry

Winter term 2014/15

Linear Programming

or

Profit Maximization

Lecture #4

Maximizing Profit

You are the boss of a small company that produces two products, P_1 and P_2 . If you produce x_1 units of P_1 and x_2 units of P_2 , your profit in € is

$$G(x_1, x_2) = 300x_1 + 500x_2$$

Your production runs on three machines M_A , M_B , and M_C with the following capacities:

$$M_A: \quad 4x_1 + 11x_2 \leq 880$$

$$M_B: \quad x_1 + x_2 \leq 150$$

$$M_C: \quad x_2 \leq 60$$

Which choice of (x_1, x_2) maximizes your profit?

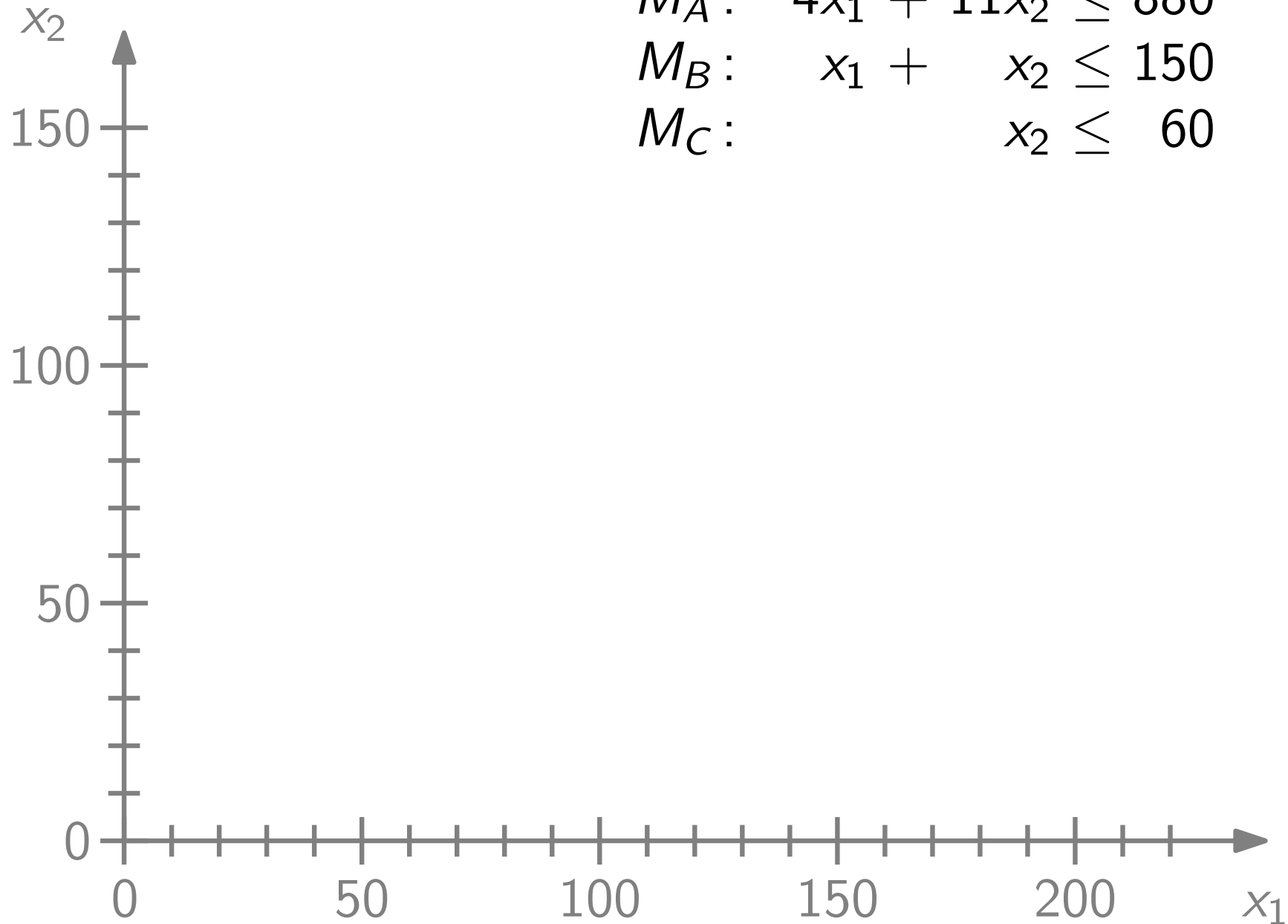
The Answer

linear constraints:

$$M_A: 4x_1 + 11x_2 \leq 880$$

$$M_B: x_1 + x_2 \leq 150$$

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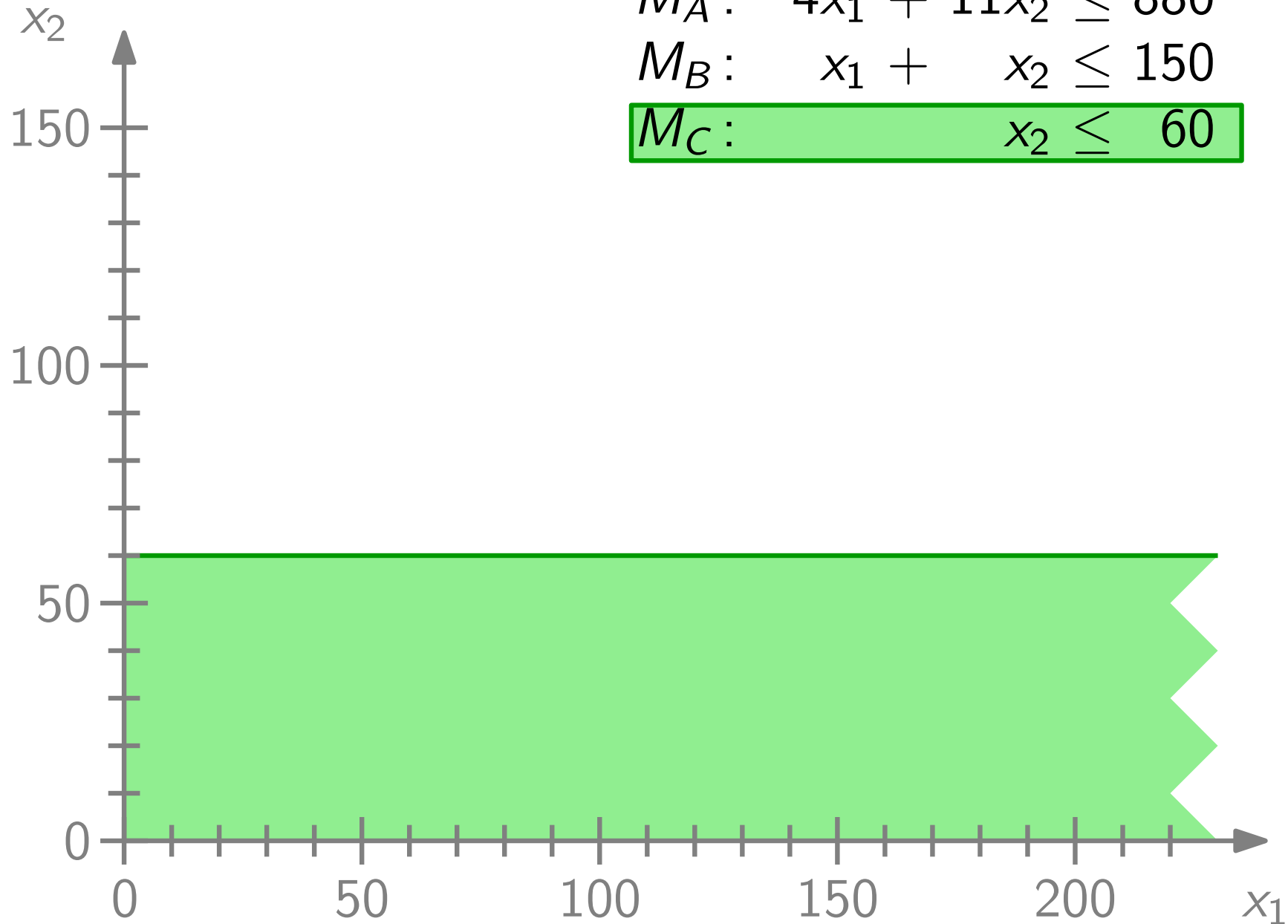
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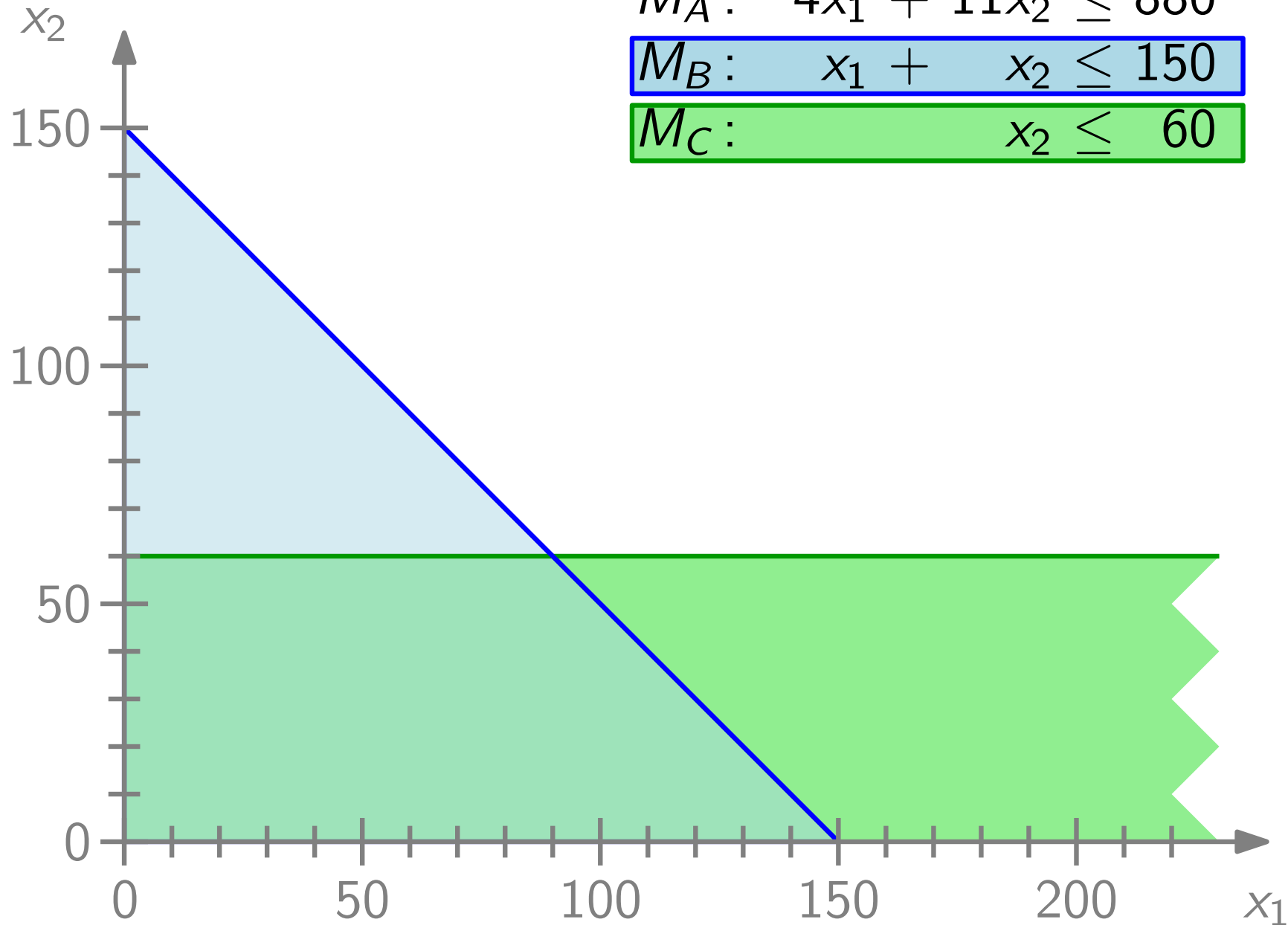
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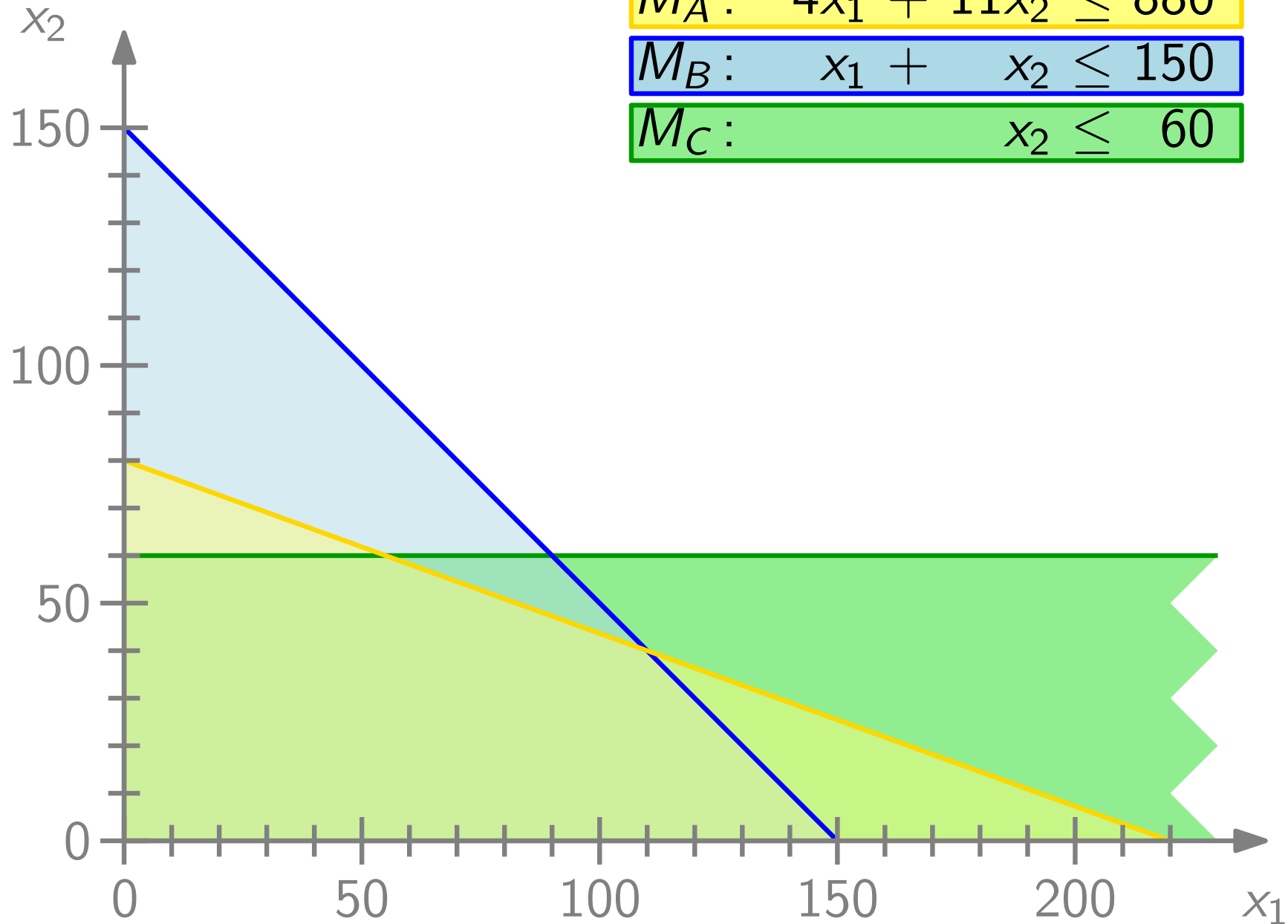
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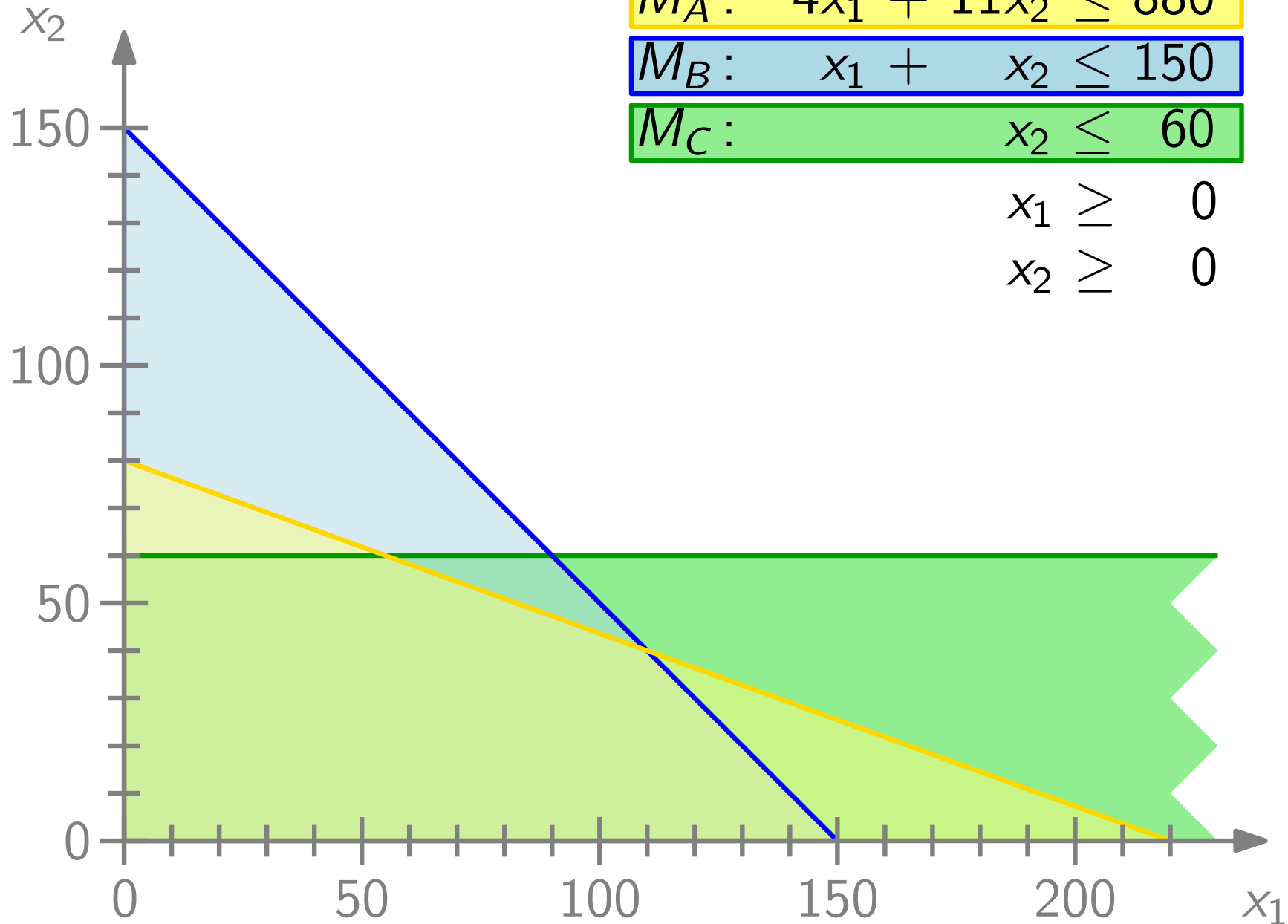
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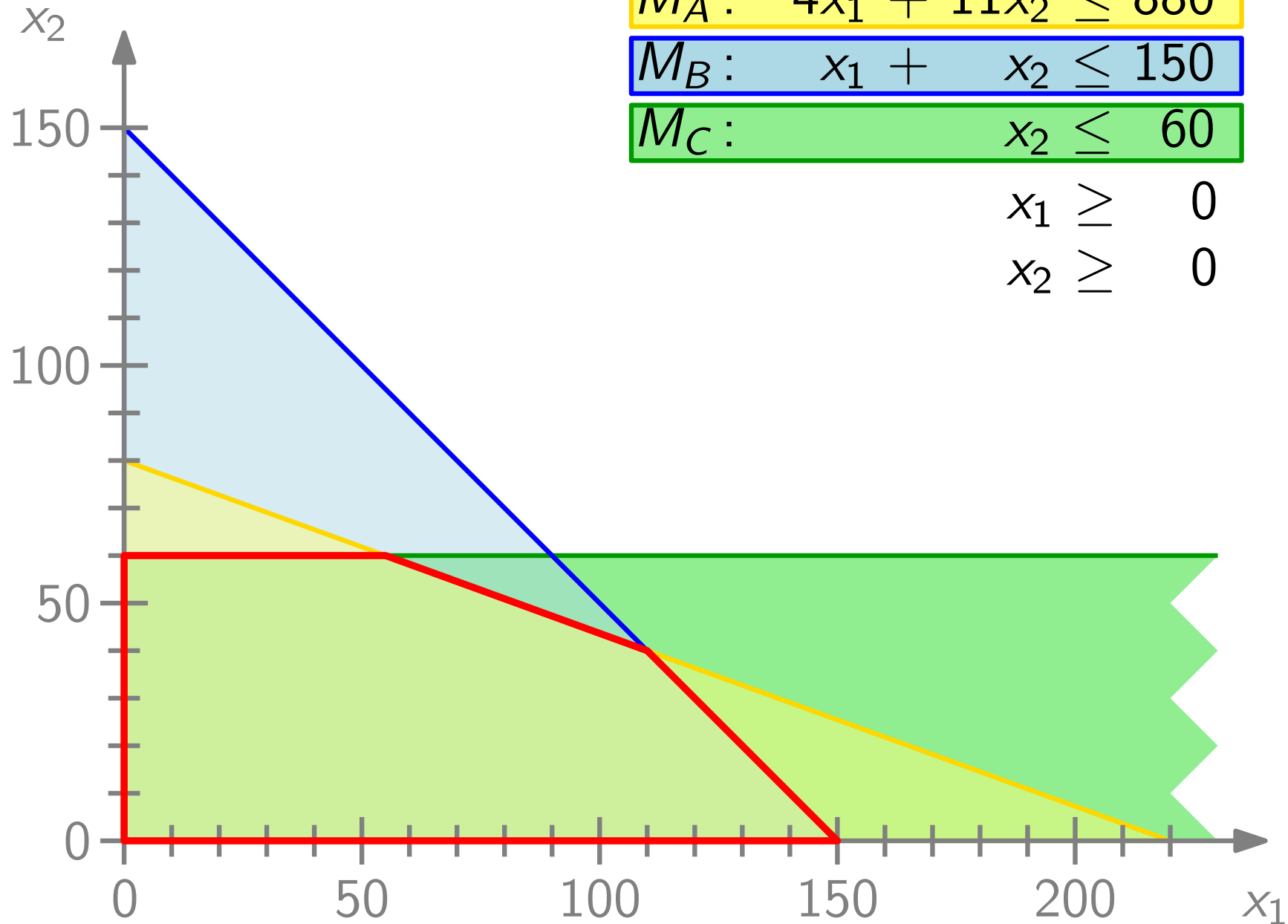
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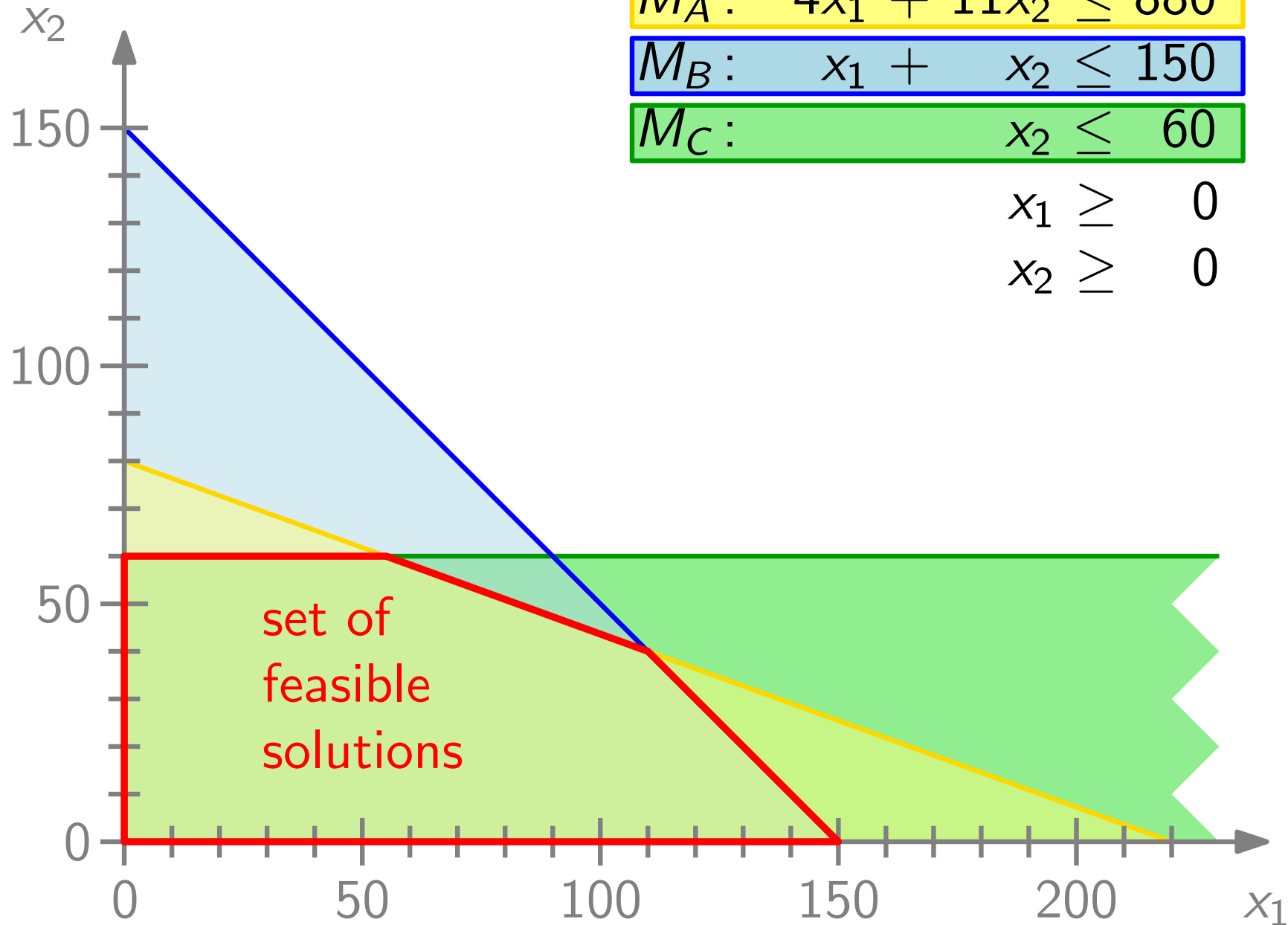
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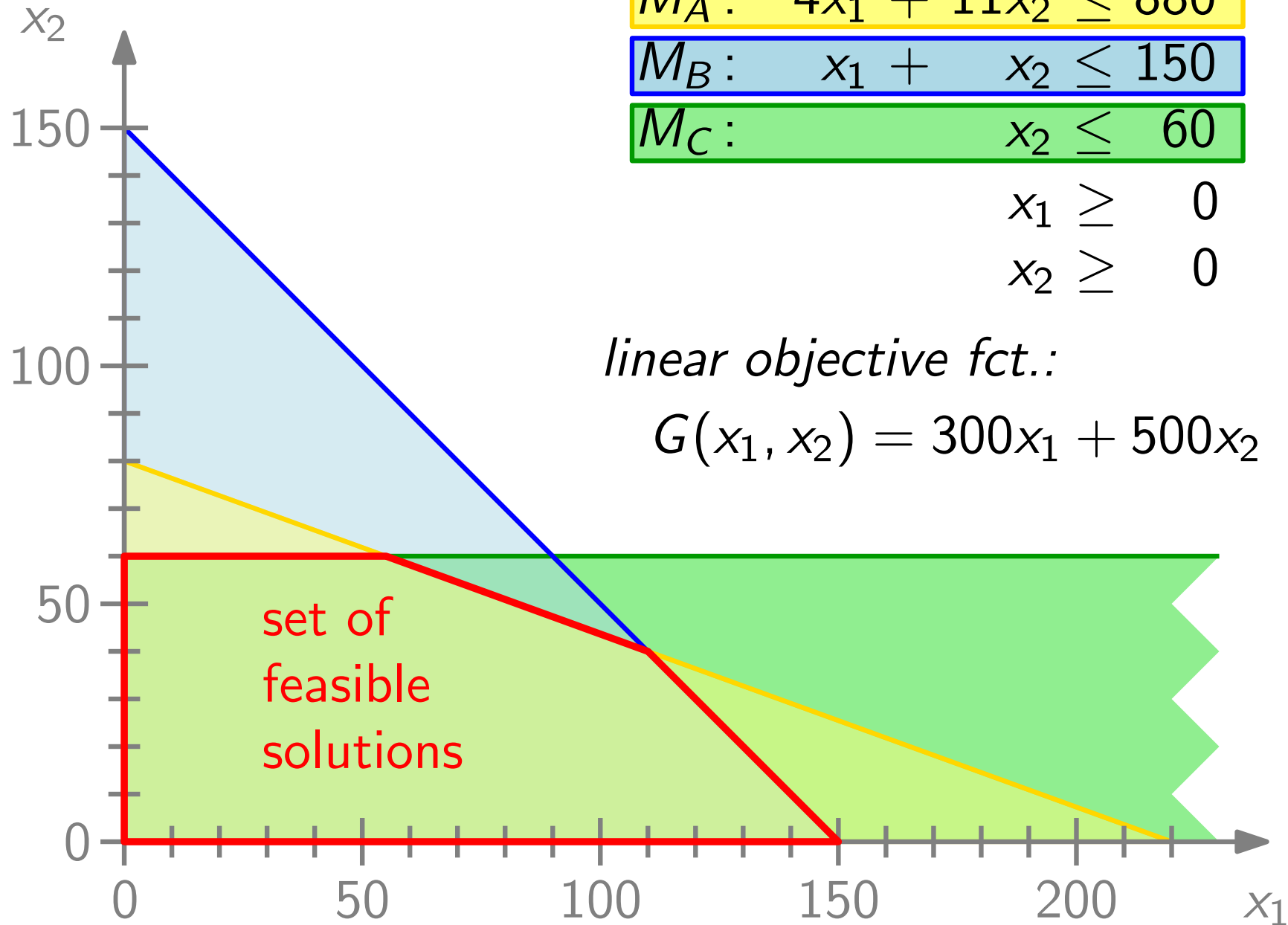
$$M_C: x_2 \leq 60$$

$$x_1 \geq 0$$

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linear objective fct.:

$$G(x_1, x_2) = 300x_1 + 500x_2$$



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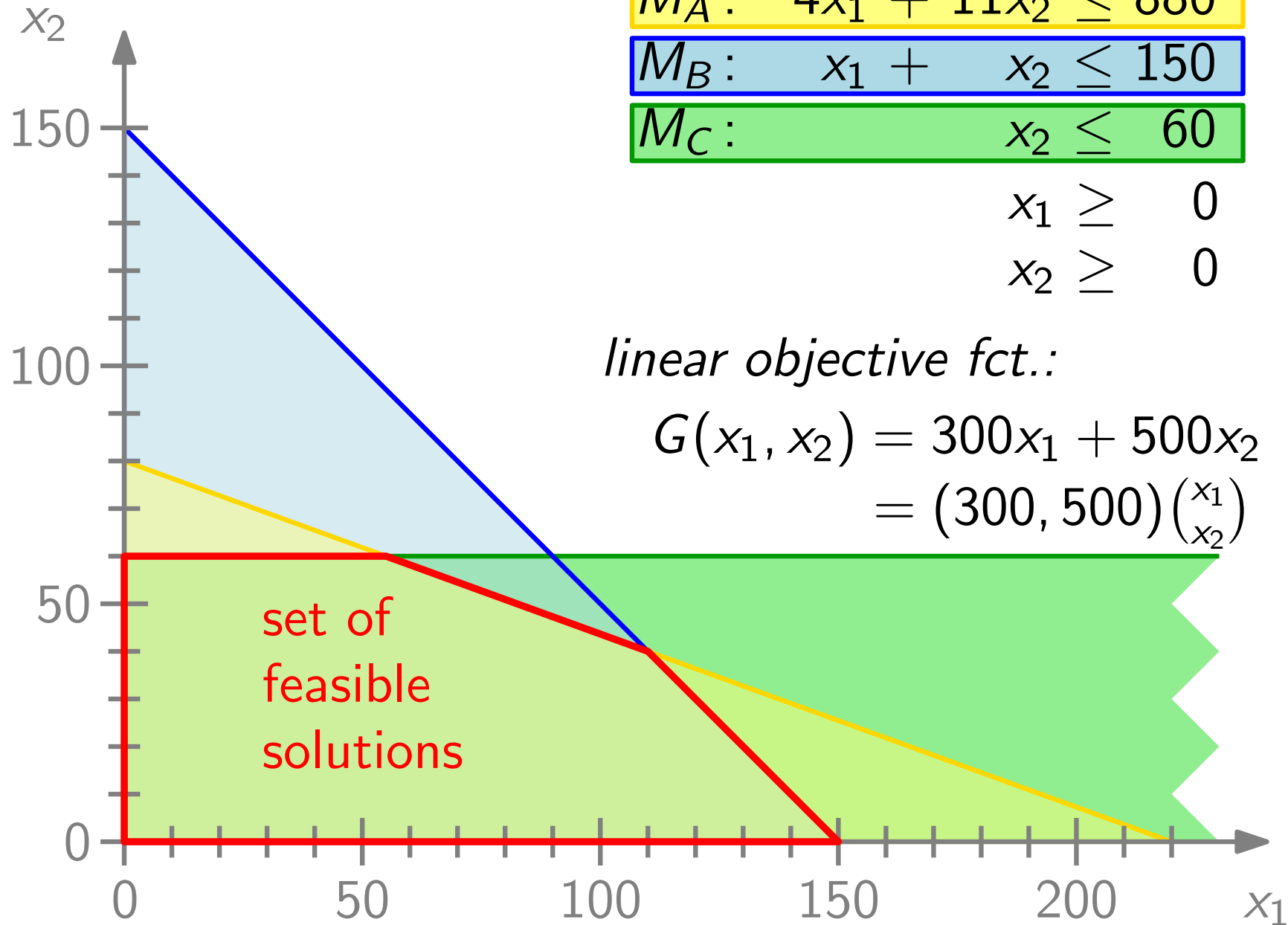
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$$x_1 \geq 0$$

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linear objective fct.:

$$\begin{aligned} G(x_1, x_2) &= 300x_1 + 500x_2 \\ &= (300, 500) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$



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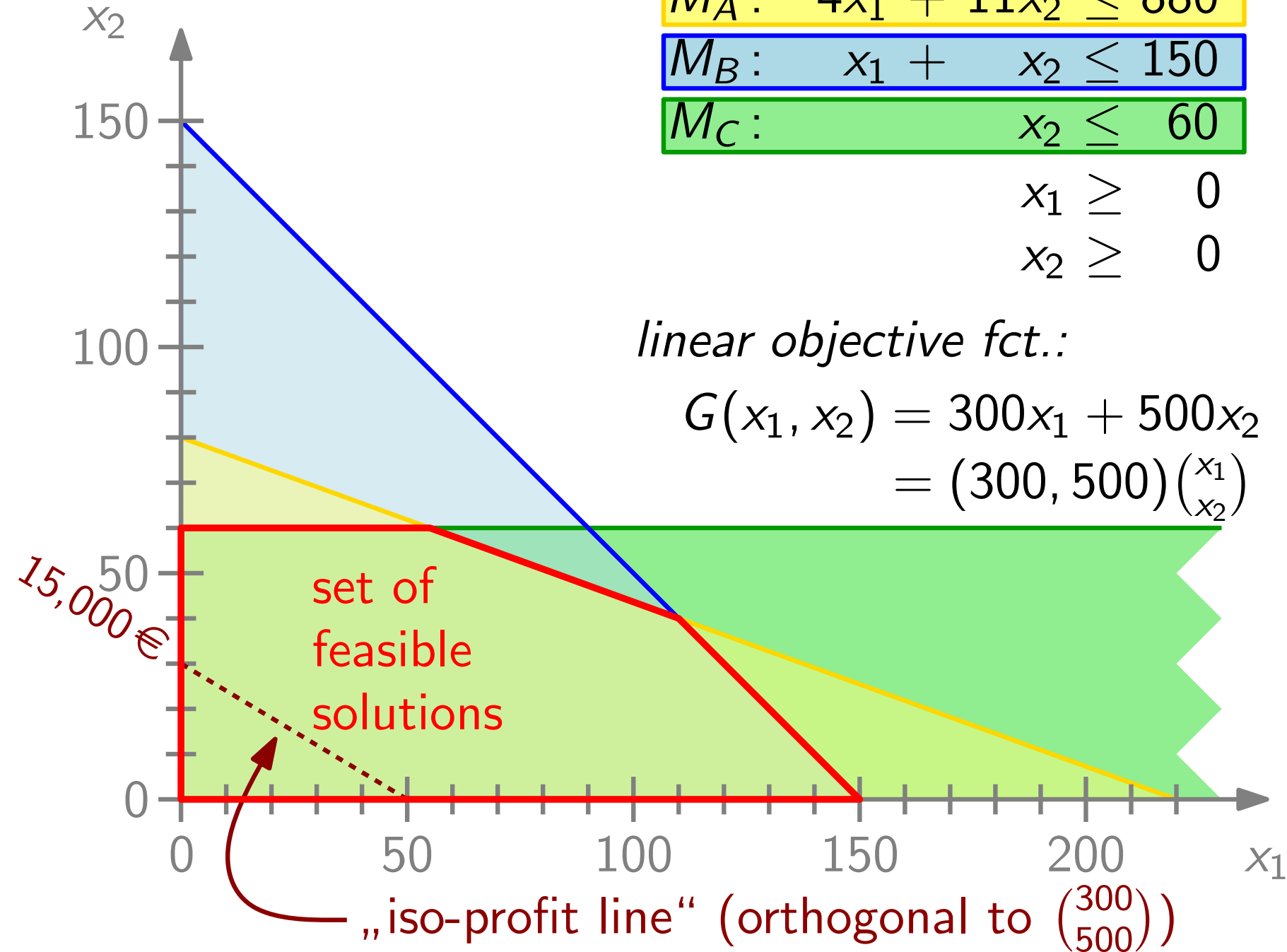
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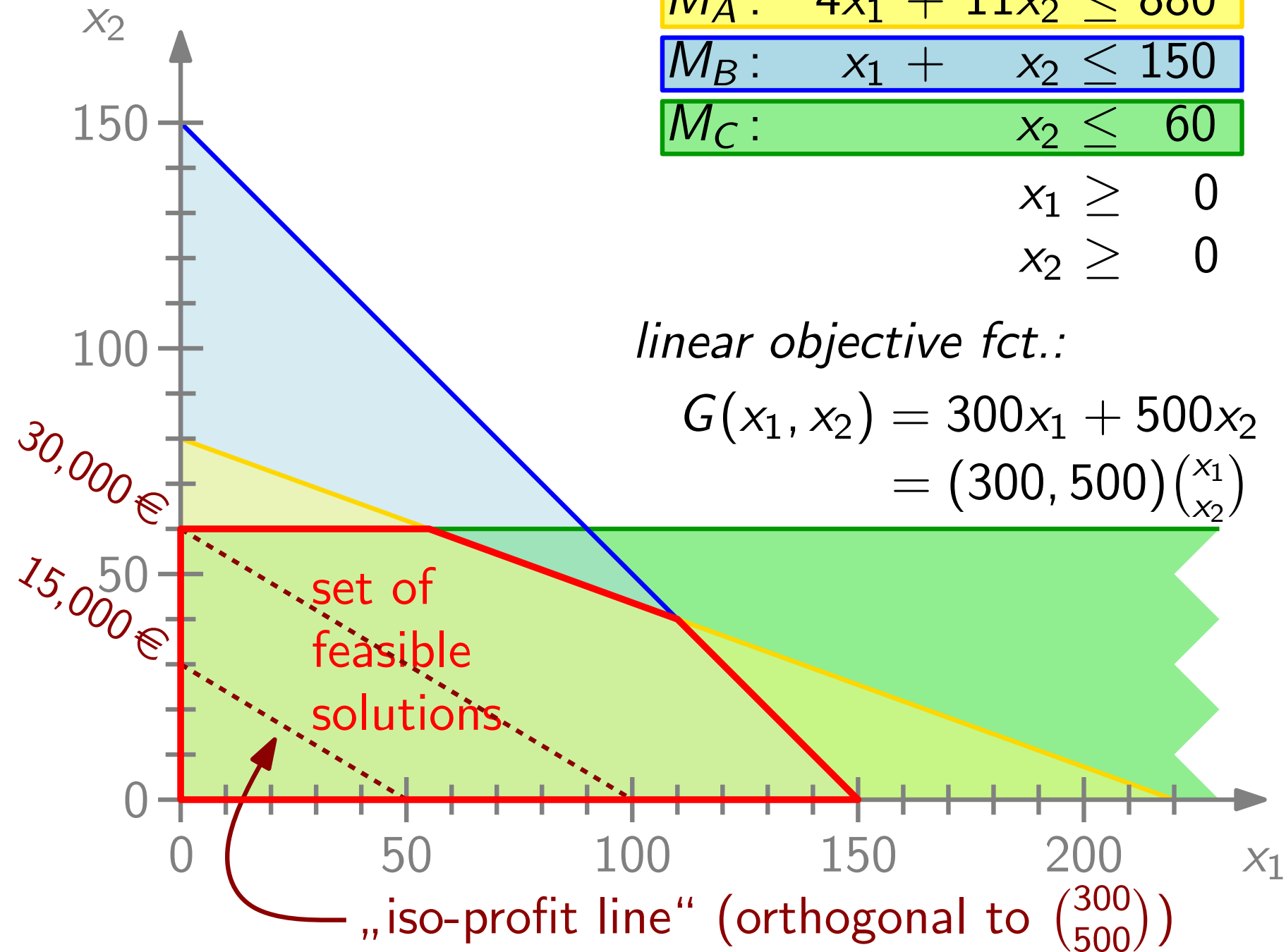
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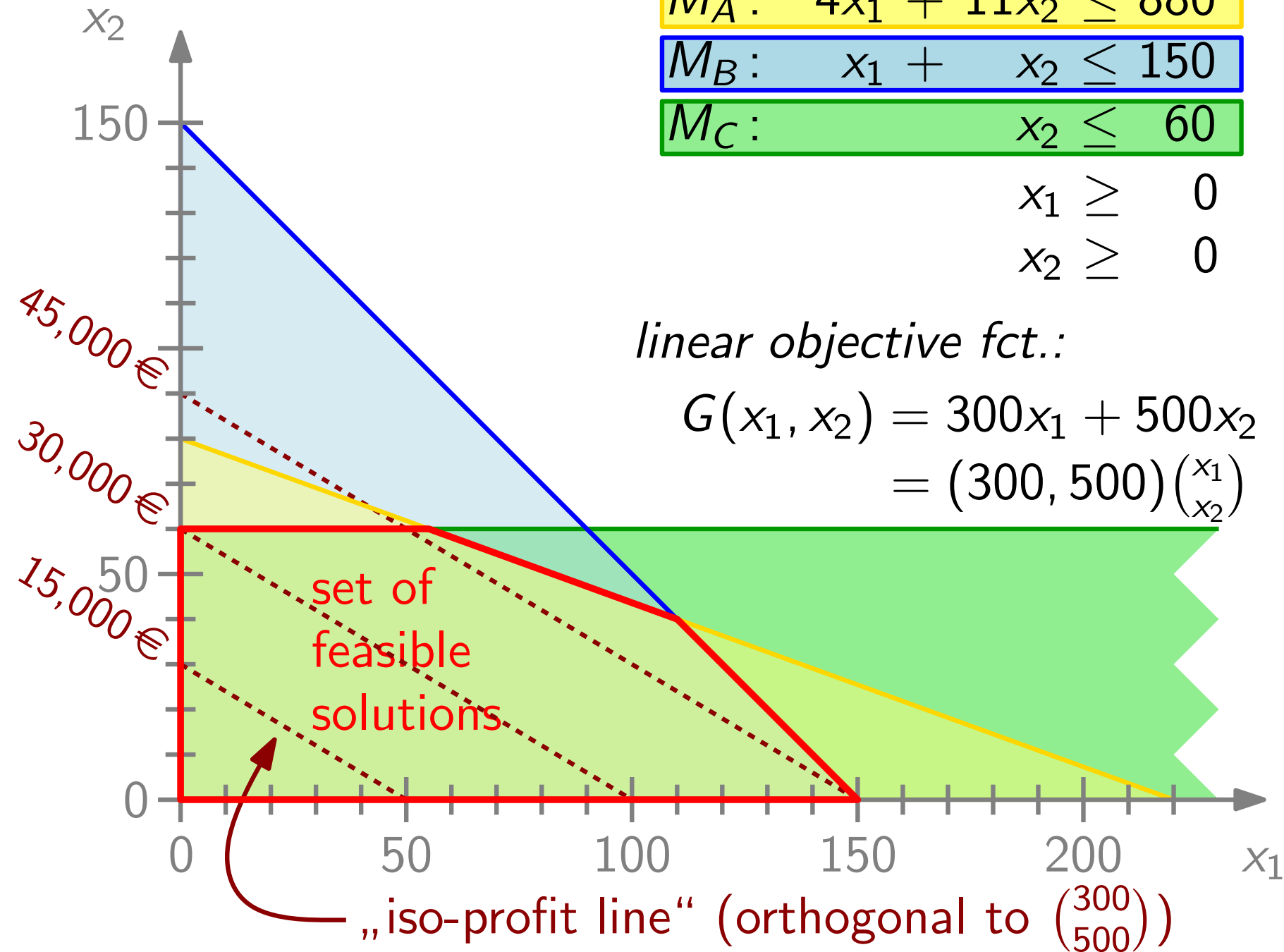
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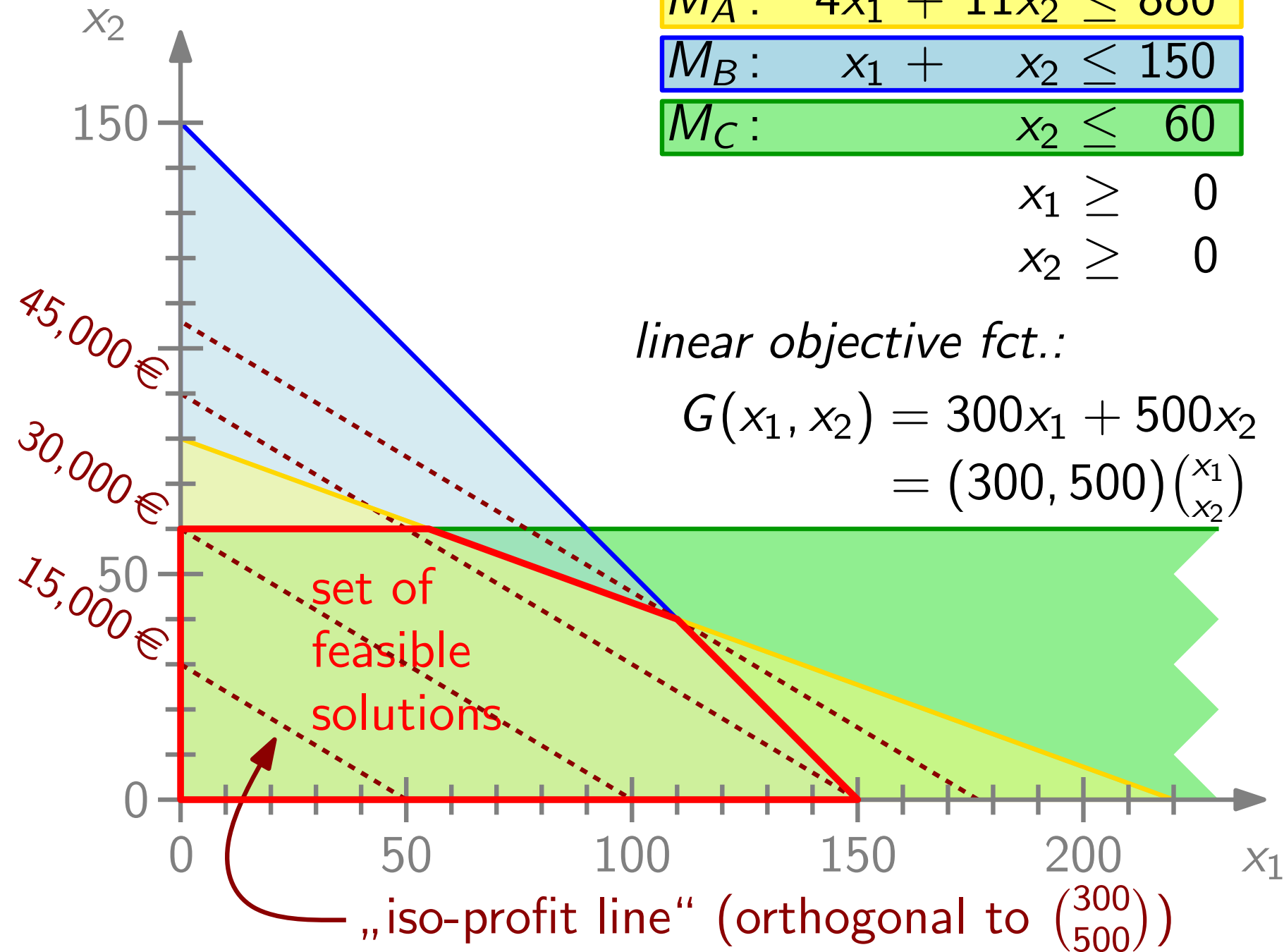
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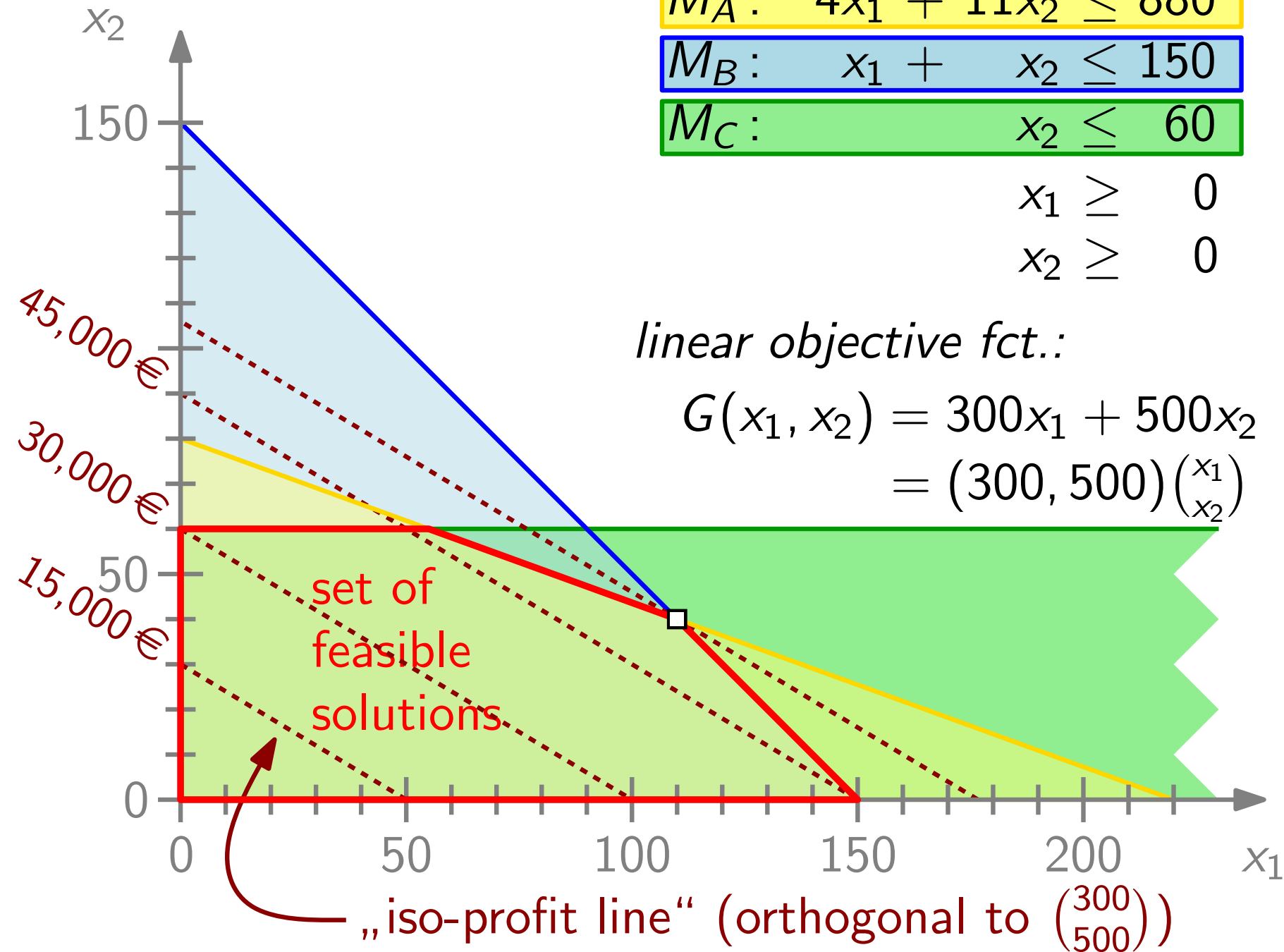
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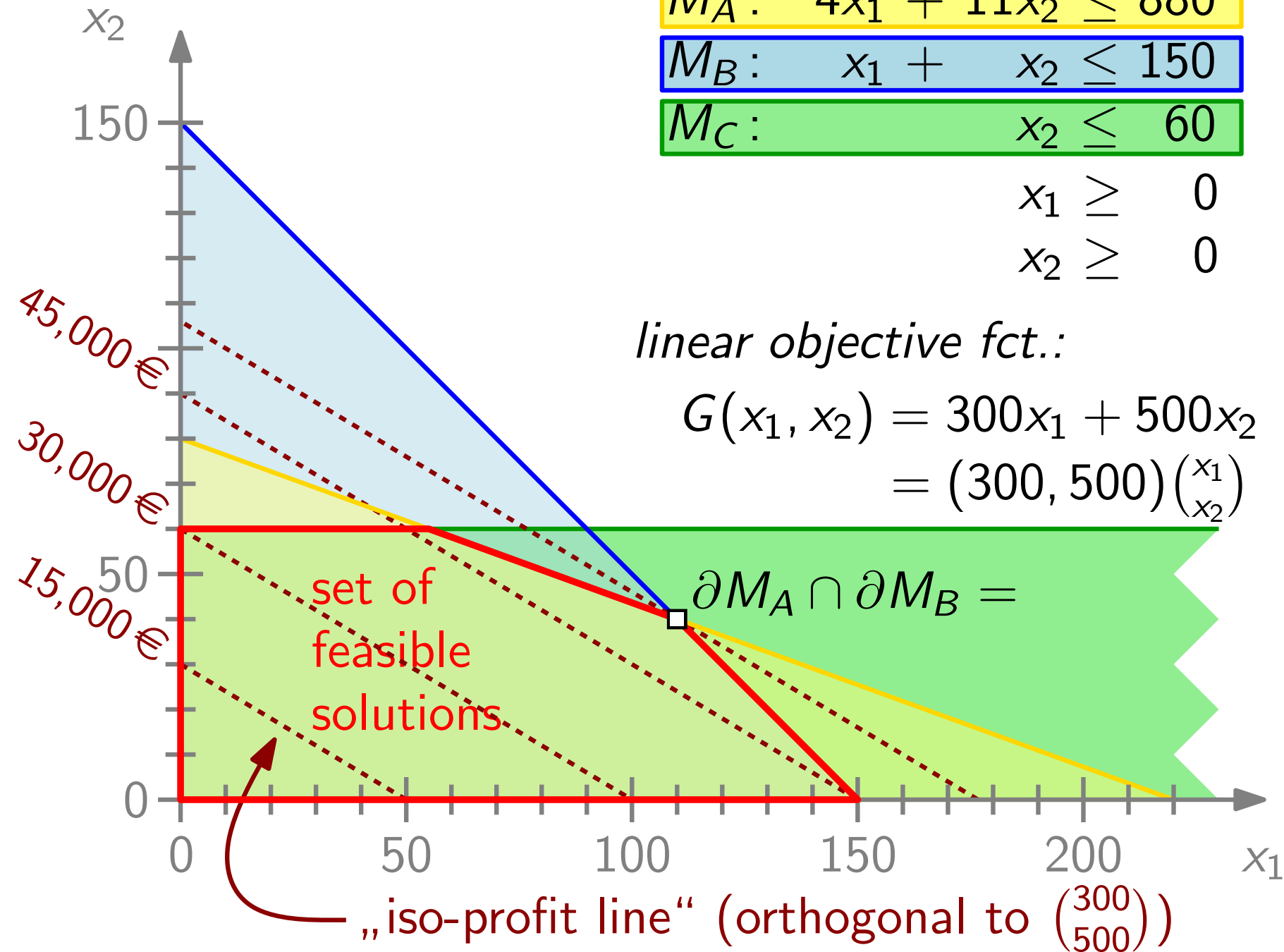
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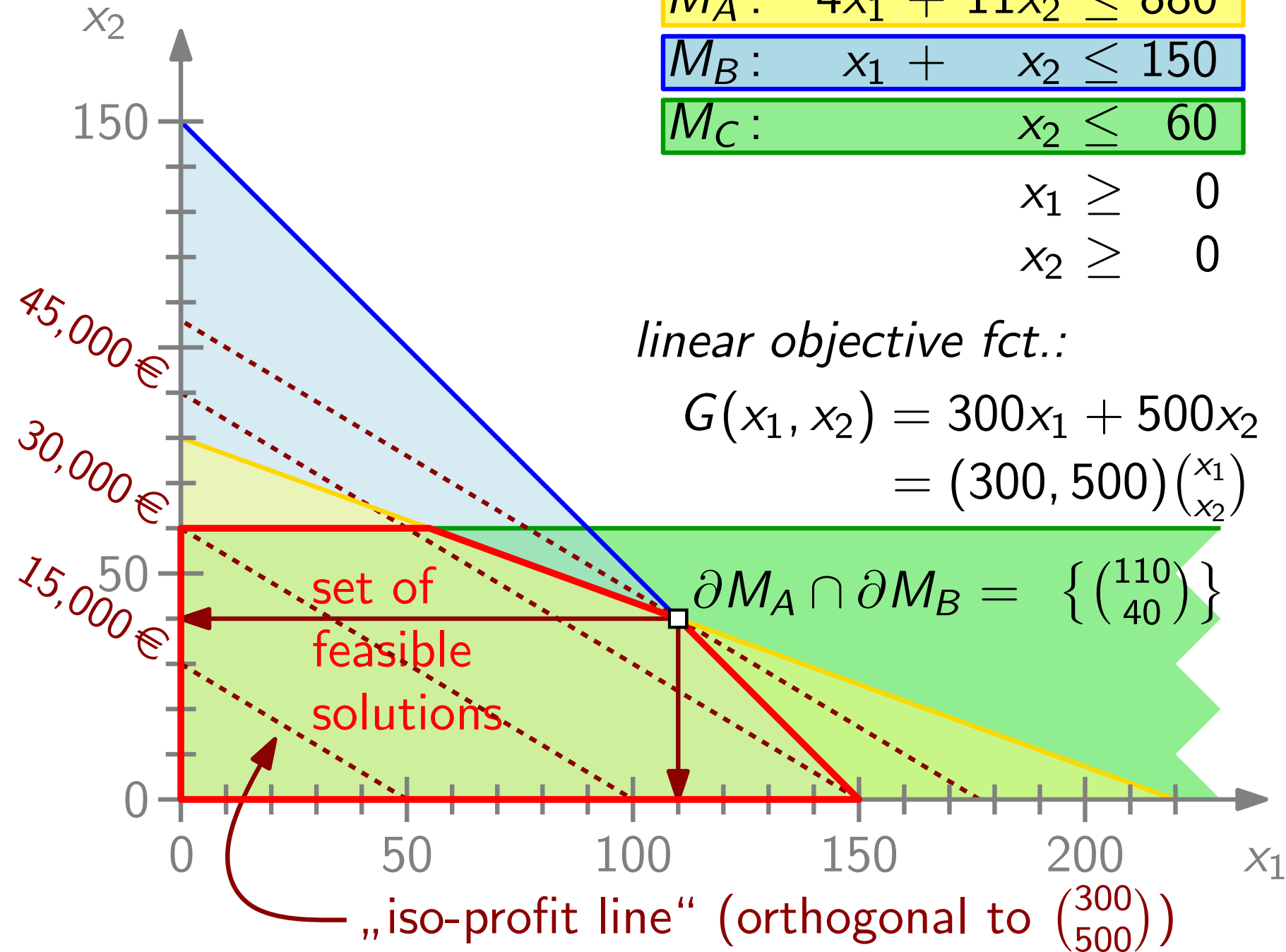
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linear objective fct.:

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$$\partial M_A \cap \partial M_B = \left\{ \begin{pmatrix} 110 \\ 40 \end{pmatrix} \right\}$$



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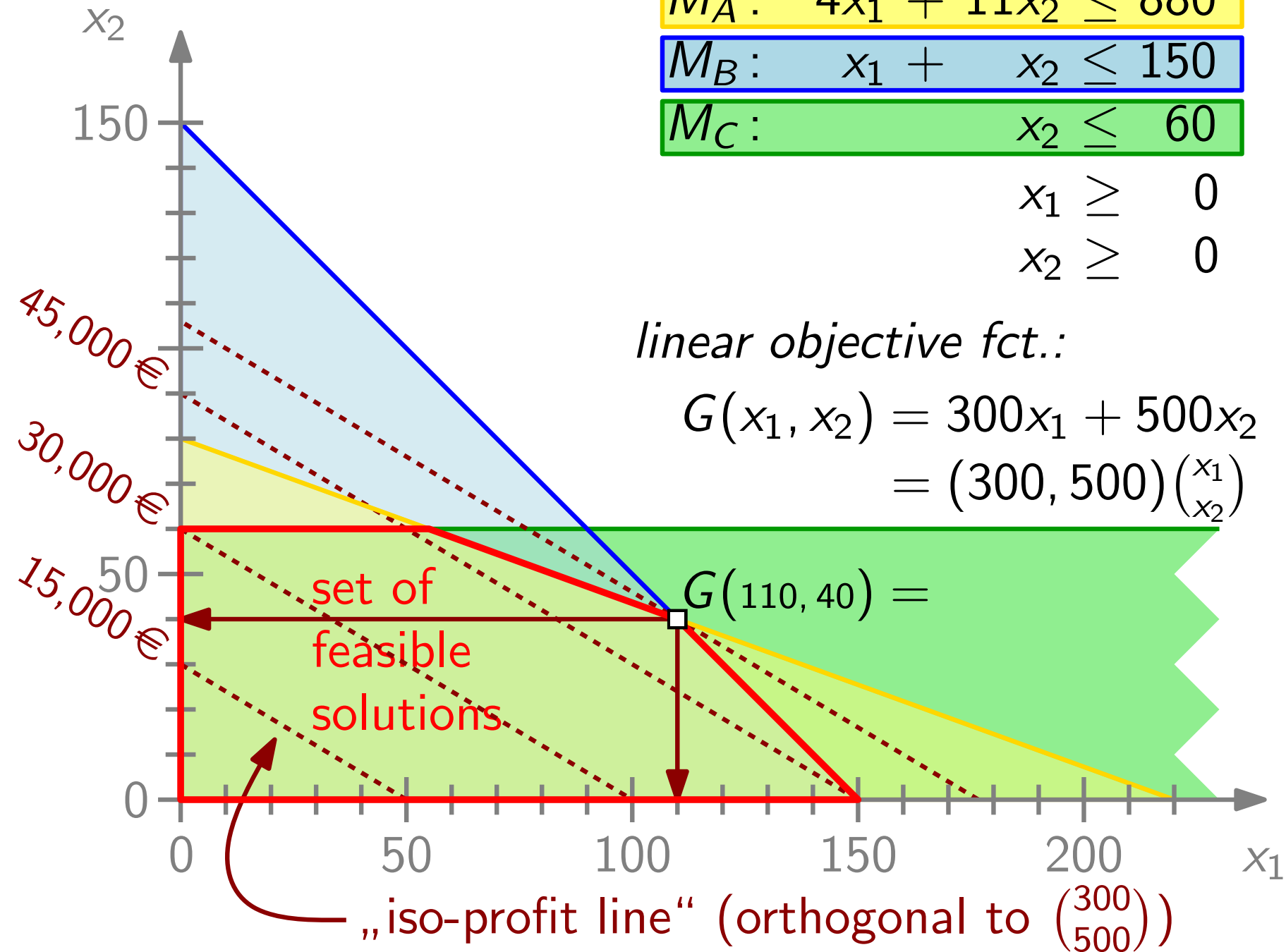
$$x_2 \geq 0$$

linear objective fct.:

$$G(x_1, x_2) = 300x_1 + 500x_2$$

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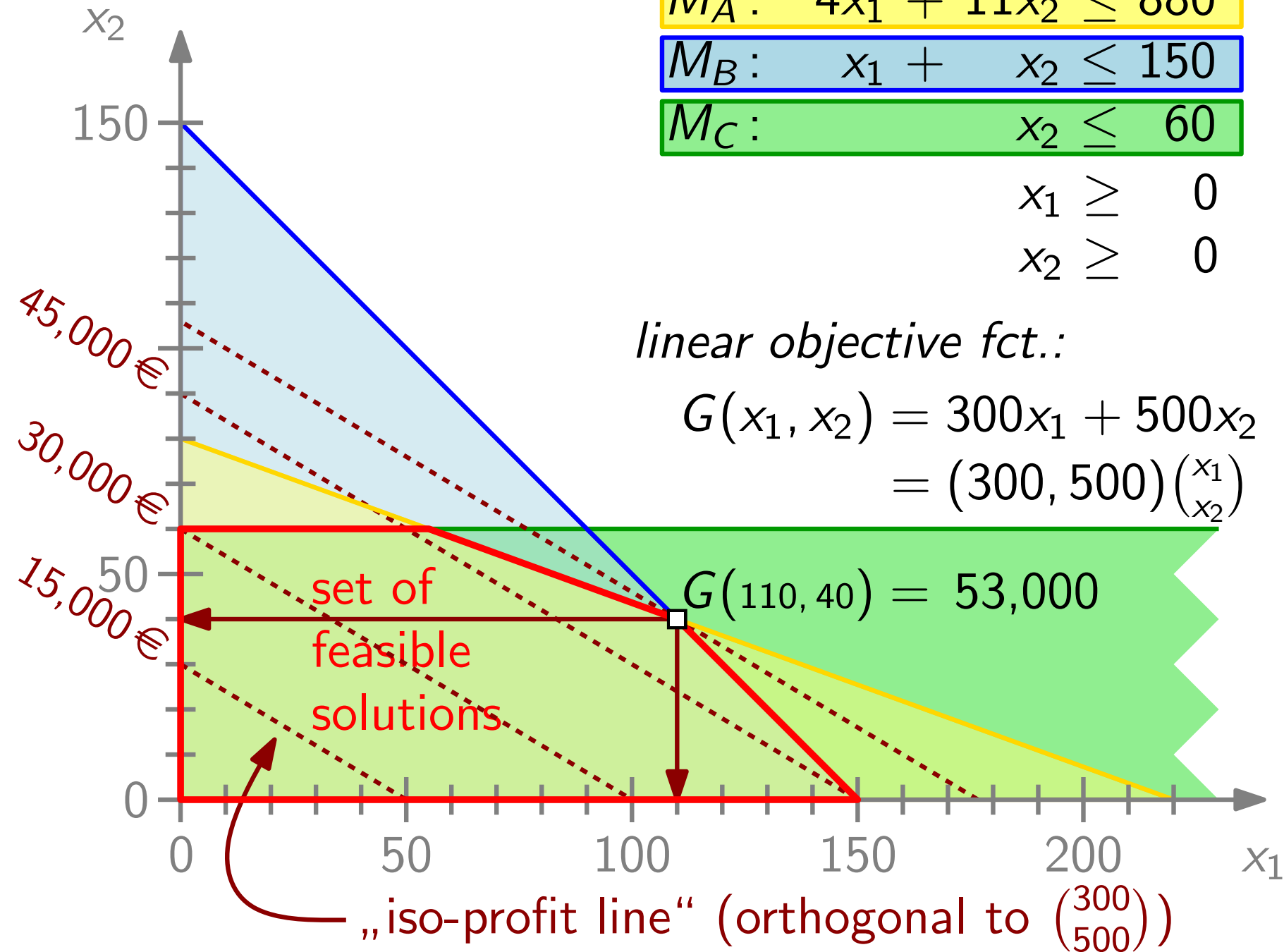
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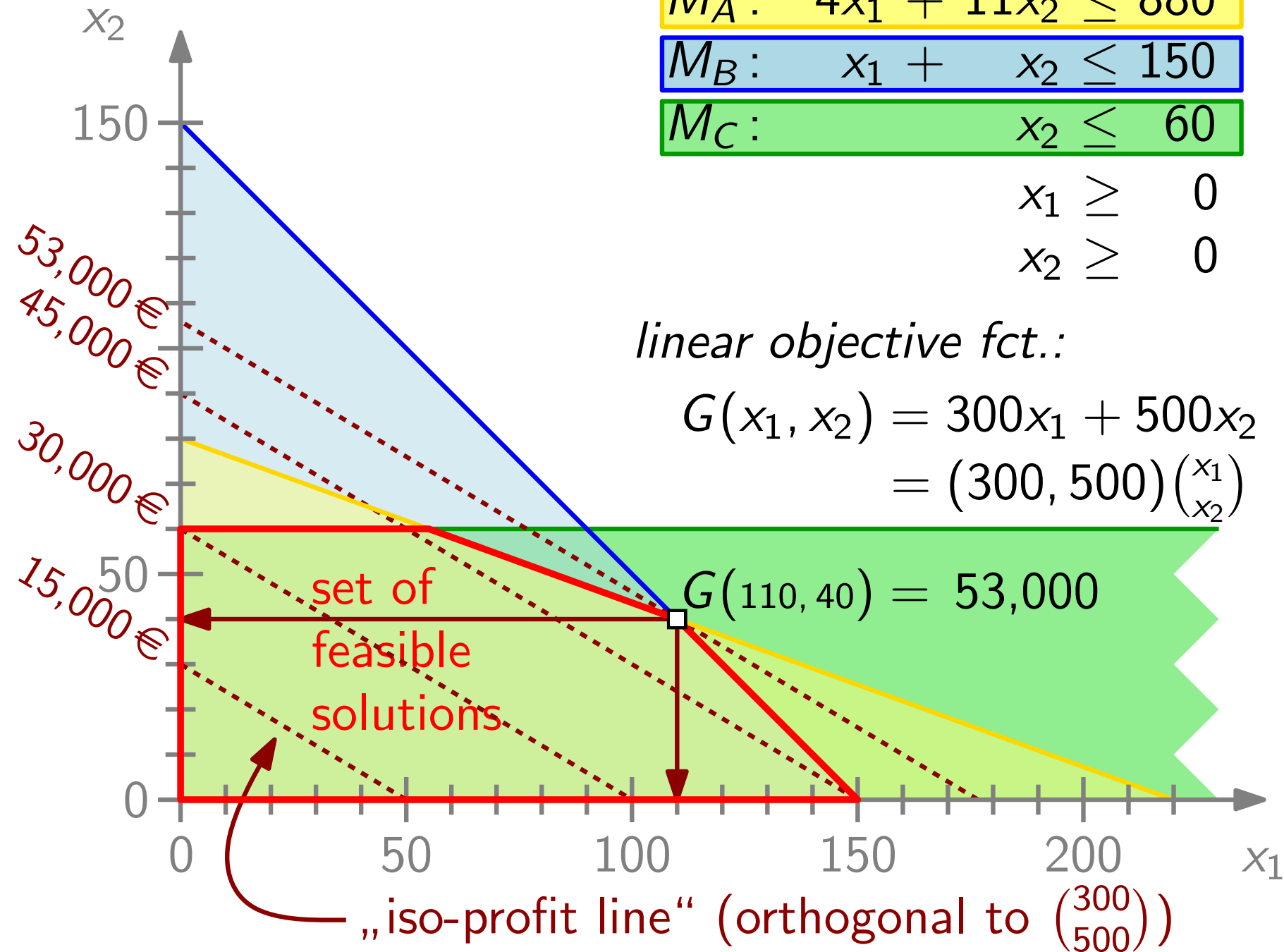
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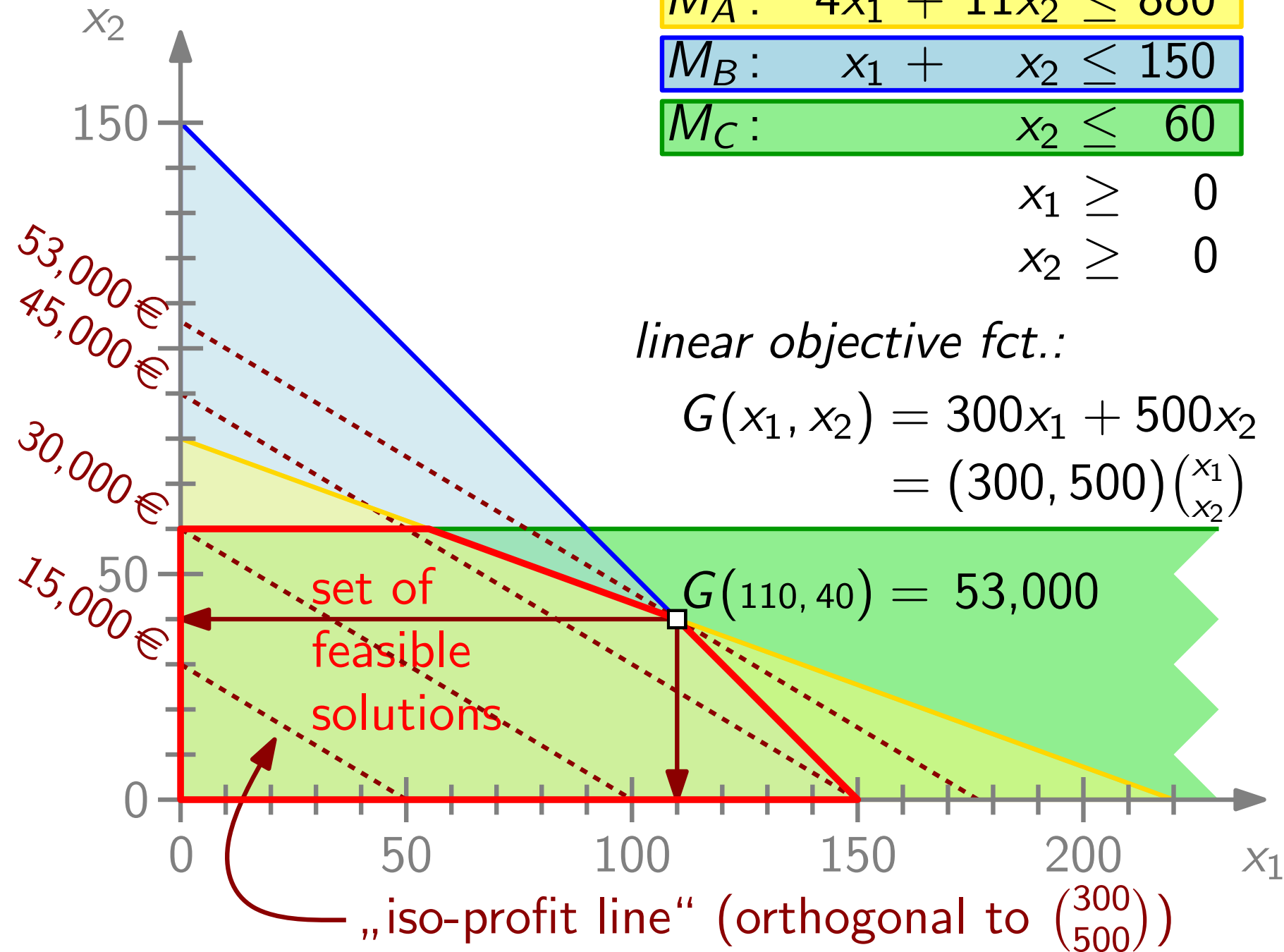
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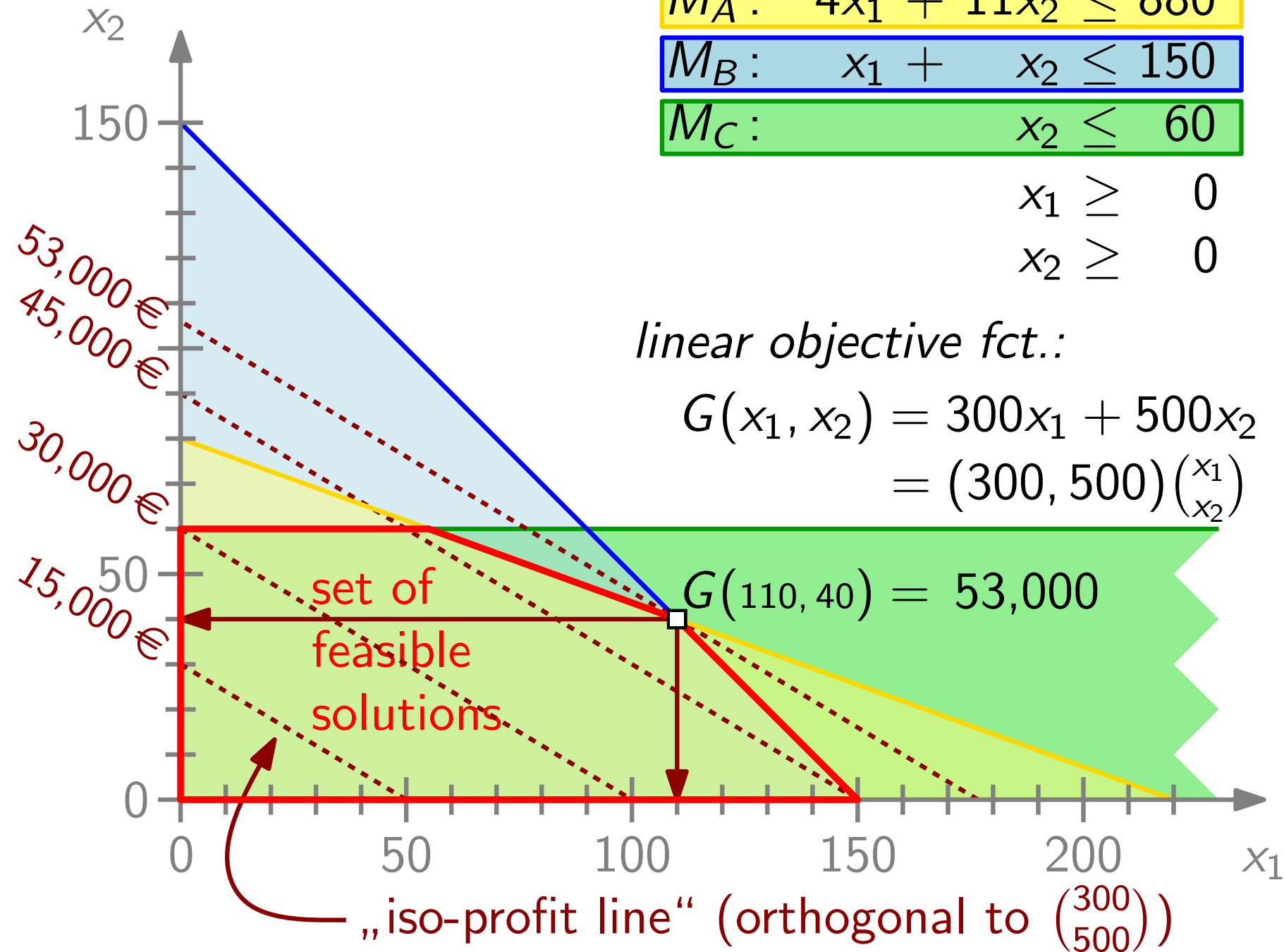
$$x_2 \geq 0$$

$$Ax \leq b$$

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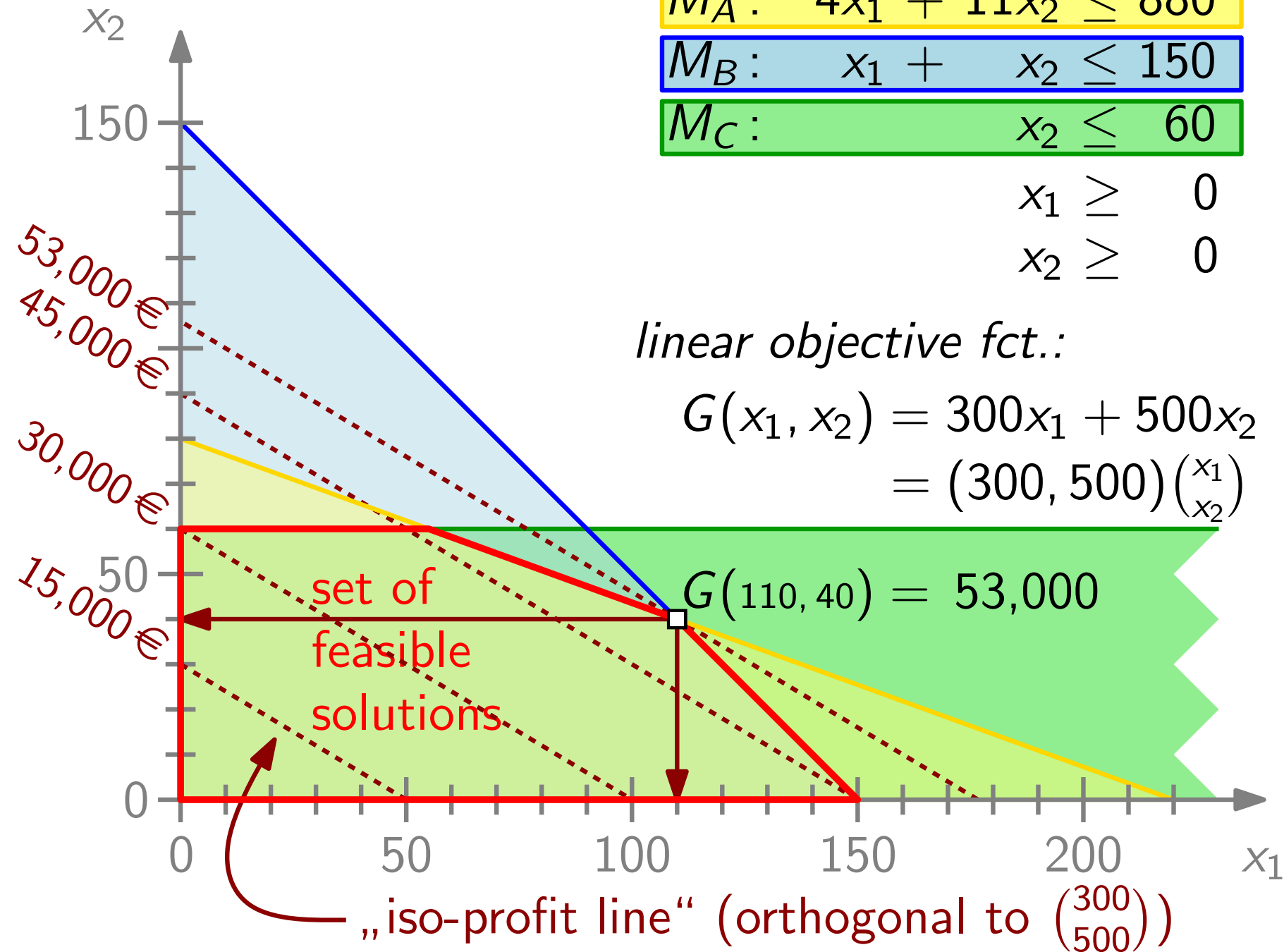
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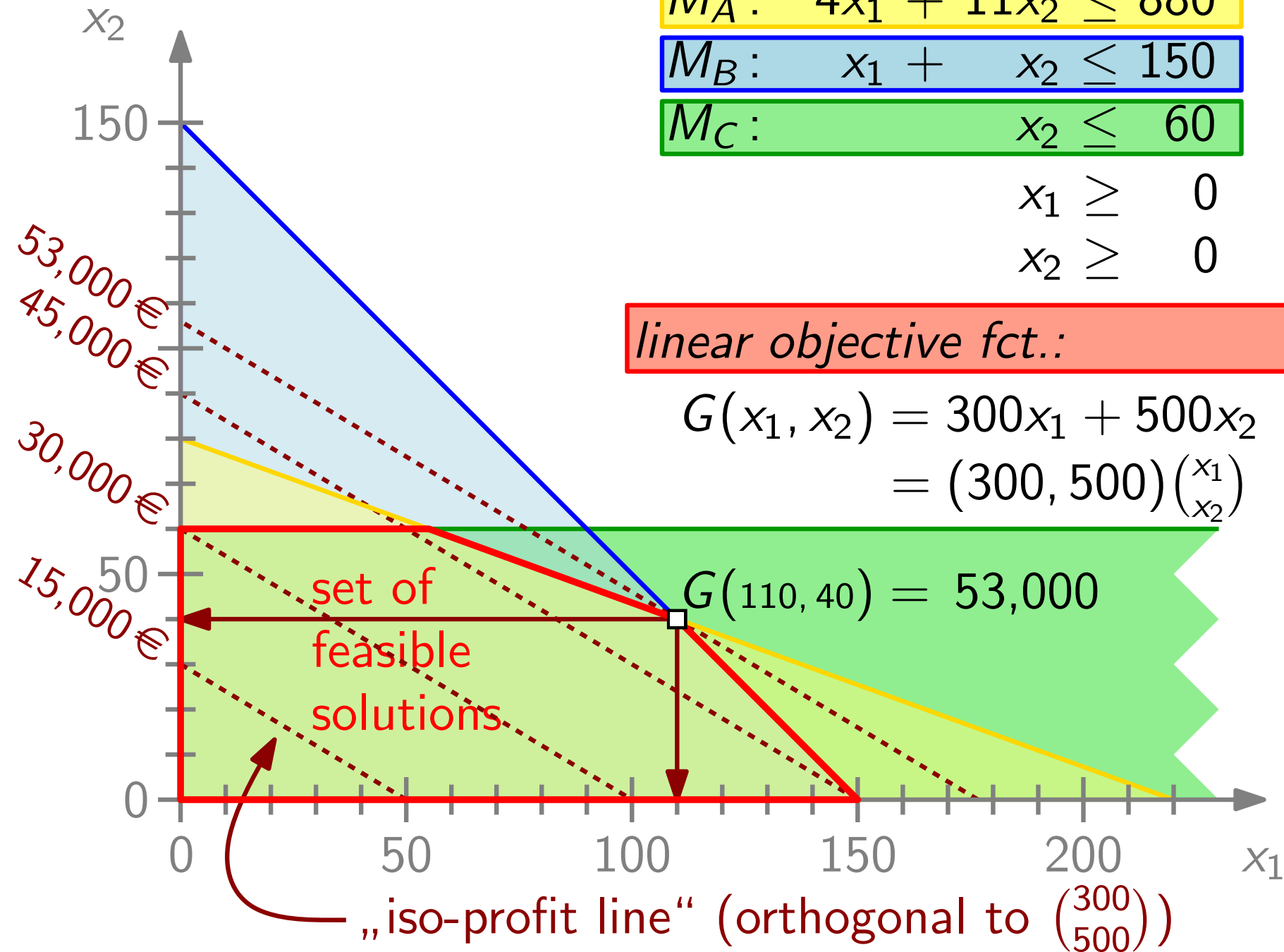
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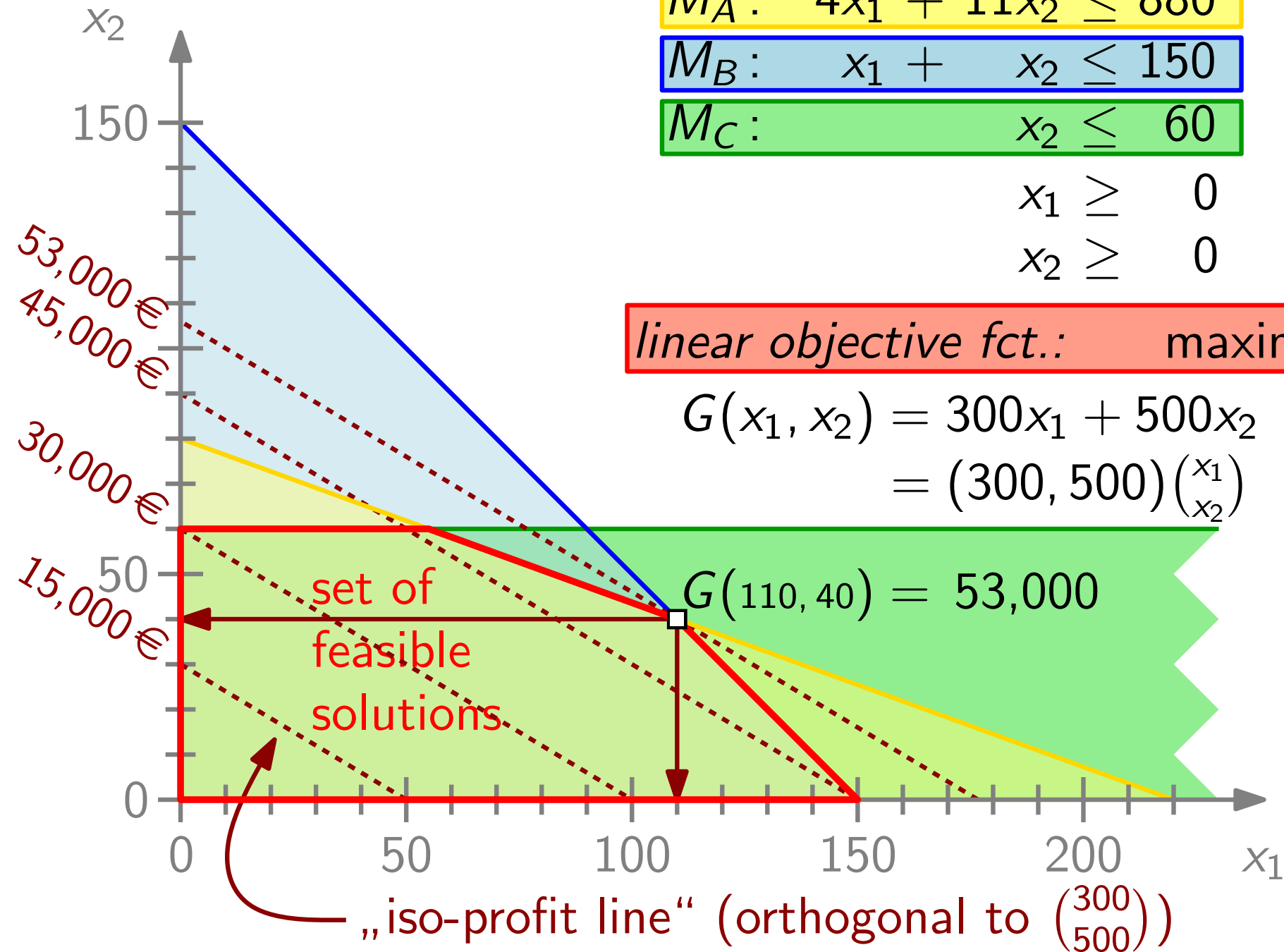
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linear objective fct.: maximize $c^T x$

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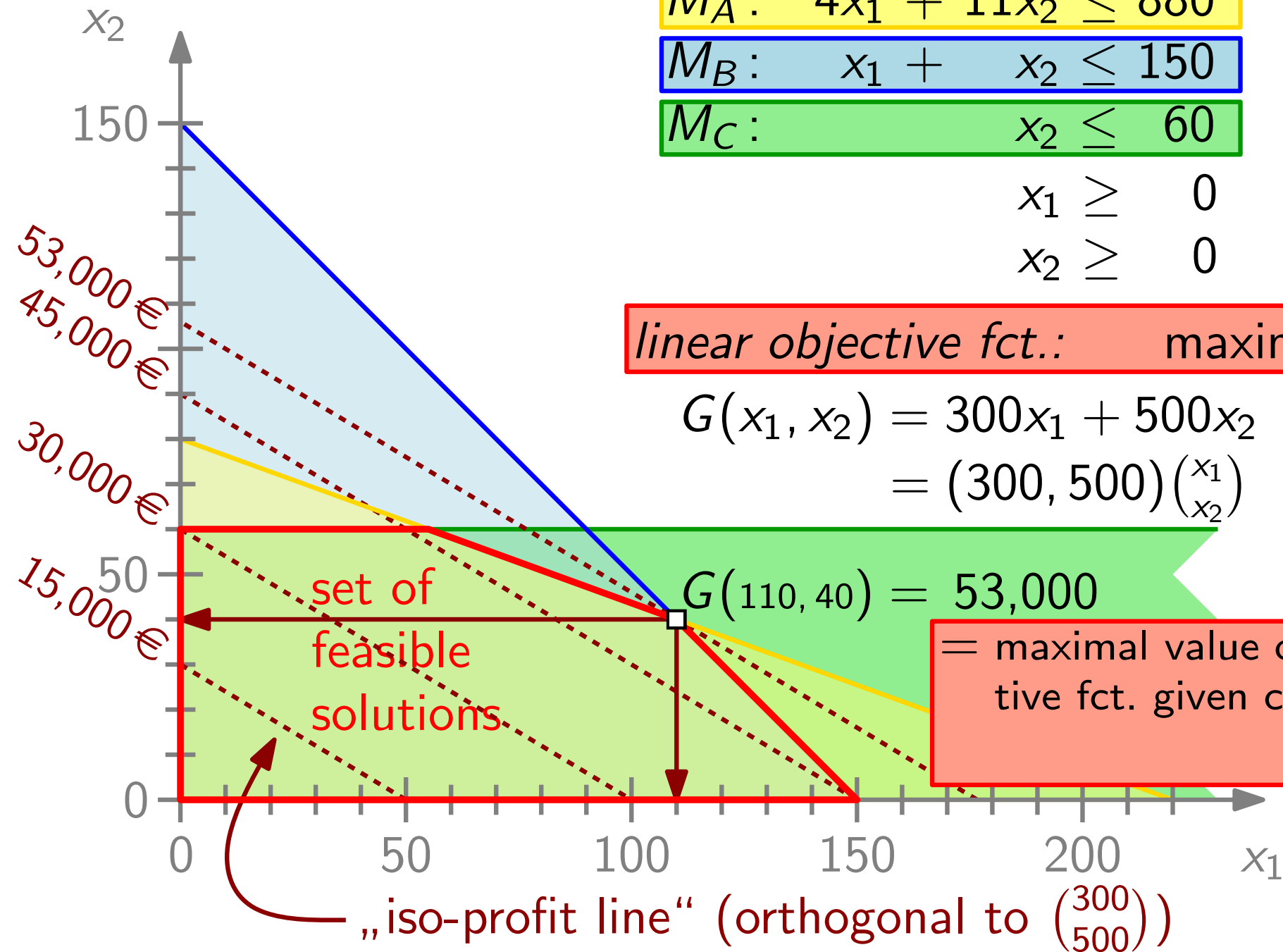
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= maximal value of objective fct. given constraints



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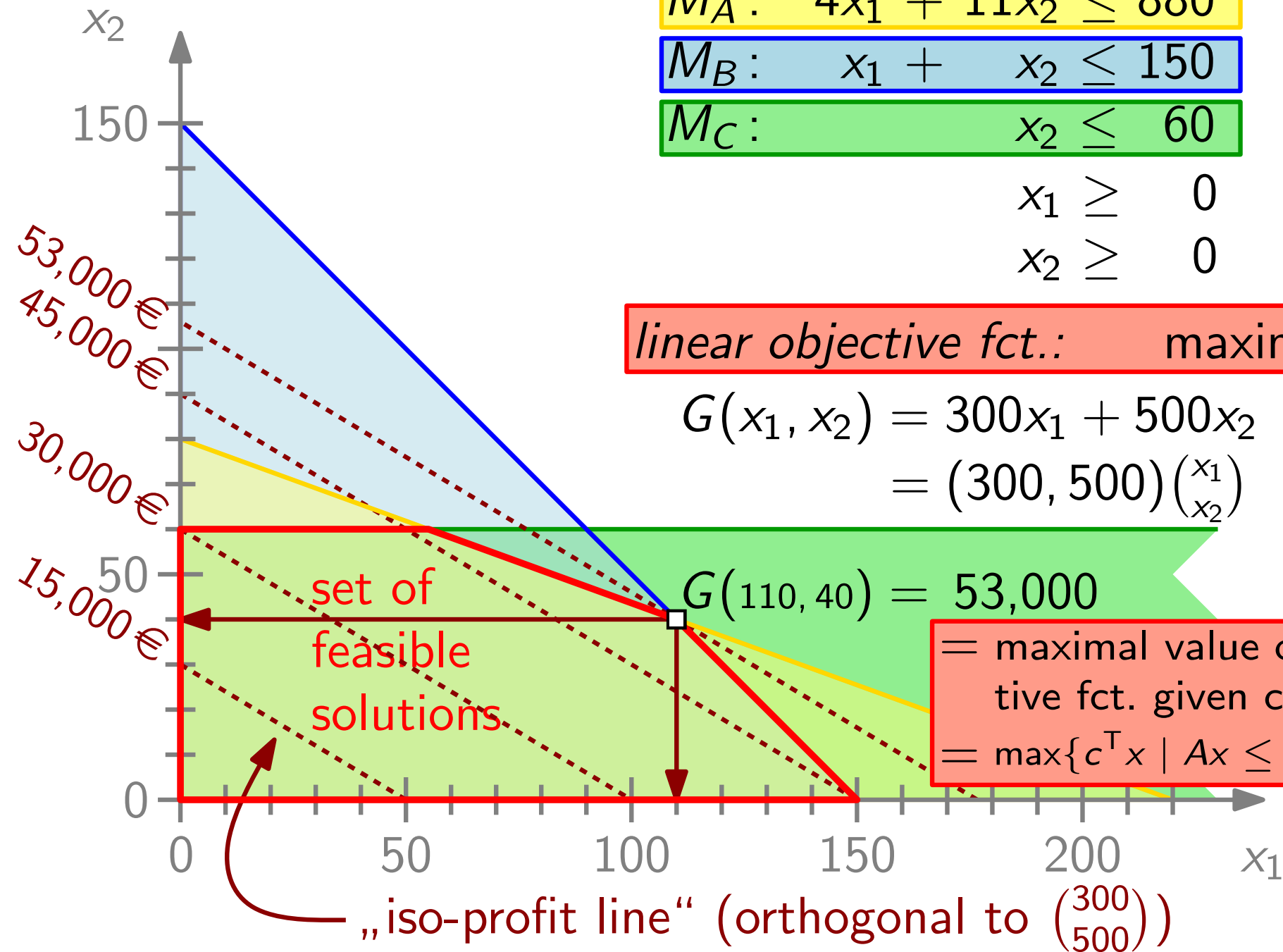
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= maximal value of objective fct. given constraints
= $\max\{c^T x \mid Ax \leq b, x \geq 0\}$



Definition and Known Algorithms

Given a set H of n halfspaces in \mathbb{R}^d and a direction c , find a point $x \in \bigcap H$ such that cx is maximum (or minimum).

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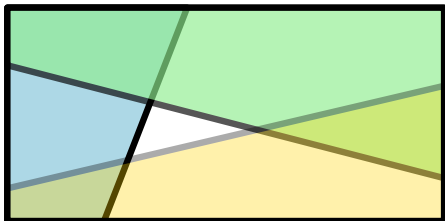
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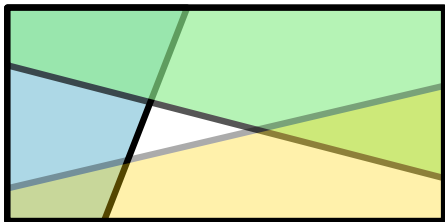
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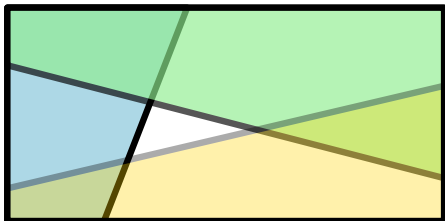
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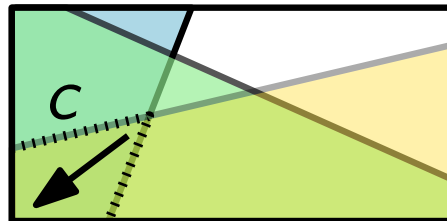
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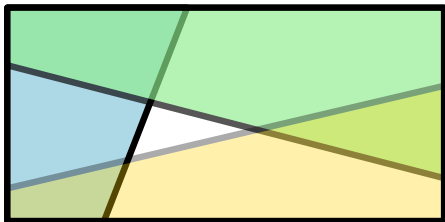
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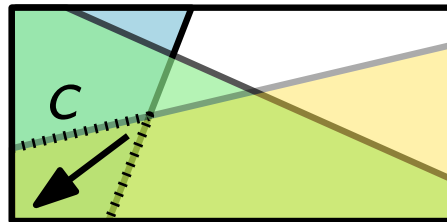
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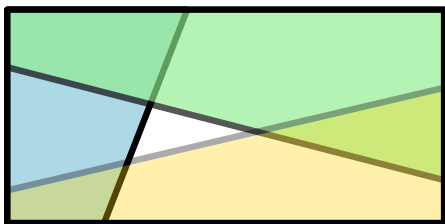
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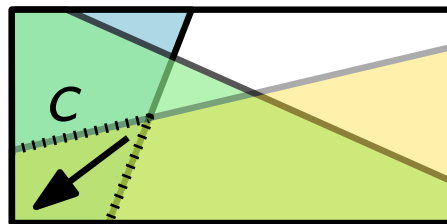
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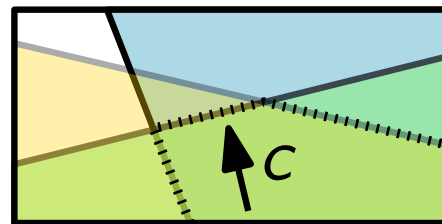
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$\bigcap H$ unbnd. in dir. c



Definition and Known Algorithms

Given a set H of n halfspaces in \mathbb{R}^d and a direction c , find a point $x \in \bigcap H$ such that cx is maximum (or minimum).

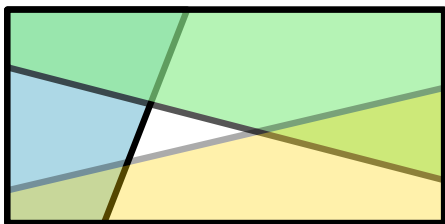
VERY important problem, for example, in Operations Research.

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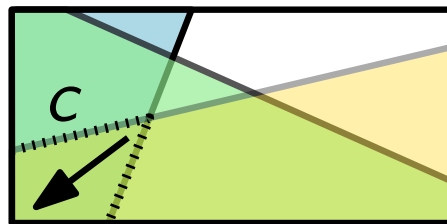
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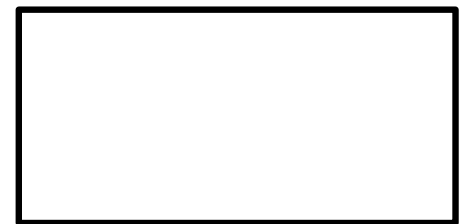
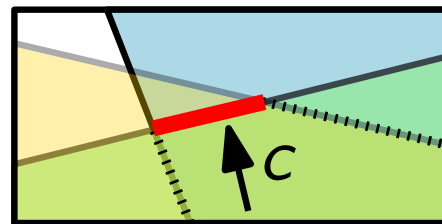
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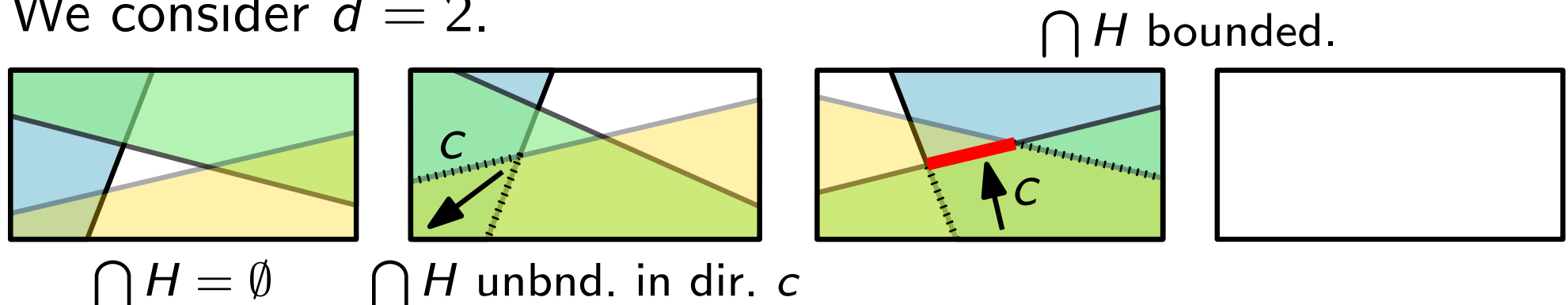
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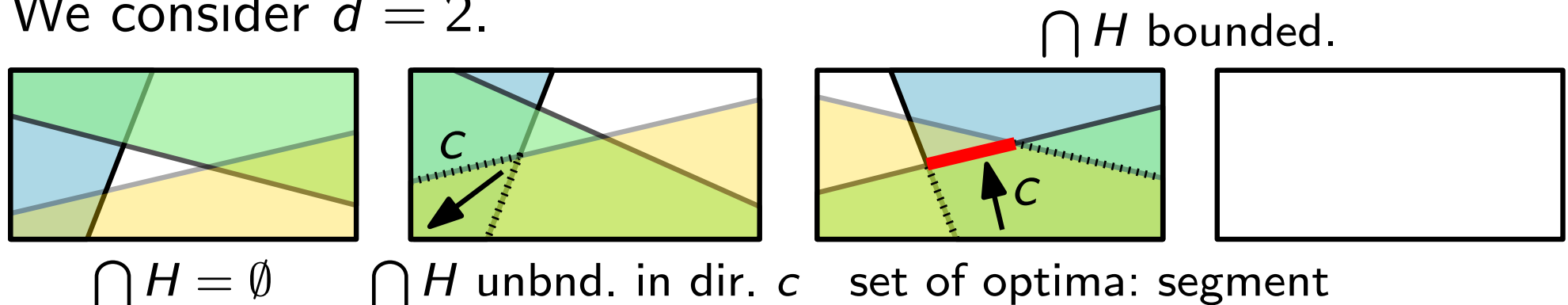
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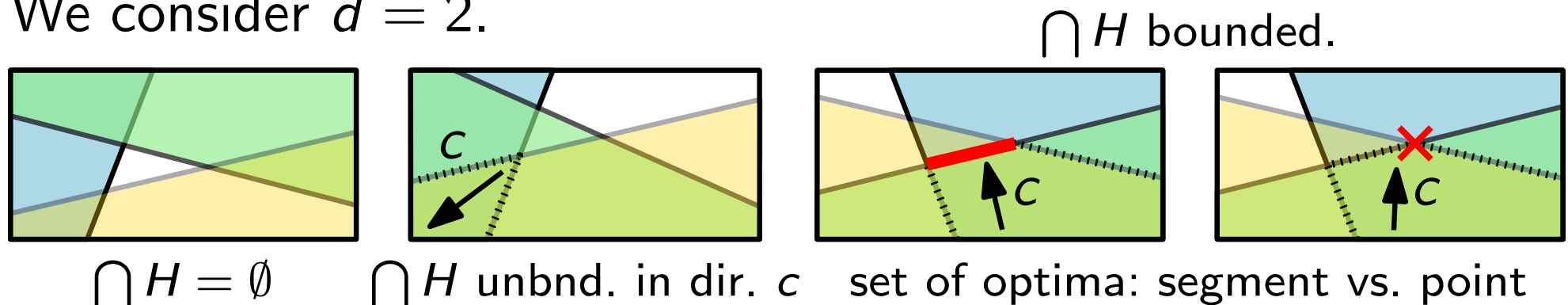
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 split H into sets H_1 and H_2 with $|H_1|, |H_2| \approx |H|/2$

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Running time: $T_{IH}(n) = 2T_{IH}(n/2) + T_{ICR}(n)$

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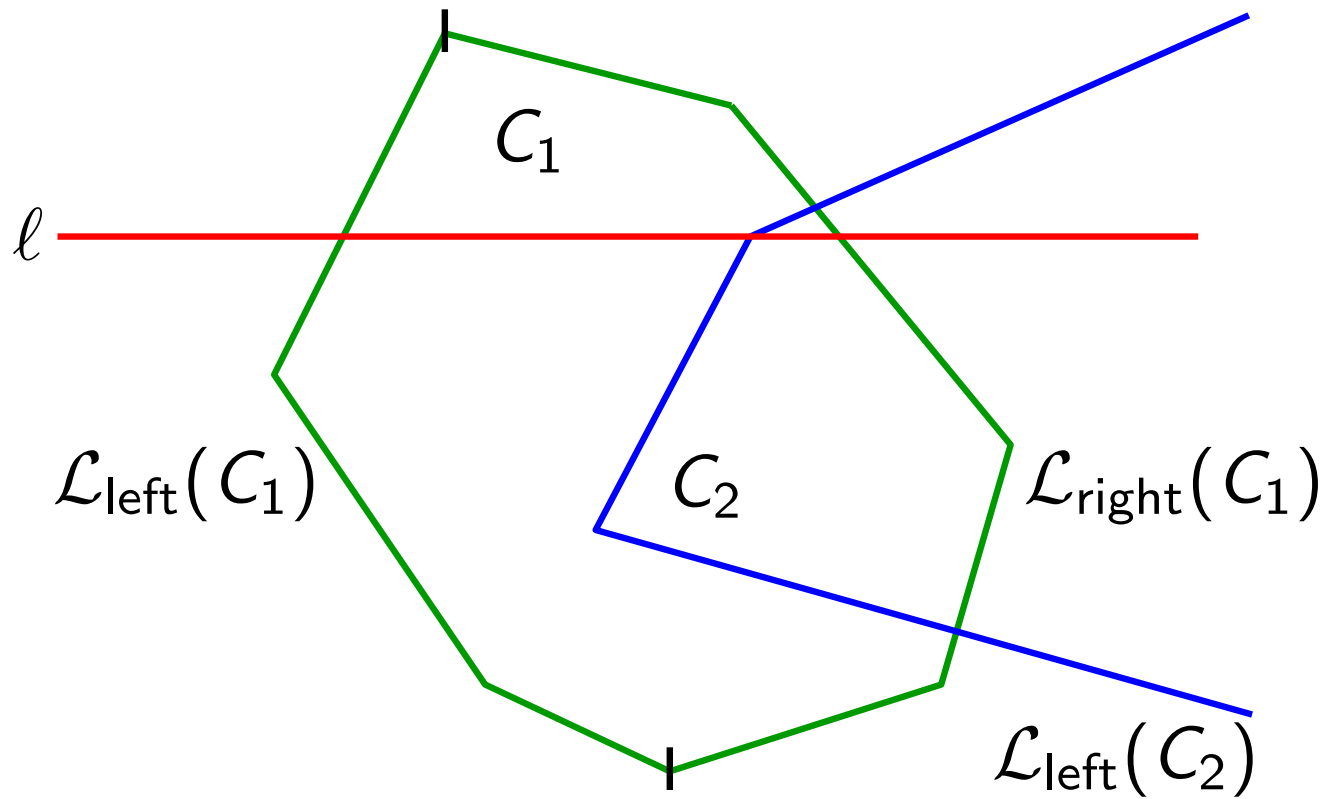
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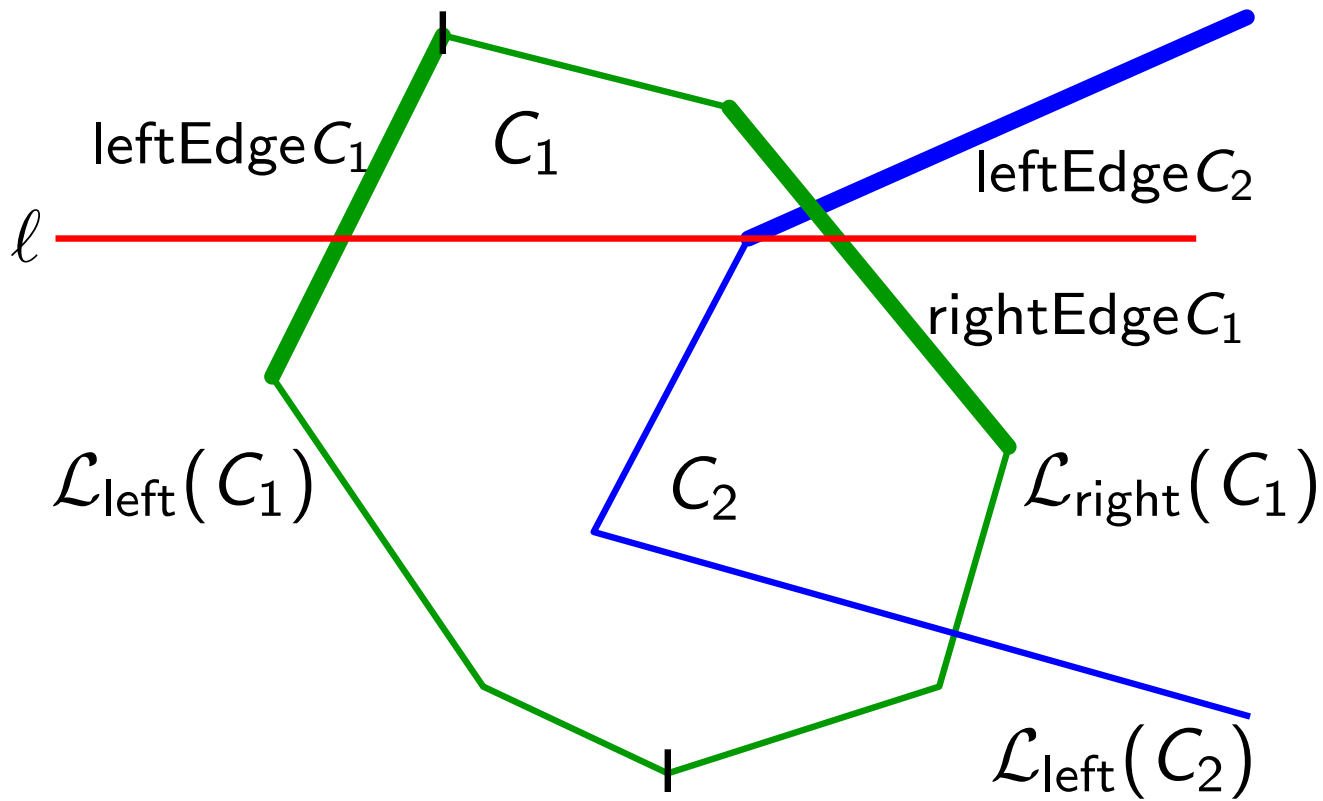
Better ideas?

Use specialized algorithm for intersecting *convex* regions/polyg.!

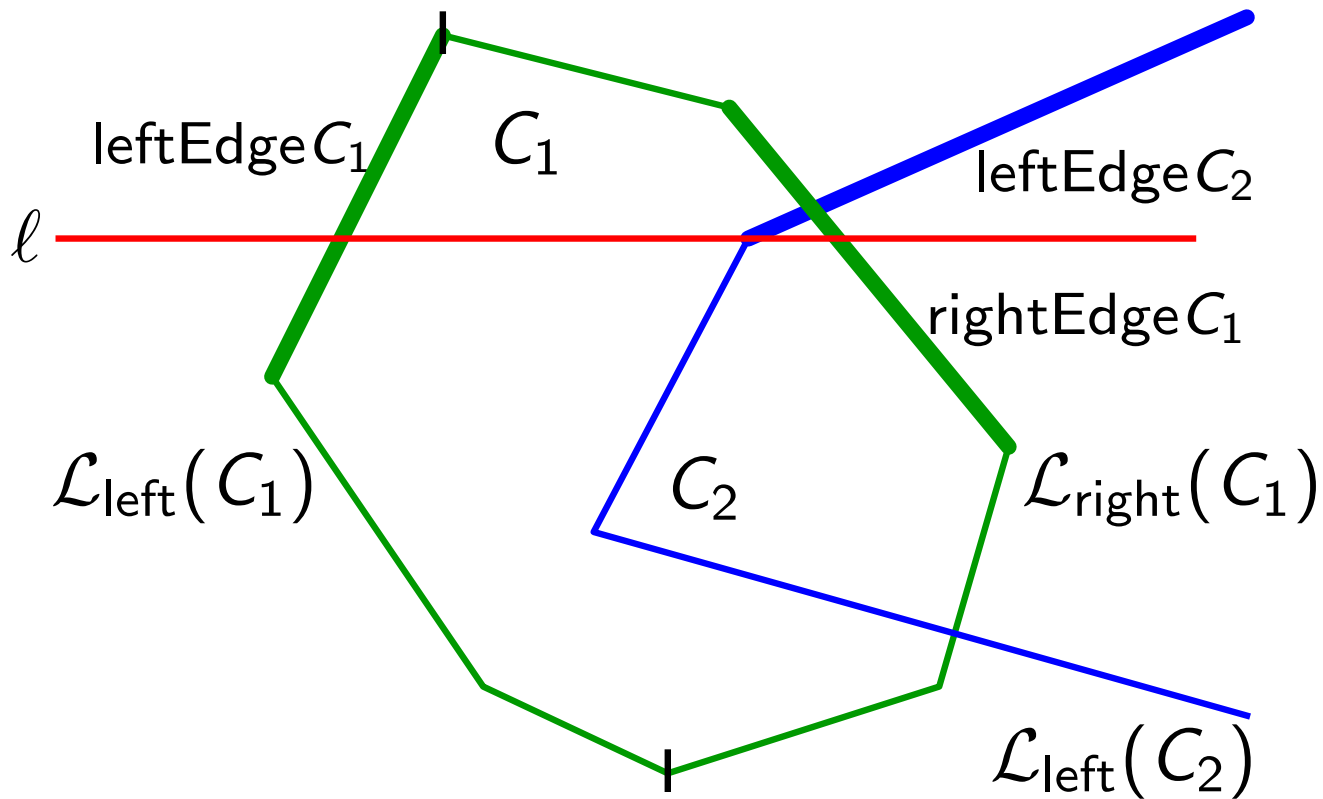
Intersecting Convex Regions Faster



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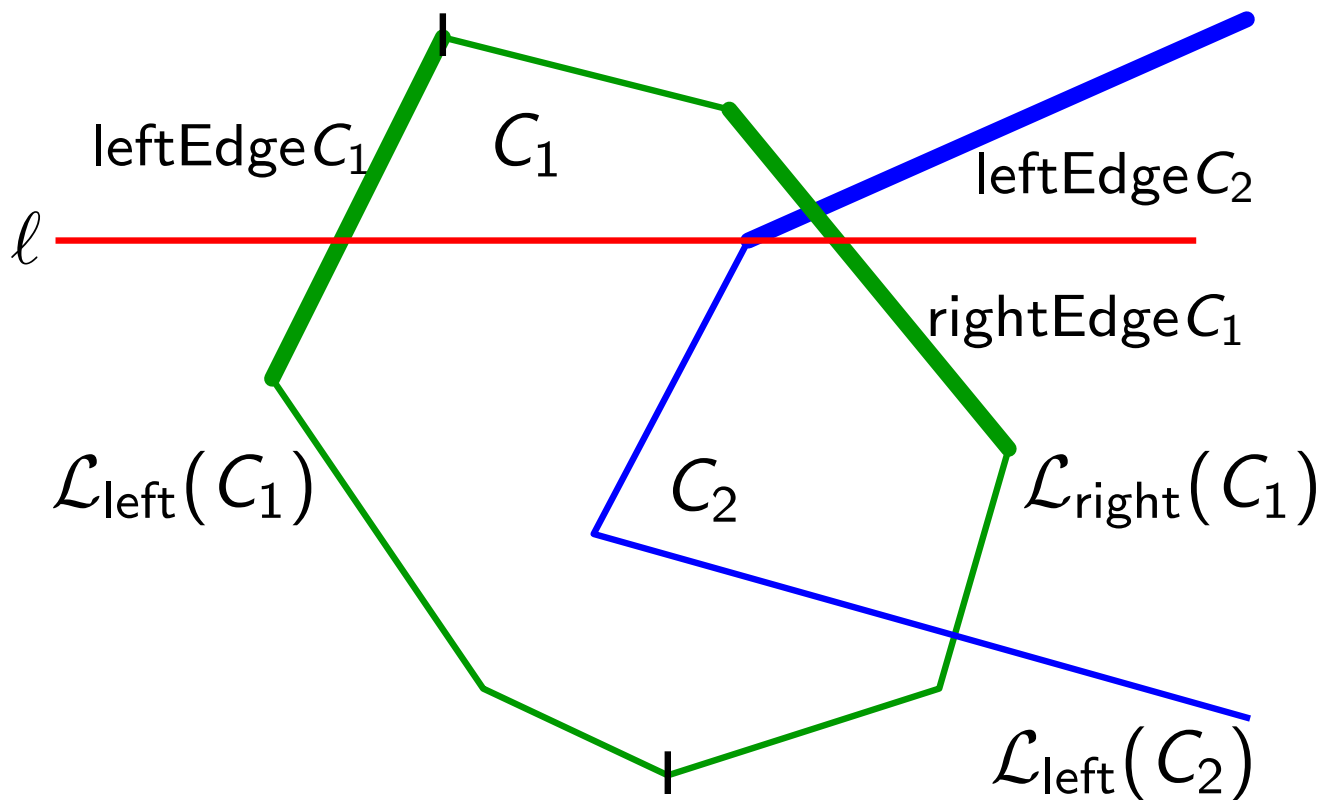


Intersecting Convex Regions Faster



Theorem. The intersection of two convex polygonal regions can be computed in linear time.

Intersecting Convex Regions Faster

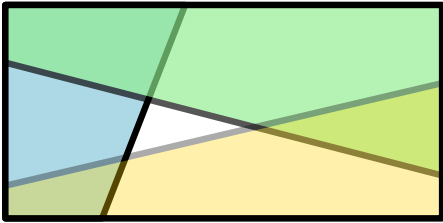


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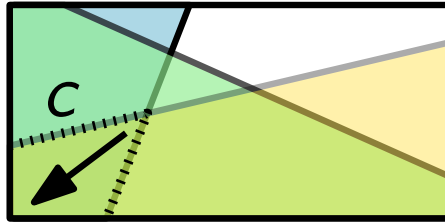
Corollary. The intersection of n half planes can be computed in $O(n \log n)$ time.

A Small Trick: Make Solution Unique

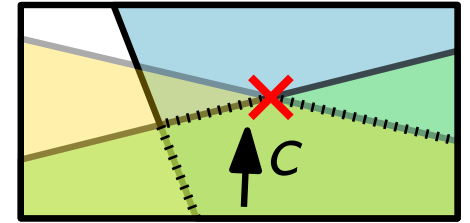
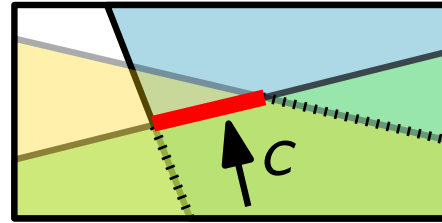
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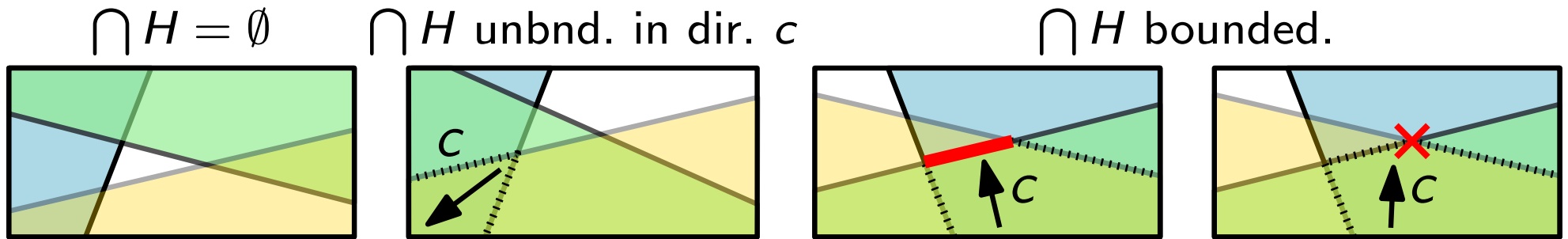
$\bigcap H$ unbnd. in dir. c



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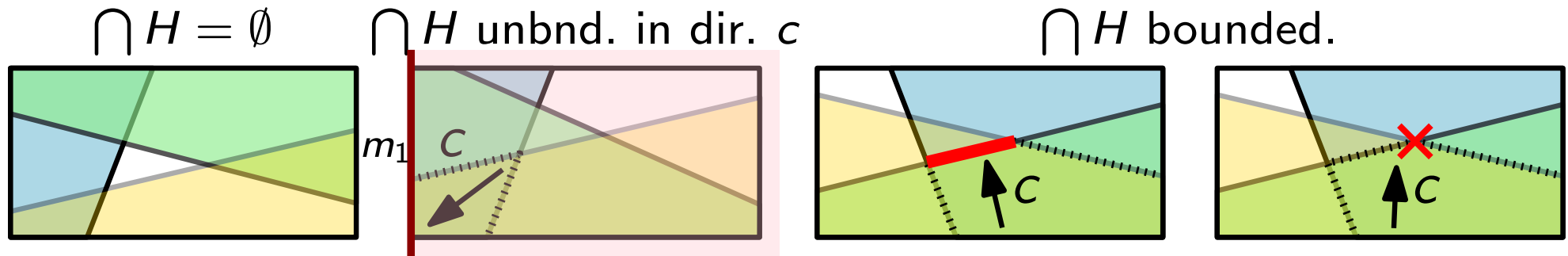


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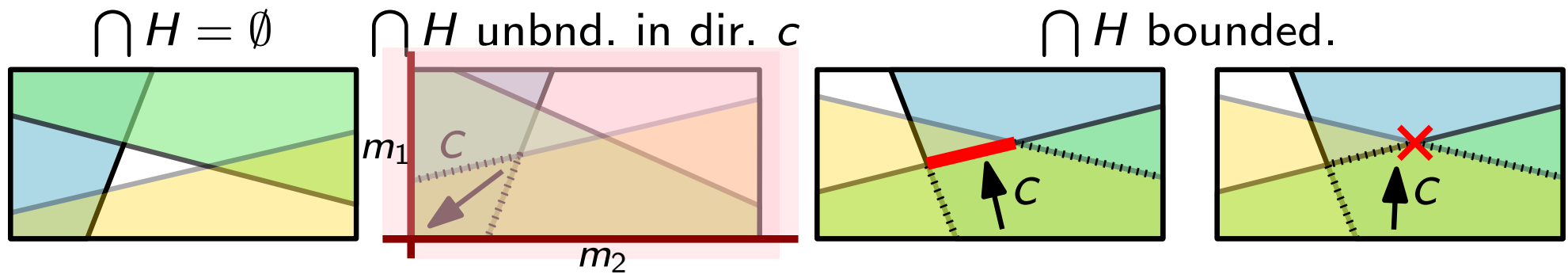
- Add two bounding halfplanes m_1 and m_2

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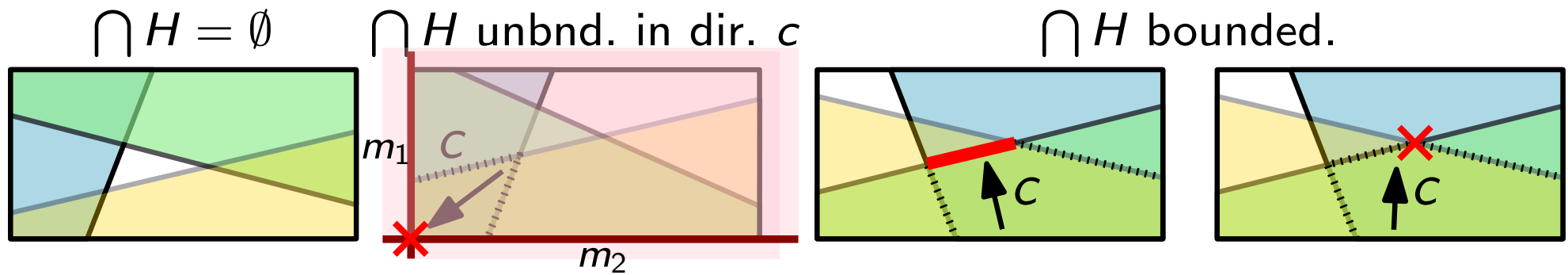
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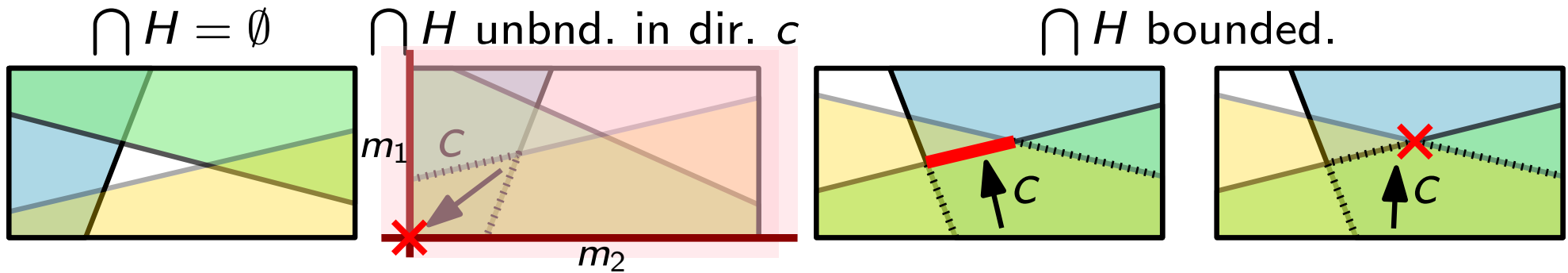
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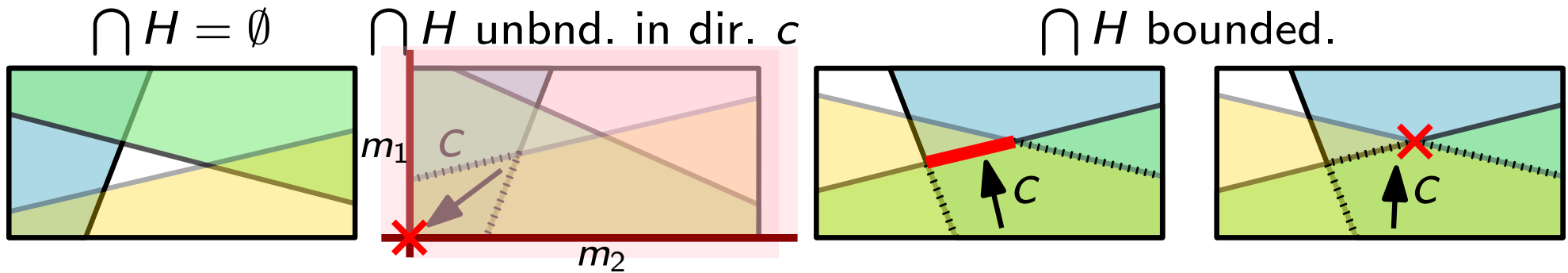
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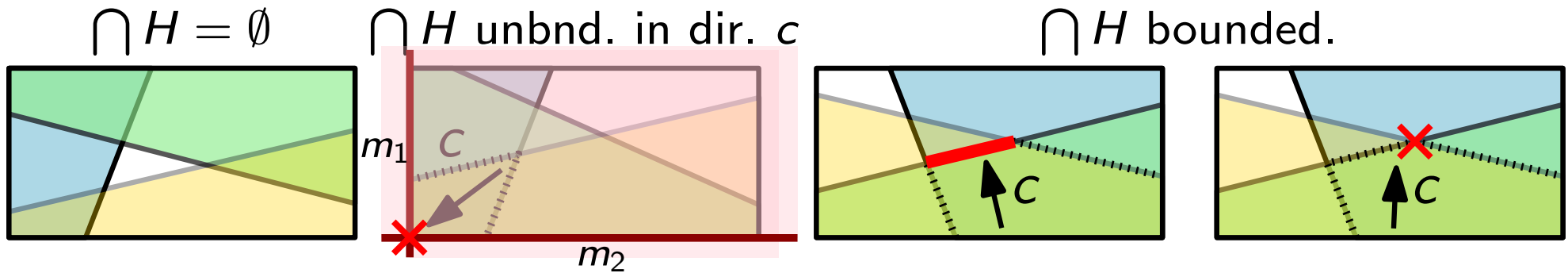
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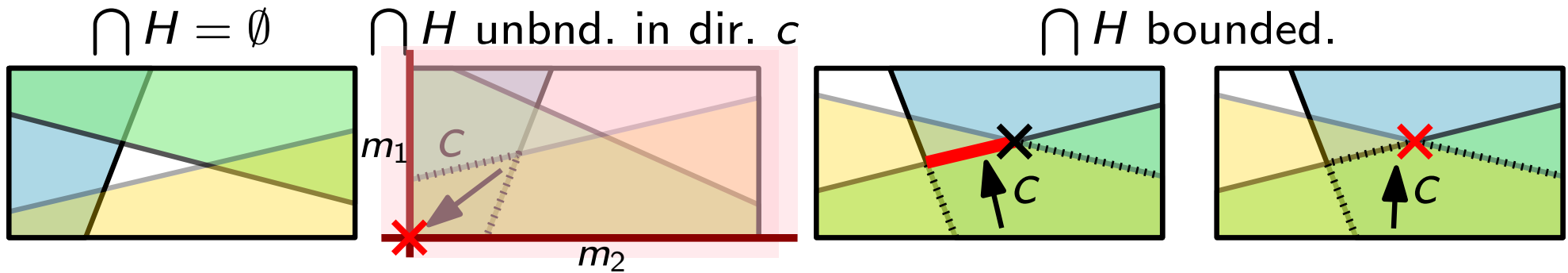
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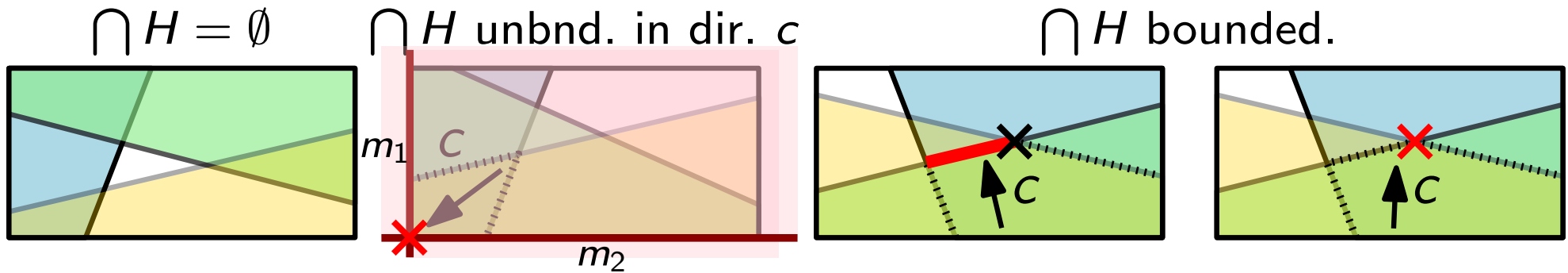
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\Rightarrow Set of solutions is either empty or a uniquely defined point.

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2dBoundedLP(H, c, m_1, m_2)

$H_0 = \{m_1, m_2\}; C_0 \leftarrow m_1 \cap m_2$

$v_0 \leftarrow$ unique vertex of C_0

for $i \leftarrow 1$ **to** n **do**

$H_i = H_{i-1} \cup \{h_i\}; C_i = C_{i-1} \cap h_i$

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w-c running time:

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Randomized

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Randomized

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compute random permutation of H

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Result

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Proof. Let $X_i = \left\{ \begin{array}{ll} 1 & \text{if } v_{i-1} \notin h_i, \\ 0 & \text{else.} \end{array} \right\}$ (indicator random variable).

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Proof technique:

Backward analysis!

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