Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Computational Geometry

Winter term 2014/15

Triangulating Polygons

or

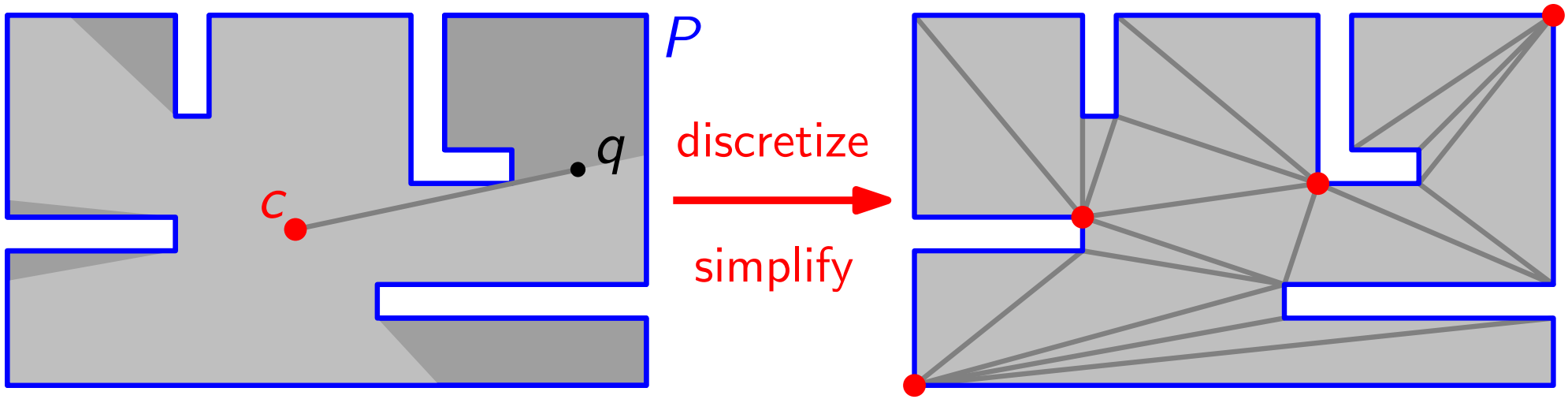
Guarding Art Galleries

Lecture #3

Prof. Dr. Alexander Wolff *Chair for Informatics I*

Guarding an Art Gallery

Given a *simple* polygon P (i.e., no holes, no self-intersection)...



Observation. Camera c "sees" a star-shaped region

Definition. A pt $q \in P$ is *visible* from $c \in P$ if $\overline{qc} \subseteq P$.

Aim: Use few cameras!

Theorem. Every simple polygon can be triangulated.

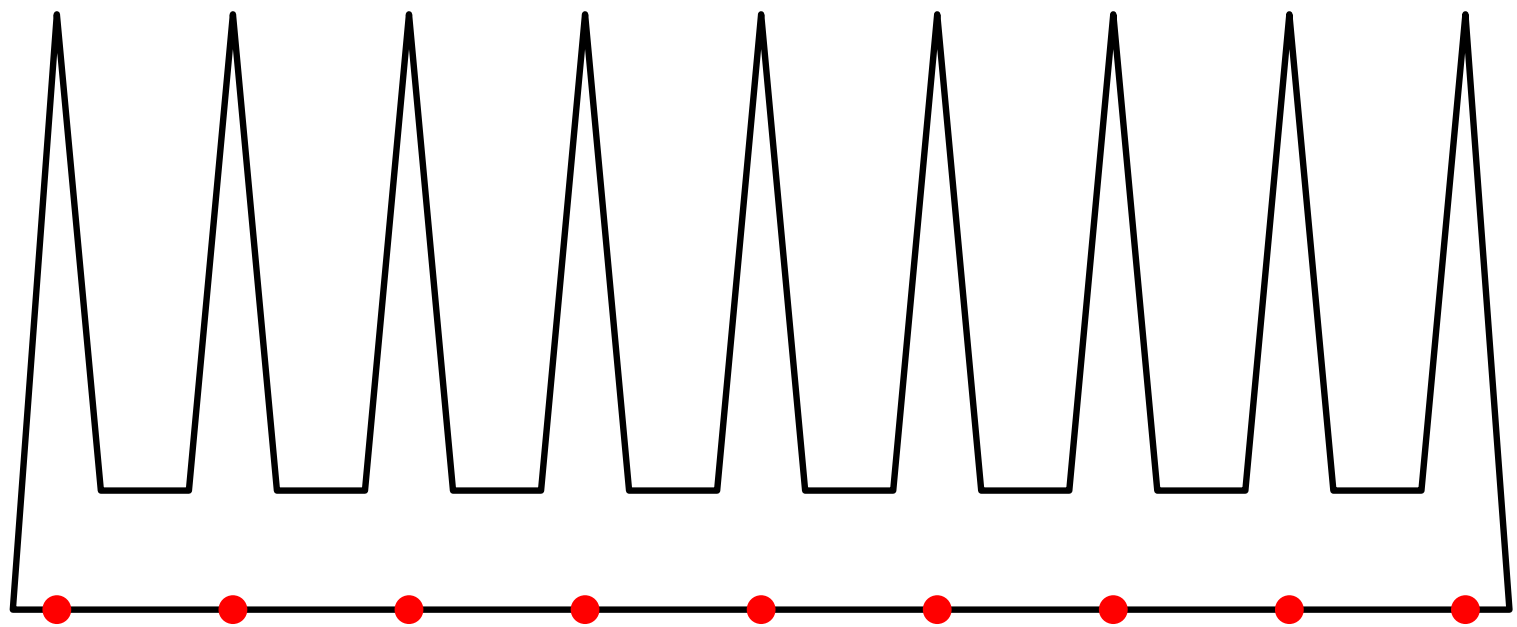
Any triangulation of a simple polygon with n vertices consists of $n - 2$ triangles.

The Art Gallery Theorem [Chvátal '75]

Theorem. For surveilling a simple polygon with n vertices, $\lfloor n/3 \rfloor$ cameras are sometimes necessary and always sufficient.

Exercise. Find, for arbitrarily large n, a polygon with n vertices, where $\approx n/3$ cameras are necessary.

[2 minutes]



[dBCvKO'08]

The Art Gallery Theorem [Chvátal '75]

Theorem. For surveilling a simple polygon with n vertices, $\lfloor n/3 \rfloor$ cameras are sometimes necessary and always sufficient. [black board]

To do: Find algorithm for triangulating a simple polygon!

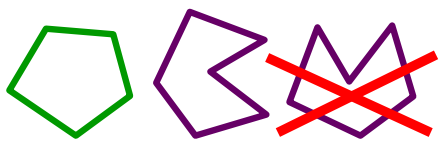
Brute force: follow existence proof, using recursion

running time: $O(n^2)$

Faster triangulation in two steps:

n-vtx polygon $\xrightarrow[O(n \log n)]{}$ "nice" pieces, n' vtc $\xrightarrow[O(n')]{}$ n'' triangles

Definition. A polygon P is *y-monotone* if, for any horizontal line ℓ, $\ell \cap P$ is connected.



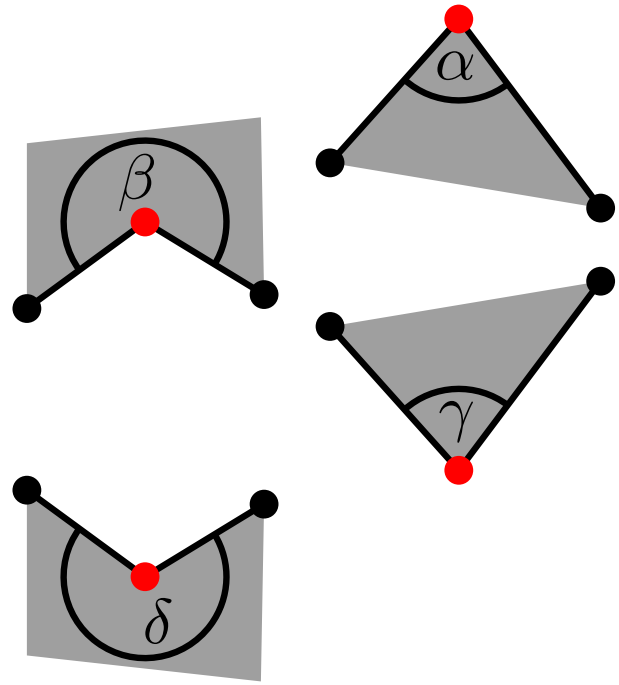
Partitioning a Polygon into Monotone Pieces

Idea: Classify vertices of given simple polygon P

– *turn* vertices:

vertical component of walking direction changes

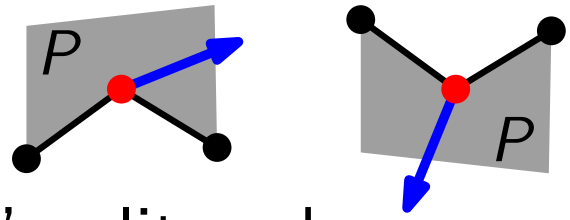
- *start* vertex — if $\alpha < 180°$
- *split* vertex — if $\beta > 180°$
- *end* vertex — if $\gamma < 180°$
- *merge* vertex — if $\delta > 180°$



– *regular* vertices

Lemma: Let P be a simple polygon. Then P is y-monotone $\Leftrightarrow$ P has neither split vertices nor merge vertices.

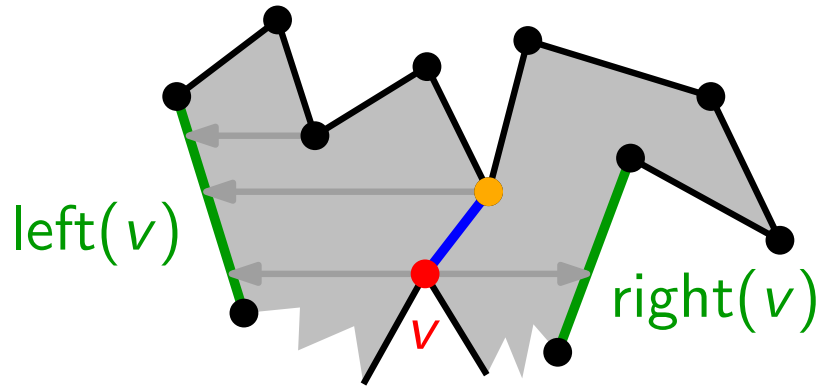
Towards an Algorithm



Idea: Add diagonals to "destroy" split and merge vertices.

Problem: Diagonals must not cross – each other
– edges of P

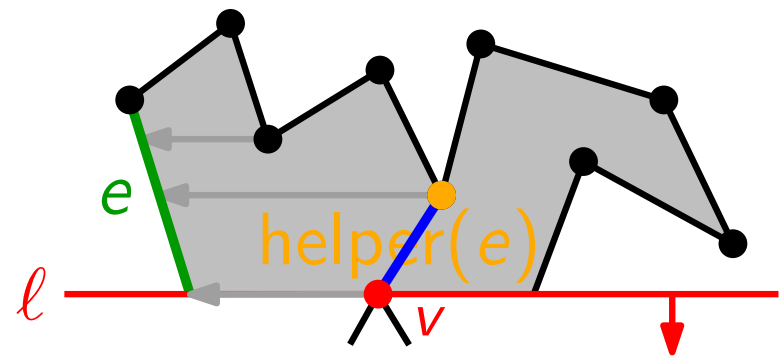
1) Treating split vertices



Connect v to vertex $w^\star$ having minimum y-coordinate among all vertices w above v and with $\text{left}(w) = \text{left}(v)$.

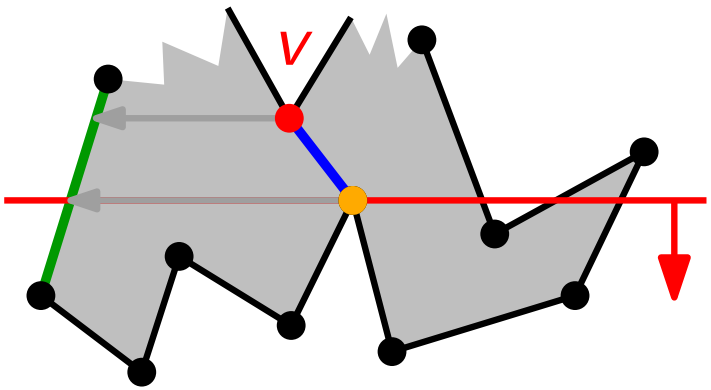
Think of a sweep-line algorithm:

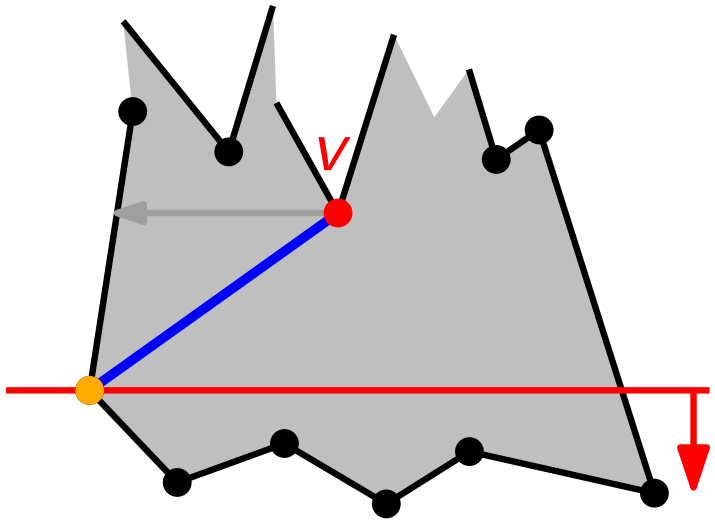
Connect v to helper(left(v)).



An Algorithm

2) Treating merge vertices





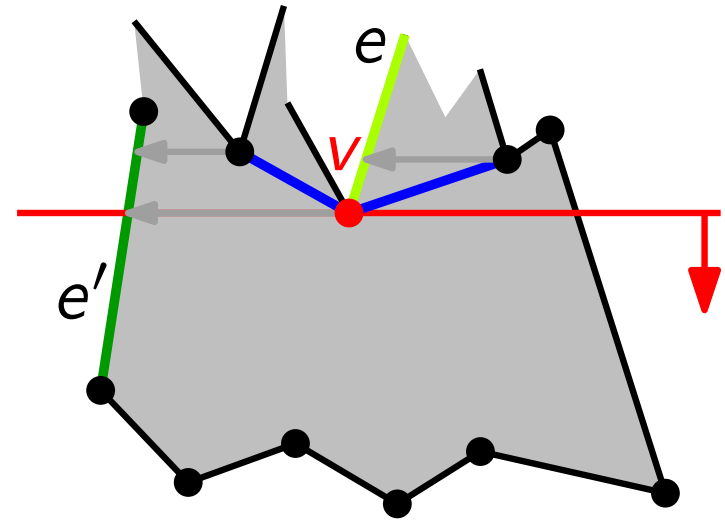
makeMonotone(polygon P**)**
$\mathcal{D} \leftarrow$ DCEL$(V(P), E(P))$
$\mathcal{Q} \leftarrow$ priority queue on $V(P)$
$\mathcal{T} \leftarrow$ empty bin. search tree
while $\mathcal{Q} \neq \emptyset$ **do**
  $v \leftarrow \mathcal{Q}$.extractMax()
  type $\leftarrow$ type of vertex v
  handleTypeVertex(v)
return DCEL $\mathcal{D}$

doubly-connected edge list:
data structure for planar subdivisions

$(x, y) \prec (x', y') \;:\Leftrightarrow$
$y > y' \;\vee\; (y = y' \wedge x < x')$

An Algorithm

2) Treating merge vertices



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  handleTypeVertex(v)
return DCEL $\mathcal{D}$

handleMergeVertex(vertex v**)**
$e \leftarrow$ edge following v cw
if helper(e) merge vtx **then**
  $\mathcal{D}$.insert(diag(v, helper(e)))
$\mathcal{T}$.delete(e)
$e' \leftarrow \mathcal{T}$.edgeLeftOf($v$)
if helper(e') merge vtx **then**
  $\mathcal{D}$.insert(diag(v, helper(e')))
helper(e') $\leftarrow v$

Analysis

Lemma. makeMonotone() adds a set of non-intersecting diagonals to P such that P is partitioned into y-monotone subpolygons.

Lemma. A simple polygon with n vertices can be subdivided into y-monotone polygons in $O(n \log n)$ time.

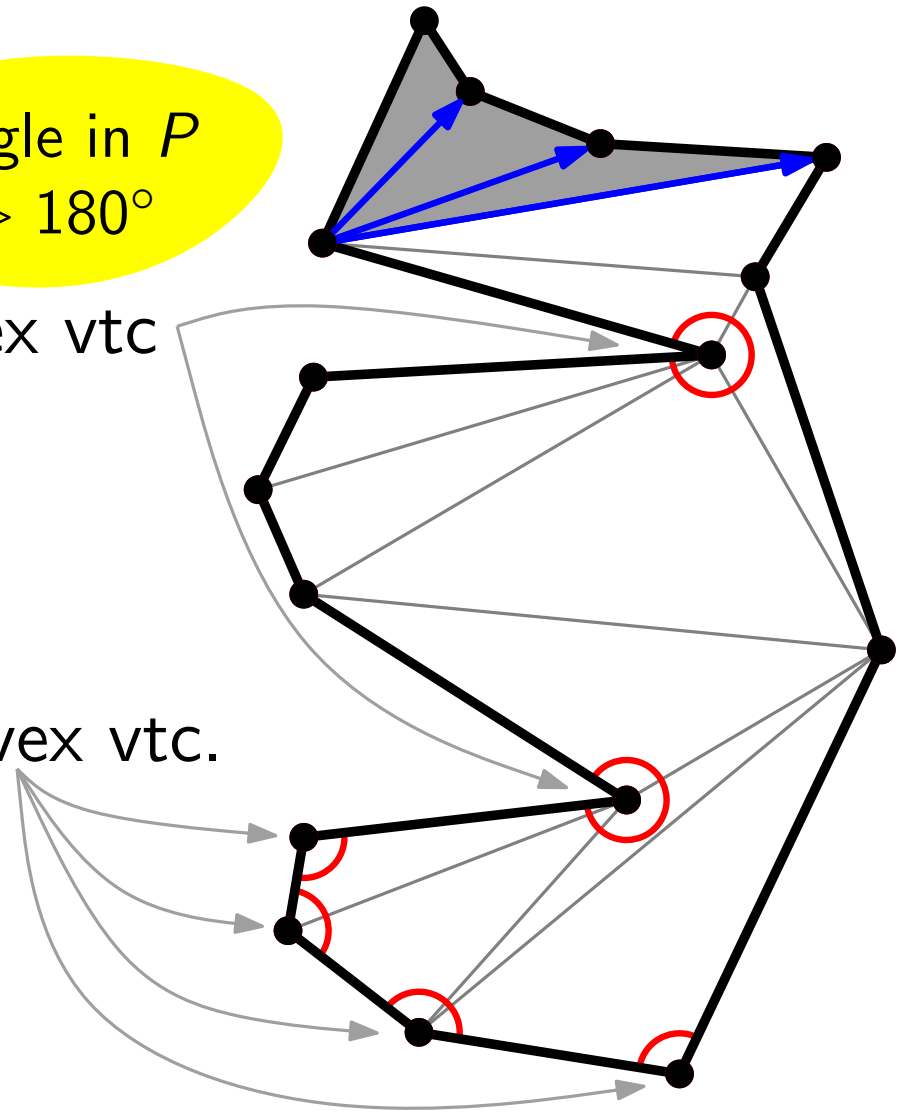
Triangulating a y-Monotone Polygon P

Approach: greedy, going from top to bottom

angle in P $> 180°$

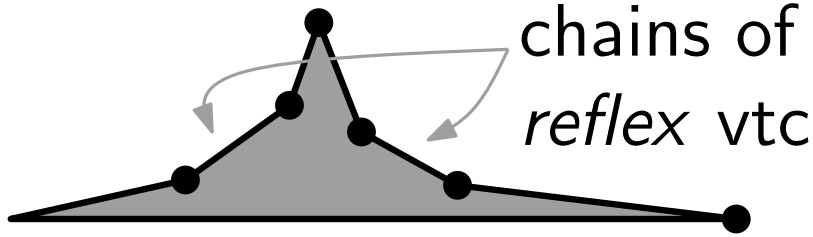
reflex vtc

convex vtc.

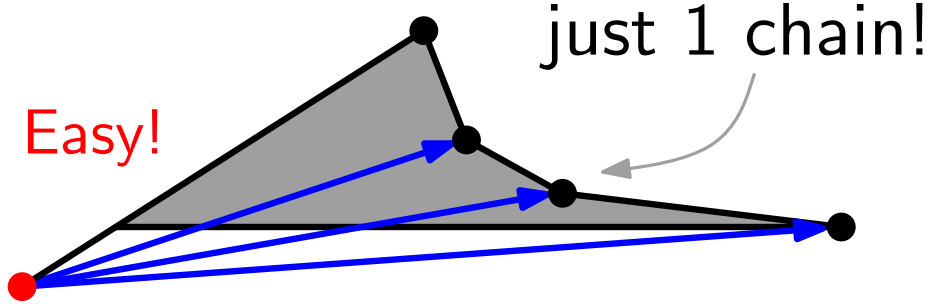


Invariant?

The part of P that we have seen but not yet triangulated is a *funnel*.



Our funnels are special:



Algorithm

Running time? $\Theta(n)$

TriangulateMonotonePolygon(Polygon P as circular vertex list)

merge left and right chain $\rightarrow$ sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$

Stack S; S.push(u_1); S.push(u_2)

for $j \leftarrow 3$ **to** $n-1$ **do**

  if u_j and S.top() lie on different chains **then**

    while not S.empty() **do**

      $v \leftarrow S$.pop()

      if not S.empty() **then** draw diag. (u_j, v)

    S.push(u_{j-1}); S.push(u_j)

  else

    $v \leftarrow S$.pop()

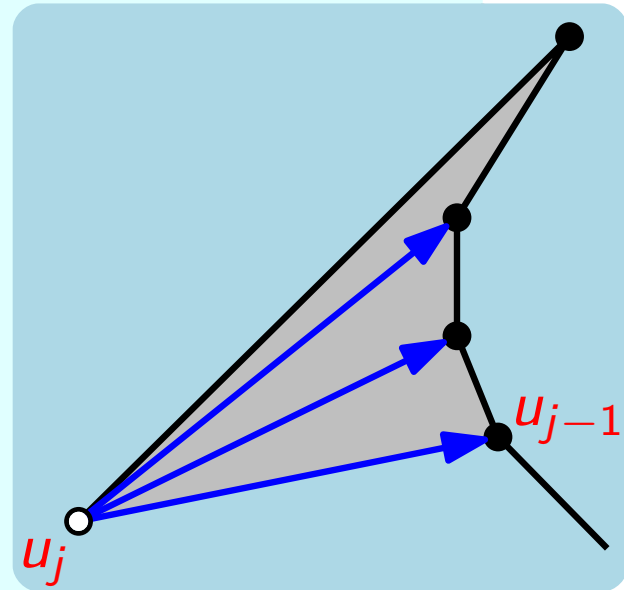
    while not S.empty() **and** u_j sees S.top() **do**

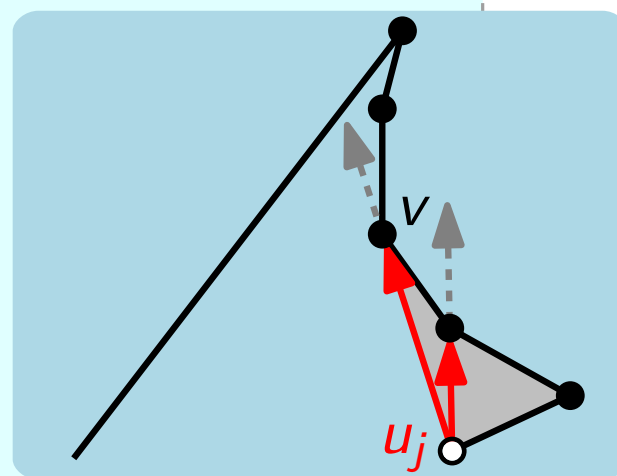
      $v \leftarrow S$.pop()

      draw diagonal (u_j, v)

    S.push(v); S.push(u_j)

draw diagonals from u_n to all vtc on S except first and last one





Summary

Lemma. A simple polygon with n vertices can be subdivided into y-monotone polygons in $O(n \log n)$ time.

old

Lemma. A y-monotone polygon with n vertices can be triangulated in $O(n)$ time.

new

Lemma. Subdividing a simple polygon with n vertices by drawing d (pairwise non-crossing) diagonals yields $d+1$ simple polygons of total complexity $O(n)$.

home-work

Theorem. A simple polygon with n vertices can be triangulated in $O(n \log n)$ time.

Is this it?

Tarjan & van Wyk [1988]:	$O(n \log \log n)$	Kirkpatrick, Klawe, Tarjan [1992]
Clarkson, Tarjan, van Wyk [1989]:	$O(n \log^* n)$	Seidel [1991]: *randomized*
Chazelle [1991]:	$O(n)$	