Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Computational Geometry

Winter term 2014/15

Triangulating Polygons

or

Guarding Art Galleries

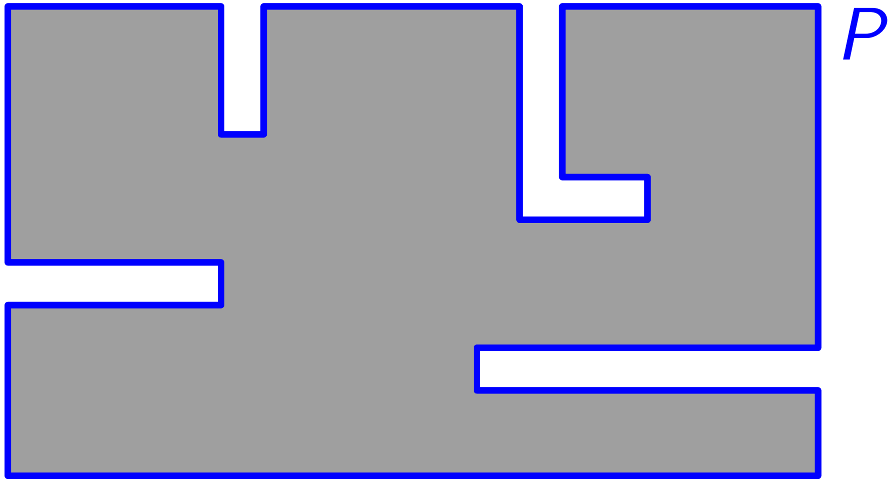
Lecture #3

Prof. Dr. Alexander Wolff

Chair for Informatics I

Guarding an Art Gallery

Given a *simple* polygon P (i.e., no holes, no self-intersection)...



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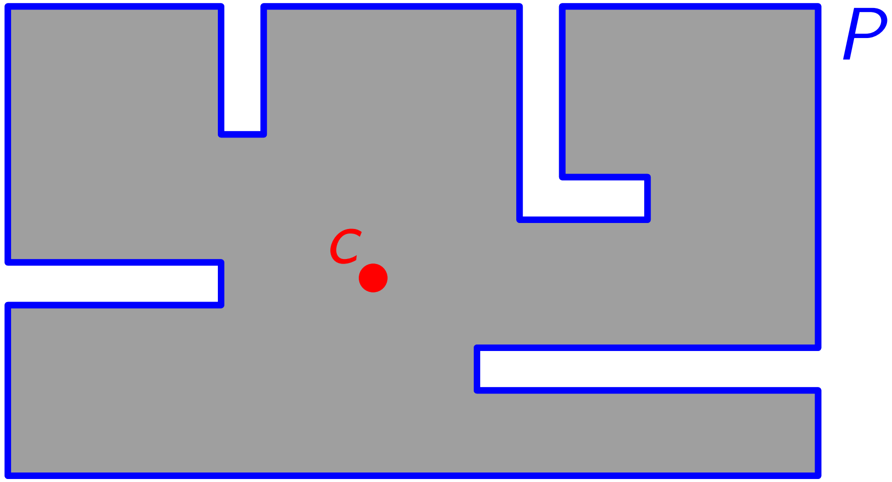

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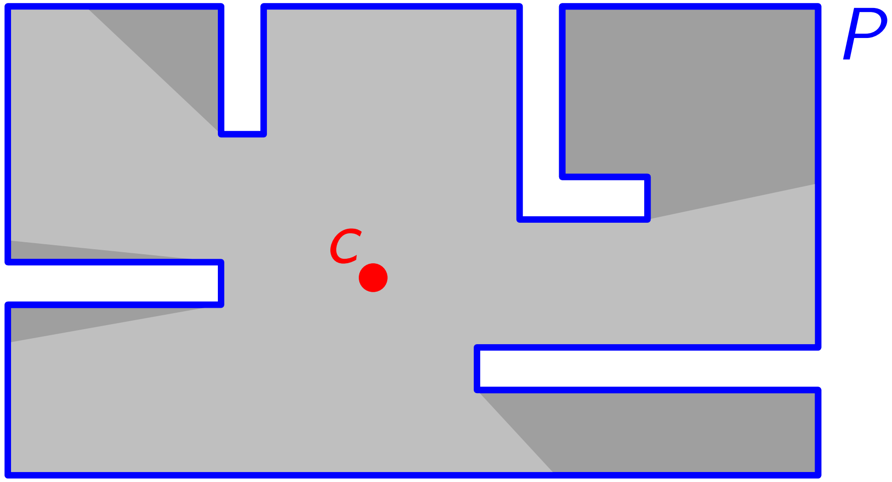

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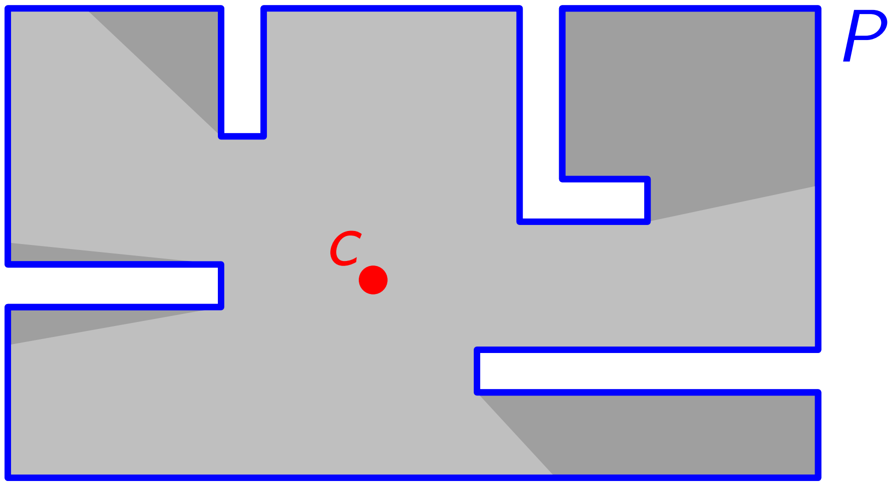

Observation. Camera c "sees" a star-shaped region

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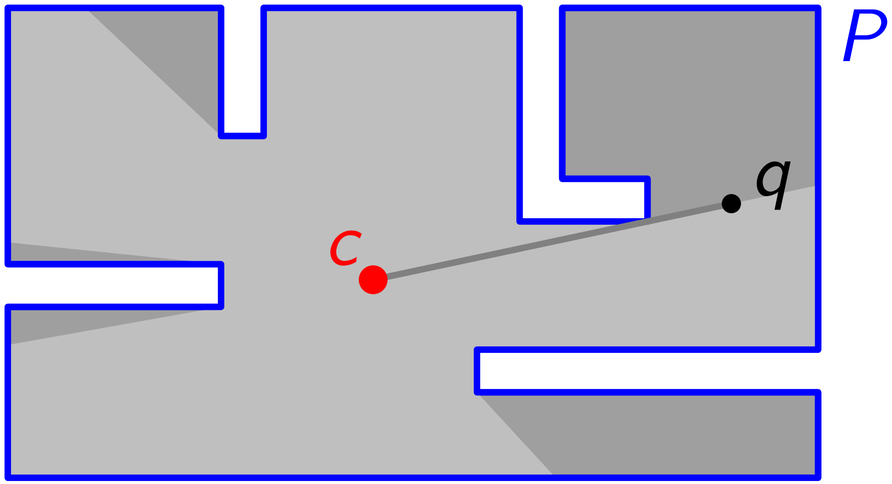

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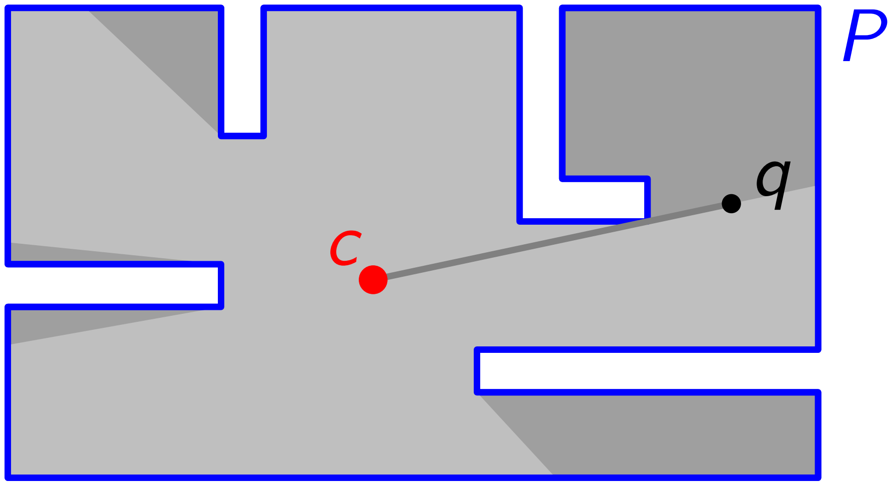

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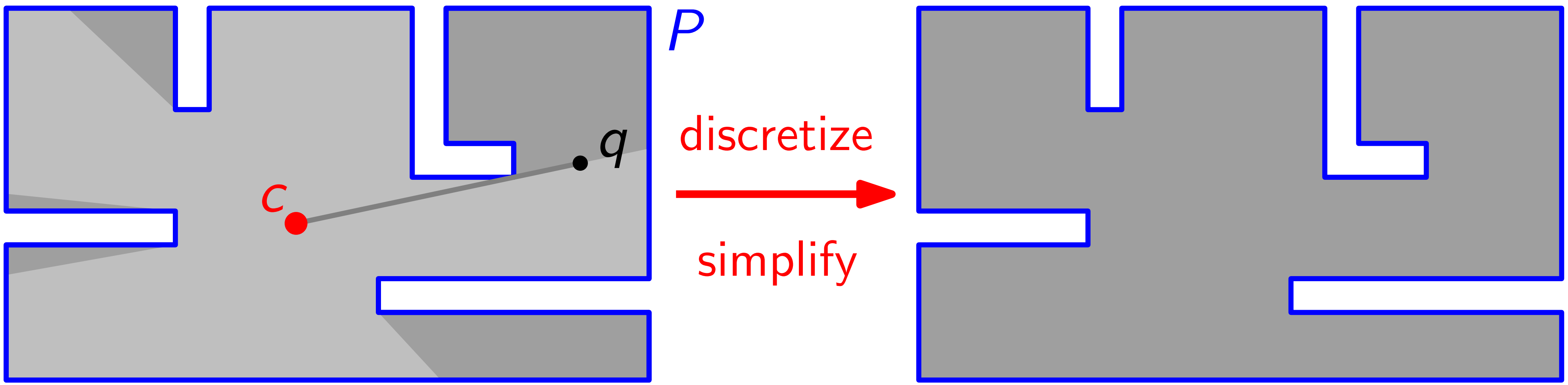

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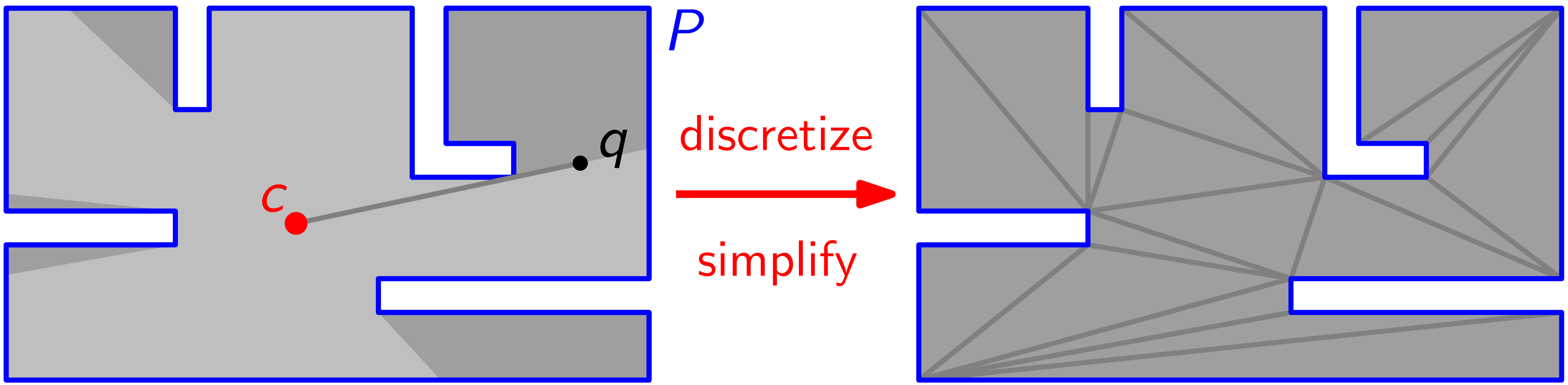

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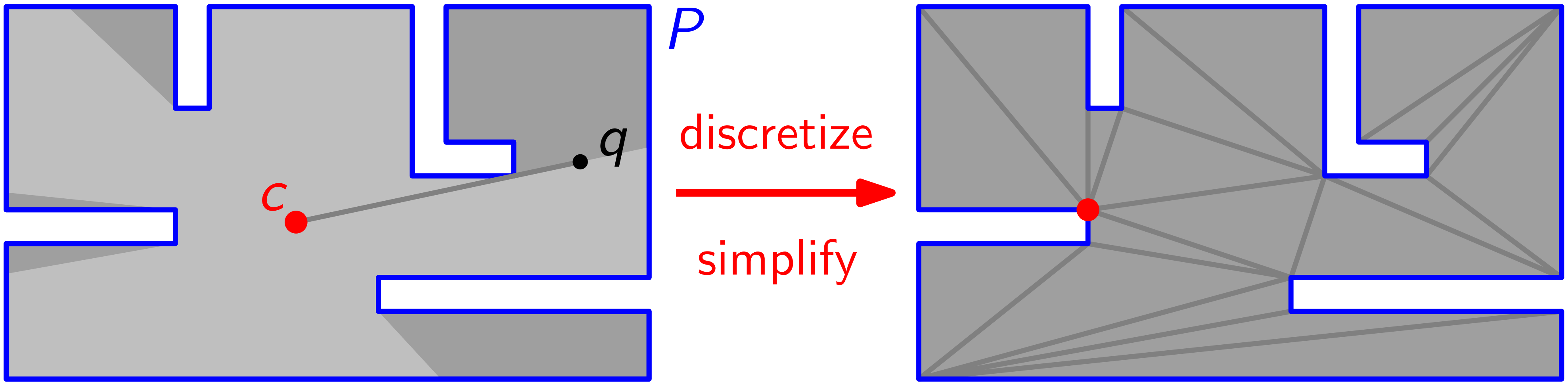

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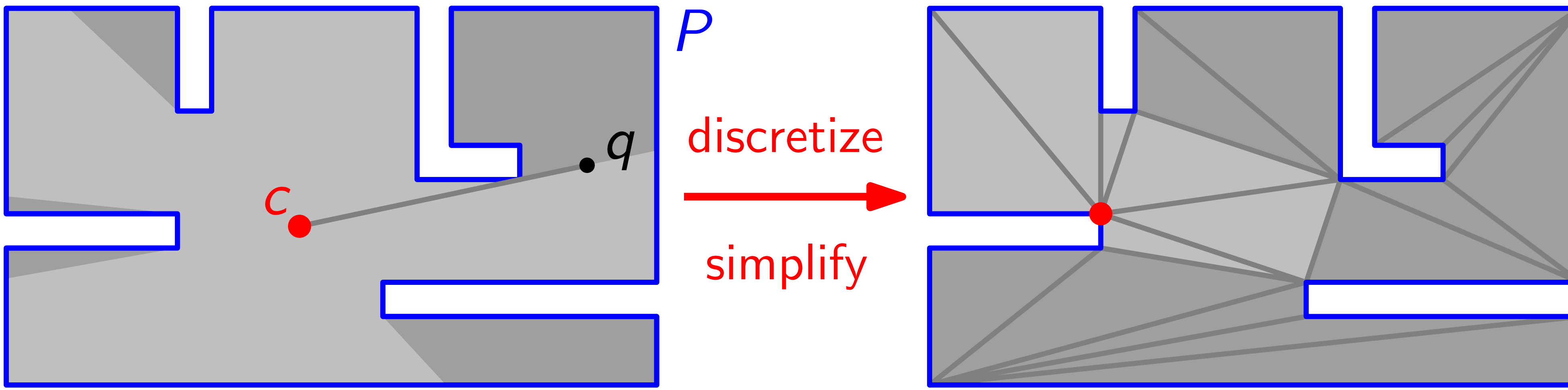

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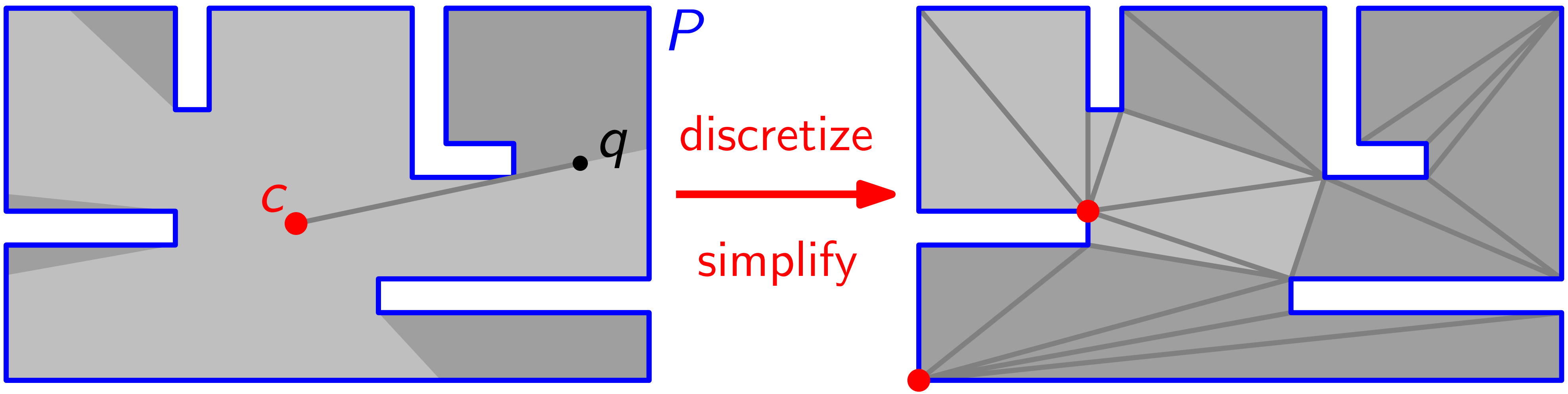

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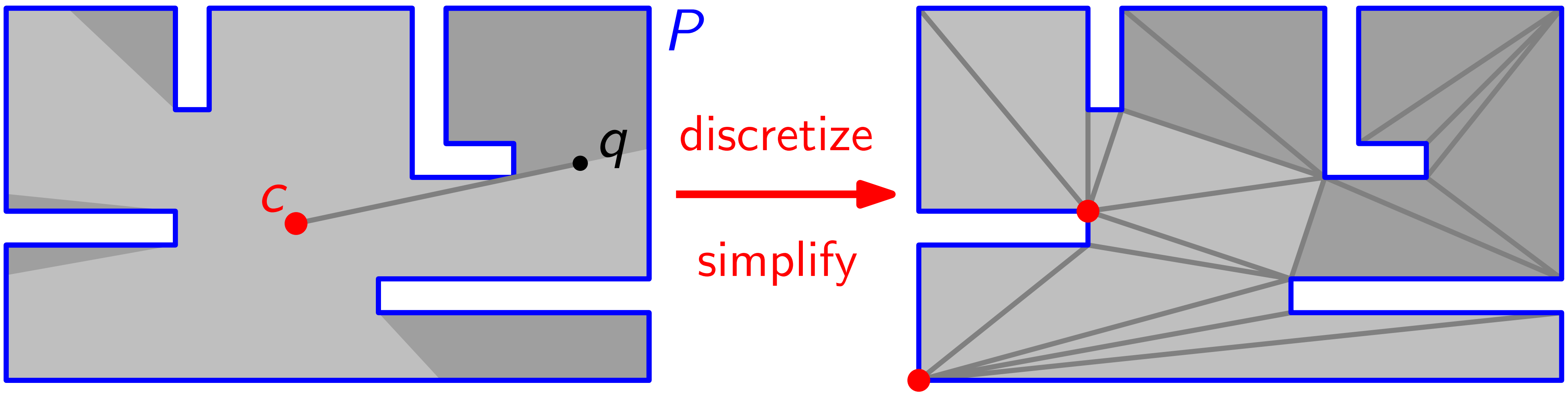

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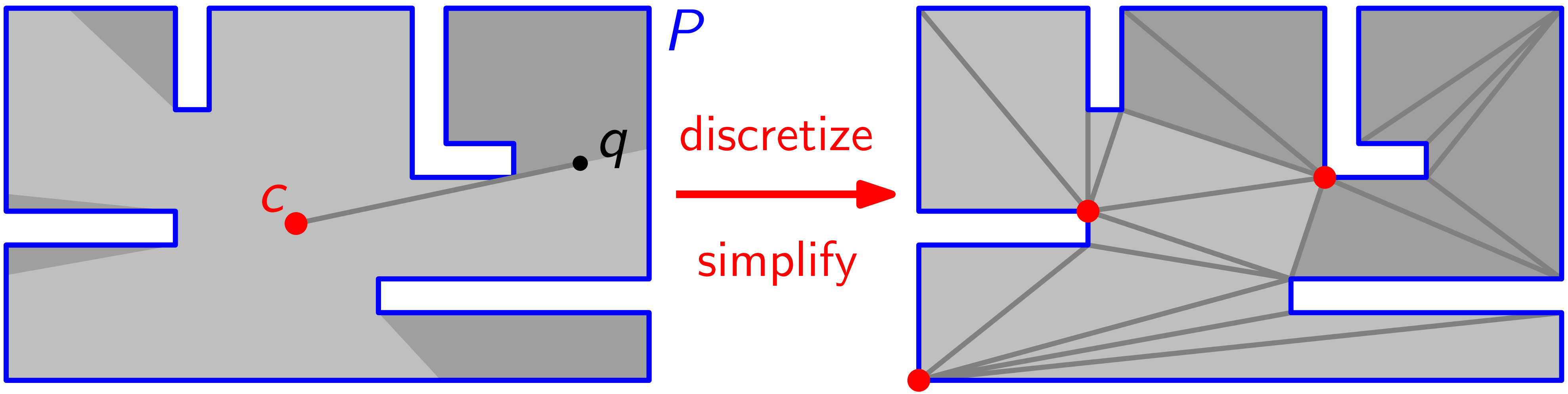

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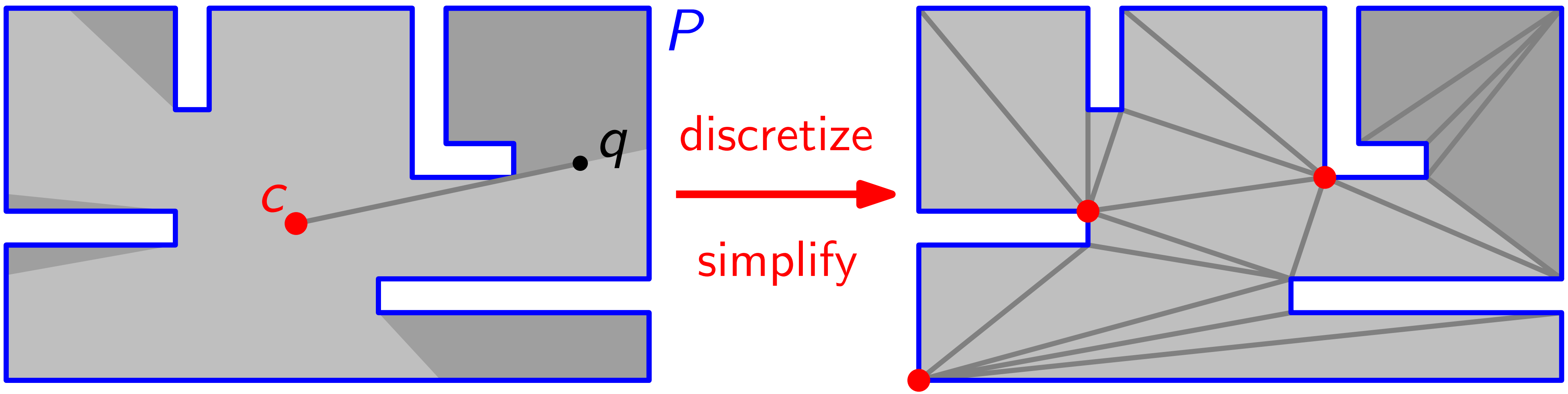

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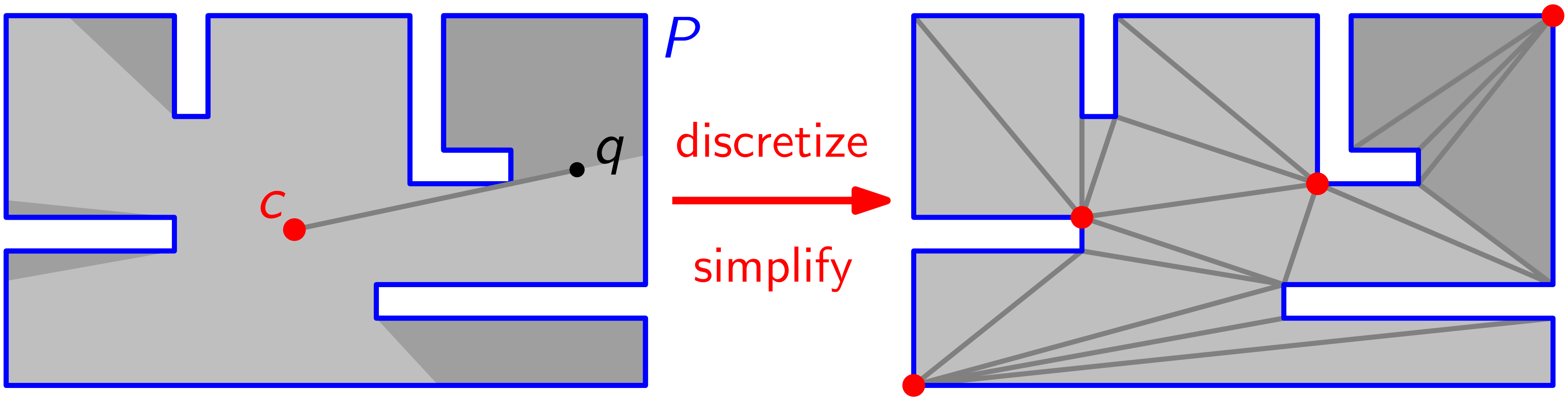

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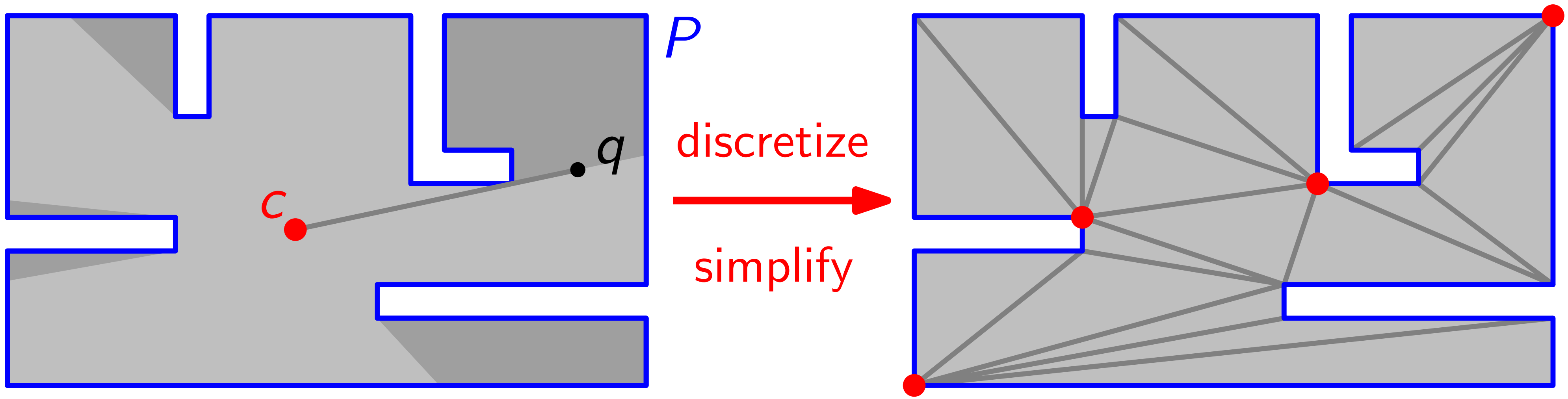

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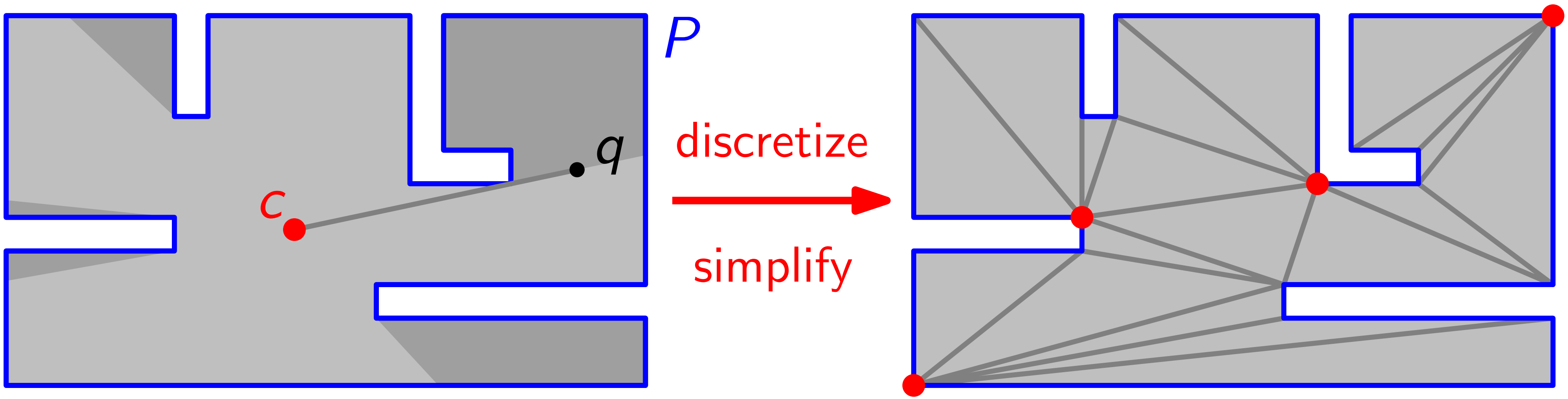

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Theorem. Every simple polygon can be triangulated.

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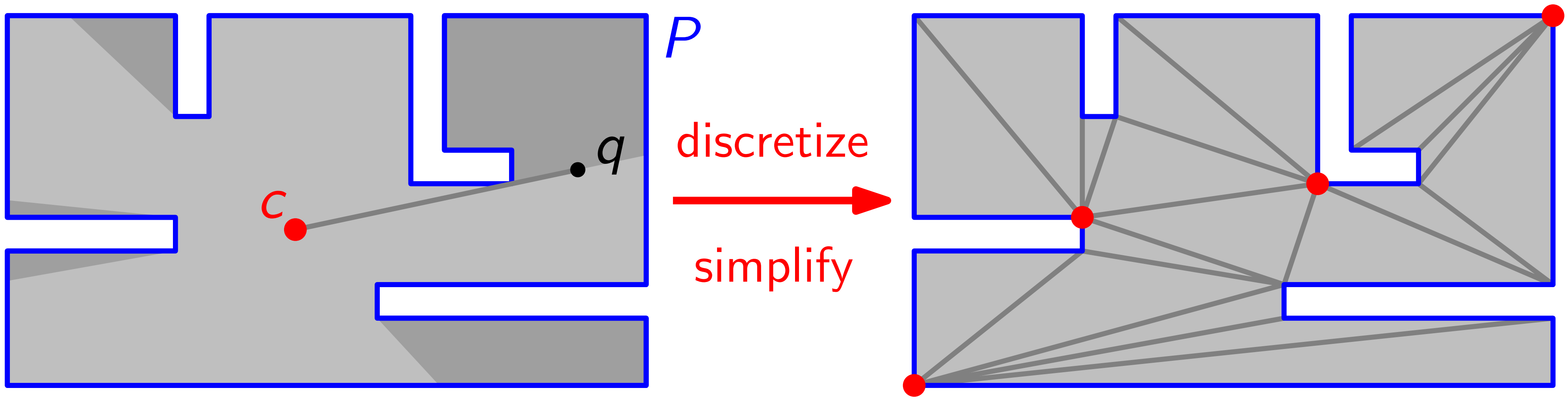

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Any triangulation of a simple polygon with n vertices consists of $n-2$ triangles.

The Art Gallery Theorem [Chvátal '75]

Theorem. For surveilling a simple polygon with n vertices, $\lfloor n/3 \rfloor$ cameras are sometimes necessary and always sufficient.

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Exercise. Find, for arbitrarily large n, a polygon with n vertices, where $\approx n/3$ cameras are necessary.

[2 minutes]

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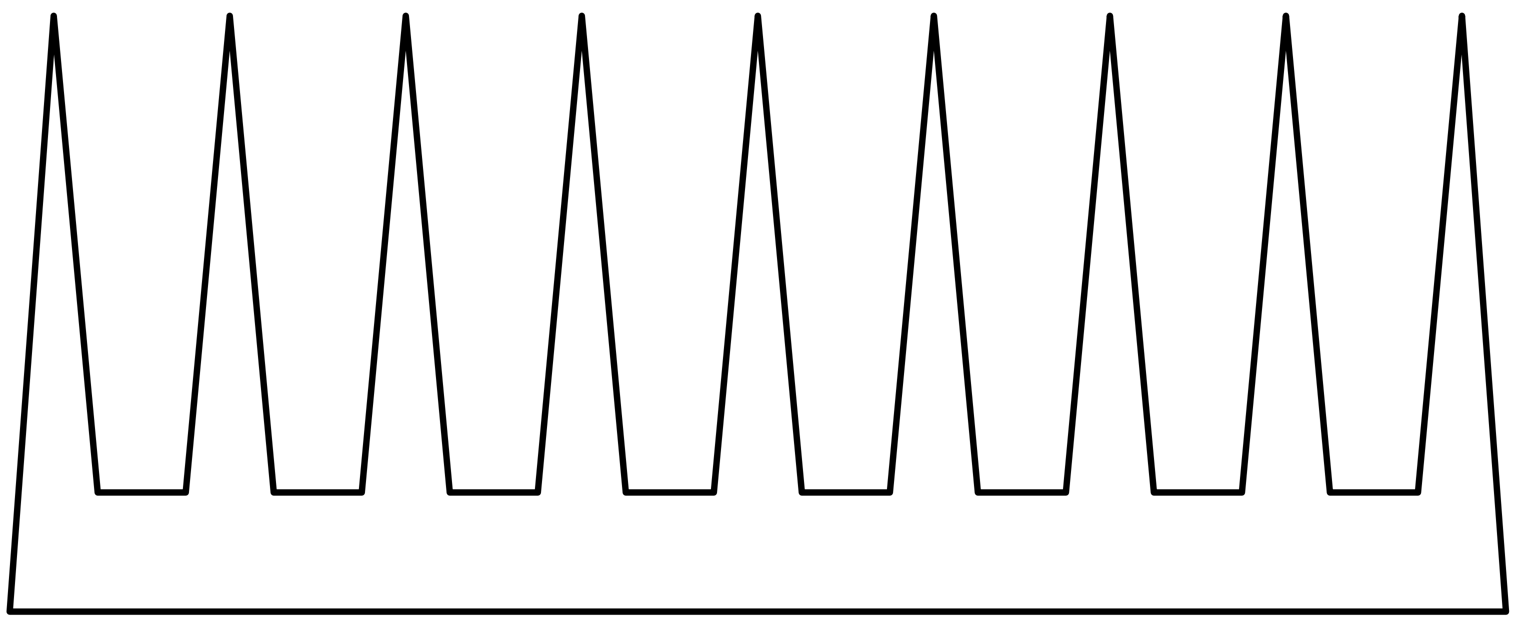

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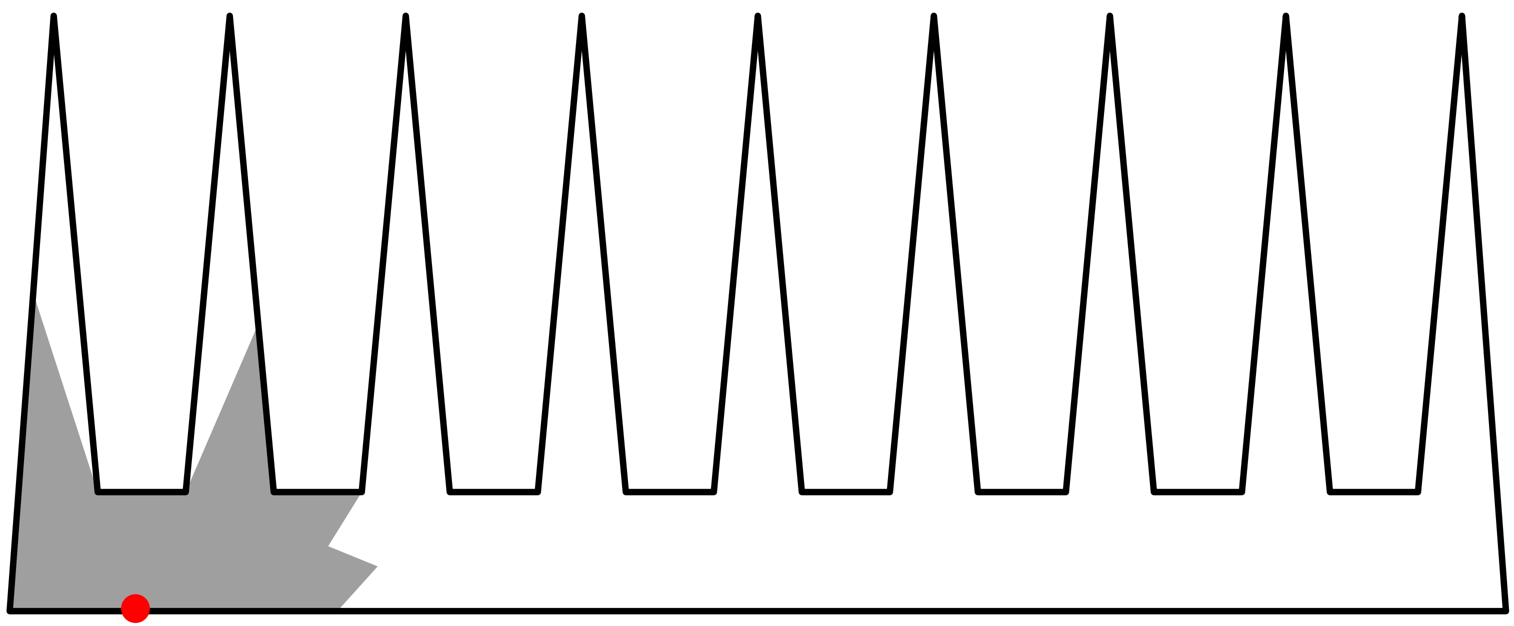

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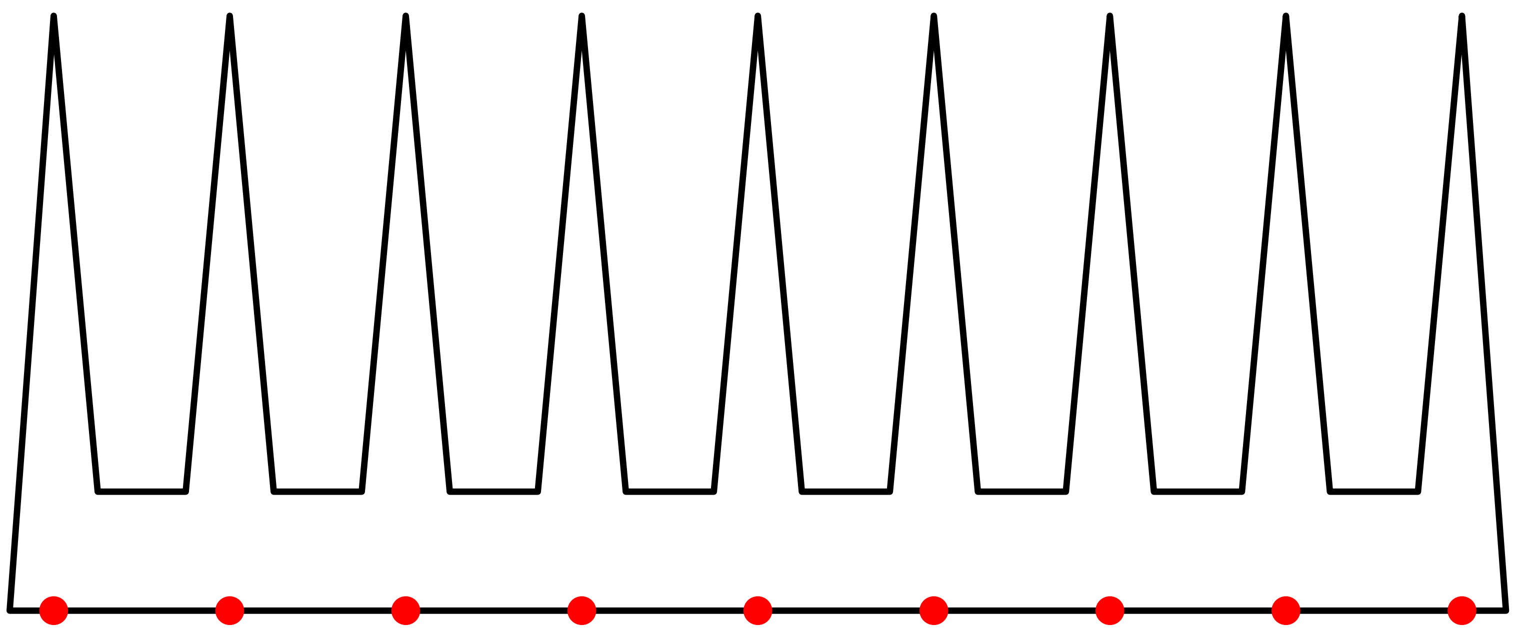

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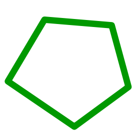

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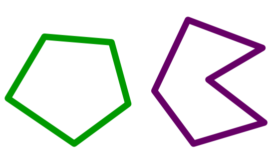

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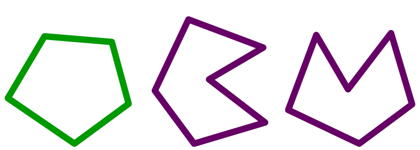

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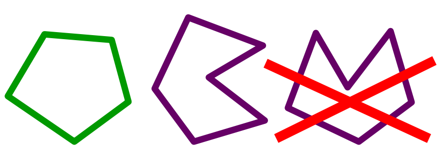

Partitioning a Polygon into Monotone Pieces

Idea: Classify vertices of given simple polygon P

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– *turn* vertices:

– *regular* vertices

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vertical component of walking direction changes

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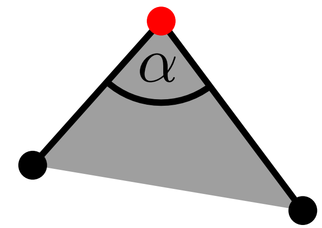

if $\alpha < 180°$

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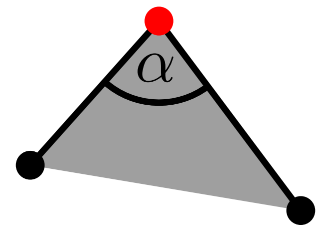
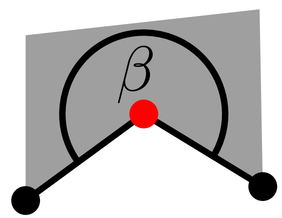


if $\alpha < 180^\circ$

- *split* vertex



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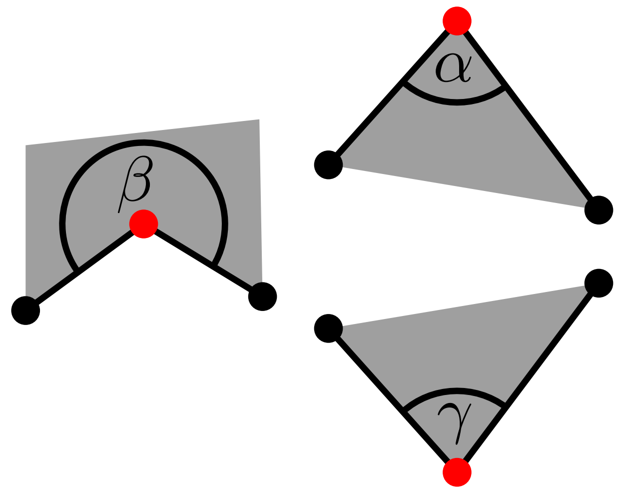

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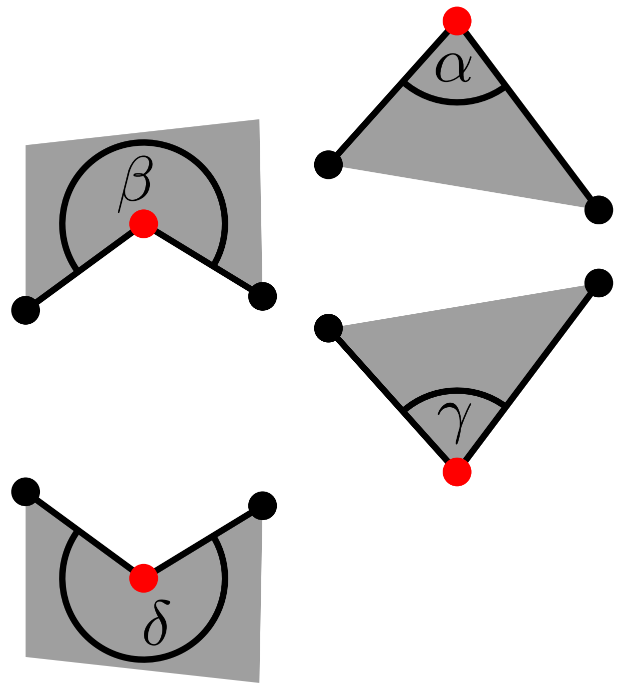

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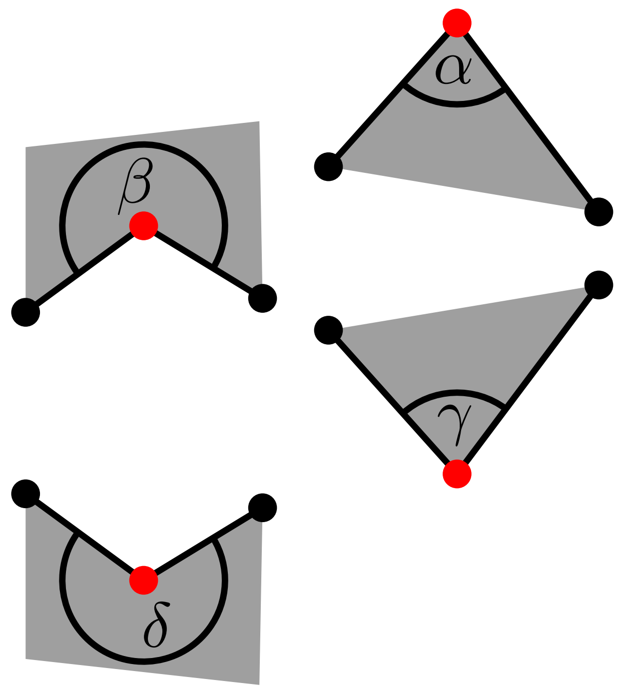
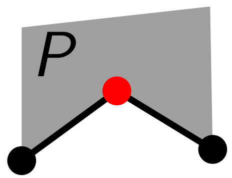
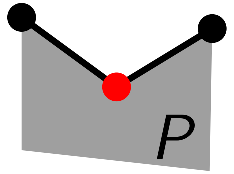


– *regular* vertices

Lemma: Let P be a simple polygon. Then P is y-monotone $\Leftrightarrow$ P has neither split vertices nor merge vertices.

Towards an Algorithm





Idea: Add diagonals to “destroy” split and merge vertices.

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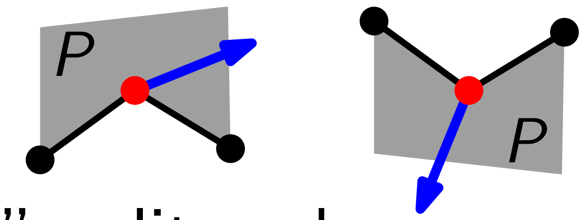

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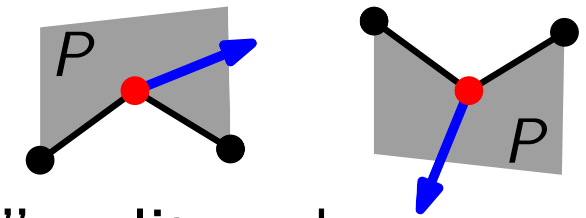
Towards an Algorithm



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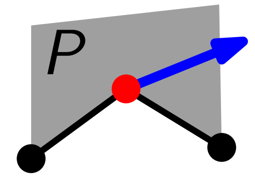
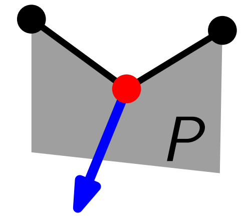
Problem: Diagonals must not cross

Towards an Algorithm

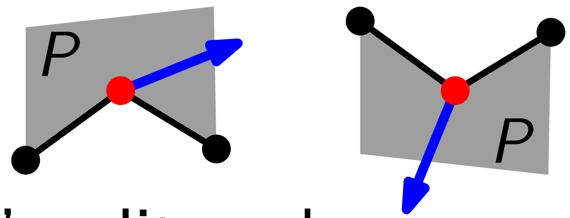




Idea: Add diagonals to “destroy” split and merge vertices.

Problem: Diagonals must not cross
– each other
– edges of P

Towards an Algorithm



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Problem: Diagonals must not cross – each other
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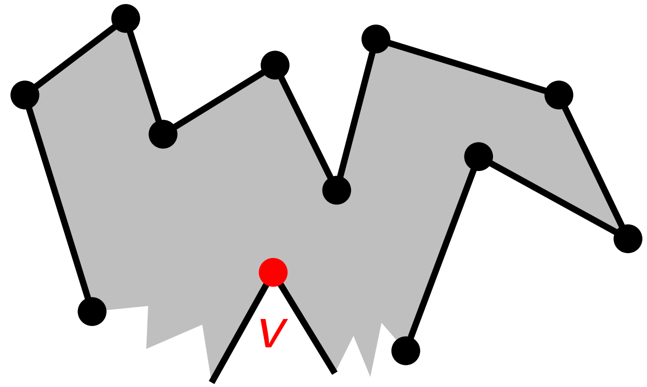
1) Treating split vertices



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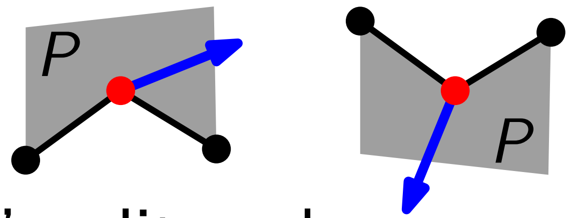

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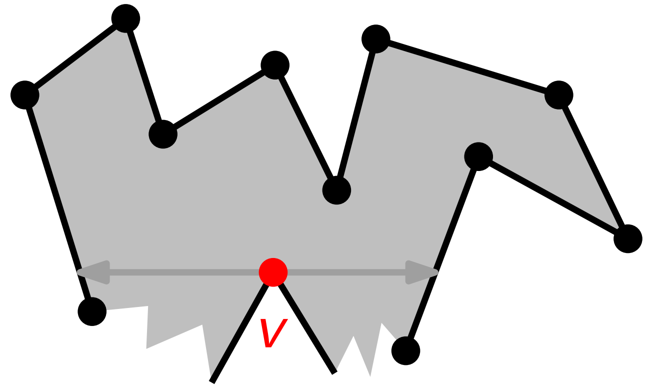

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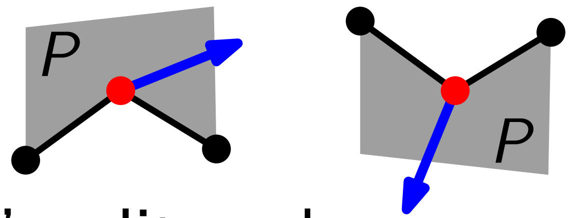

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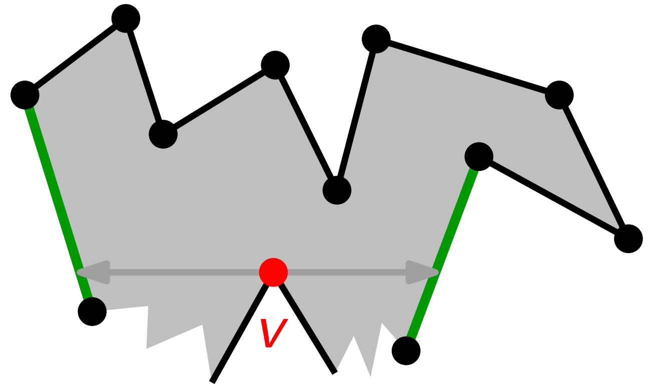

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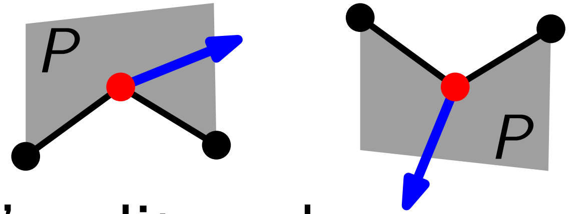

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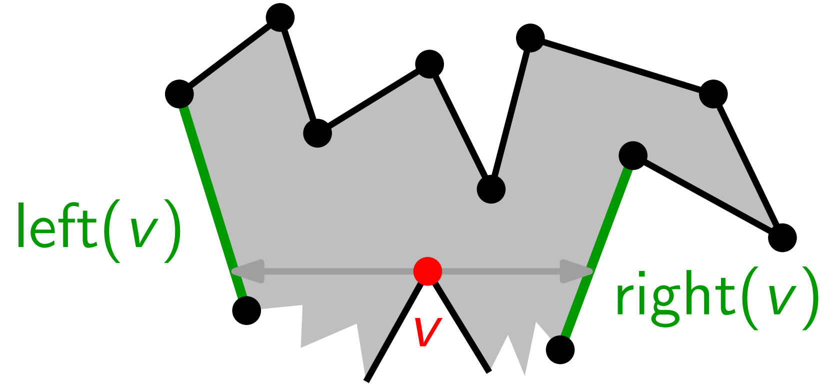
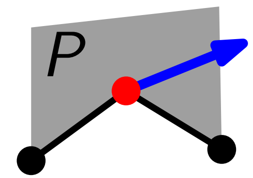
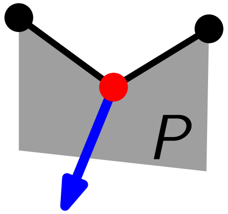


Towards an Algorithm





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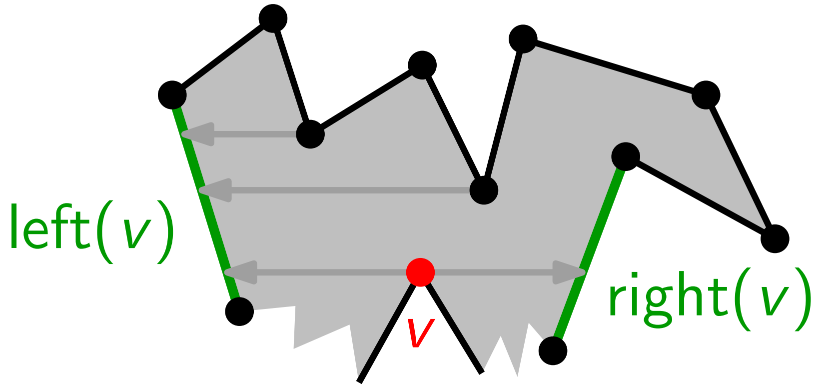

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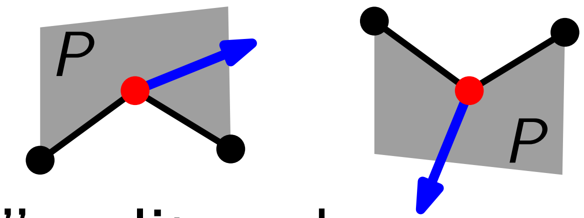

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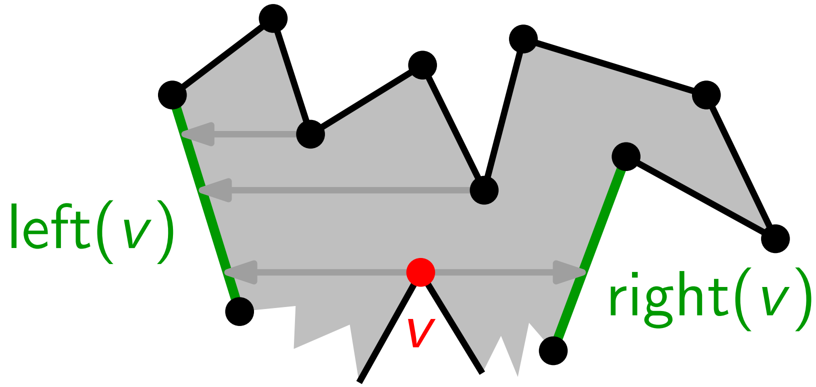
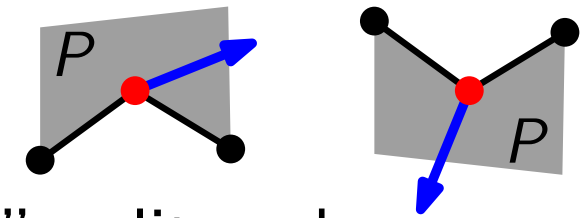


Connect v to vertex $w^\star$ having minimum y-coordinate among all vertices w above v and with $\text{left}(w) = \text{left}(v)$.

Towards an Algorithm



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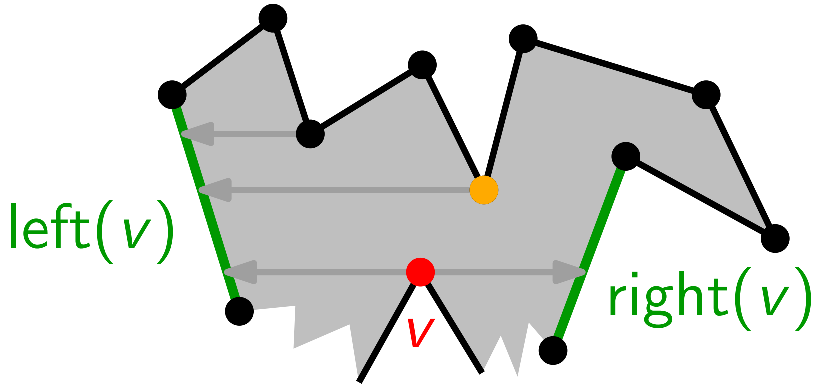

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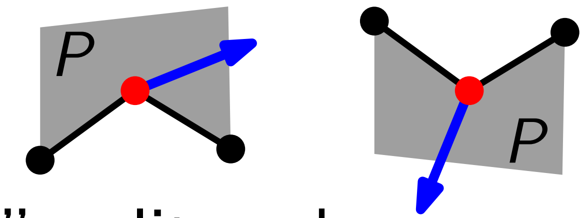

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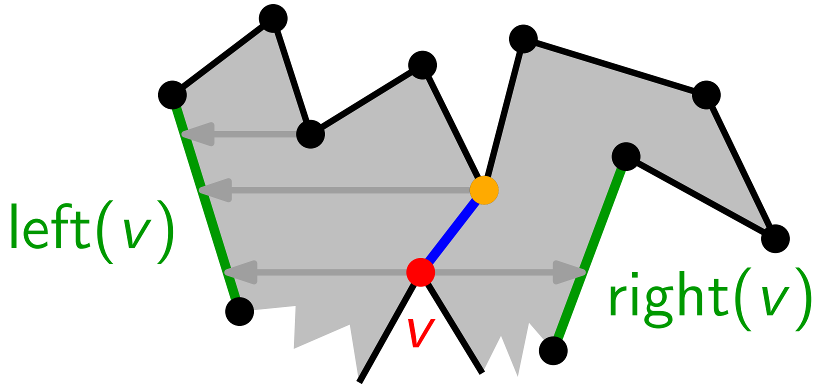

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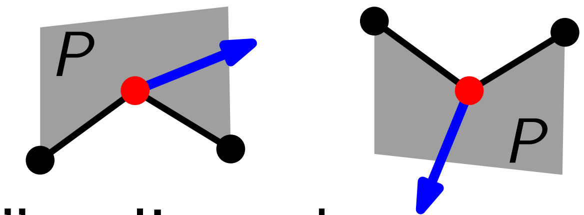

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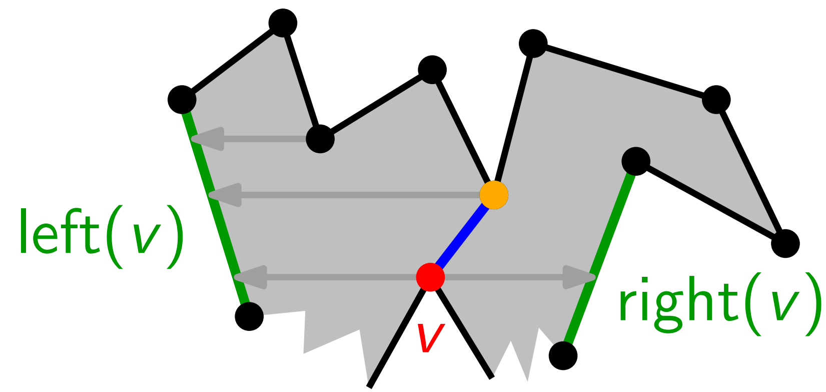

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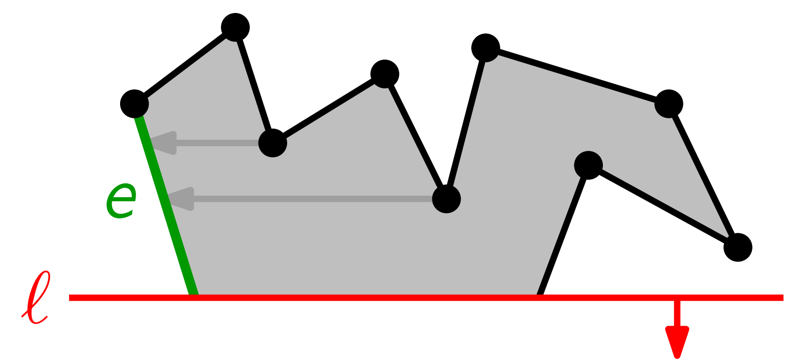

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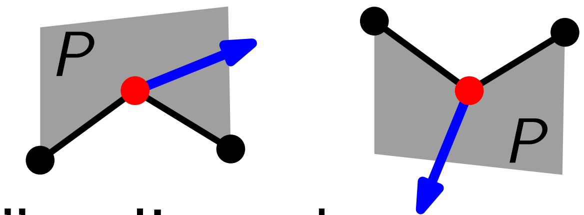

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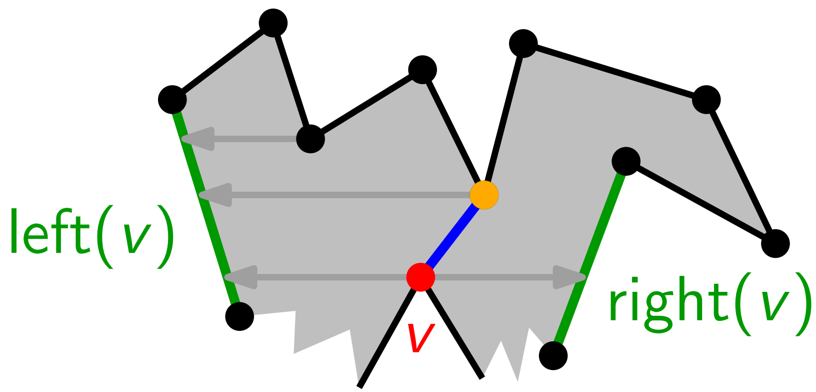

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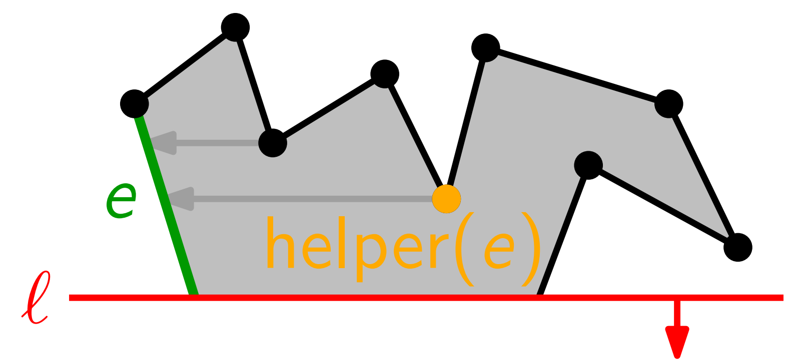
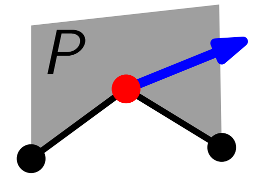
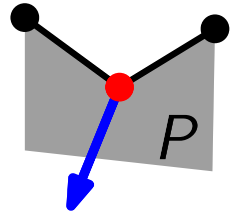


Towards an Algorithm





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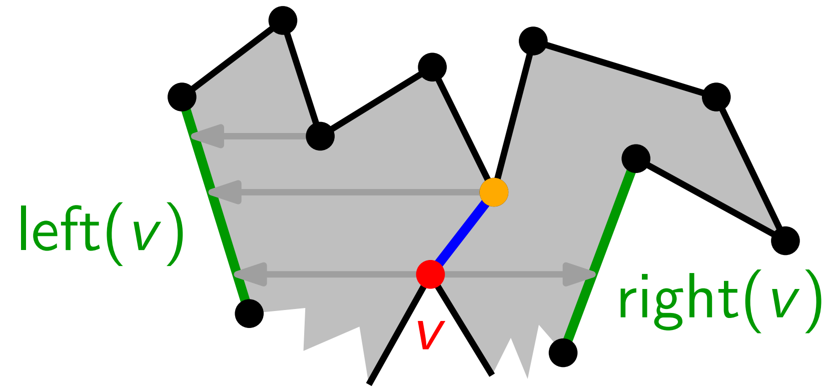

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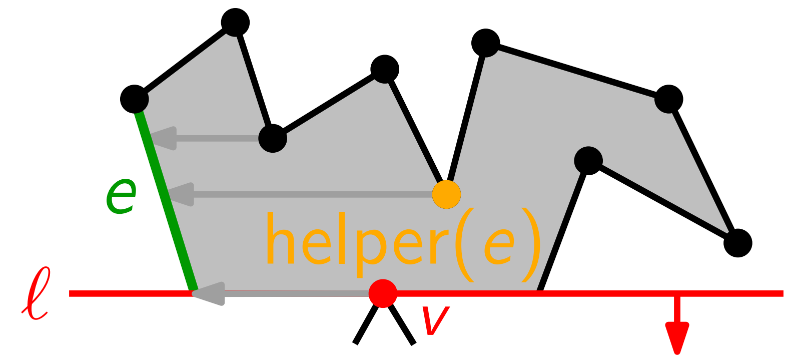

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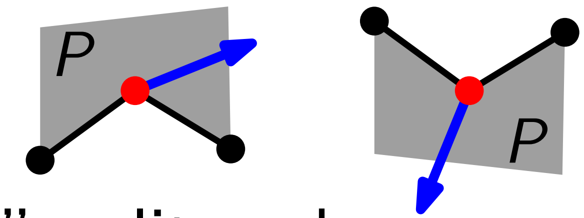

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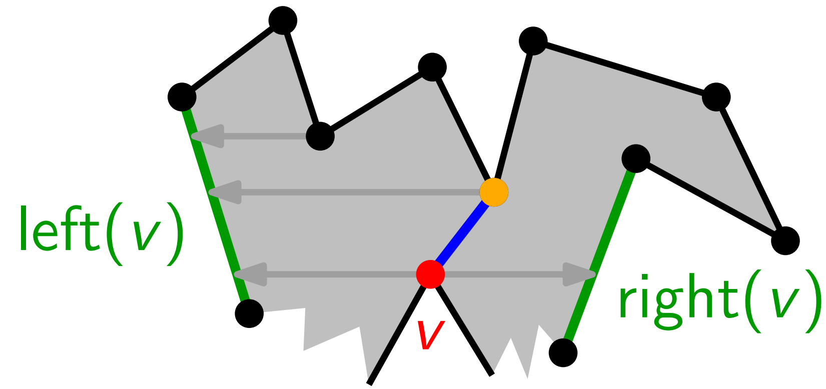

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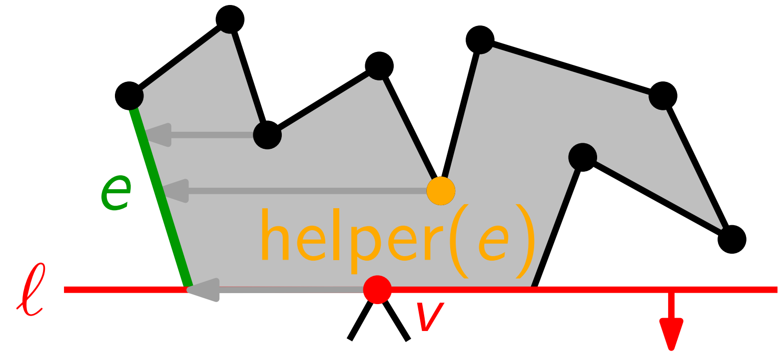
Connect v to helper(left(v)).



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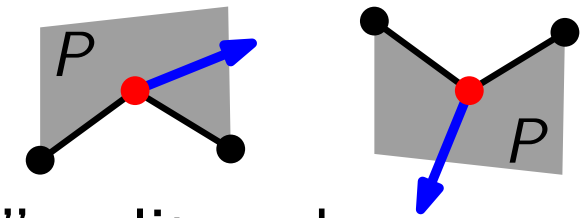

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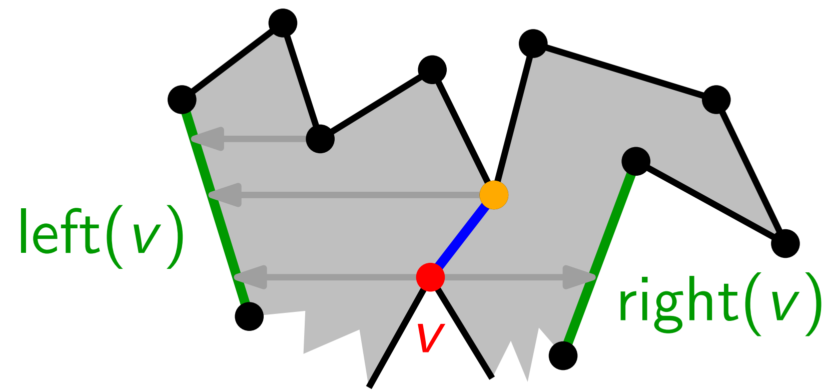

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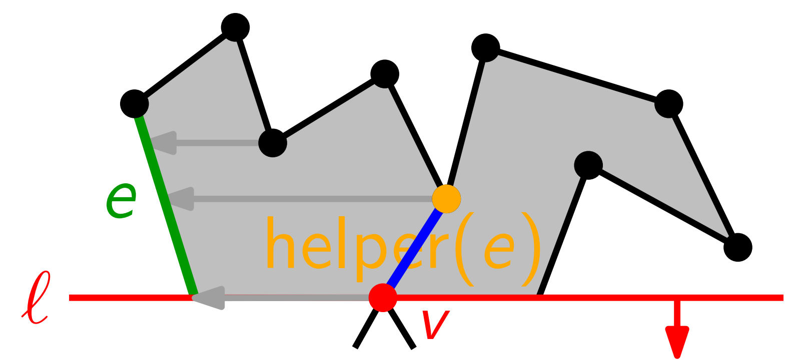
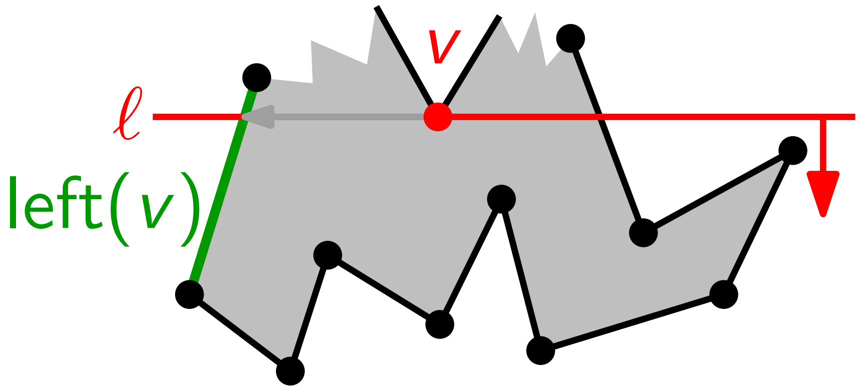


An Algorithm

2) Treating merge vertices



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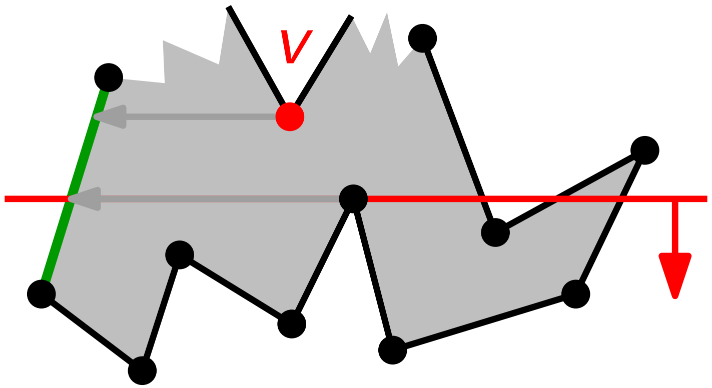

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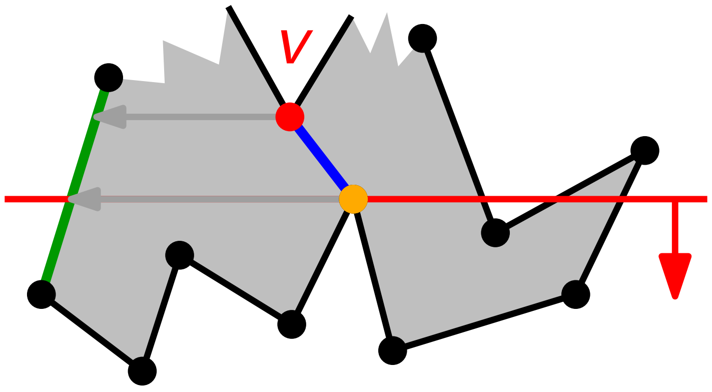

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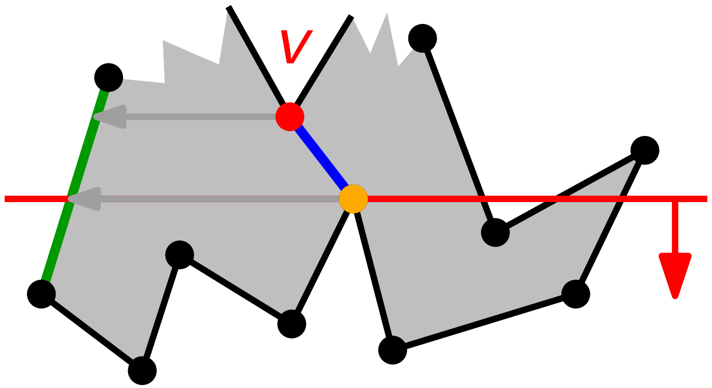

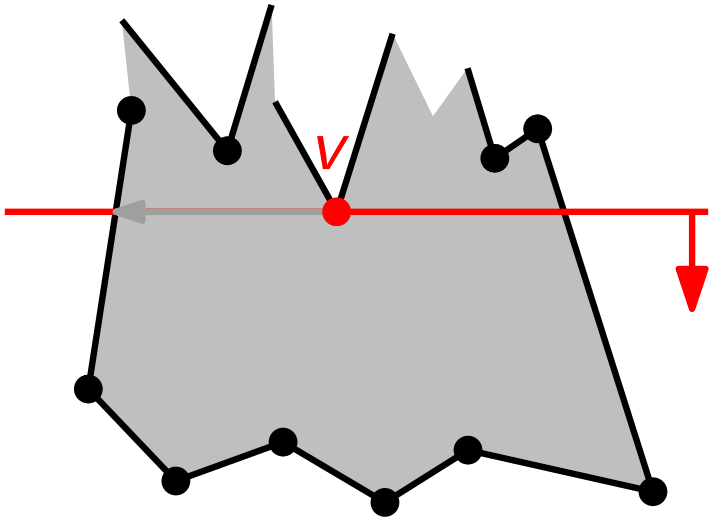

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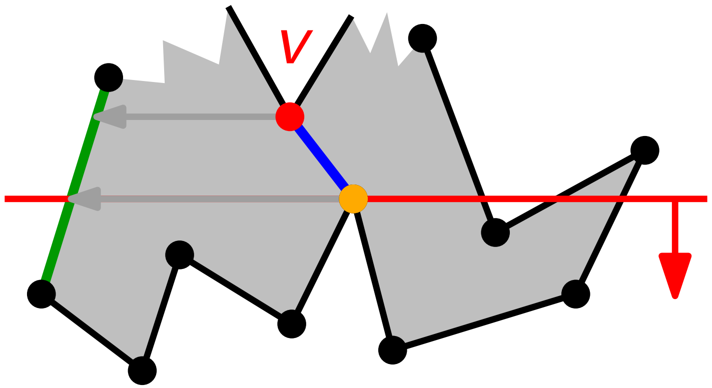

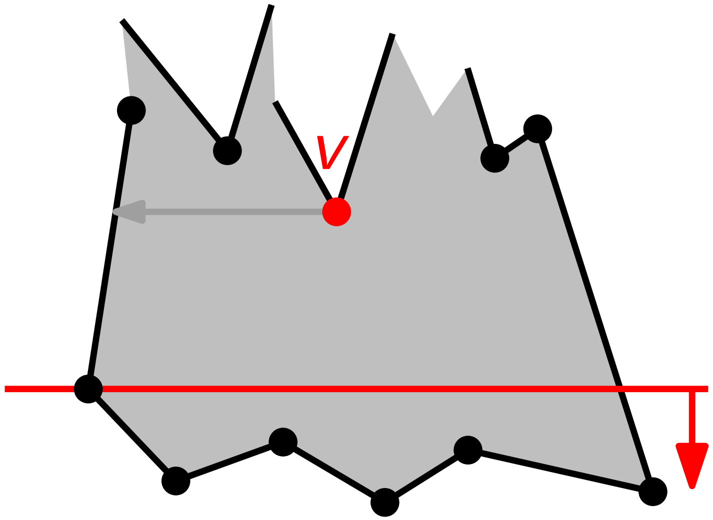

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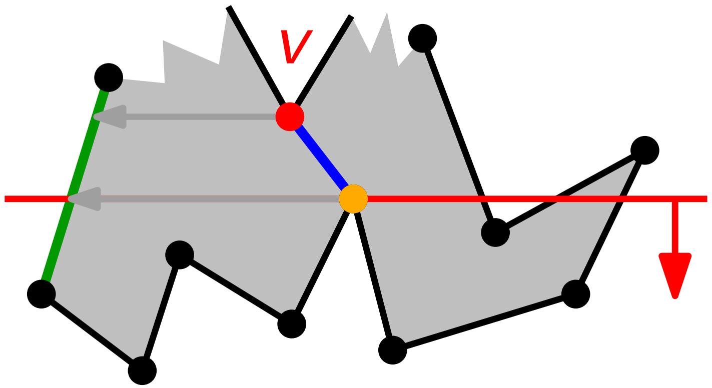

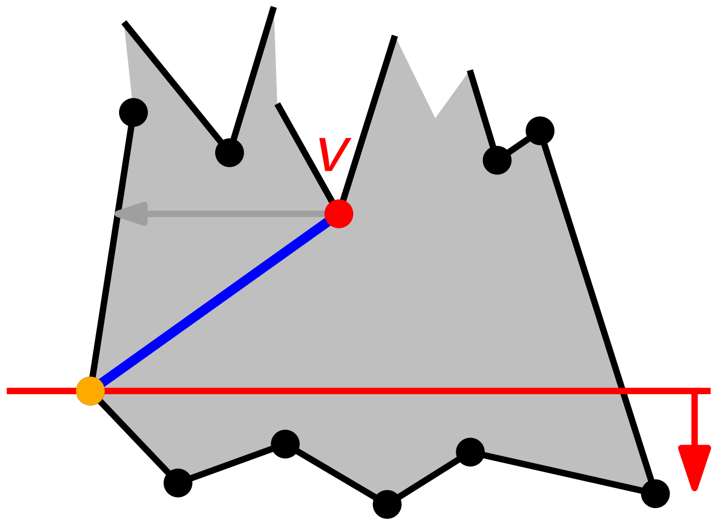

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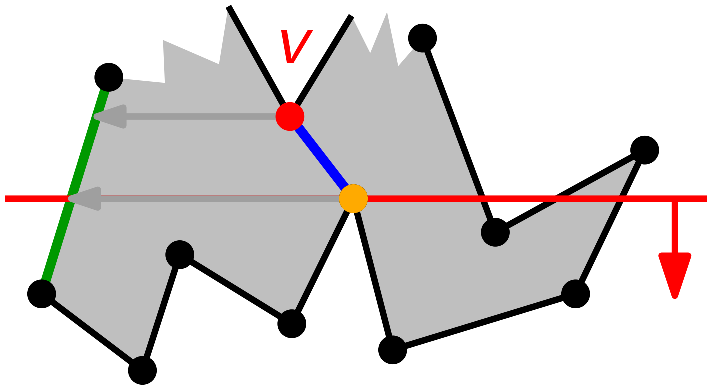

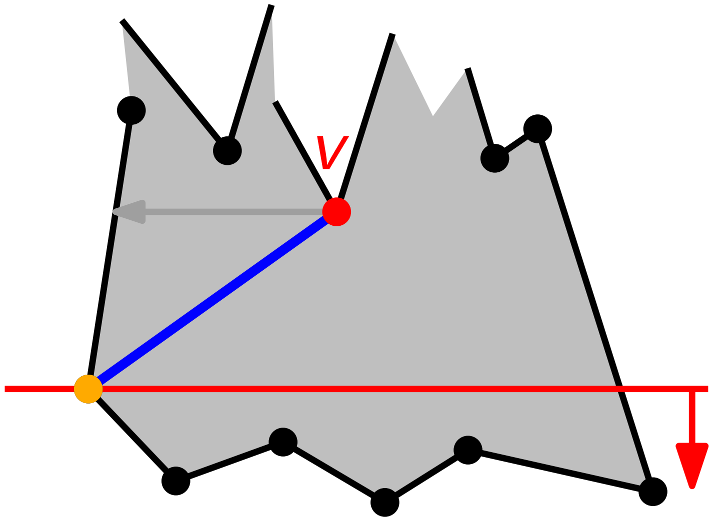

makeMonotone(polygon P**)**
$\mathcal{D} \leftarrow \mathrm{DCEL}(V(P), E(P))$
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An Algorithm

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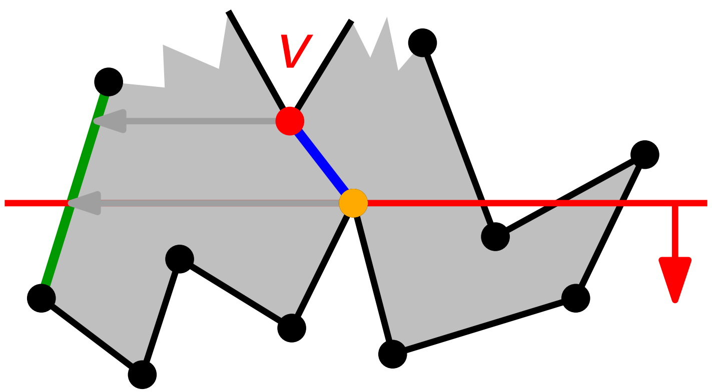

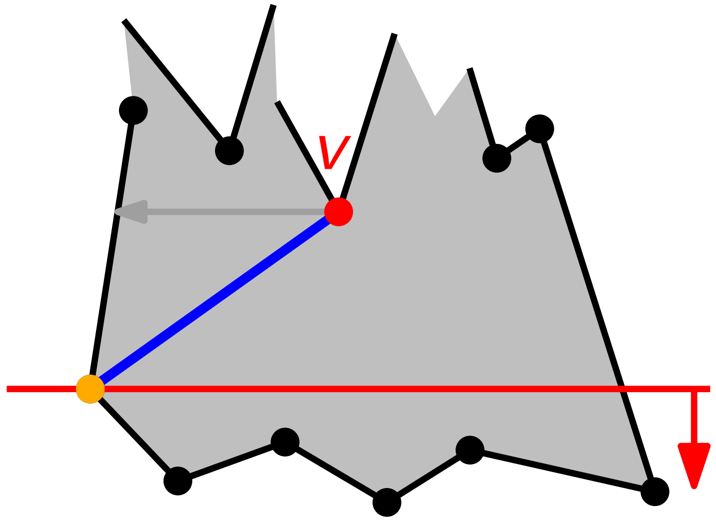

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doubly-connected edge list:
data structure for planar subdivisions

An Algorithm

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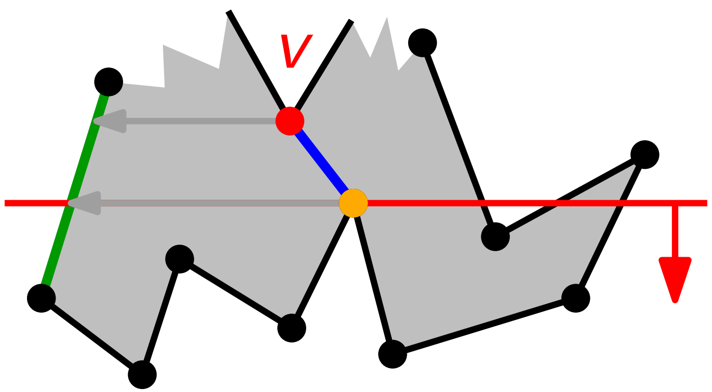

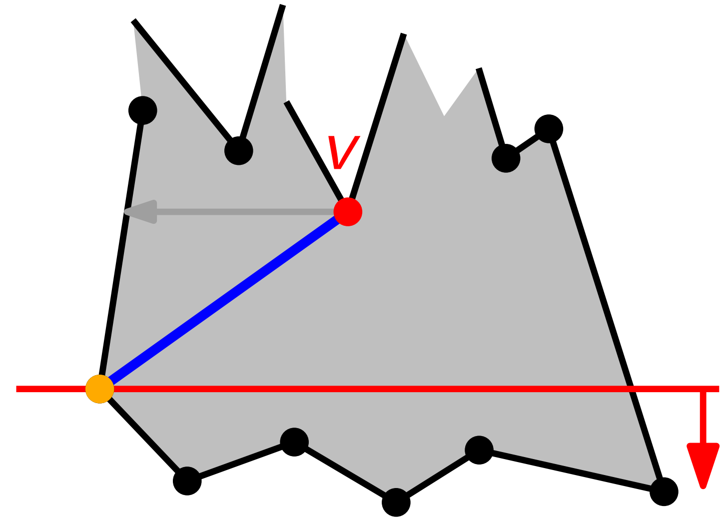

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$(x, y) \prec (x', y') \;:\Leftrightarrow$
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An Algorithm

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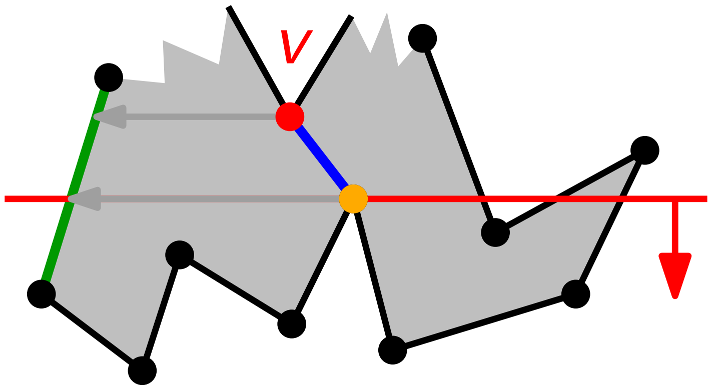

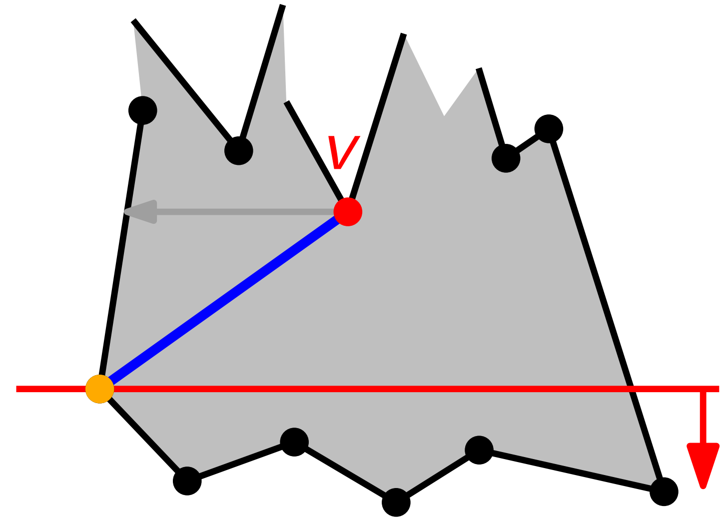

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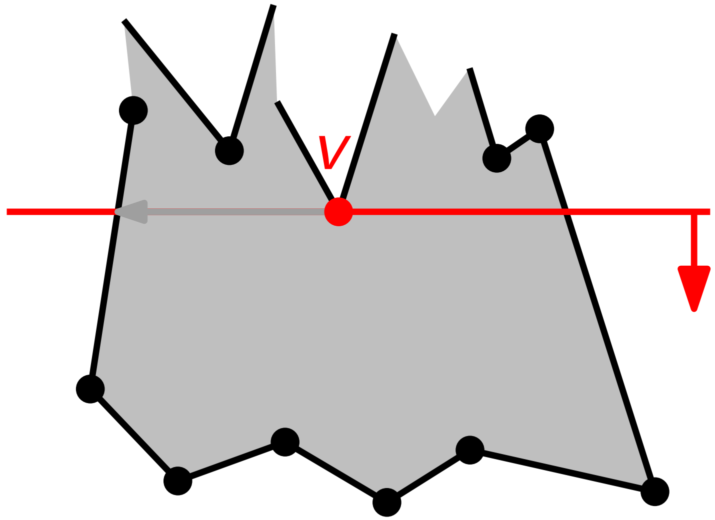

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$e \leftarrow$ edge following v cw
if helper(e) merge vtx **then**
  $\mathcal{D}$.insert(diag(v, helper(e)))
$\mathcal{T}$.delete(e)
$e' \leftarrow \mathcal{T}$.edgeLeftOf($v$)
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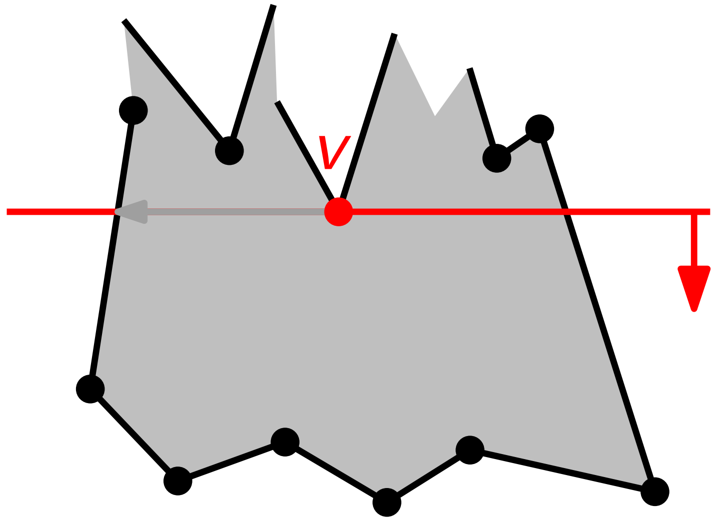
An Algorithm

2) Treating merge vertices



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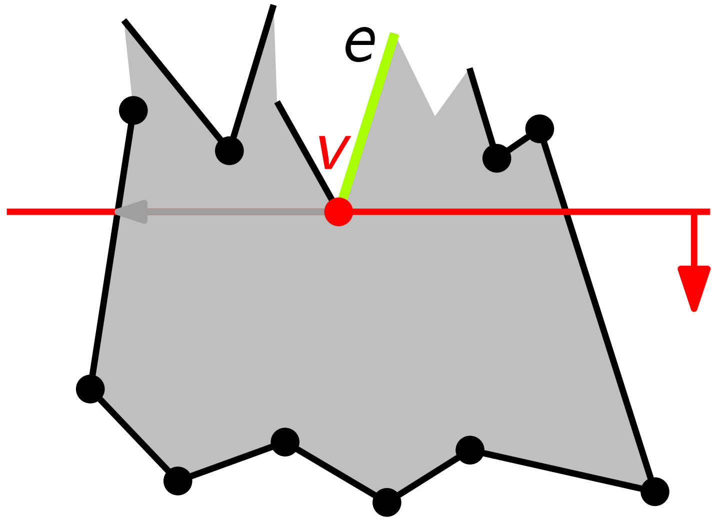

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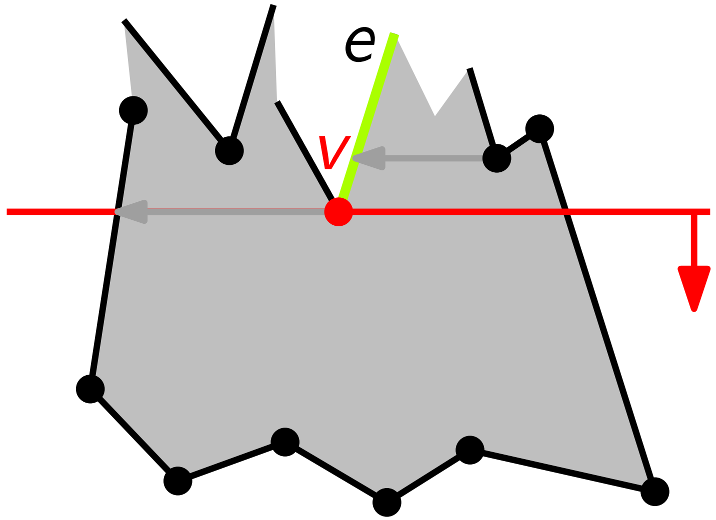
An Algorithm

2) Treating merge vertices



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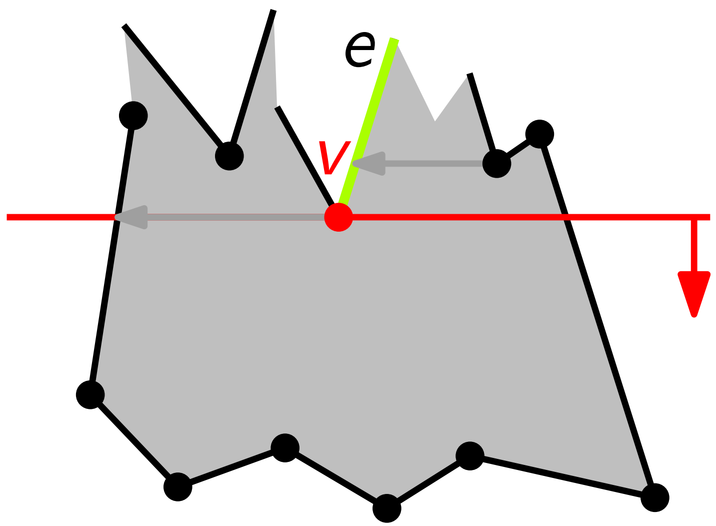

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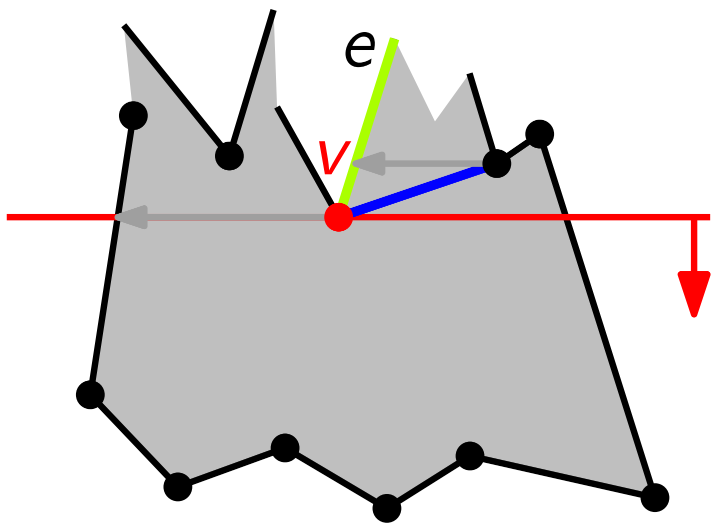

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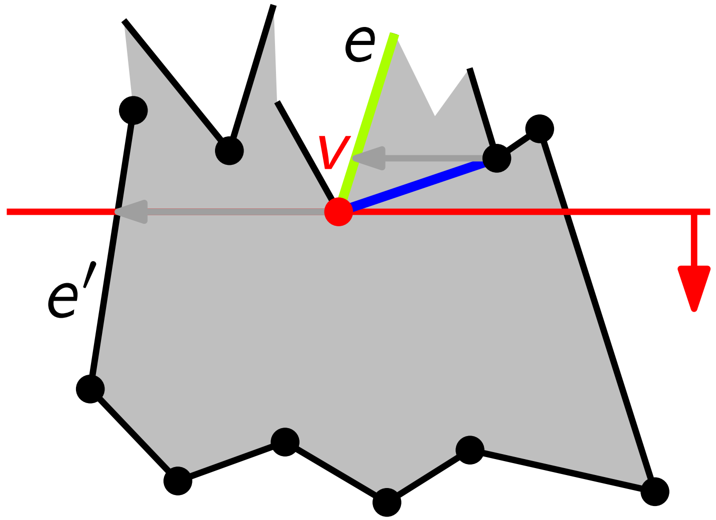

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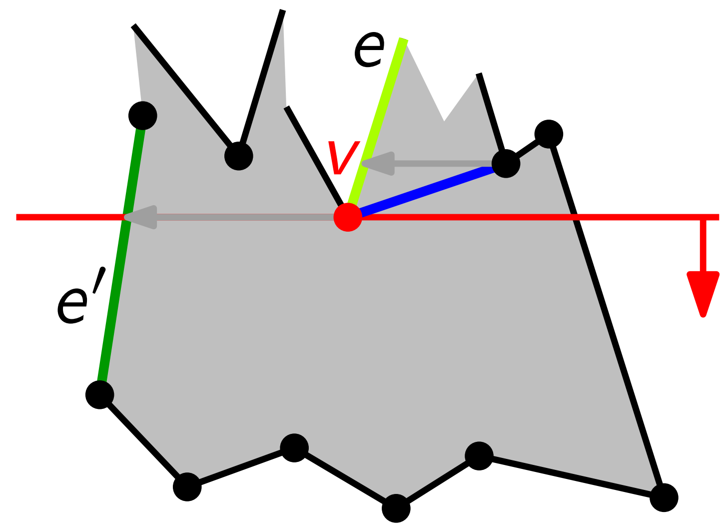

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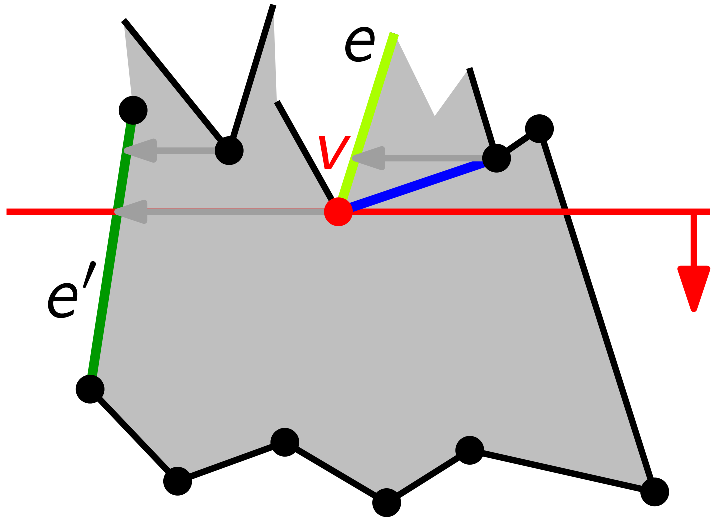

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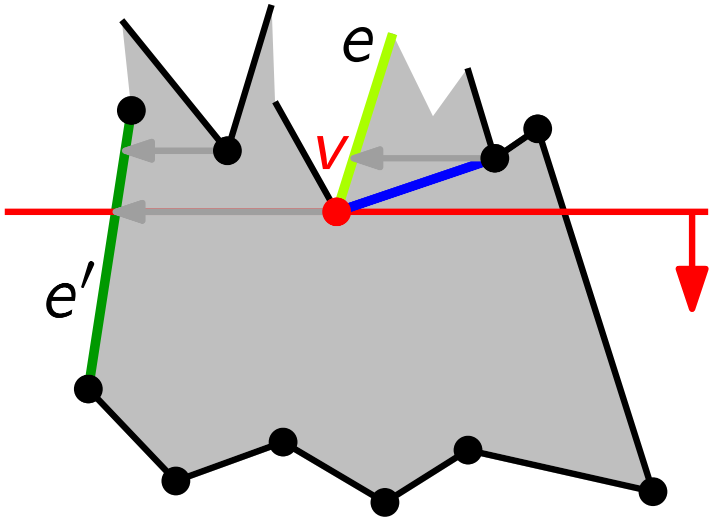

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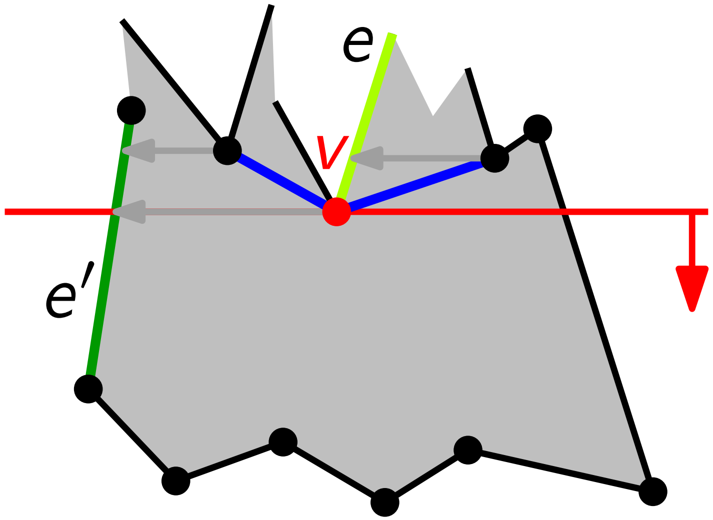

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Analysis

Lemma. makeMonotone() adds a set of non-intersecting diagonals to P such that P is partitioned into y-monotone subpolygons.

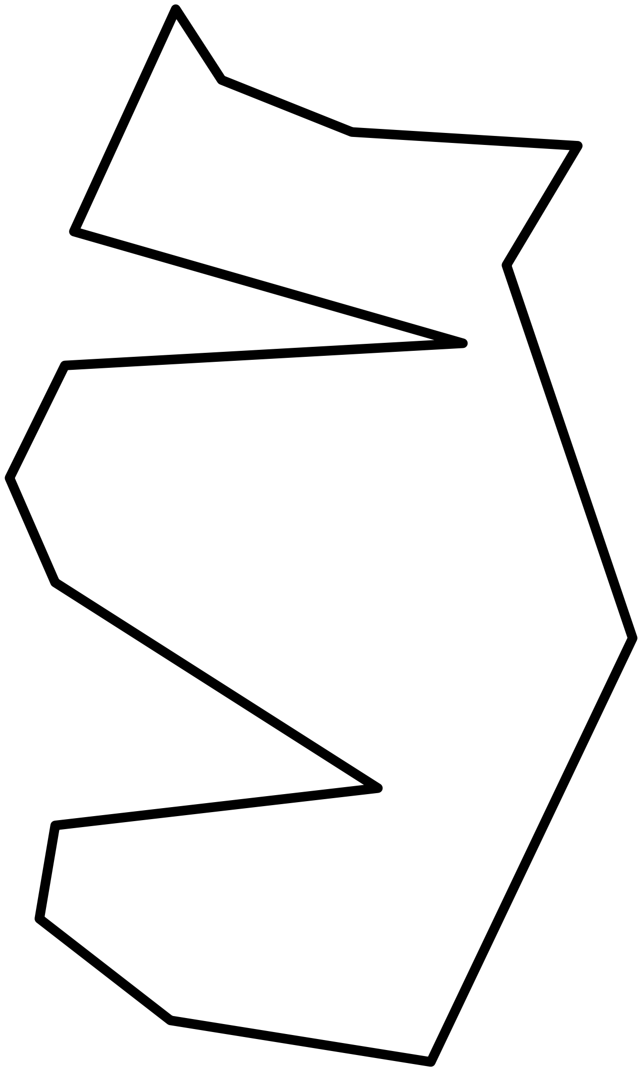
Analysis

Lemma. makeMonotone() adds a set of non-intersecting diagonals to P such that P is partitioned into y-monotone subpolygons.

Lemma. A simple polygon with n vertices can be subdivided into y-monotone polygons in $O(n \log n)$ time.

Triangulating a y-Monotone Polygon P

Approach: greedy, going from top to bottom



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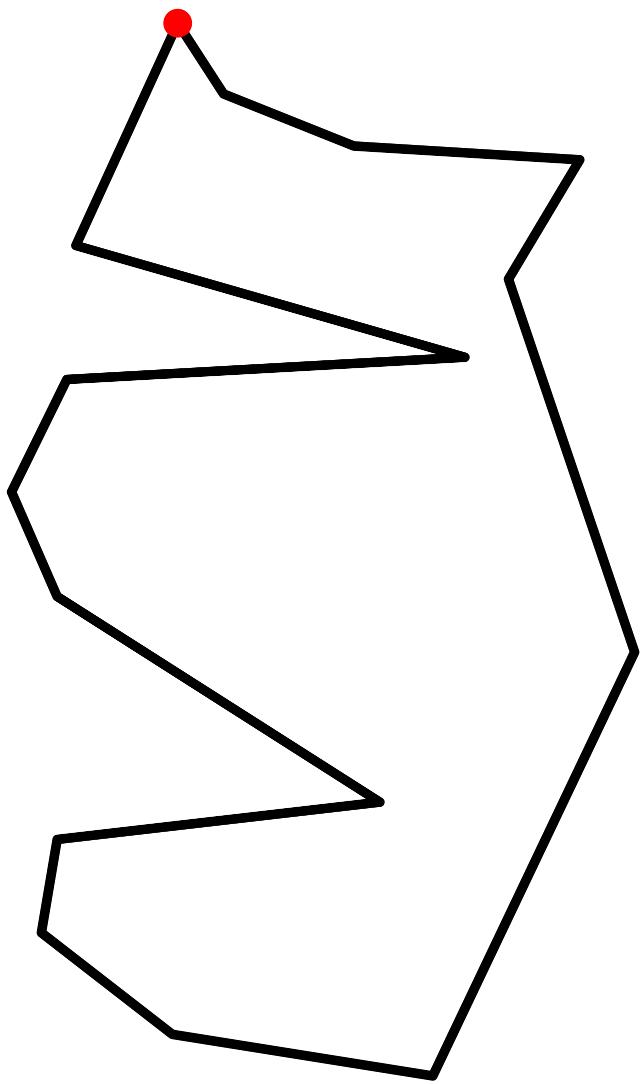

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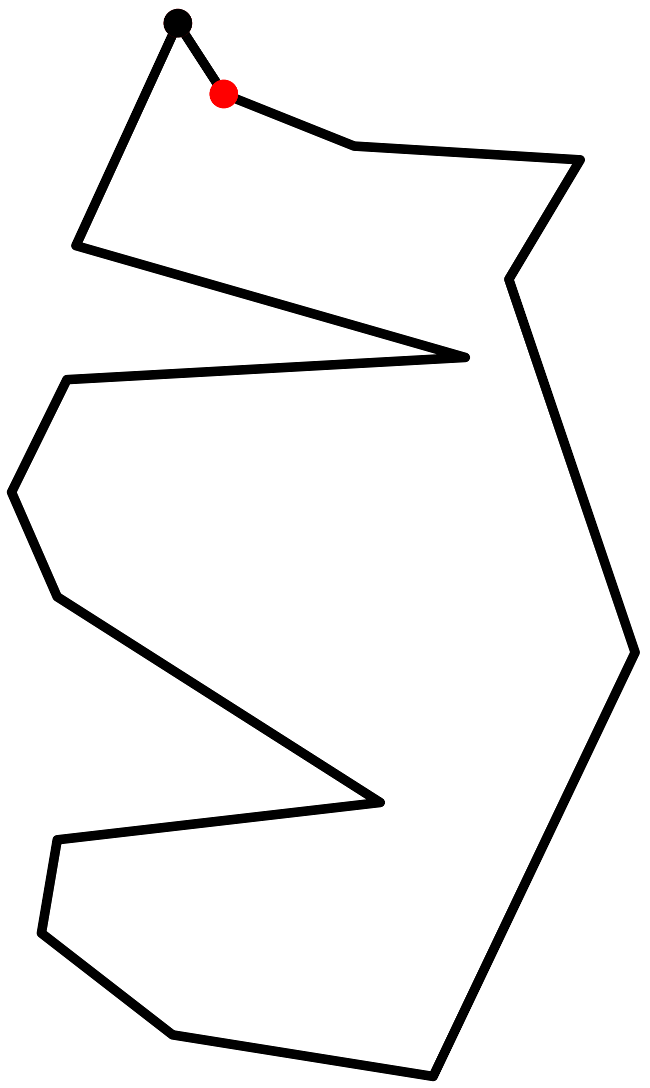

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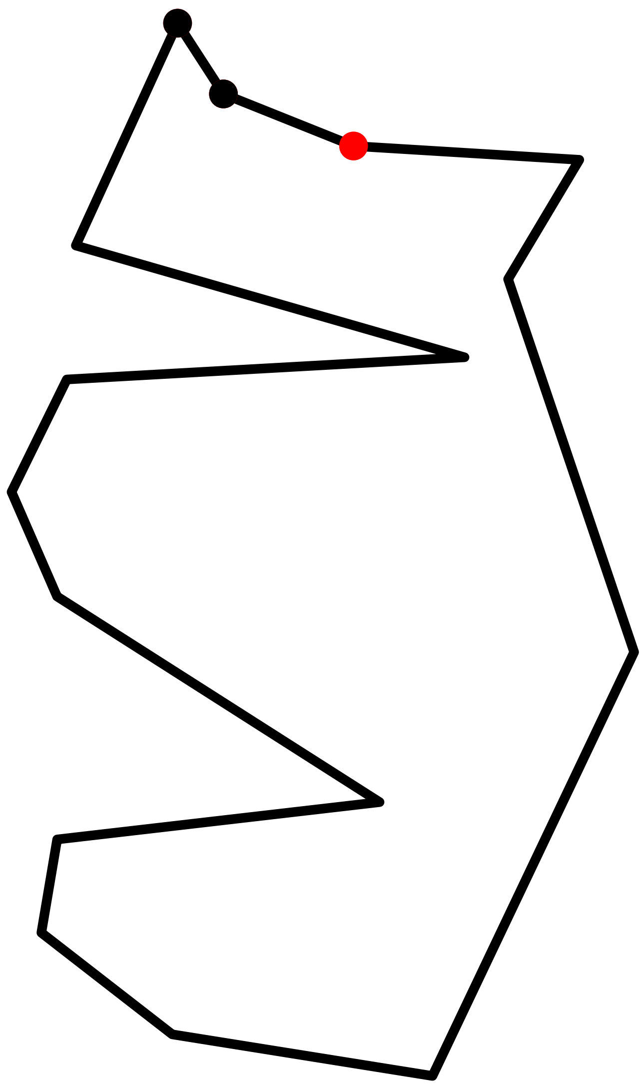

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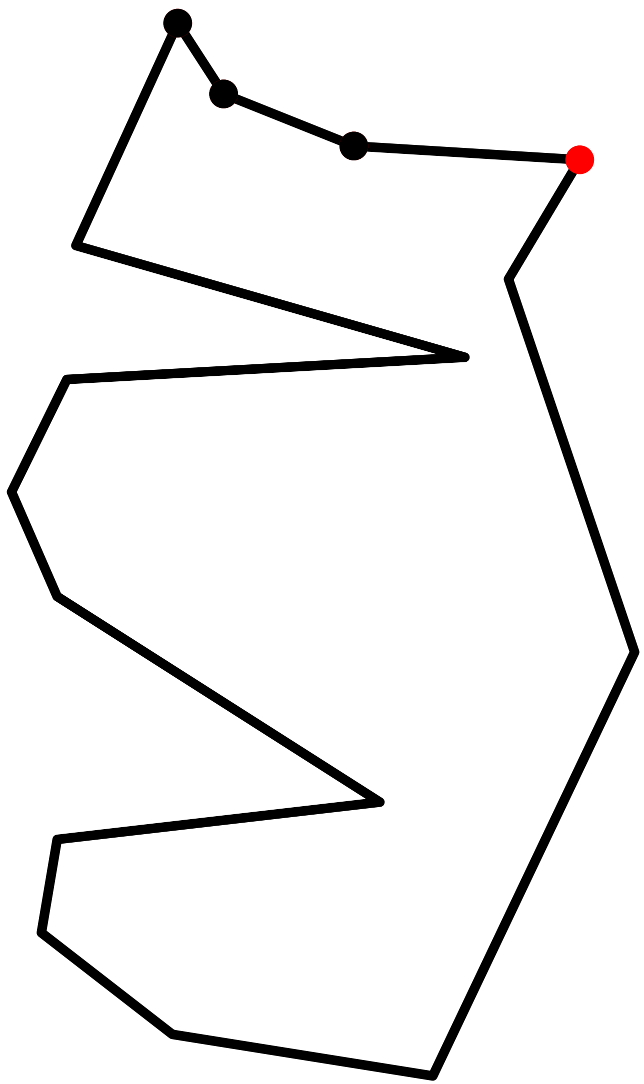

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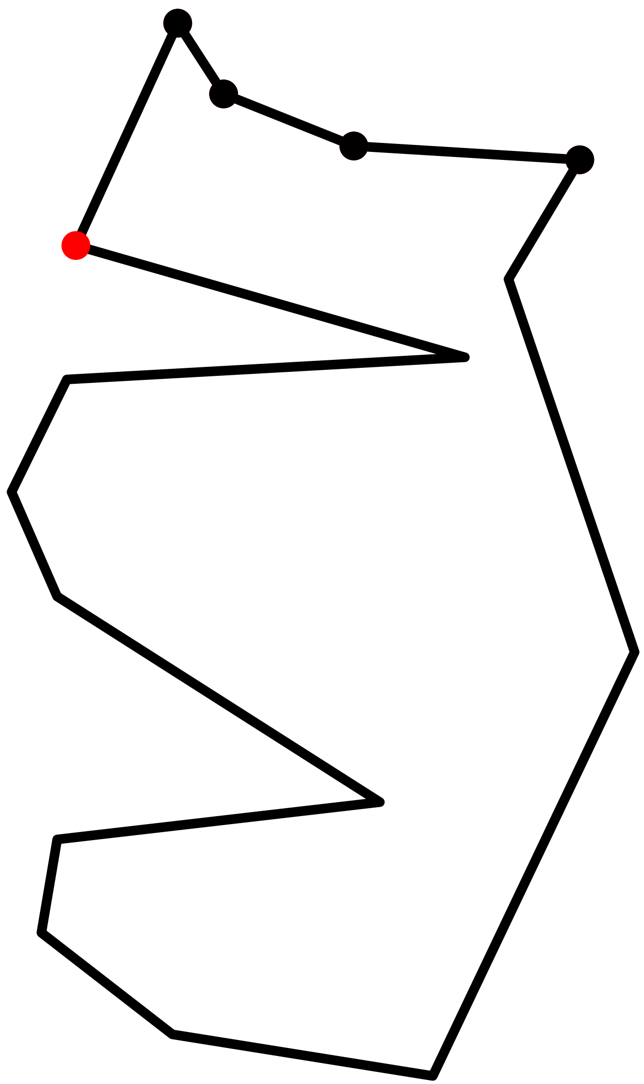

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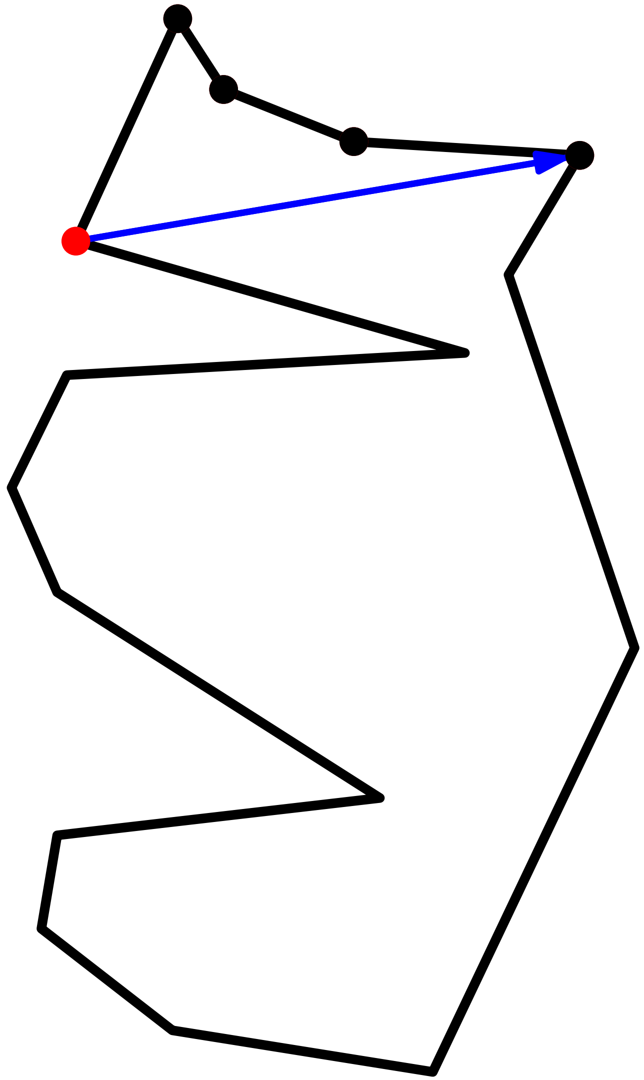

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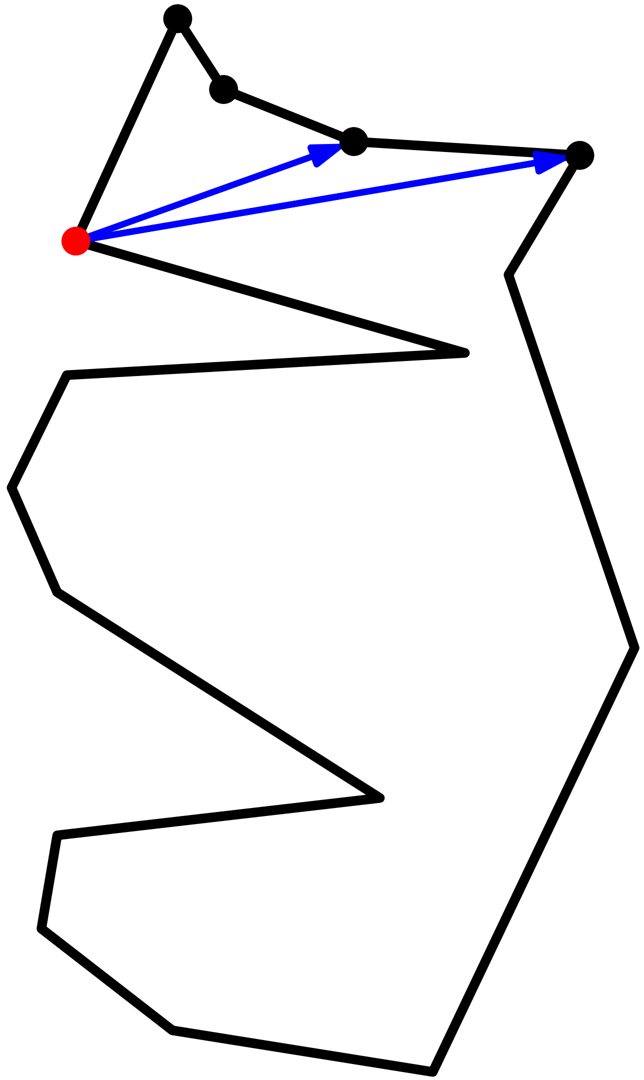

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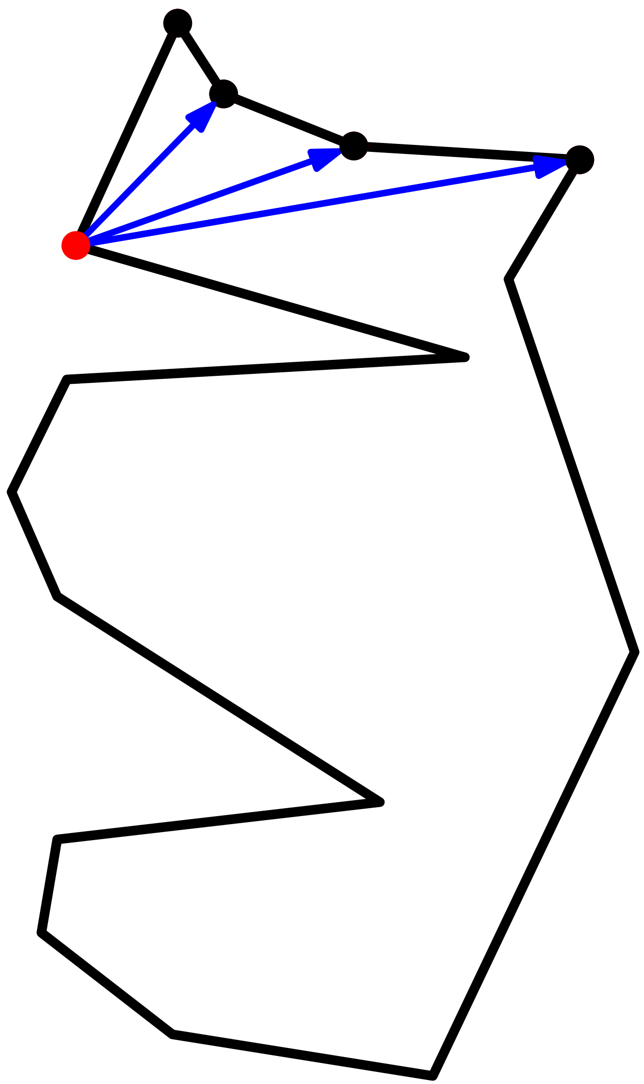

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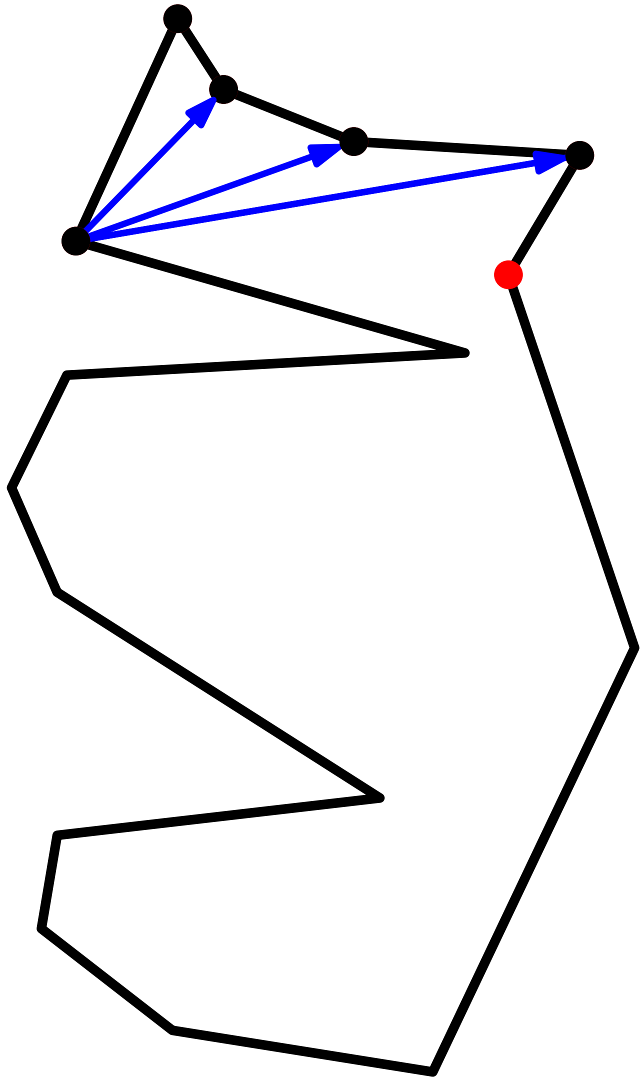

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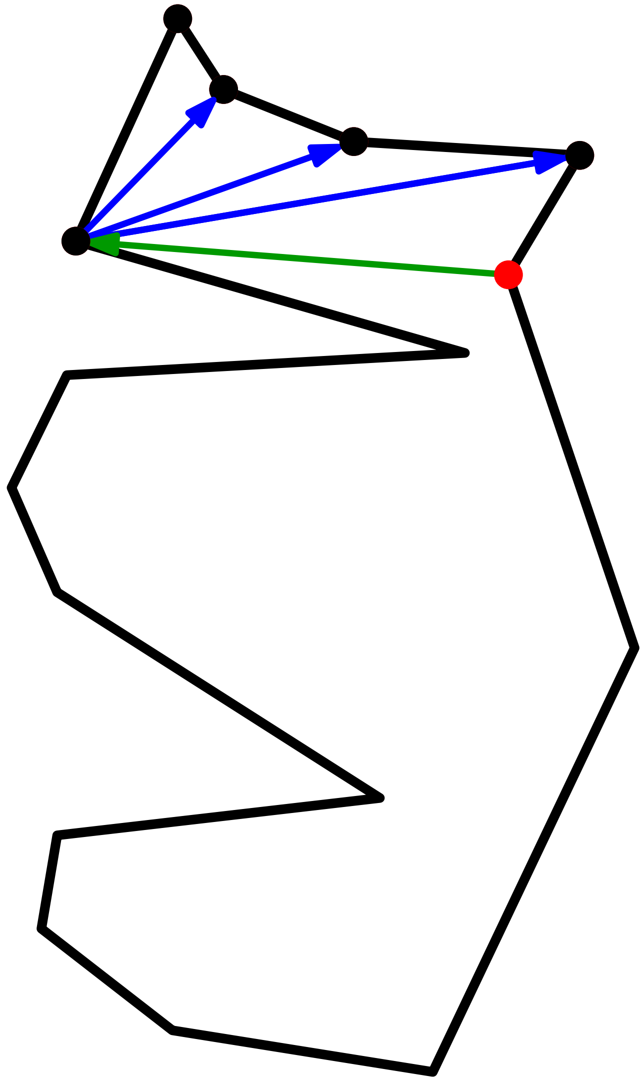

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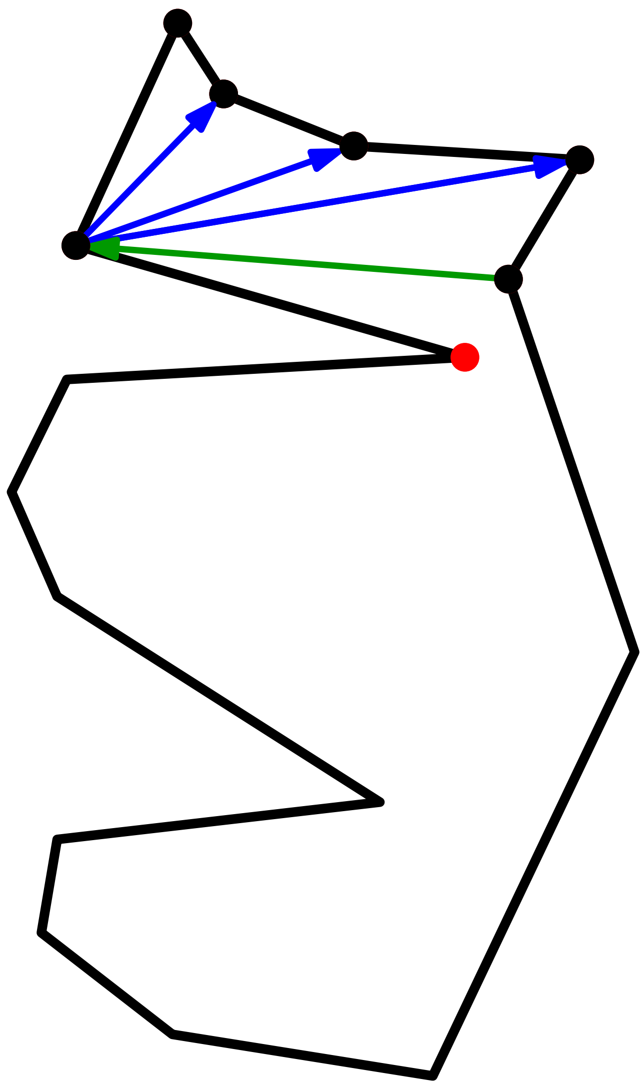

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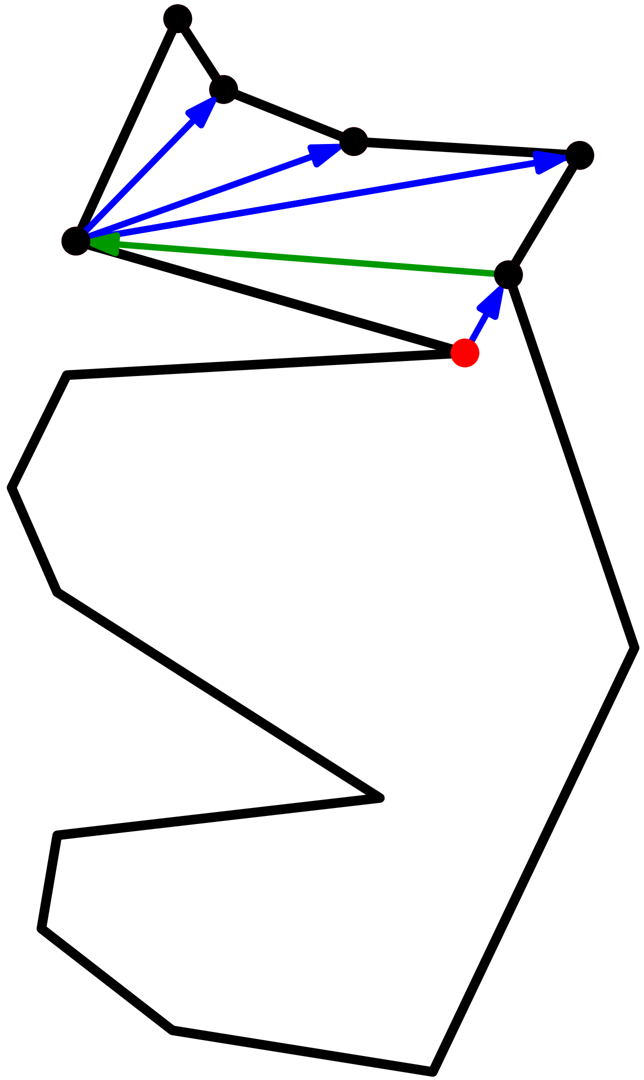

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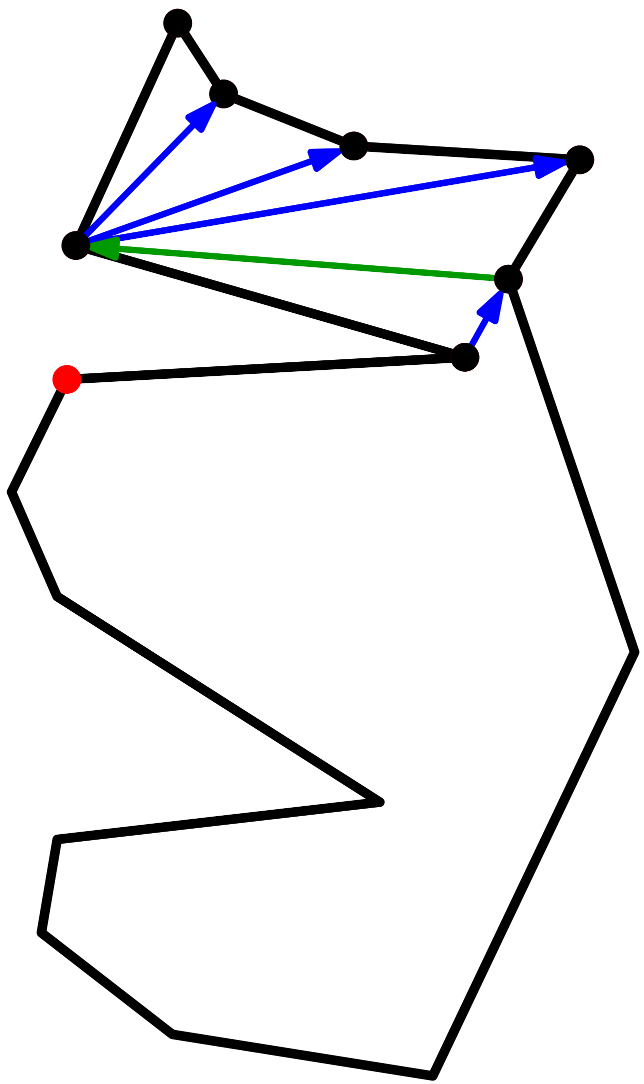

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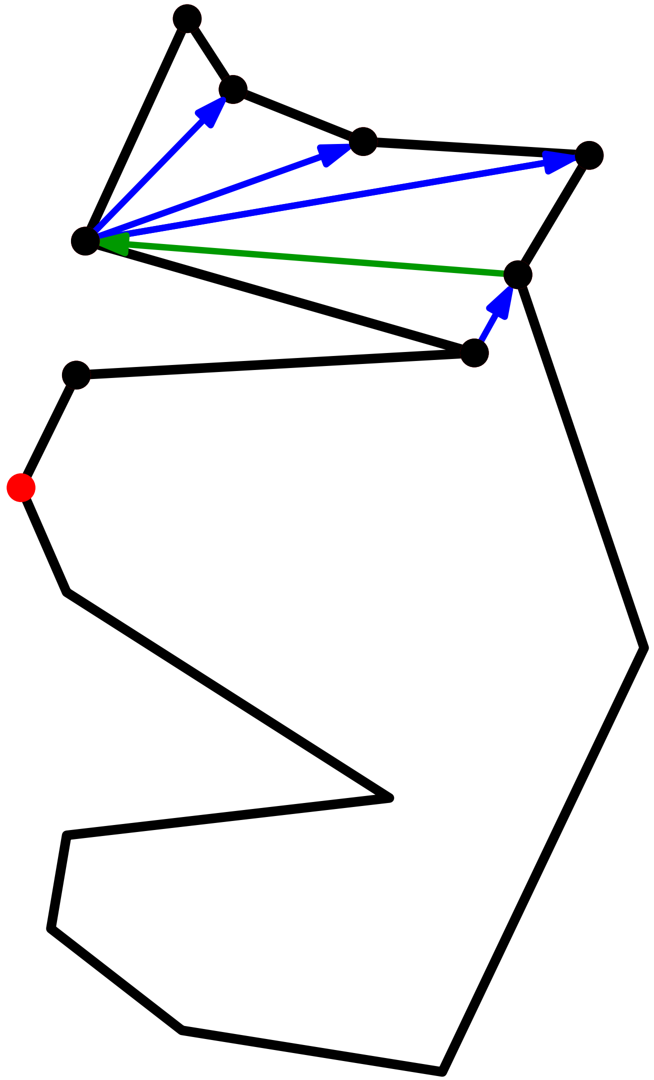

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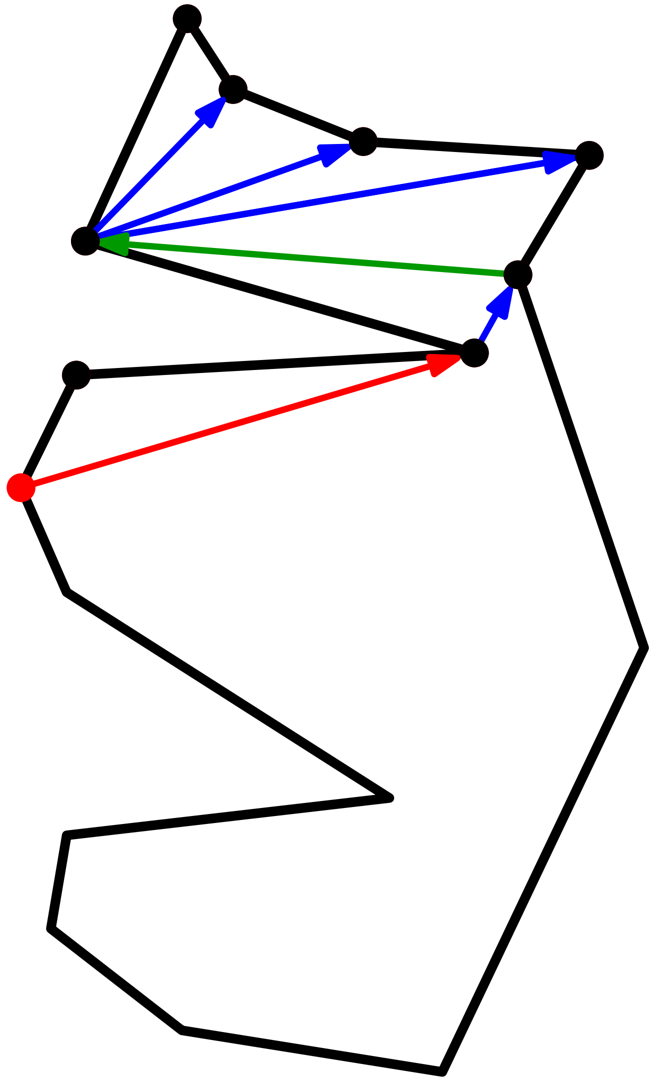

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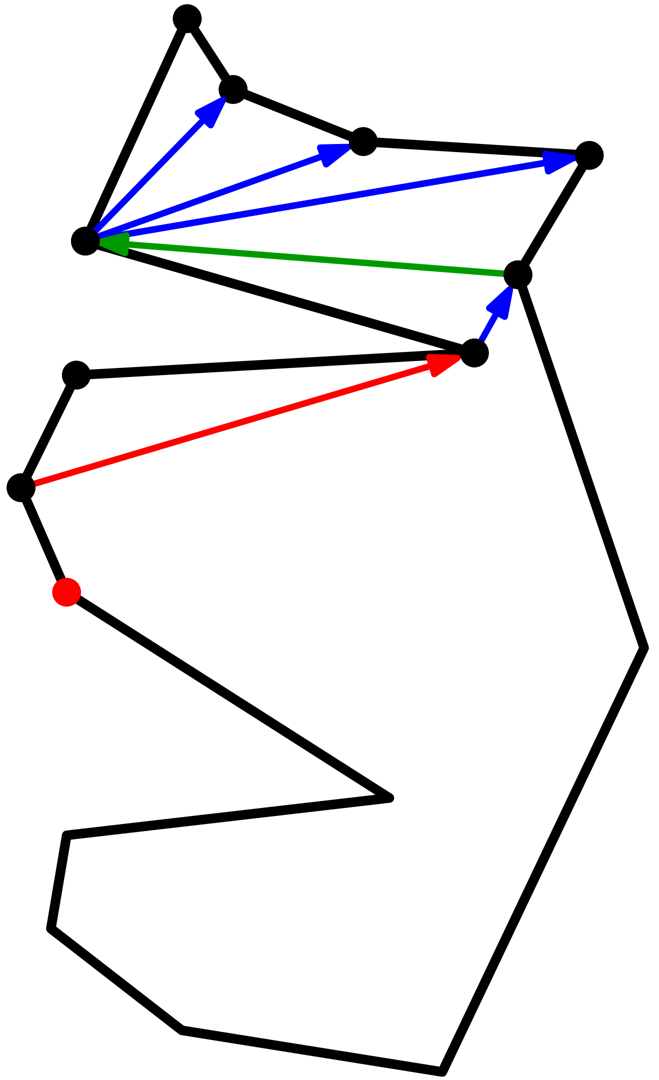

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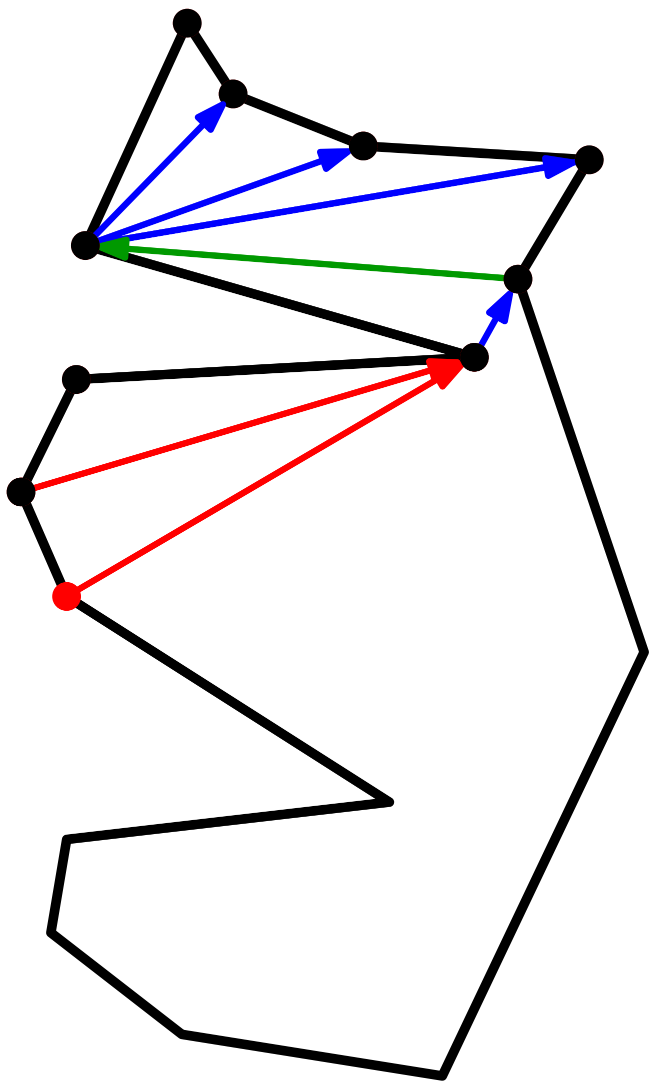

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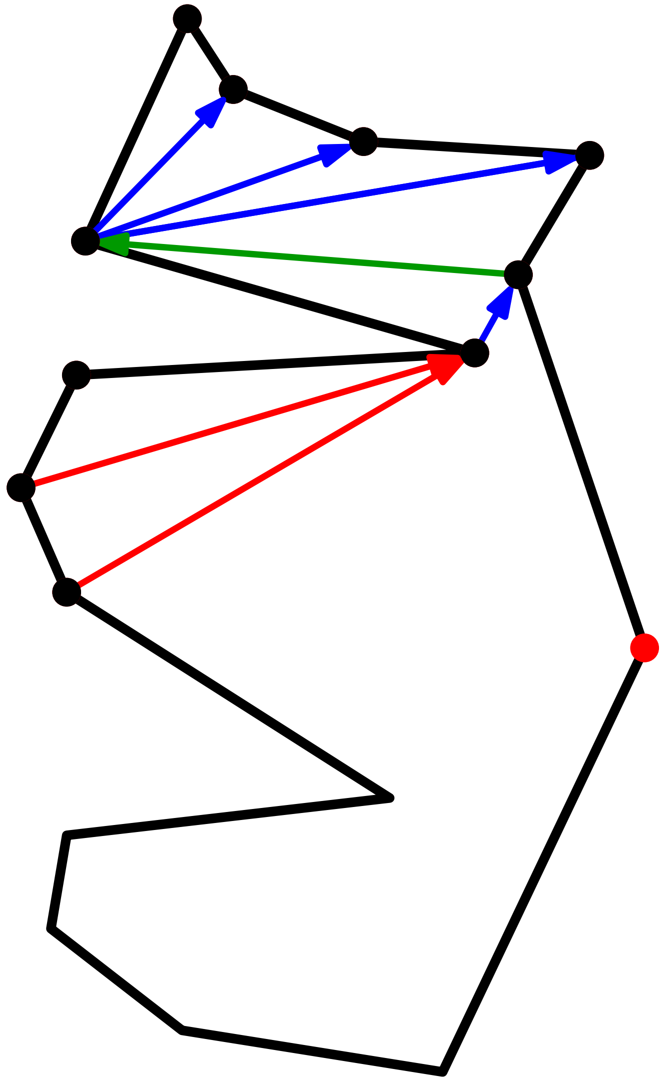

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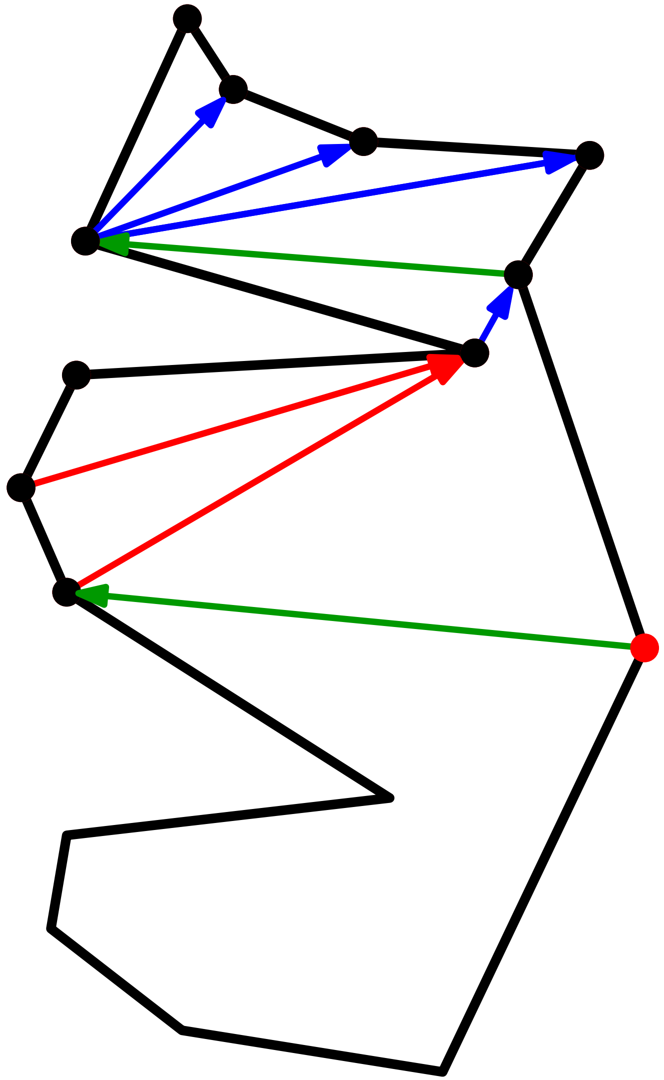

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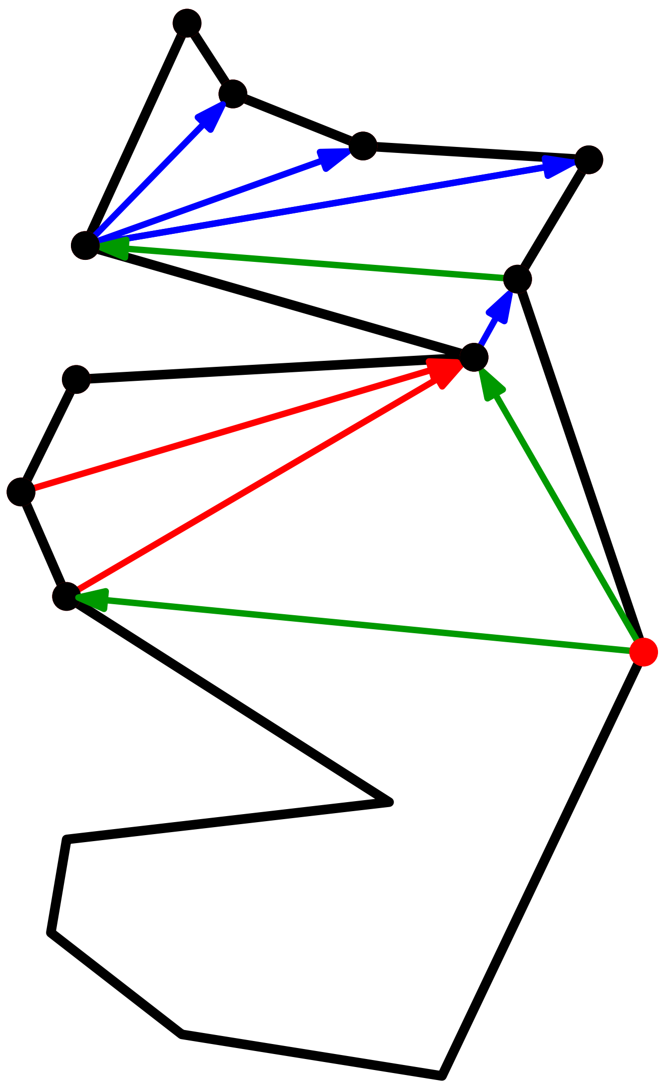

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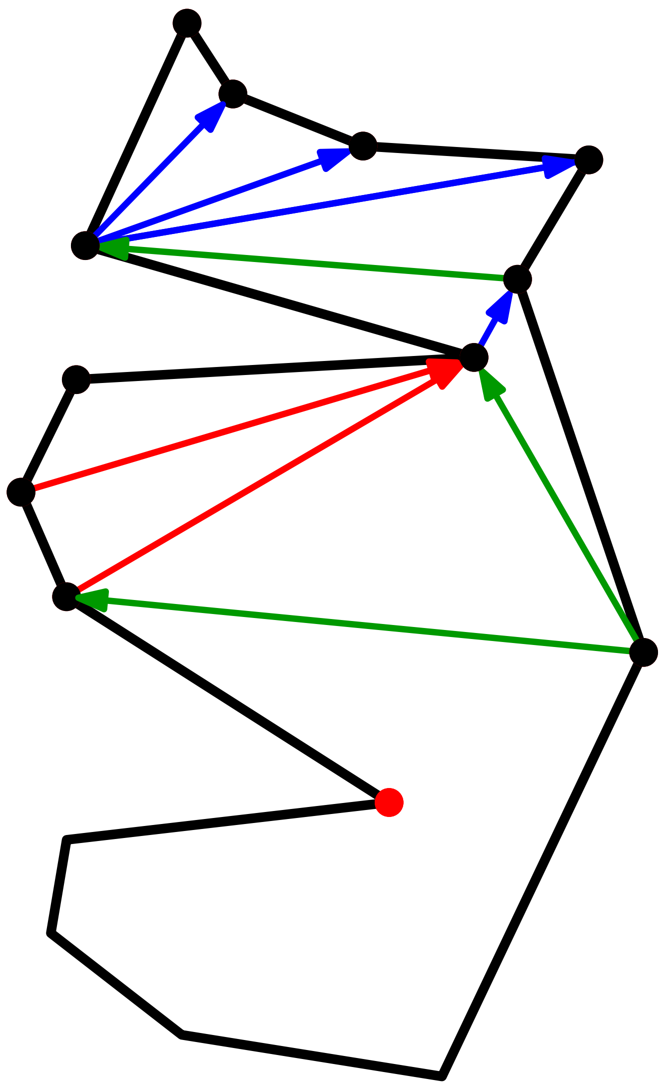

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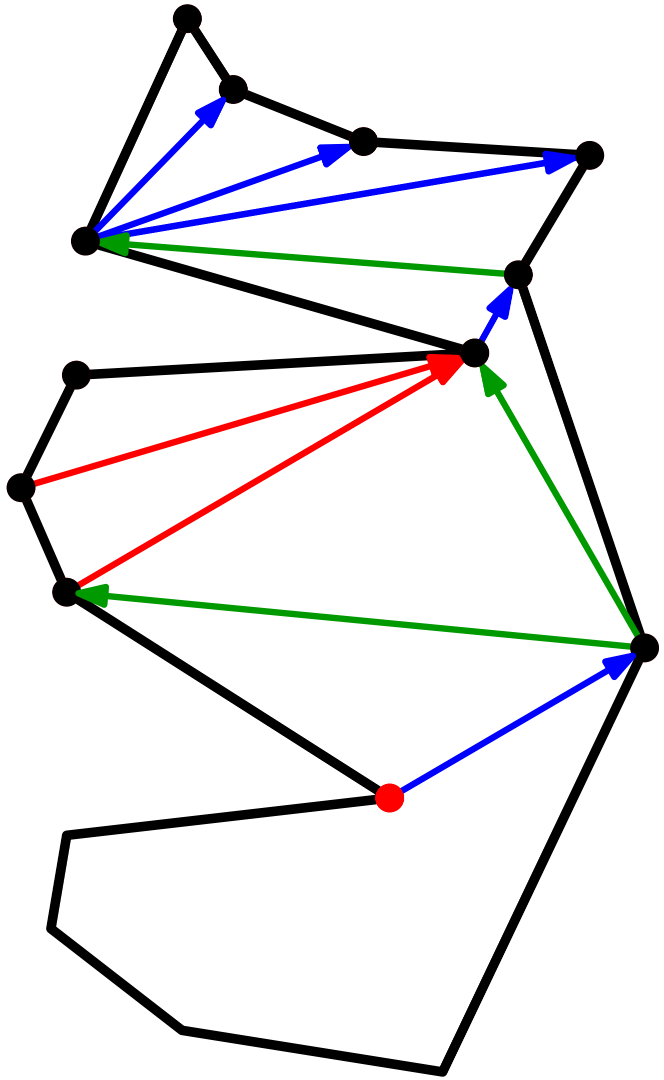

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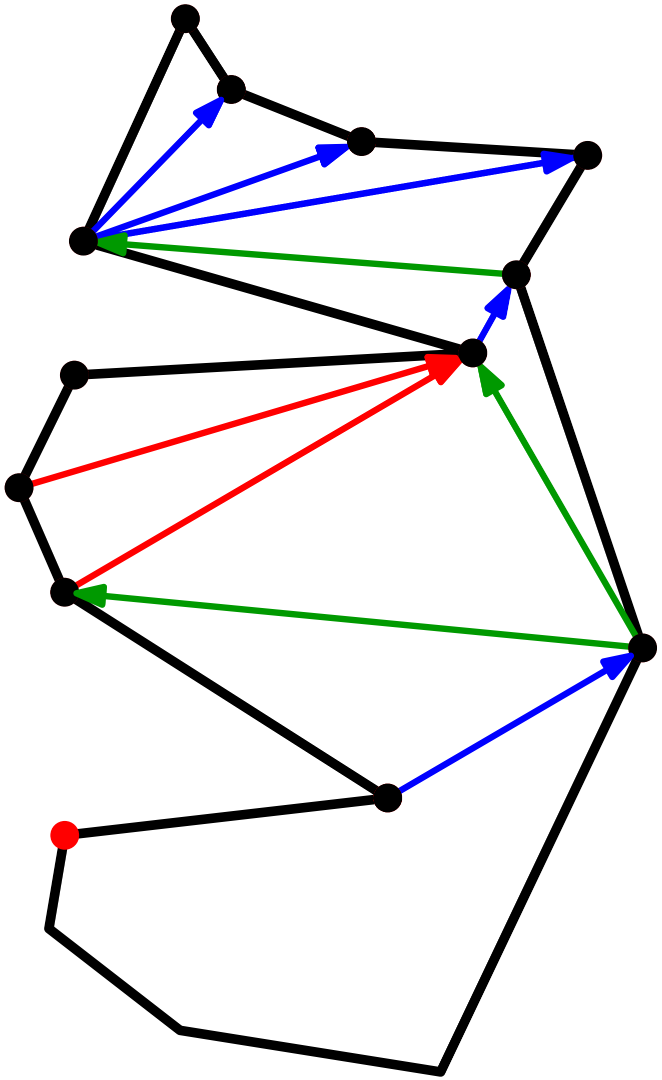

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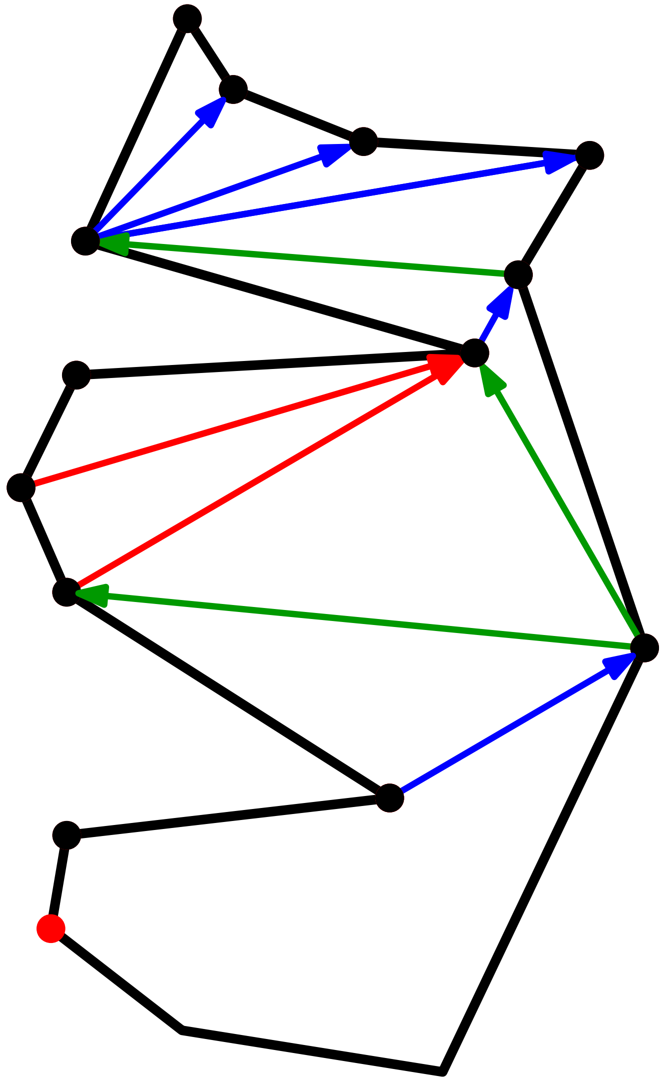

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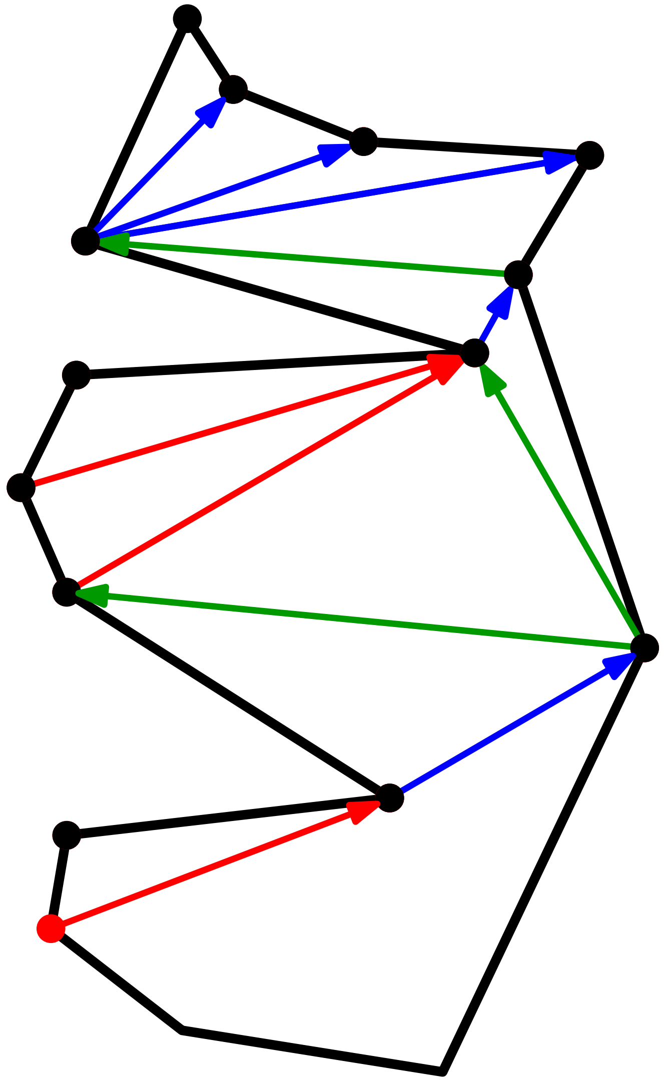

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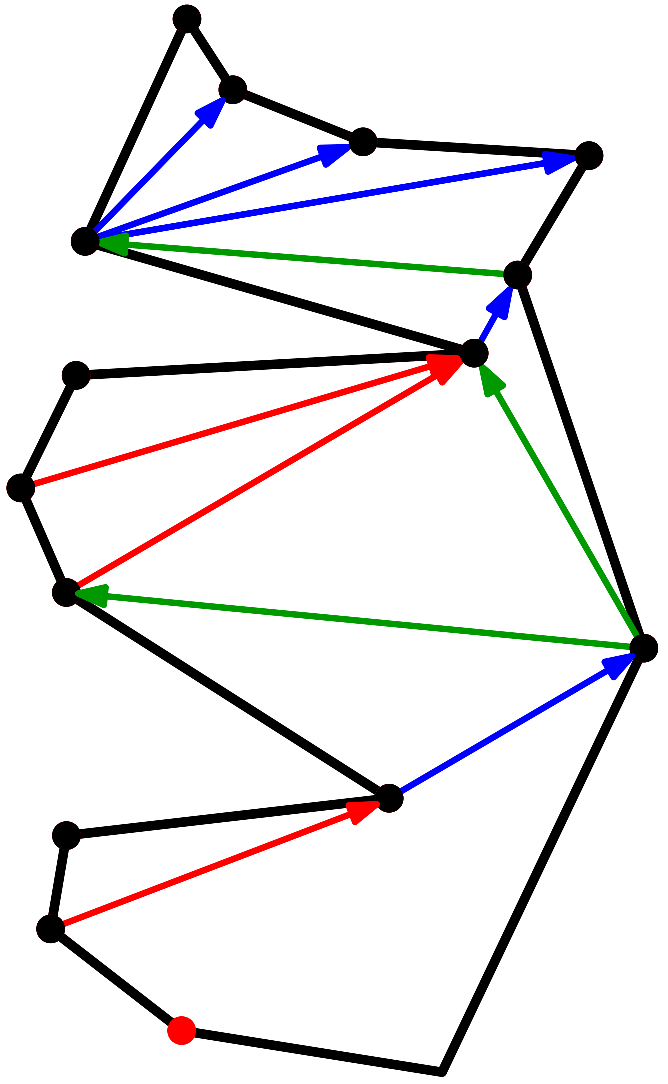

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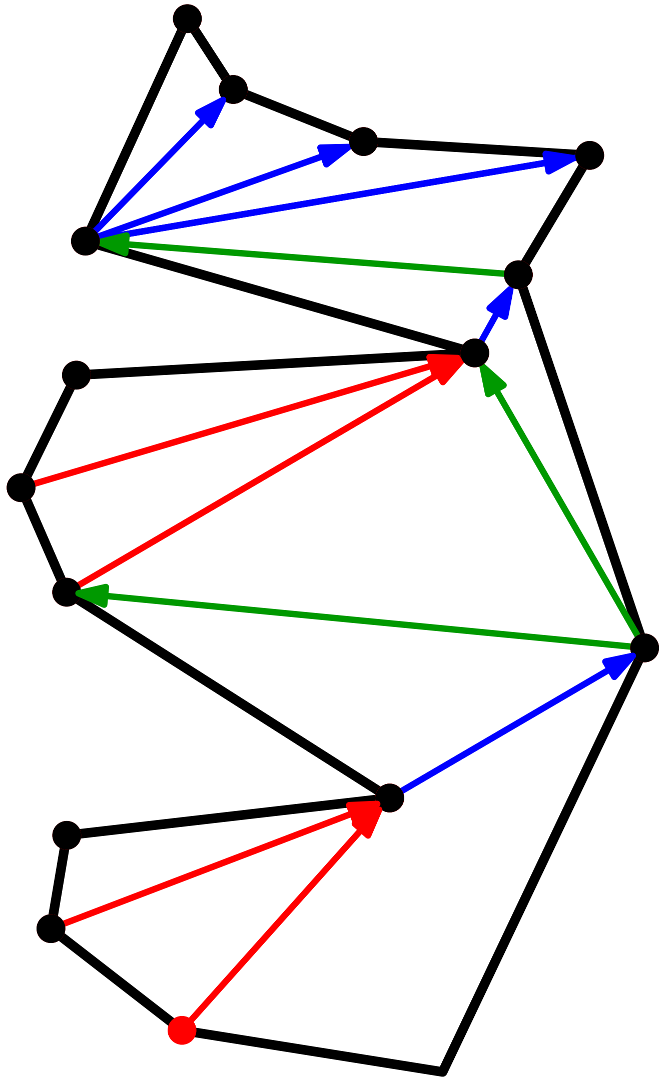

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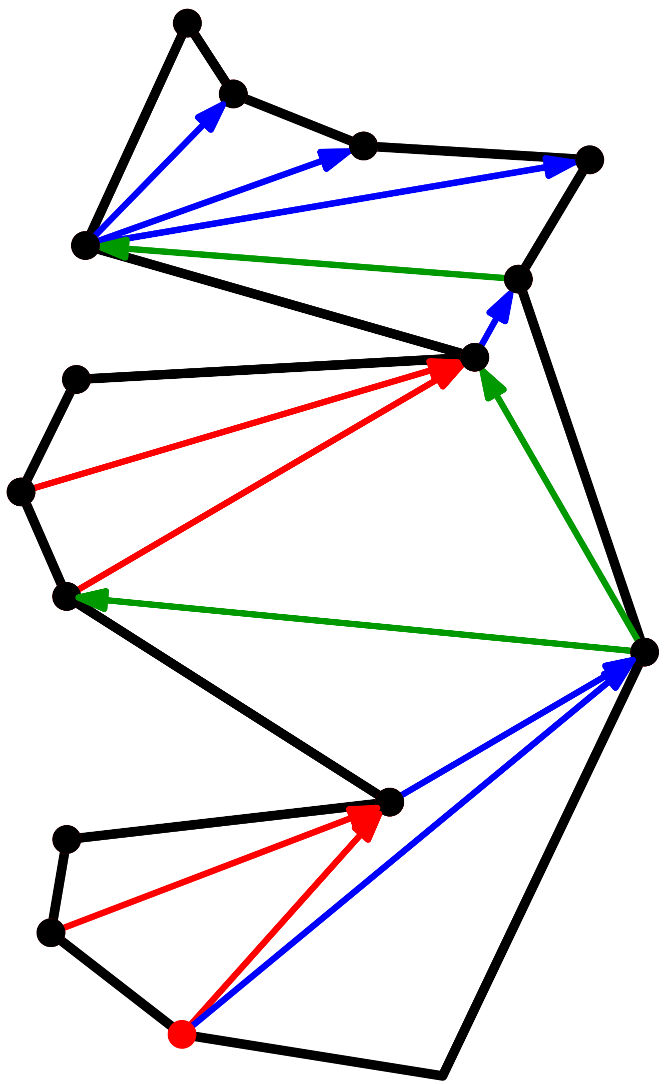

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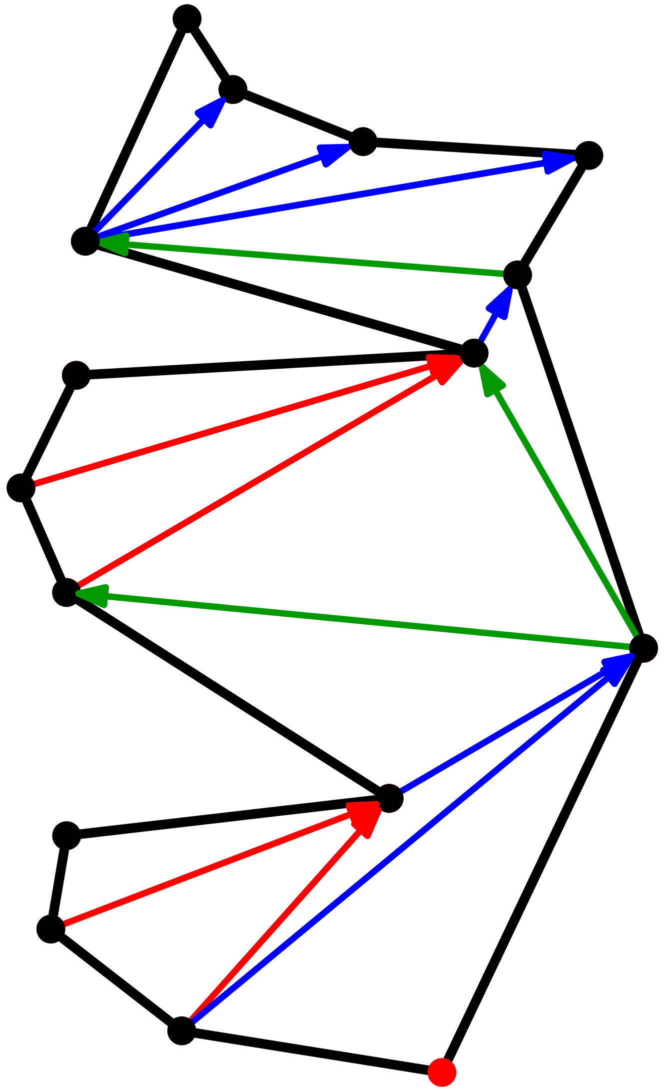

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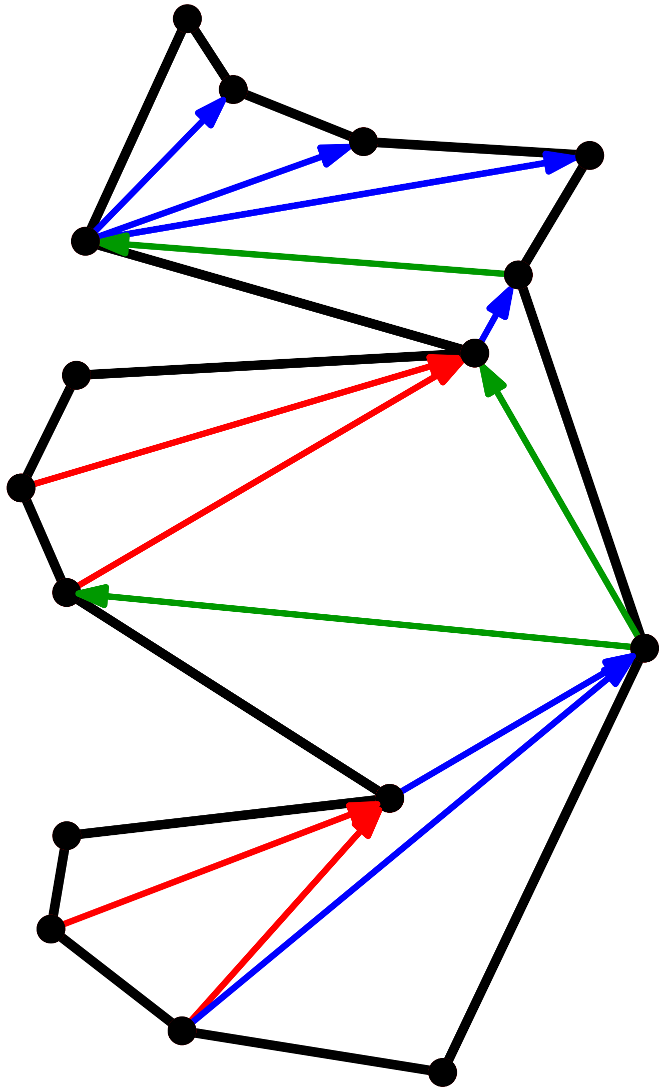

Invariant?

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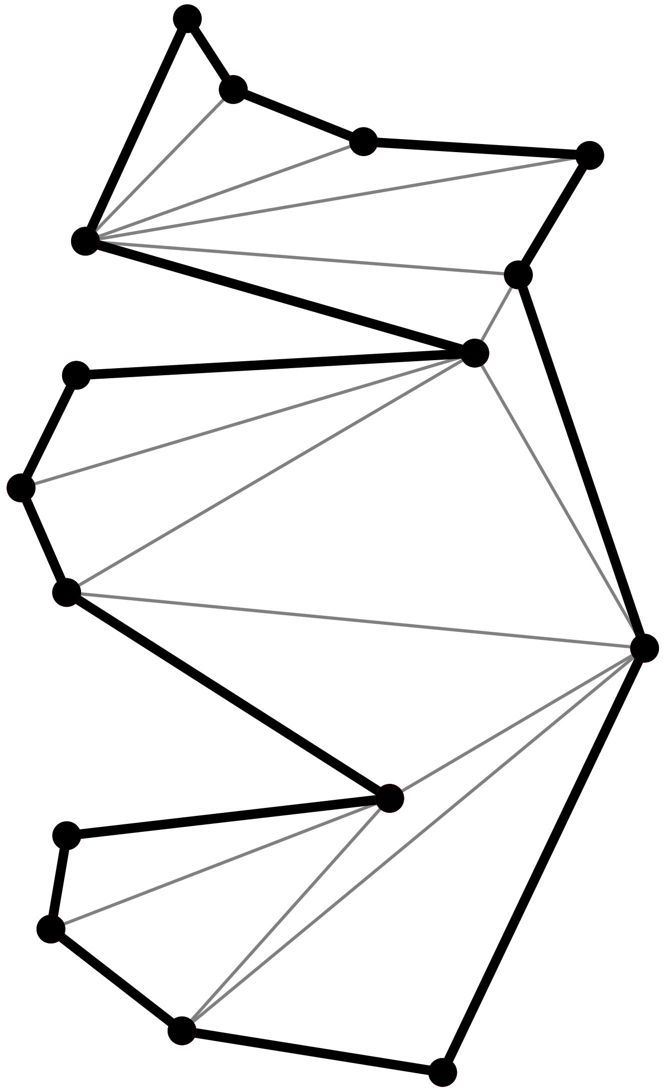

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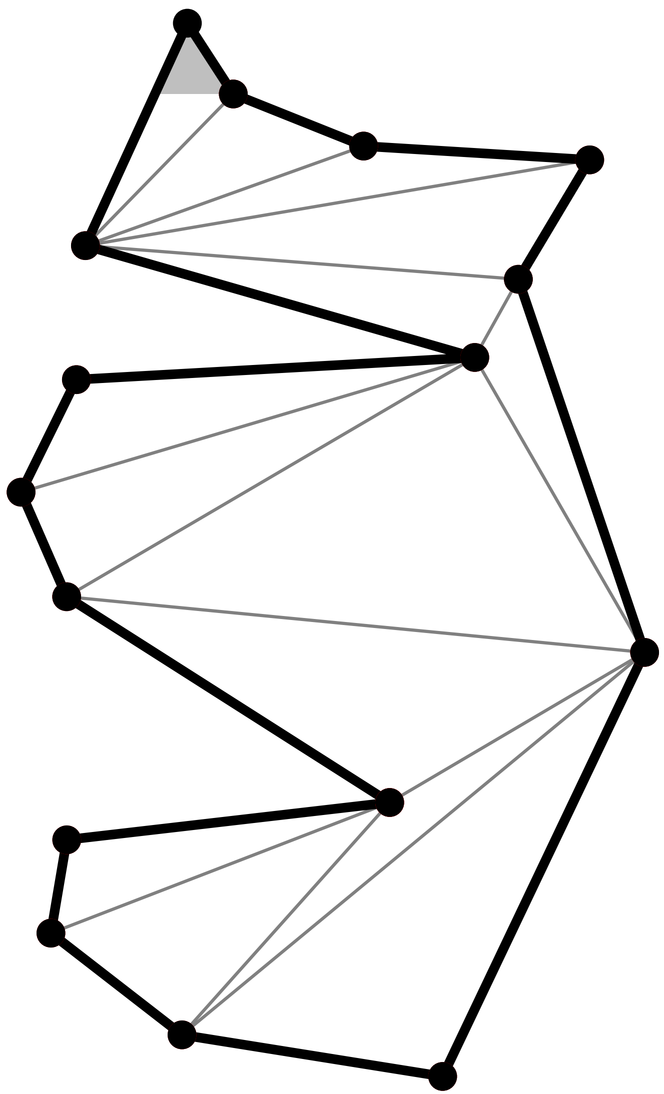

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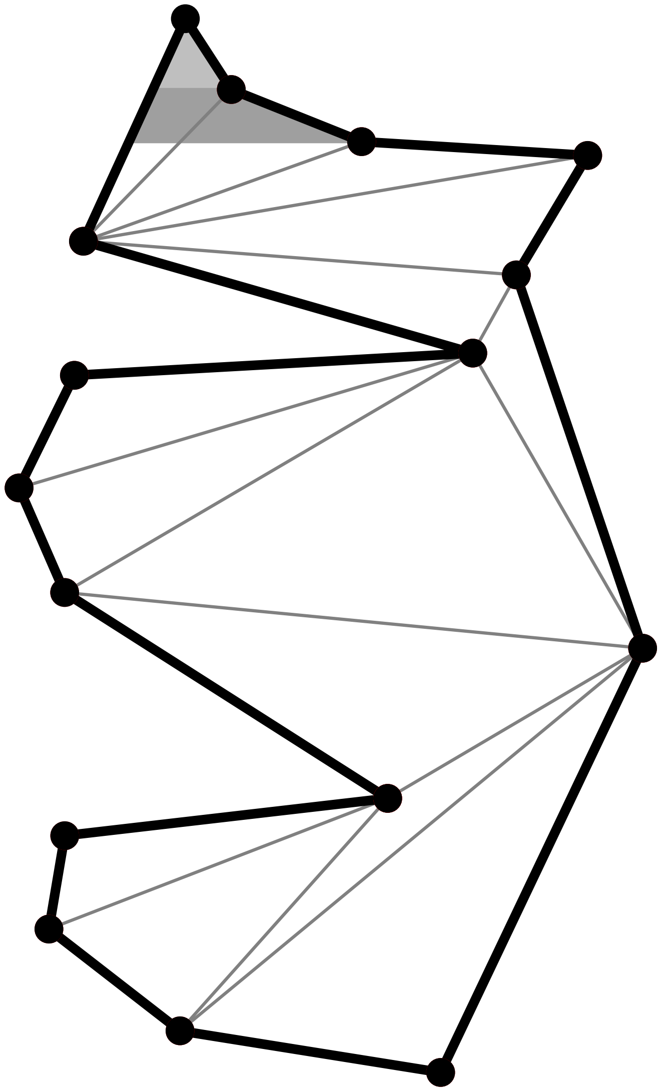

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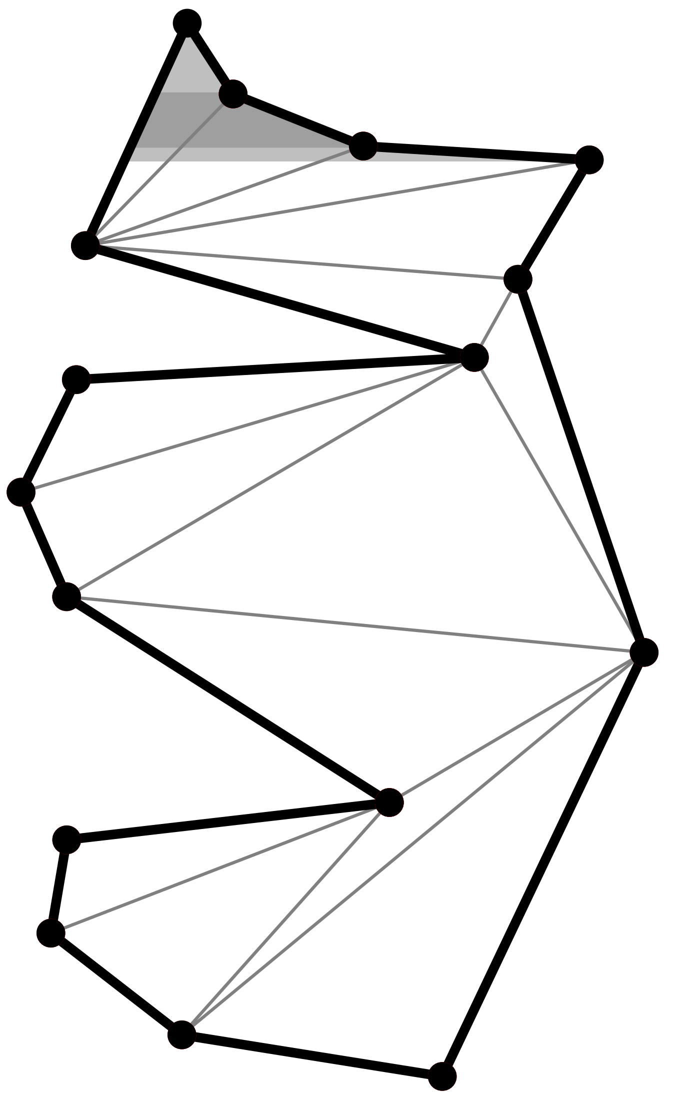

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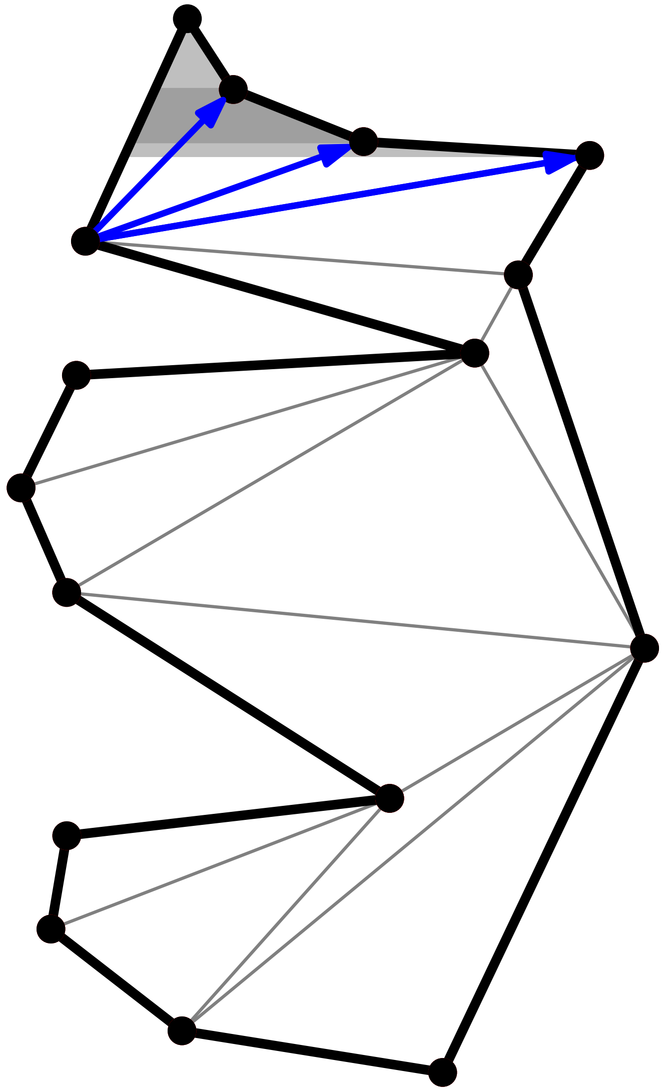

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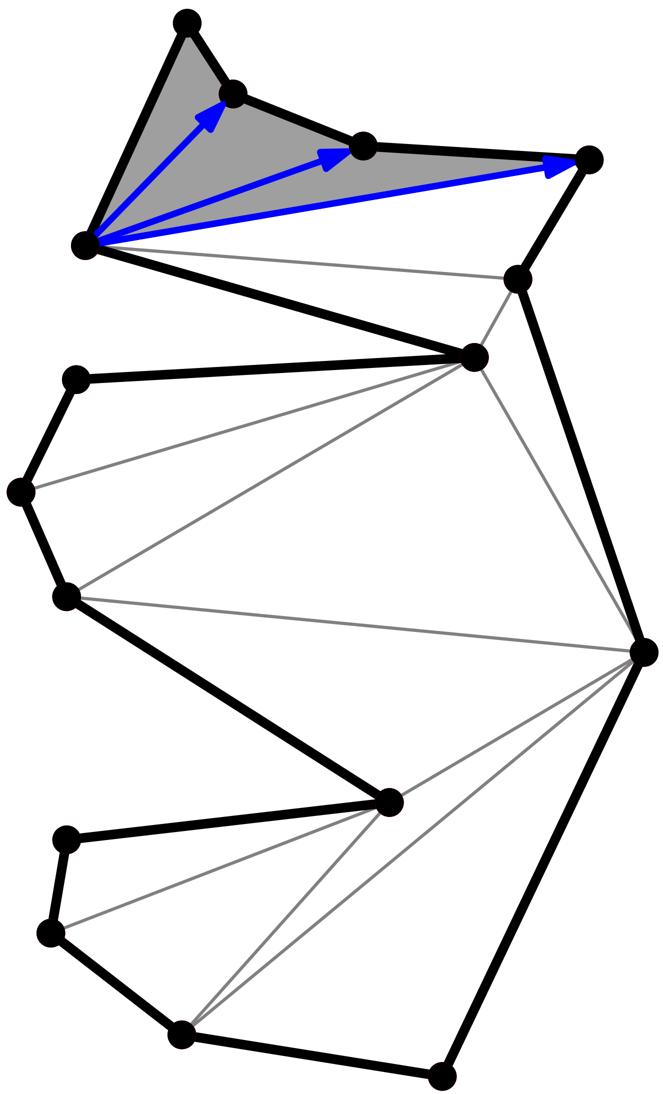
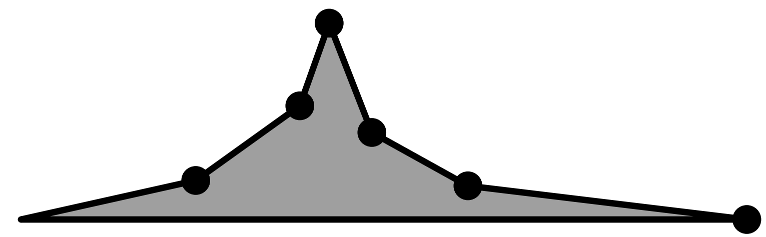


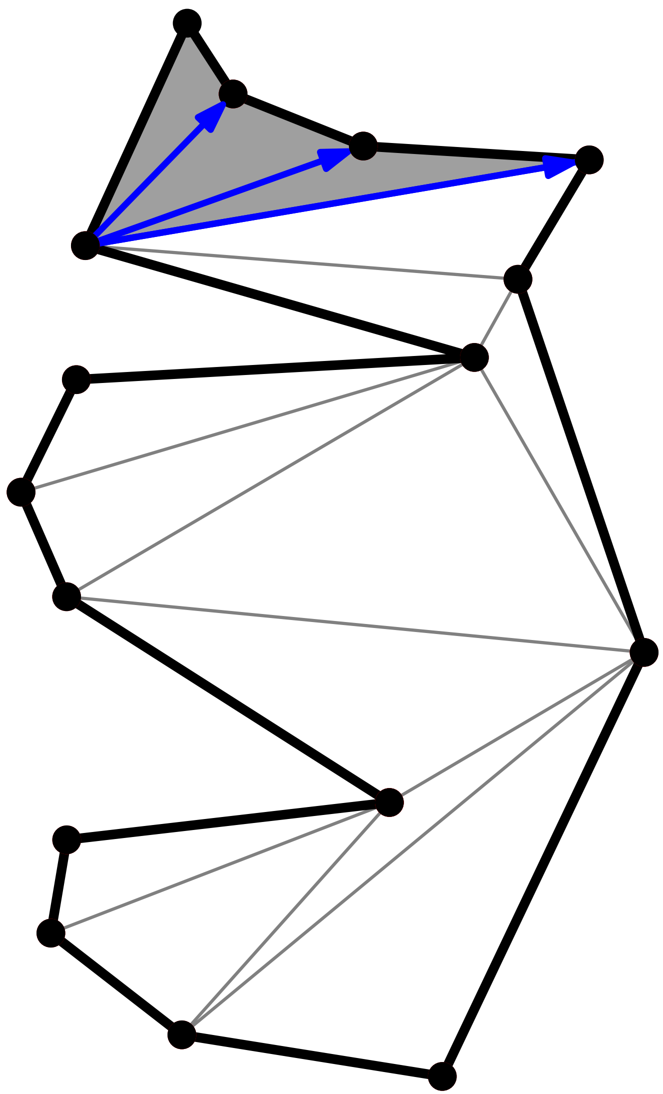
Invariant?

The part of P that we have seen but not yet triangulated is a *funnel*.



Triangulating a y-Monotone Polygon P

Approach: greedy, going from top to bottom



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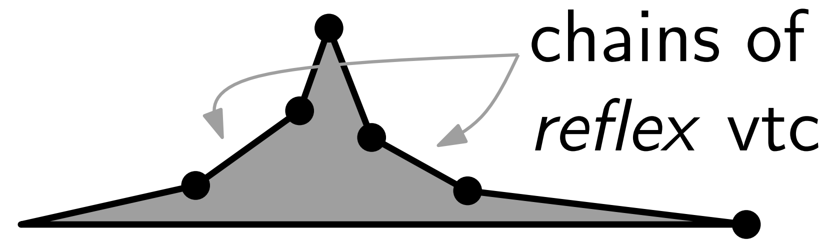

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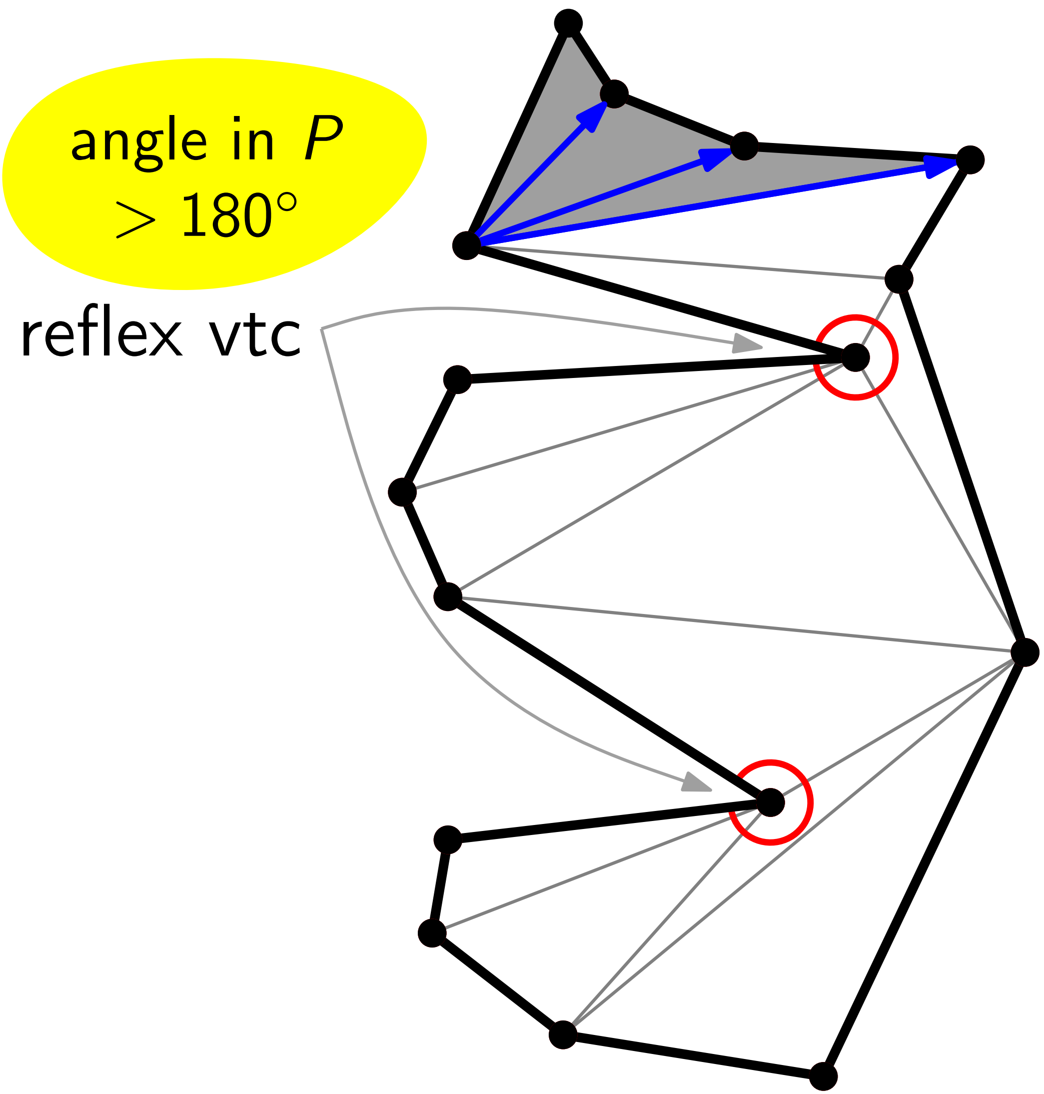

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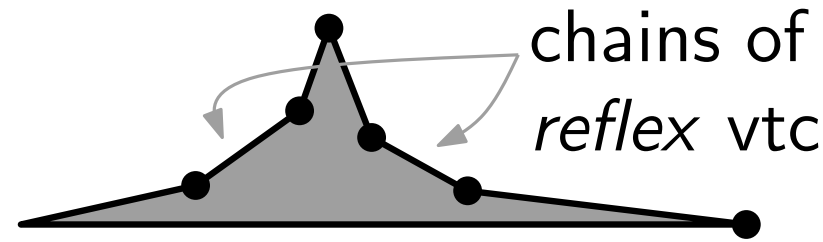

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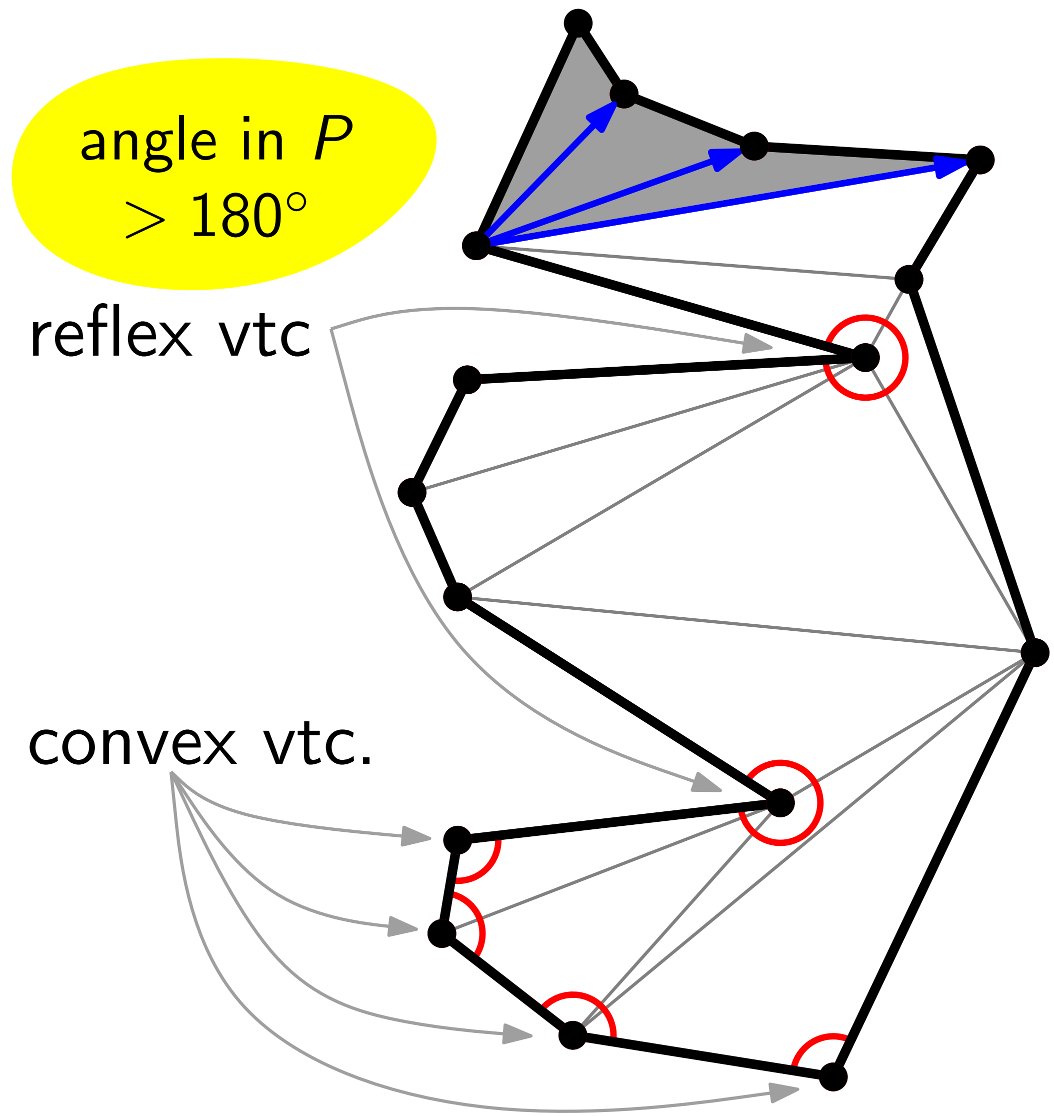

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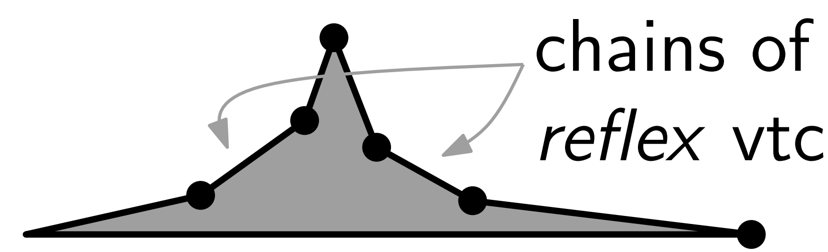

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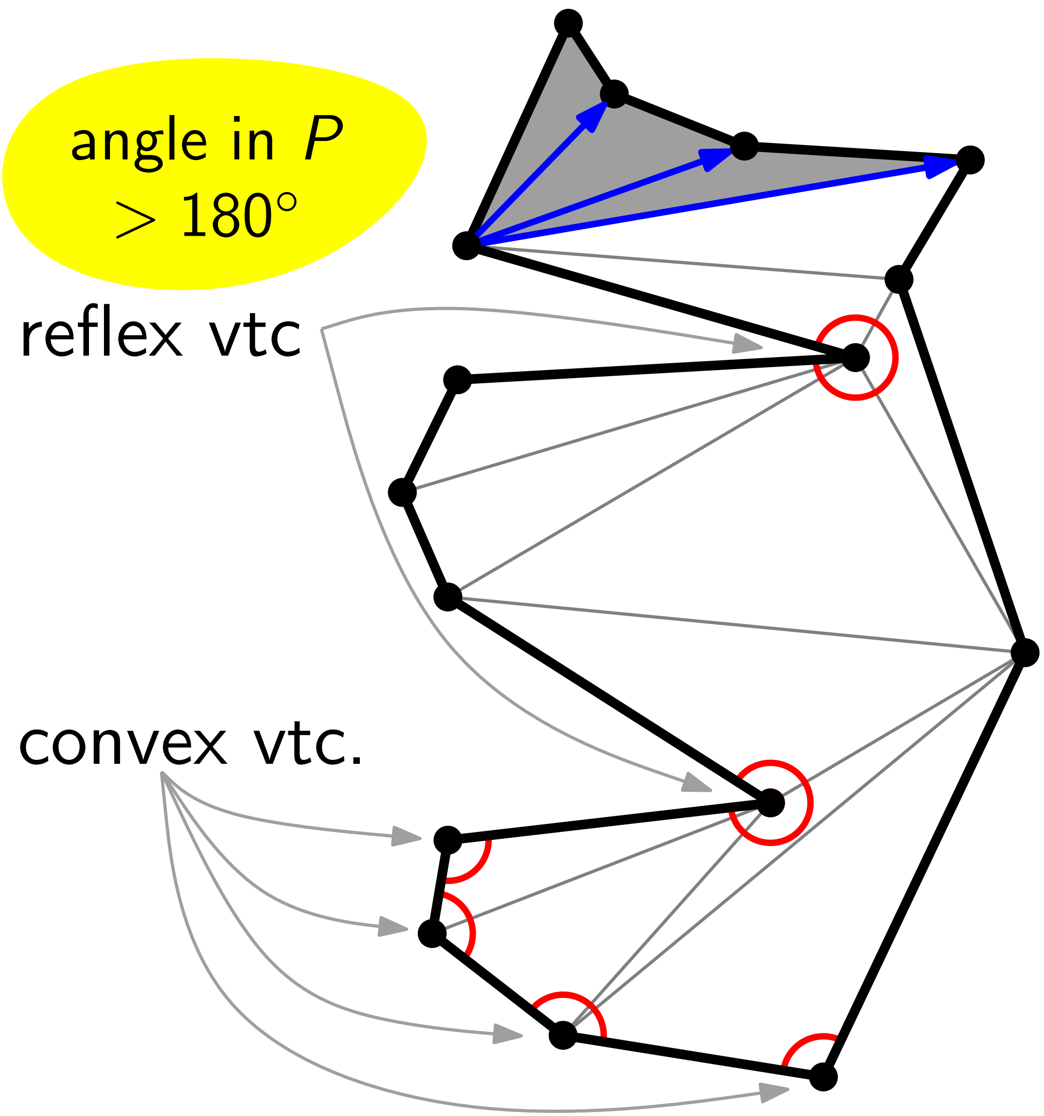

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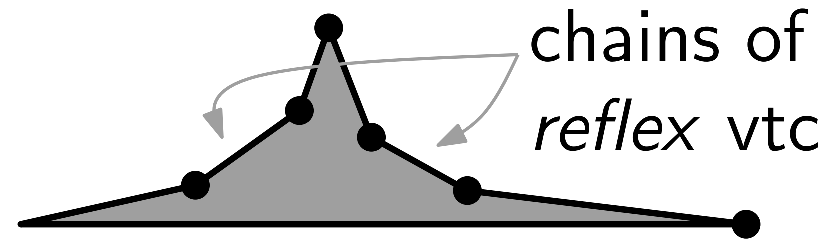

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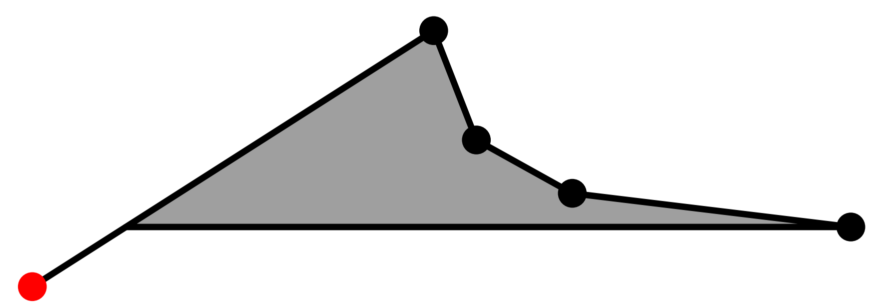

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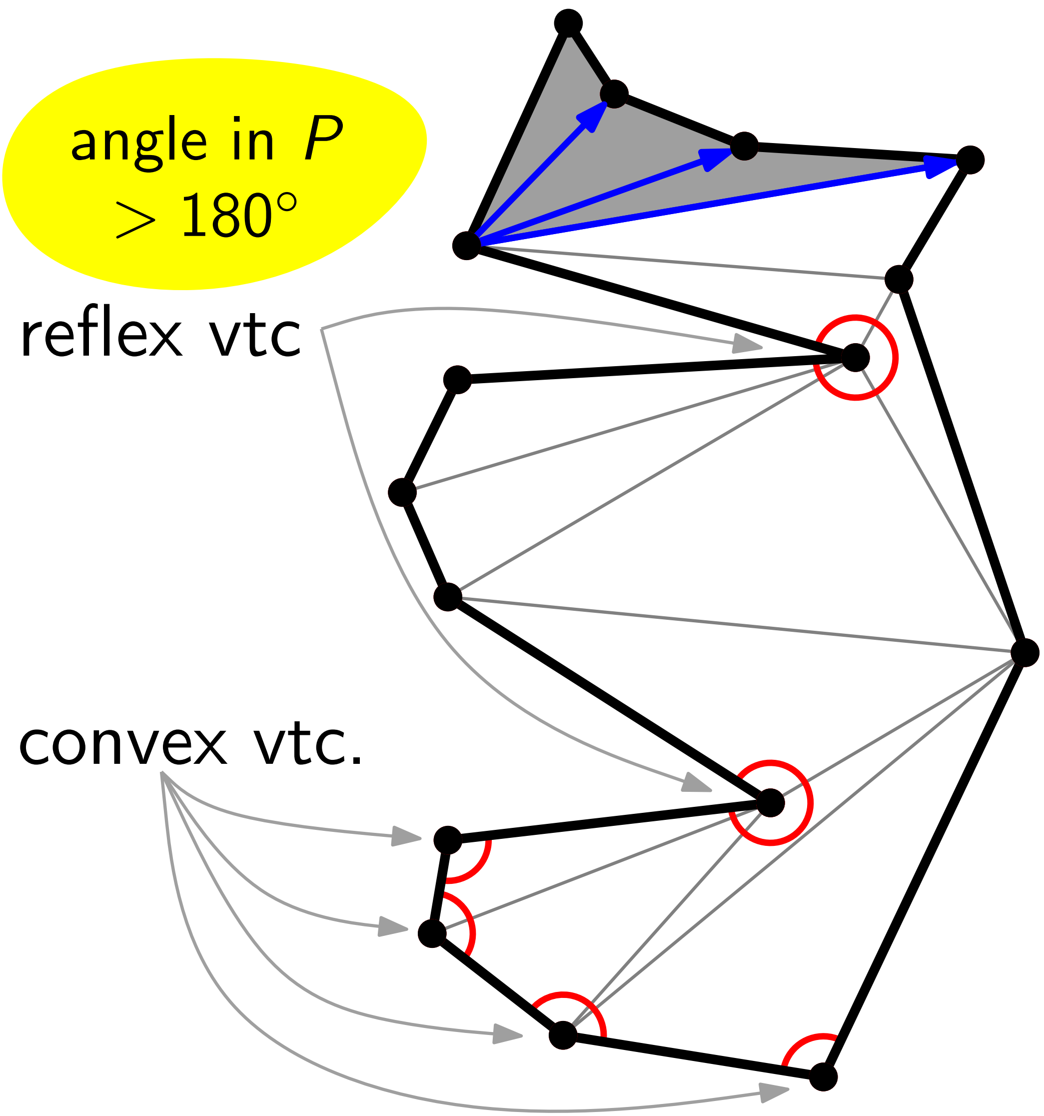

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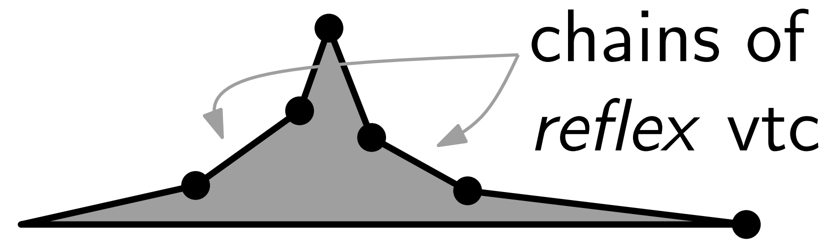

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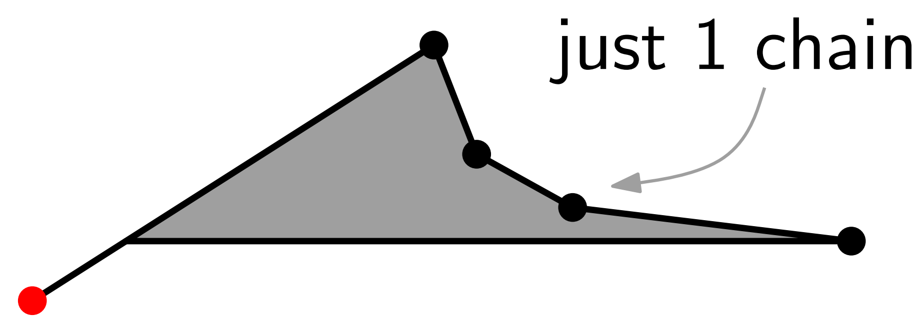

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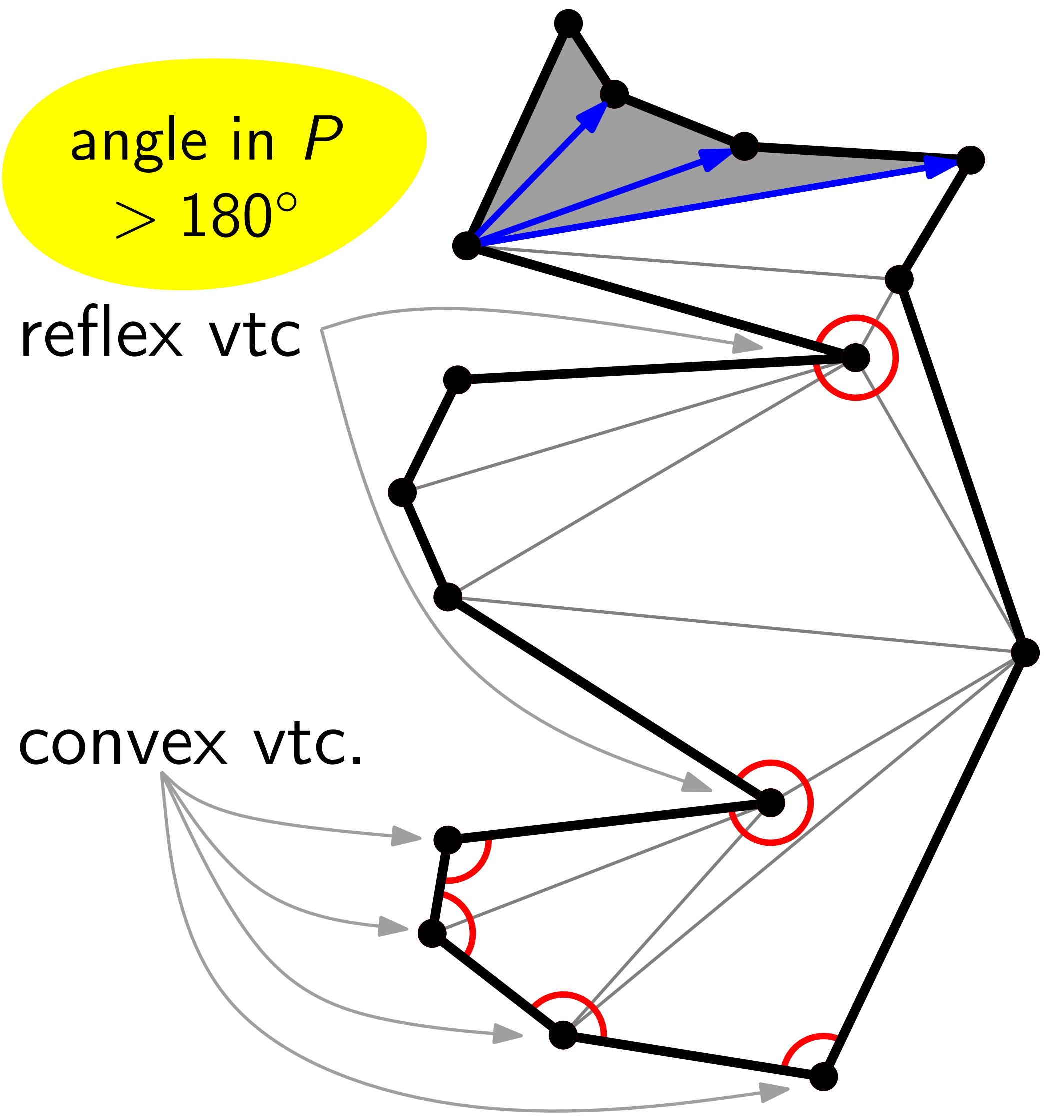

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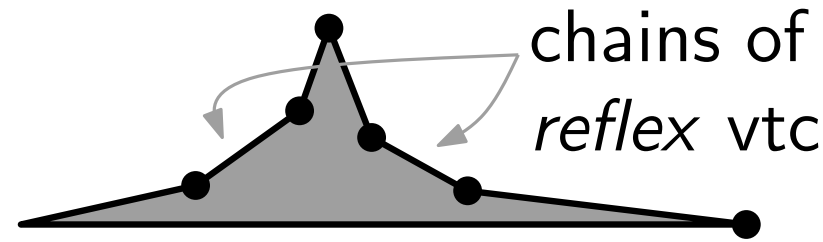

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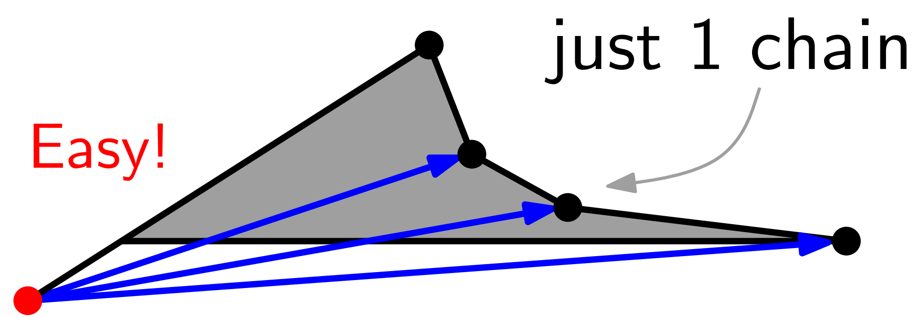

Algorithm

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merge left and right chain $\rightarrow$ sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$
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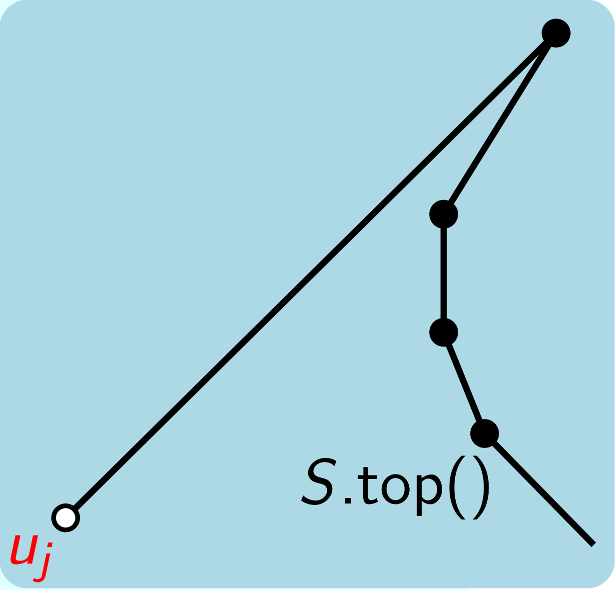

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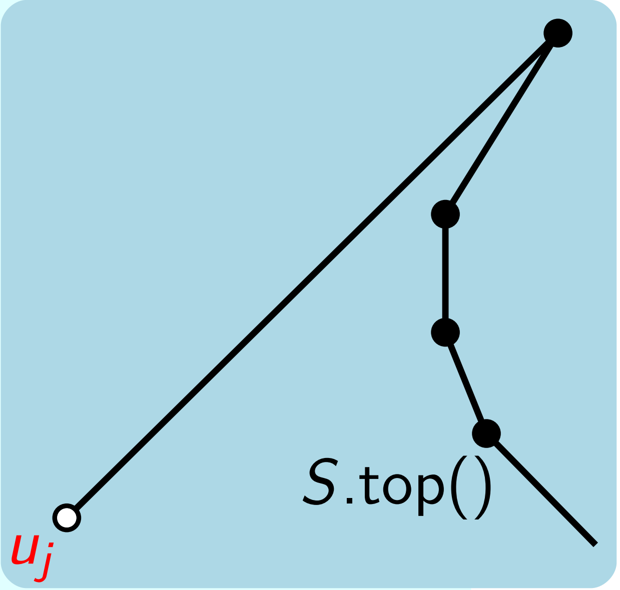

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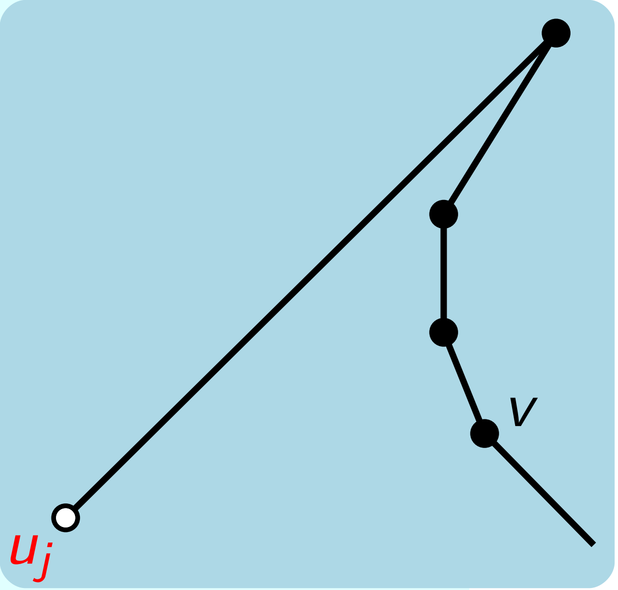

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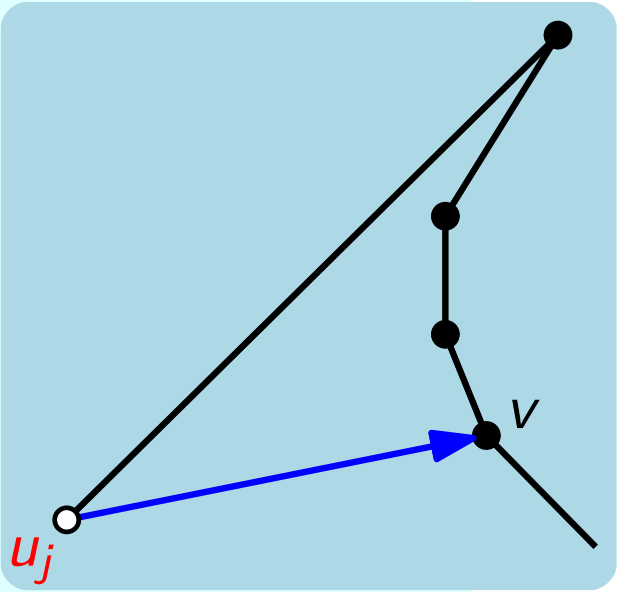

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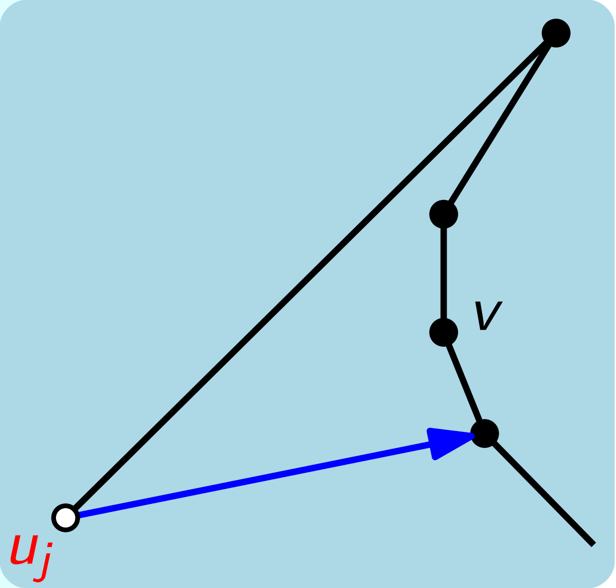

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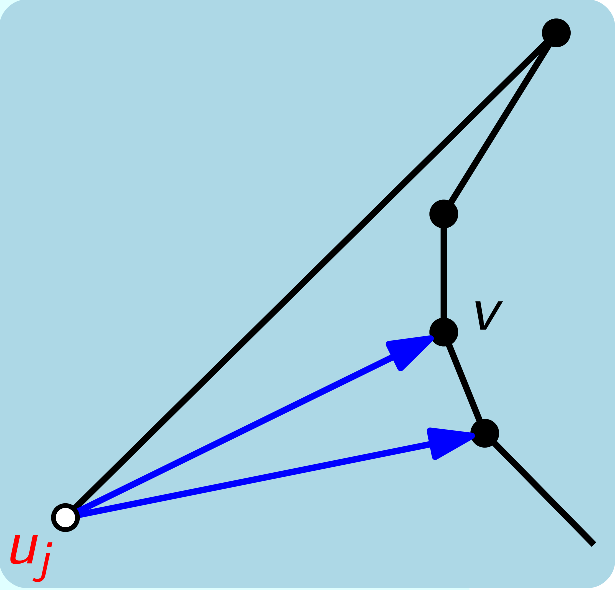

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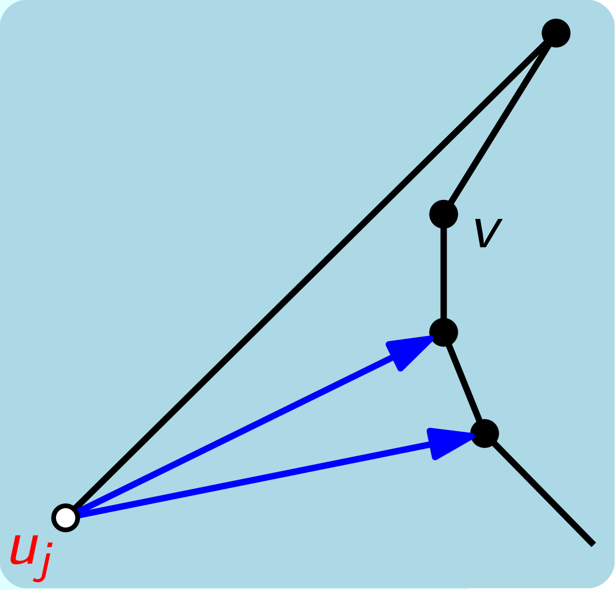

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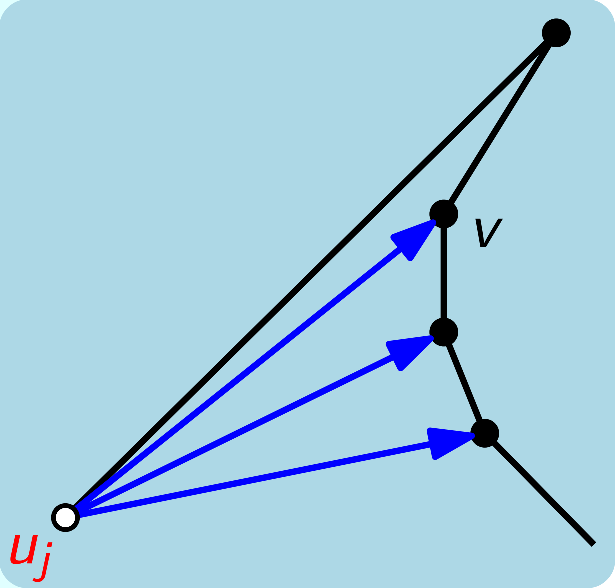

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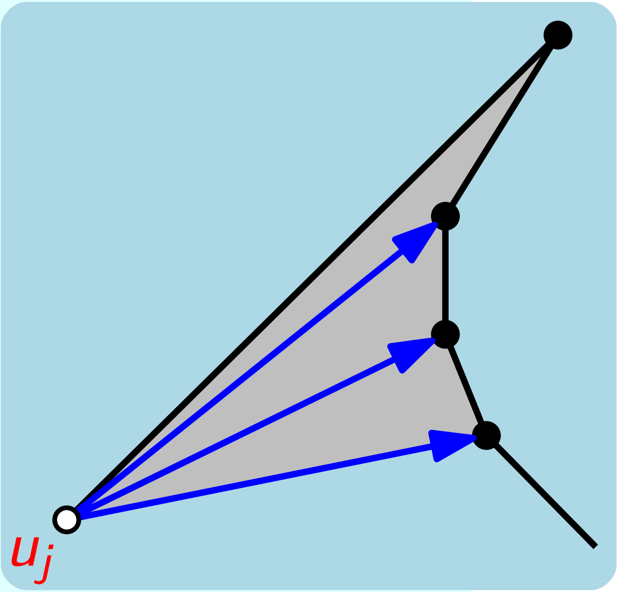

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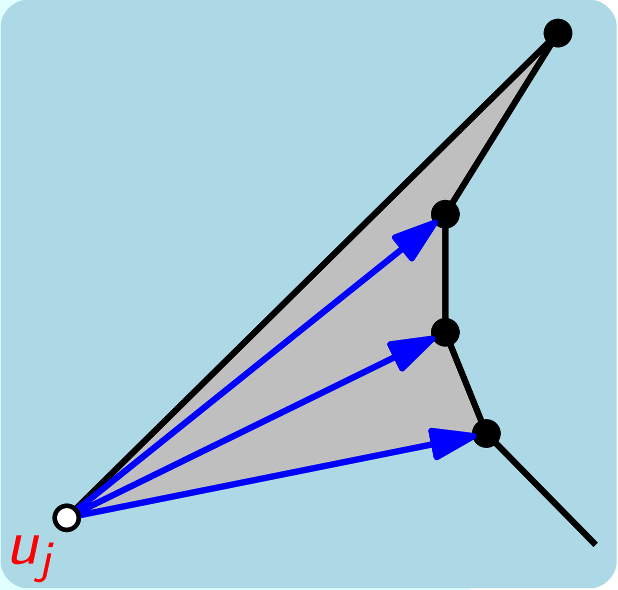

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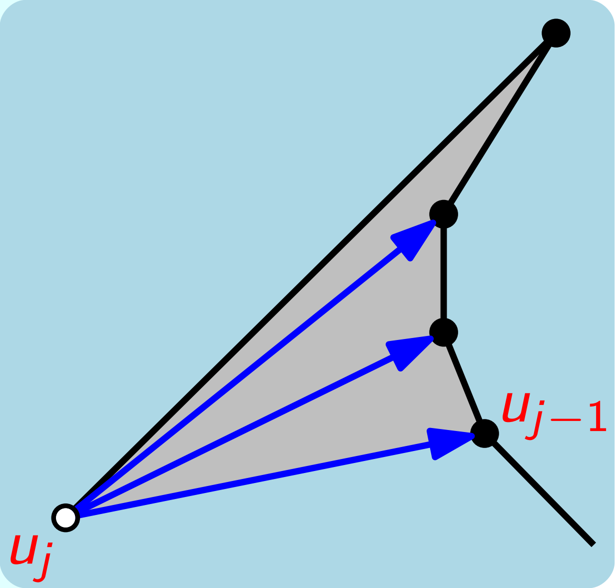

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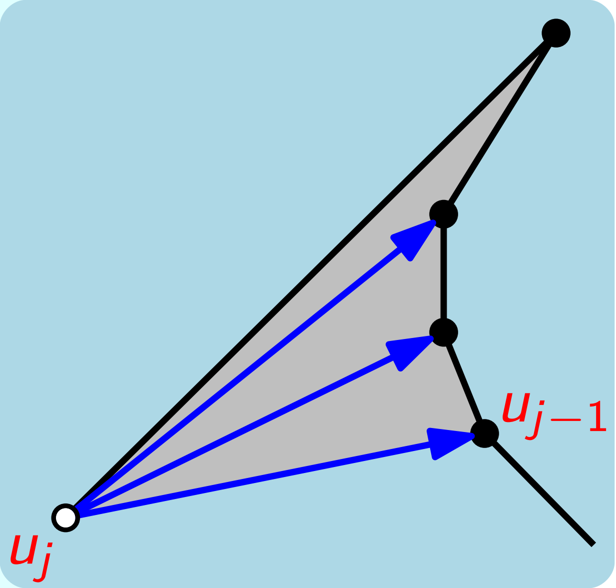

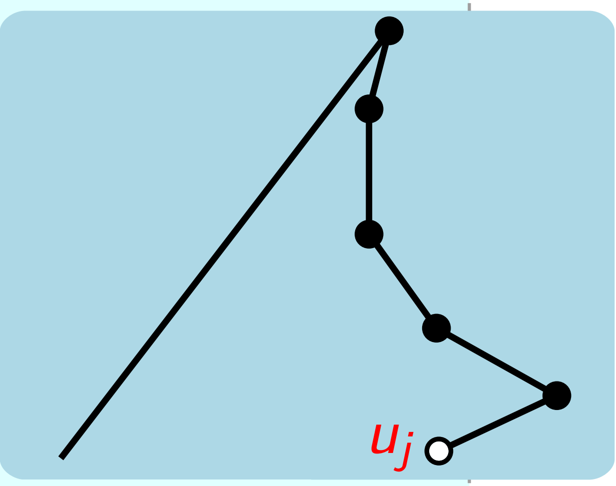

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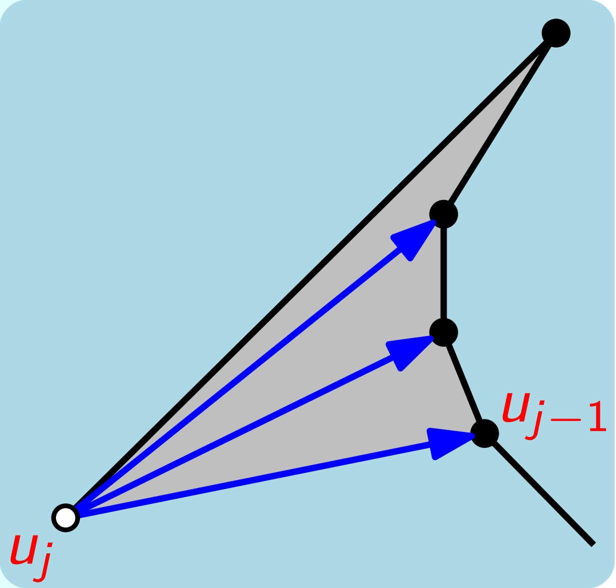

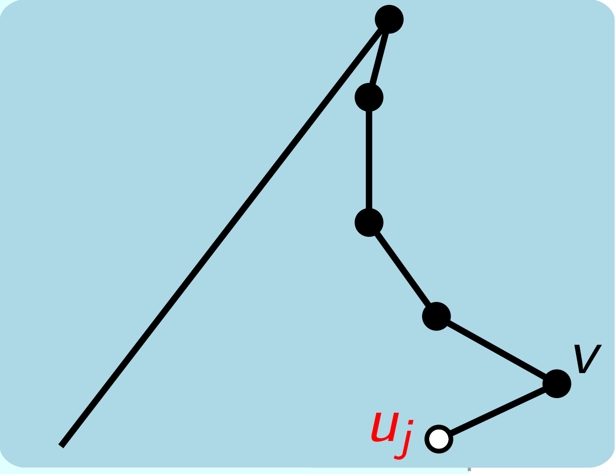

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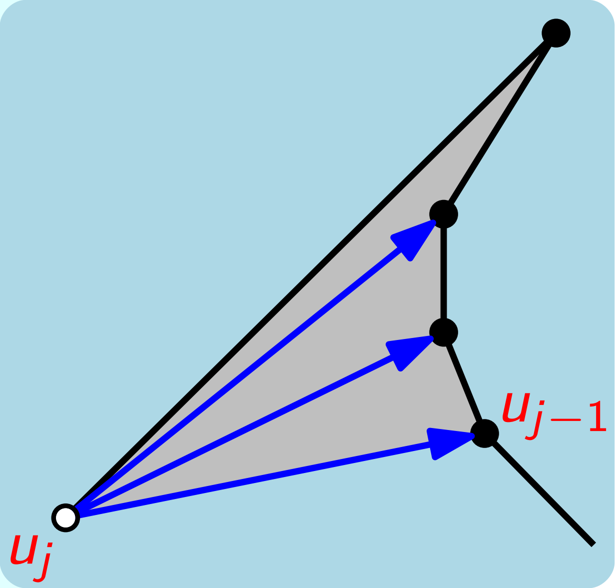

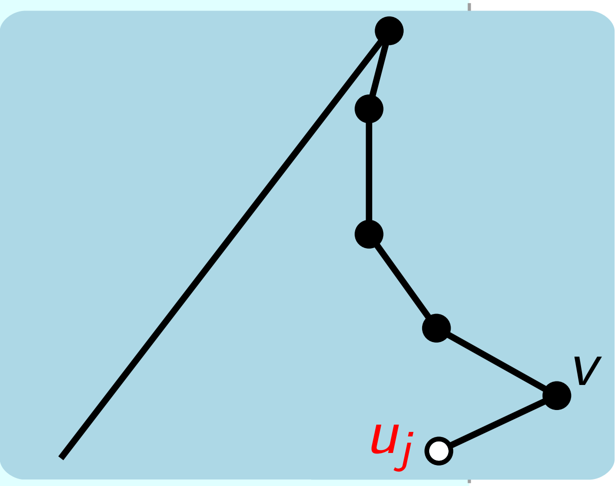

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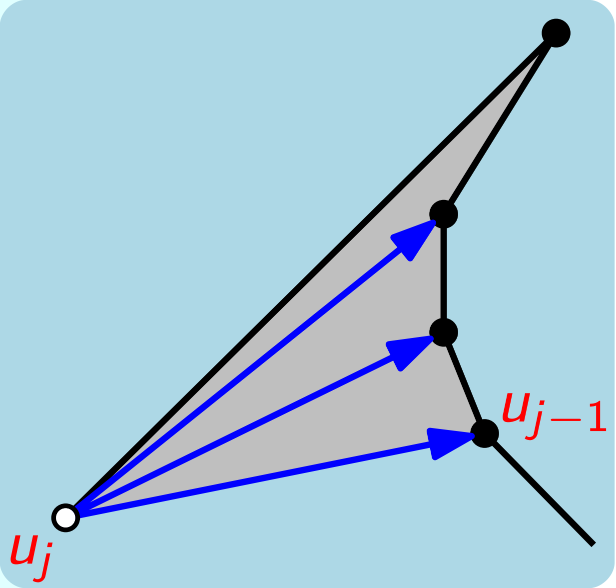

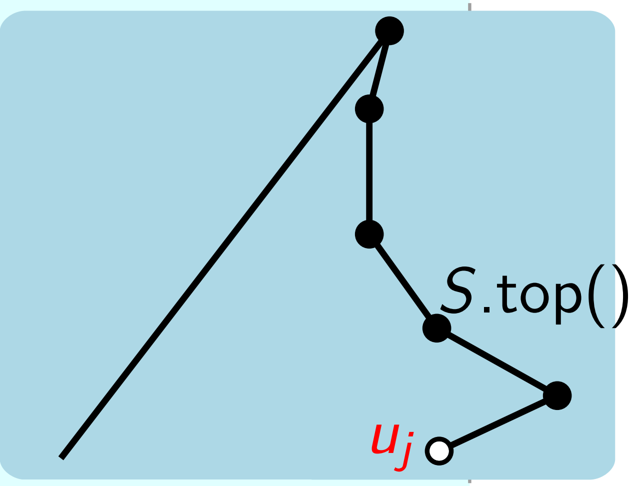

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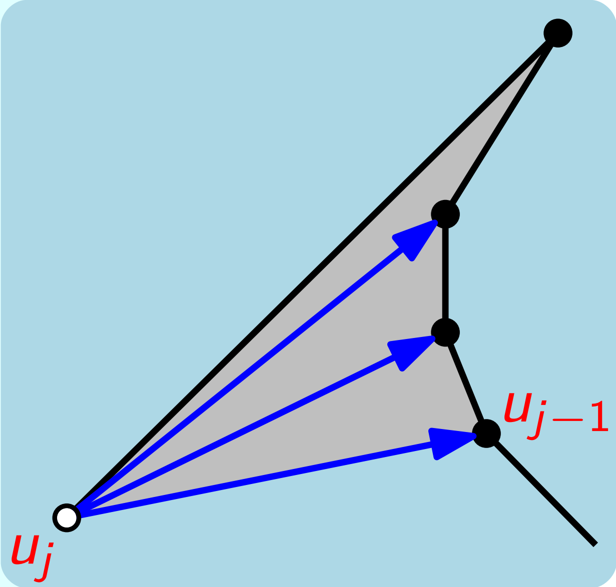

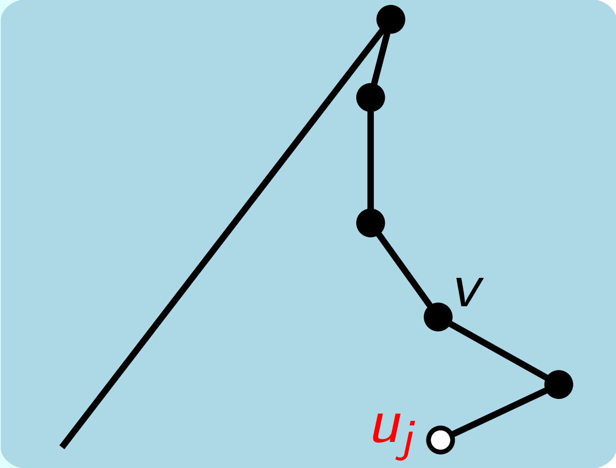

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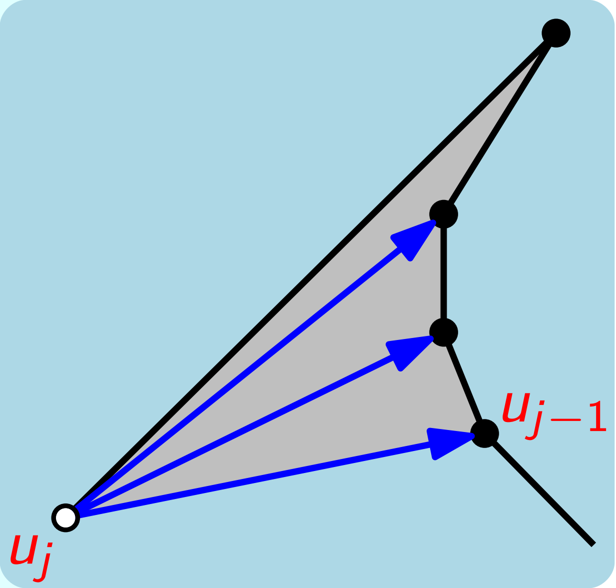

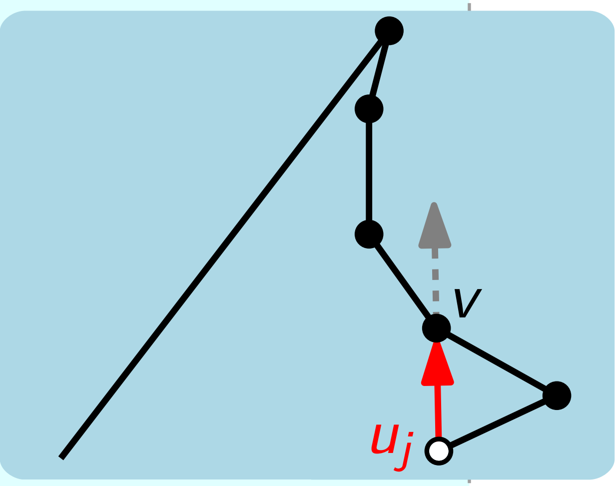

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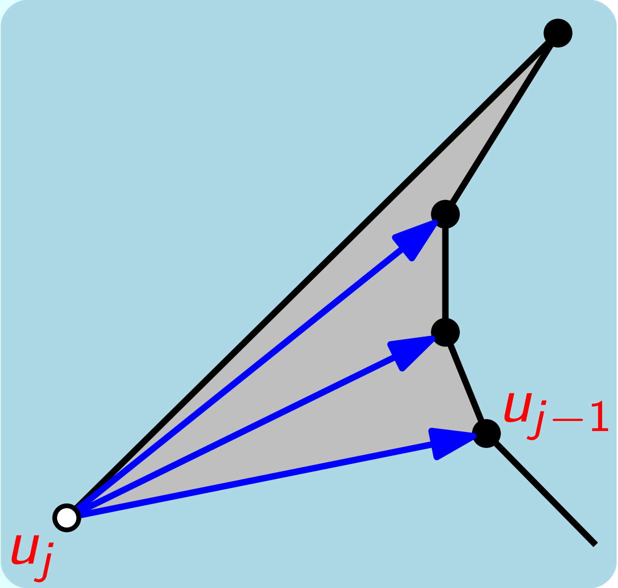

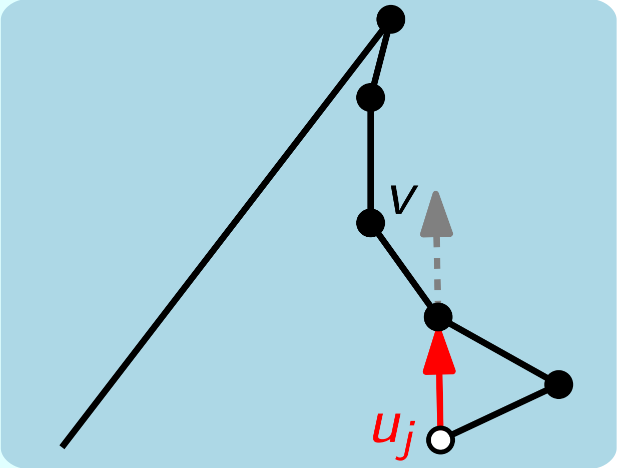

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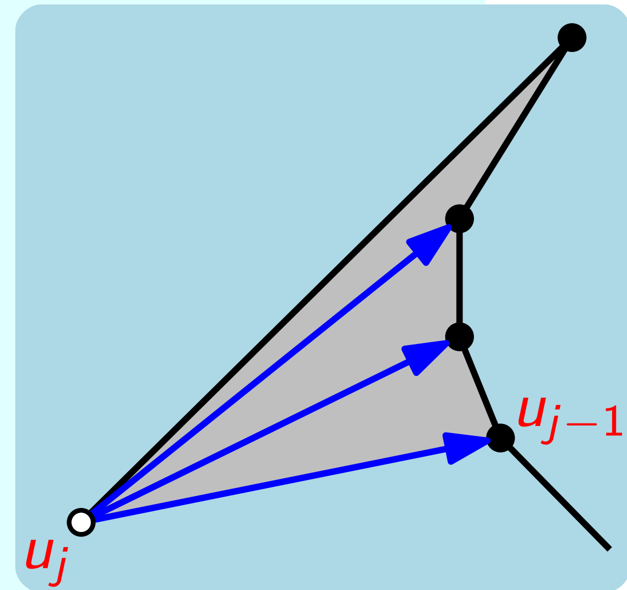
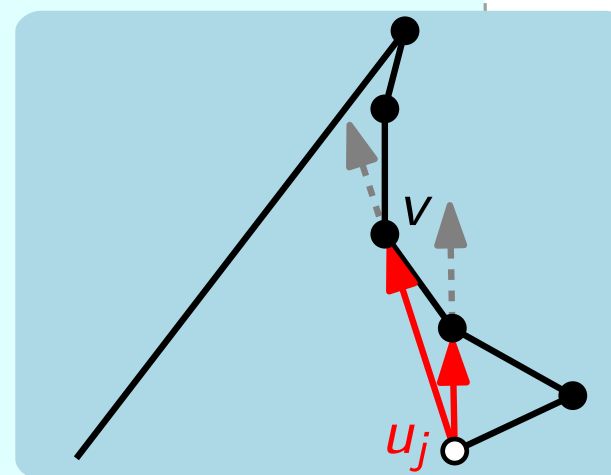
TriangulateMonotonePolygon(Polygon P as circular vertex list)
merge left and right chain $\rightarrow$ sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$
Stack S; S.push(u_1); S.push(u_2)
for $j \leftarrow 3$ **to** $n-1$ **do**
    if u_j and S.top() lie on different chains **then**
        while not S.empty() **do**
            $v \leftarrow S$.pop()
            if not S.empty() **then** draw diag. (u_j, v)
        S.push(u_{j-1}); S.push(u_j)
    else
        $v \leftarrow S$.pop()
        while not S.empty() **and** u_j sees S.top() **do**
            $v \leftarrow S$.pop()
            draw diagonal (u_j, v)





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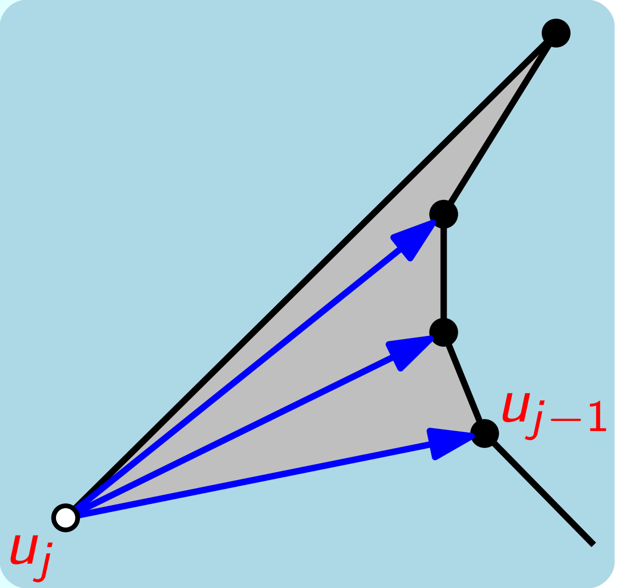

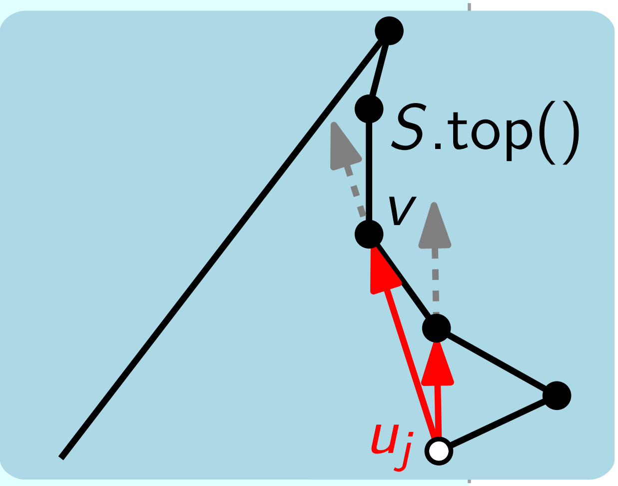

Algorithm

TriangulateMonotonePolygon(Polygon P as circular vertex list)

merge left and right chain $\rightarrow$ sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$

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  else

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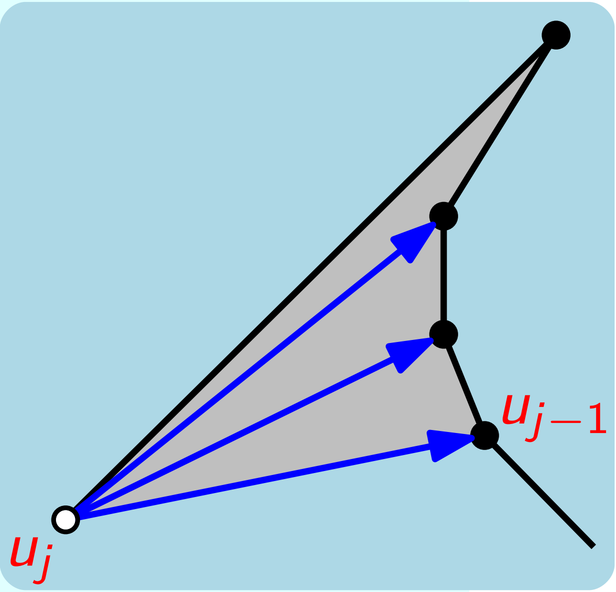
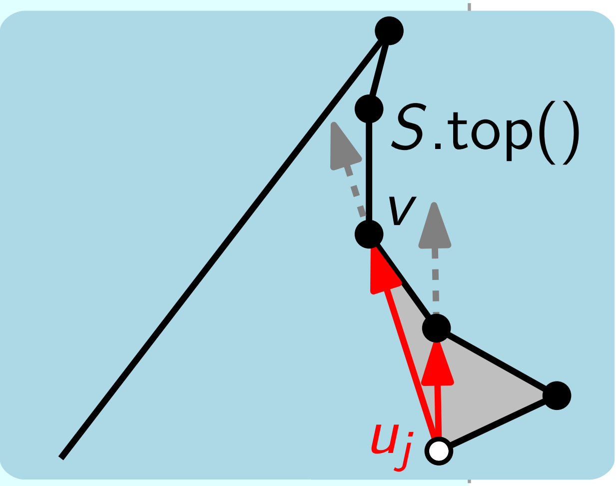
    while not S.empty() **and** u_j sees S.top() **do**

      $v \leftarrow S$.pop()

      draw diagonal (u_j, v)

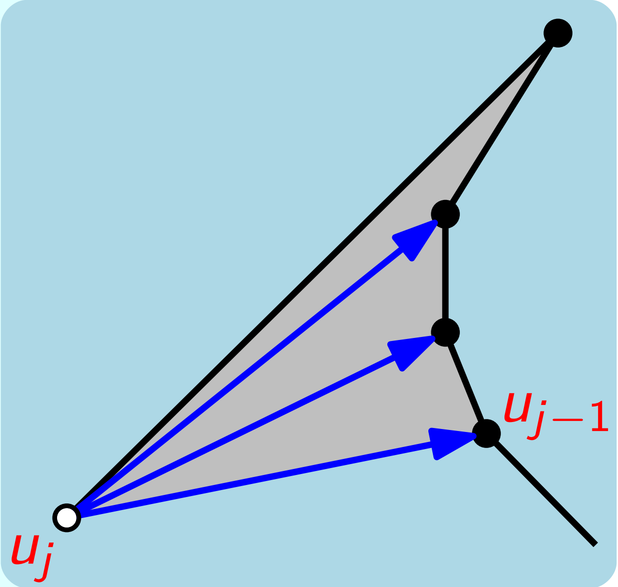


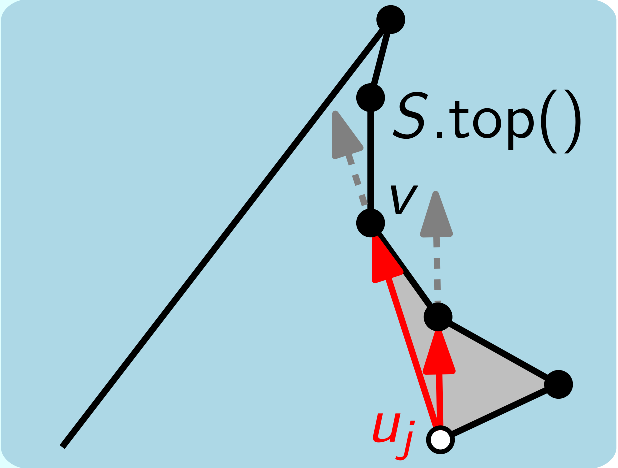


Algorithm

TriangulateMonotonePolygon(Polygon P as circular vertex list)

```
merge left and right chain → sequence u_1, ..., u_n with y_1 ≥ ··· ≥ y_n
Stack S; S.push(u_1); S.push(u_2)
for j ← 3 to n − 1 do
    if u_j and S.top() lie on different chains then
        while not S.empty() do
            v ← S.pop()
            if not S.empty() then draw diag. (u_j, v)
        S.push(u_{j−1}); S.push(u_j)
    else
        v ← S.pop()
        while not S.empty() and u_j sees S.top() do
            v ← S.pop()
            draw diagonal (u_j, v)
        S.push(v); S.push(u_j)
```





Algorithm

TriangulateMonotonePolygon(Polygon P as circular vertex list)

merge left and right chain $\rightarrow$ sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$

Stack S; S.push(u_1); S.push(u_2)

for $j \leftarrow 3$ **to** $n-1$ **do**

  if u_j and S.top() lie on different chains **then**

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      if not S.empty() **then** draw diag. (u_j, v)

    S.push(u_{j-1}); S.push(u_j)

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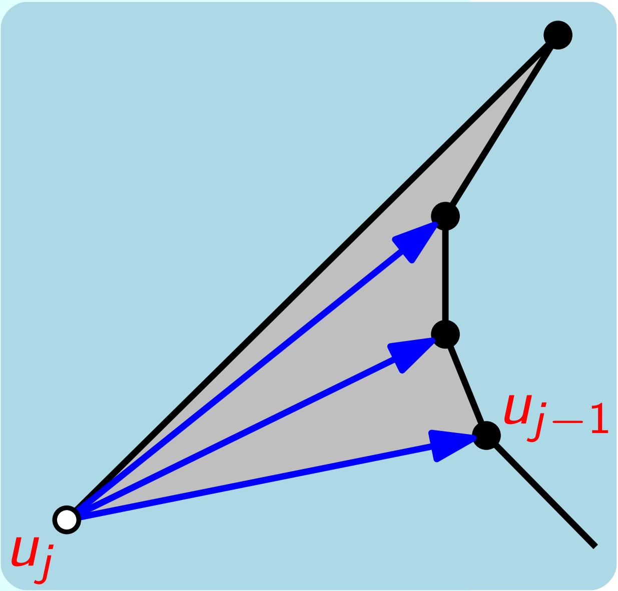
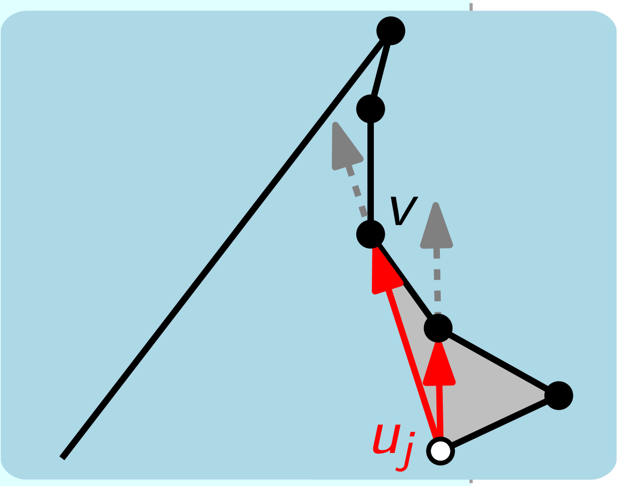
    while not S.empty() **and** u_j sees S.top() **do**

      $v \leftarrow S$.pop()

      draw diagonal (u_j, v)

    S.push(v); S.push(u_j)





Algorithm

TriangulateMonotonePolygon(Polygon P as circular vertex list)

merge left and right chain $\rightarrow$ sequence $u_1, \dots, u_n$ with $y_1 \geq \dots \geq y_n$

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      $v \leftarrow S$.pop()

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    S.push(u_{j-1}); S.push(u_j)

  else

    $v \leftarrow S$.pop()

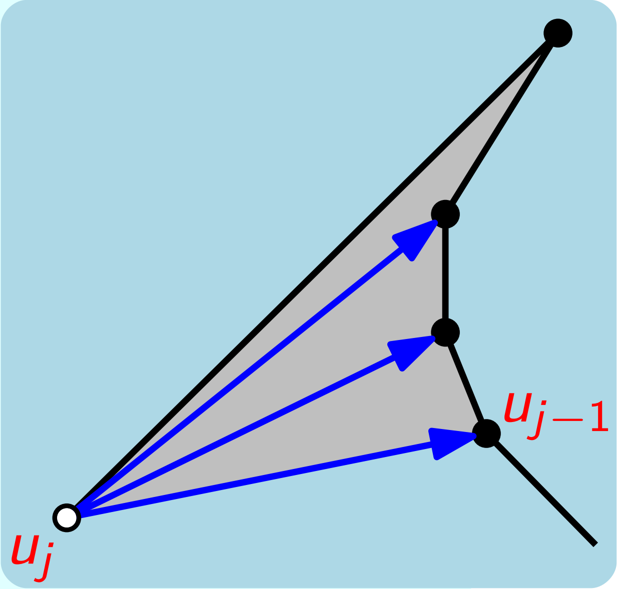
    while not S.empty() **and** u_j sees S.top() **do**

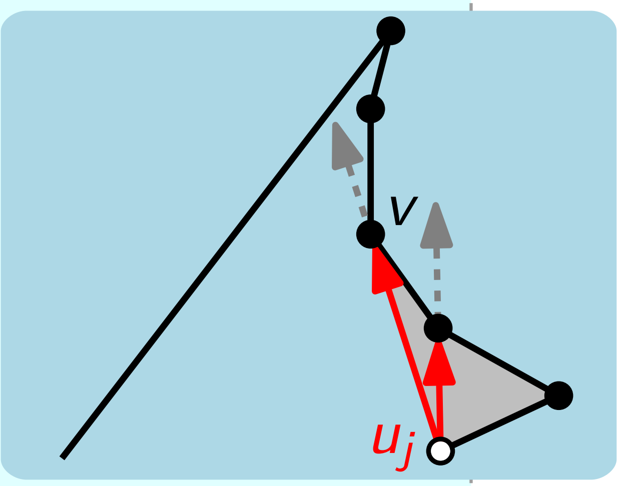
      $v \leftarrow S$.pop()

      draw diagonal (u_j, v)

    S.push(v); S.push(u_j)

draw diagonals from u_n to all vtc on S except first and last one





Algorithm

Running time?

TriangulateMonotonePolygon(Polygon P as circular vertex list)

merge left and right chain $\rightarrow$ sequence $u_1, \ldots, u_n$ with $y_1 \geq \cdots \geq y_n$

Stack S; S.push(u_1); S.push(u_2)

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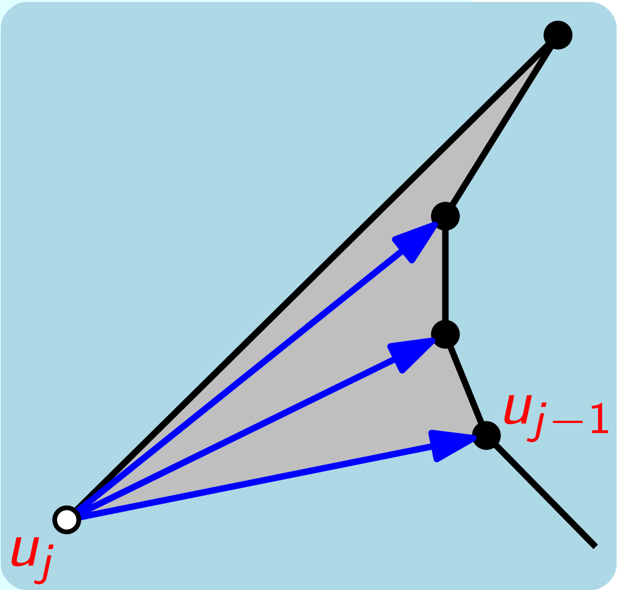
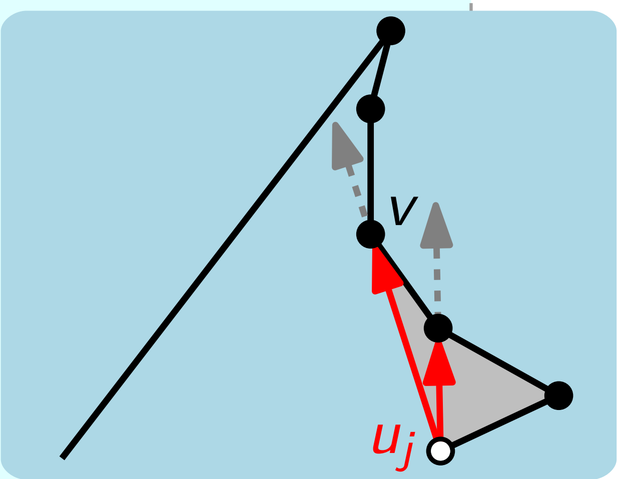
draw diagonals from u_n to all vtc on S except first and last one





Algorithm

Running time? $\Theta(n)$

TriangulateMonotonePolygon(Polygon P as circular vertex list)
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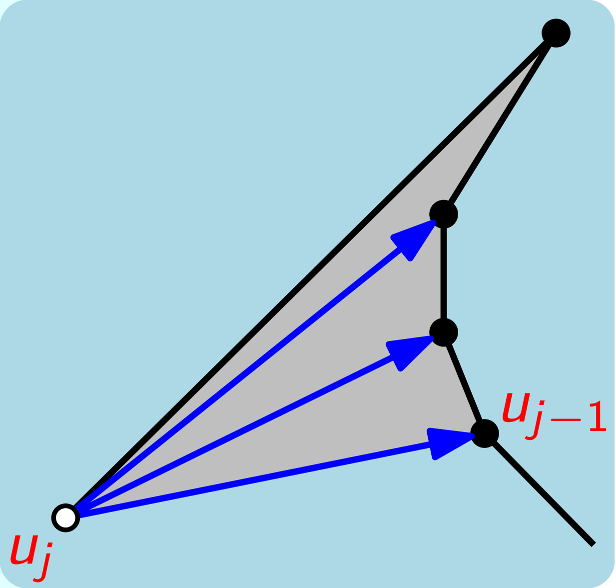
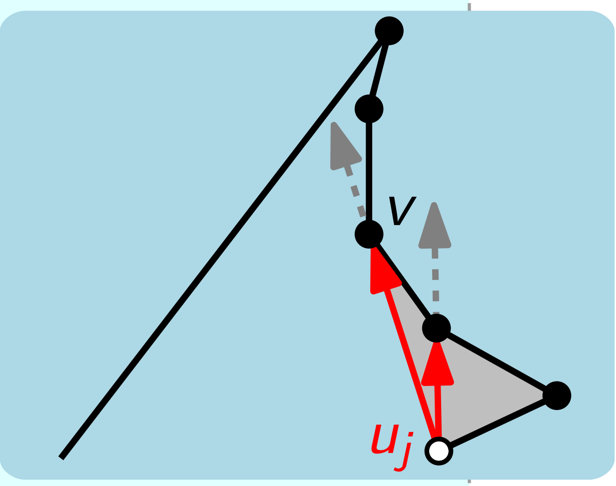
draw diagonals from u_n to all vtc on S except first and last one





Summary

n-vtx polygon $\xrightarrow[O(n \log n)]{}$ "nice" pieces, n' vtc $\xrightarrow[O(n')]{}$ n'' triangles

Summary

n-vtx polygon $\xrightarrow[O(n \log n)]{}$ "nice" pieces | n' vtc $\xrightarrow[O(n')]{}$ n'' triangles

Lemma. A simple polygon with n vertices can be subdivided into y-monotone polygons in $O(n \log n)$ time.

old

Summary

n-vtx polygon $\xrightarrow[O(n \log n)]{}$ "nice" pieces | n' vtc $\xrightarrow[O(n')]{}$ n'' triangles

Lemma. A simple polygon with n vertices can be subdivided into y-monotone polygons in $O(n \log n)$ time.

old

Lemma. A y-monotone polygon with n vertices can be triangulated in $O(n)$ time.

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n-vtx polygon $\xrightarrow[O(n \log n)]{}$ "nice" pieces | n' vtc $\xrightarrow[O(n')]{}$ n'' triangles

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A simple polygon with n vertices can be subdivided into y-monotone polygons in $O(n \log n)$ time.

Lemma. new

A y-monotone polygon with n vertices can be triangulated in $O(n)$ time.

Lemma. home-work

Subdividing a simple polygon with n vertices by drawing d (pairwise non-crossing) diagonals yields $d+1$ simple polygons of total complexity $O(n)$.

Summary

n-vtx polygon $\xrightarrow{O(n \log n)}$ "nice" pieces | n' vtc $\xrightarrow{O(n')}$ n'' triangles

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home-work

Theorem. A simple polygon with n vertices can be triangulated in $O(n \log n)$ time.

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Is this it?

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Is this it? Tarjan & van Wyk [1988]:

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Is this it? Tarjan & van Wyk [1988]: $O(n \log \log n)$

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Clarkson, Tarjan, van Wyk [1989]: $O(n \log^* n)$

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Tarjan & van Wyk [1988]:	$O(n \log \log n)$	Kirkpatrick, Klawe, Tarjan [1992]
Clarkson, Tarjan, van Wyk [1989]:	$O(n \log^* n)$	Seidel [1991]: *randomized*
Chazelle [1991]:	$O(n)$	