Approximation Algorithms

Lecture 6:  
k-CENTER via Parametric Pruning

Part I:  
Metric k-CENTER

Alexander Wolff  
Winter 2022/23
**Metric $k$-Center**

**Given:** A complete graph $G = (V, E)$ with edge costs $c: E \to \mathbb{Q}_{\geq 0}$ satisfying the triangle inequality and a natural number $k \leq |V|$.

For each vertex set $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from $v$ to a vertex in $S$.

**Find:** A $k$-element vertex set $S$ such that $\text{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.
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![Diagram showing $S_1$, $S_2$, and $S_3$.]
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Approximation Algorithms

Lecture 6:
$k$-CENTER via Parametric Pruning

Part II:
Parametric Pruning
Parametric Pruning

Let \( E = \{e_1, \ldots, e_m\} \) with \( c(e_1) \leq \ldots \leq c(e_m) \).

Suppose we know that \( \text{OPT} = c(e_j) \).

\( G \)
Let $E = \{e_1, \ldots, e_m\}$ with $c(e_1) \leq \ldots \leq c(e_m)$.
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![Diagram of a graph $G$ with vertices $s_1, s_2, s_3$ and edge $e_j$.]
Parametric Pruning

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$$G := (V, \{e_1, \ldots, e_j\})$$

$$G_j := (V, \{e_1, \ldots, e_j\})$$
Parametric Pruning

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\[ G_j := (V, \{e_1, \ldots, e_j\}) \]

\[ s_1 \quad s_2 \quad s_3 \]

\[ e_j \]

\[ G \]

\[ \text{... try each } G_j. \]
... try each $G_j$.

Def.

$G_j := (V, \{e_1, \ldots, e_j\})$
...try each $G_j$.

**Def.** A vertex set $D$ of a graph $H$ is **dominating** if each vertex is either in $D$ or adjacent to a vertex in $D$.

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\text{dom}(G_j) \leq k
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\text{dom}(G_j) \leq k
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$$G_j := (V, \{e_1, \ldots, e_j\})$$

...but computing $\text{dom}(H)$ is NP-hard.
Approximation Algorithms

Lecture 6:
\( k\)-Center via Parametric Pruning

Part III:
Square of a Graph
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**Idea:** Find a small dominating set in a "coarsened" $G_j$. 

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Obs. A dominating set with at most $k$ elements in $G_j^2$ is a 2-approximation for metric $k$-Center.
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Why? $\max_{e \in E(G_j)} c(e) = \text{OPT}$!
Independent Sets

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Obs. Maximal independent sets are dominating sets :-)

$I$
Lemma. For a graph $H$ and an independent set $I$ in $H^2$, $|I| \leq \text{dom}(H)$. 
Independent Sets in $H^2$

**Lemma.** For a graph $H$ and an independent set $I$ in $H^2$, $|I| \leq \text{dom}(H)$.

What does a dominating set of $H$ look like in $H^2$?
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Part IV:  
Factor-2 Approximation for Metric-k-Center

Alexander Wolff  
Winter 2022/23
Factor-2 Approx. for Metric $k$-CENTER

Metric-$k$-CENTER($G = (V, E; c), k$)
Sort the edges of $G$ by cost: $c(e_1) \leq \ldots \leq c(e_m)$
Factor-2 Approx. for Metric $k$-Center

Metric-$k$-Center($G = (V, E; c), k$)

Sort the edges of $G$ by cost: $c(e_1) \leq \ldots \leq c(e_m)$

for $j = 1$ to $m$ do

...
Factor-2 Approx. for Metric $k$-Center

**Metric-$k$-Center**($G = (V, E; c), k$)

Sort the edges of $G$ by cost: $c(e_1) \leq \ldots \leq c(e_m)$

for $j = 1$ to $m$ do

   Construct $G^2_j$

   Find a maximal independent set $I_j$ in $G^2_j$

   if $|I_j| \leq k$ then
      return $I_j$
Factor-2 Approx. for Metric $k$-Center

Metric-$k$-Center($G = (V, E; c), k$)

Sort the edges of $G$ by cost: $c(e_1) \leq \ldots \leq c(e_m)$

for $j = 1$ to $m$ do

Construct $G_j^2$

Find a maximal independent set $I_j$ in $G_j^2$

return $I_j$
Metric-k-CENTER(G = (V, E; c), k)

Sort the edges of G by cost: \( c(e_1) \leq \ldots \leq c(e_m) \)

for \( j = 1 \) to \( m \) do
  Construct \( G^2_j \)
  Find a maximal independent set \( I_j \) in \( G^2_j \)
  if \( |I_j| \leq k \) then
    return \( I_j \)
Lemma. For $j$ provided by the algorithm, it holds that $c(e_j) \leq \text{OPT}$.
Factor-2 Approx. for Metric $k$-Center

Metric-$k$-Center($G = (V, E; c), k$)

Sort the edges of $G$ by cost: $c(e_1) \leq \ldots \leq c(e_m)$

for $j = 1$ to $m$ do
    Construct $G_j^2$
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    if $|I_j| \leq k$ then
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Lemma. For $j$ provided by the algorithm, it holds that $c(e_j) \leq \text{OPT}$. 

Theorem. The above algorithm is a factor-2 approximation algorithm for the metric $k$-Center problem.
Can we do better ...?
Can we do better . . . ?

What about a tight example?
Can we do better . . .?

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What about a tight example?

**Theorem.** Assuming \( P \neq NP \), there is no factor-\((2 - \varepsilon)\) approximation algorithm for the metric \( k \)-Center problem, for any \( \varepsilon > 0 \).
Can we do better . . .?

What about a tight example?

**Theorem.** Assuming $P \neq NP$, there is no factor-$(2 - \varepsilon)$ approximation algorithm for the metric $k$-Center problem, for any $\varepsilon > 0$.

**Proof.** Reduce from dominating set to metric $k$-Center.
Can we do better . . . ?

What about a tight example?

**Theorem.** Assuming P $\neq$ NP, there is no factor-$(2 - \varepsilon)$ approximation algorithm for the metric $k$-Center problem, for any $\varepsilon > 0$.

**Proof.** Reduce from dominating set to metric $k$-Center. Given graph $G = (V, E)$ and integer $k$,
Can we do better . . .?

What about a tight example?

**Theorem.** Assuming P ≠ NP, there is no factor-\((2 - \varepsilon)\) approximation algorithm for the metric \(k\)-Center problem, for any \(\varepsilon > 0\).

**Proof.** Reduce from dominating set to metric \(k\)-Center. Given graph \(G = (V, E)\) and integer \(k\),
Can we do better . . . ?

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\[\triangle\text{-inequality holds}\]

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Approximation Algorithms

Lecture 6:
k-CENTER via Parametric Pruning

Part V:
Metric-Weighted-Center

Alexander Wolff
Winter 2022/23
Metric-$k$-Center

Given: A complete graph $G = (V, E)$ with metric edge costs $c: E \rightarrow \mathbb{Q}_{\geq 0}$ and a natural number $k \leq |V|$.

For $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from $v$ to a vertex in $S$.

Find: A $k$-element vertex set $S$ such that $\text{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.
**Metric-$k$-Center**

**Weighted**

**Given:** A complete graph $G = (V, E)$ with metric edge costs $c : E \to \mathbb{Q}_{\geq 0}$ and a natural number $k \leq |V|$.

For $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from $v$ to a vertex in $S$.

**Find:** A $k$-element vertex set $S$ such that $\text{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.
**Metric-k-Center**

**Weighted**

**Given:** A complete graph $G = (V, E)$ with metric edge costs $c: E \rightarrow \mathbb{Q}_{\geq 0}$ and a natural number $k \leq |V|$, vertex weights $w: V \rightarrow \mathbb{Q}_{\geq 0}$ and a budget $W \in \mathbb{Q}_+$.

For $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from $v$ to the vertex in $S$.

**Find:** A $k$-element vertex set $S$ such that $\text{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.
**Metric-k-Center**

**Weighted**

**Given:** A complete graph \( G = (V, E) \) with metric edge costs \( c : E \to \mathbb{Q}_{\geq 0} \) and a natural number \( k \leq |V| \), vertex weights \( w : V \to \mathbb{Q}_{\geq 0} \) and a budget \( W \in \mathbb{Q}_+ \).

For \( S \subseteq V \), \( c(v, S) \) is the cost of the cheapest edge from \( v \) to the vertex in \( S \).

**Find:** A \( k \)-element vertex set \( S \) such that the cost of \( S \) is minimized:

\[
\text{cost}(S) := \max_{v \in V} c(v, S)
\]
Algorithm for the Weighted Version

Algorithm Metric-Center

Sort the edges of $G$ by cost: $c(e_1) \leq \ldots \leq c(e_m)$

for $j = 1, \ldots, m$ do

    Construct $G^2_j$

    Find a maximal independent set $I_j$ in $G^2_j$

    if $|I_j| \leq k$ then
        return $I_j$
Algorithm for the Weighted Version

Algorithm Metric-\textbf{Weighted-Center}

Sort the edges of $G$ by cost: $c(e_1) \leq \ldots \leq c(e_m)$

\textbf{for} $j = 1, \ldots, m$ \textbf{do}

\hspace{1em} Construct $G_j^2$

\hspace{2em} Find a maximal independent set $I_j$ in $G_j^2$

\hspace{1em} \textbf{if} $|I_j| \leq k$ \textbf{then}

\hspace{2em} \textbf{return} $I_j$
Algorithm Metric-Weighted-Center

Sort the edges of $G$ by cost: $c(e_1) \leq \ldots \leq c(e_m)$

for $j = 1, \ldots, m$ do
  Construct $G_j^2$
  Find a maximal independent set $I_j$ in $G_j^2$
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what about the weights?
Algorithm Metric-Weighted-Center

Sort the edges of $G$ by cost: $c(e_1) \leq \ldots \leq c(e_m)$

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Construct $G^2_j$

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  Find a maximal independent set $I_j$ in $G^2_j$

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$s_j(u) := \text{lightest node in } N_{G_j}(u) \cup \{u\}$
Algorithm for the Weighted Version

Algorithm Metric-Weighted-Center

Sort the edges of $G$ by cost: $c(e_1) \leq \ldots \leq c(e_m)$

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end if

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Sort the edges of $G$ by cost: $c(e_1) \leq \ldots \leq c(e_m)$

for $j = 1, \ldots, m$ do

- Construct $G_j^2$
- Find a maximal independent set $I_j$ in $G_j^2$
- Compute $S_j := \{ s_j(u) \mid u \in I_j \}$

if $|I_j| \leq k$ then

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Algorithm Metric-Weighted-Center

Sort the edges of $G$ by cost: $c(e_1) \leq \ldots \leq c(e_m)$

for $j = 1, \ldots, m$ do

    Construct $G_j^2$

    Find a maximal independent set $I_j$ in $G_j^2$

    Compute $S_j := \{ s_j(u) | u \in I_j \}$

    if $|I_j| \leq k$ then

        $w(S_j) \leq W$

        return $I_j$

    end if

end for

$s_j(u) := \text{lightest node in } N_{G_j}(u) \cup \{u\}$
**Algorithm for the Weighted Version**

**Algorithm Metric-Weighted-Center**

Sort the edges of $G$ by cost: $c(e_1) \leq \ldots \leq c(e_m)$

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- If $|I_j| \leq k$ then
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Algorithm Metric-Weighted-Center

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Compute $S_j := \{ s_j(u) \mid u \in I_j \}$

if $|I_j| \leq k$ then

return $I_j$

$s_j(u) := \text{lightest node in } N_{G_j}(u) \cup \{u\}$

$w(S_j) \leq W$

$u \in I_j$

$S_j \leq 3c(e_j)$
Algorithm for the Weighted Version

Algorithm Metric-Weighted-Center

Sort the edges of $G$ by cost: $c(e_1) \leq \ldots \leq c(e_m)$

for $j = 1, \ldots, m$ do

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if $|I_j| \leq k$ then

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$s_j(u) := \text{lightest node in } N_{G_j}(u) \cup \{u\}$

$w(S_j) \leq W$

$u \in I_j$

$s_j(u) \leq 3c(e_j)$

Theorem. The above is a factor-3 approximation algorithm for Metric-Weighted-Center.
Tight Example...?

Here, we need to have a budget $W$, and edge costs satisfying the triangle inequality.
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Consider $W = 3$. 
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$w(a) = 1$
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Here, we need to have a budget $W$, and edge costs satisfying the triangle inequality.

Consider $W = 3$. 

$w(a) = 1$

$w(\cdot) = 2$
Tight Example... ?

Here, we need to have a budget $W$, and edge costs satisfying the triangle inequality.

Consider $W = 3$. 

\[ w(a) = 1 \]

\[ w(\cdot) = 2 \]
Here, we need to have a budget $W$, and edge costs satisfying the triangle inequality.

Consider $W = 3$. 

$w(\cdot) = 4$
Tight Example...?

Here, we need to have a budget $W$, and edge costs satisfying the triangle inequality.

Consider $W = 3$. 

$w(\cdot) = 4$

$w(a) = 1$

$w(\cdot) = 2$
Tight Example...?

Here, we need to have a budget $W$, and edge costs satisfying the triangle inequality.

Consider $W = 3$.

Other edge costs?
Tight Example...?

Here, we need to have a budget $W$, and edge costs satisfying the triangle inequality.

Consider $W = 3$.

Other edge costs?

$w(\cdot) = 2$

$1 + \varepsilon$

$w(a) = 1$

$w(\cdot) = 4$
Tight Example...?

Here, we need to have a budget $W$, and edge costs satisfying the triangle inequality.

Consider $W = 3$. Other edge costs?

$w(\cdot) = 2$

$w(\cdot) = 4$

$w(a) = 1$

OPT?

ALG?
Here, we need to have a budget $W$, and edge costs satisfying the triangle inequality.

Consider $W = 3$.

Other edge costs? \( \rightarrow \) metric completion!

$w(\cdot) = 2$

OPT? pick $a$ and $c \Rightarrow$ cost $1 + \varepsilon$.

ALG?
Here, we need to have a budget $W$, and edge costs satisfying the triangle inequality.

Consider $W = 3$.

Other edge costs? $\rightarrow$ metric completion!

OPT? pick $a$ and $c \Rightarrow$ cost $1 + \epsilon$.

ALG? since $N_{G^2}(b) = G$, $\{b\}$ is a maximal independent set in $G^2$
Tight Example...?

Here, we need to have a budget $W$, and edge costs satisfying the triangle inequality.

Consider $W = 3$.

Other edge costs?
→ metric completion!

OPT?
pick $a$ and $c$ ⇒ cost $1 + \epsilon$.

ALG?
since $N_{G^2}(b) = G$, $\{b\}$ is a maximal independent set in $G^2$
Thus, alg. picks only $a$ ⇒ cost 3.
Here, we need to have a budget $W$, and edge costs satisfying the triangle inequality.

Consider $W = 3$.

Other edge costs? $\to$ metric completion!

$w(\cdot) = 2$

OPT? pick $a$ and $c \Rightarrow$ cost $1 + \varepsilon$.

ALG? since $N_{G^2}(b) = G$, $\{b\}$ is a maximal independent set in $G^2$

Thus, alg. picks only $a \Rightarrow$ cost $3$.

How can we generalize this to larger $W$?
Here, we need to have a budget $W$, and edge costs satisfying the triangle inequality.

Consider $W = 3$. Other edge costs? → metric completion!

$w(\cdot) = W + 1$

OPT? pick $a$ and $c \Rightarrow$ cost $1 + \varepsilon$.

ALG? since $N_{G^2}(b) = G$, $\{b\}$ is a maximal independent set in $G^2$.

Thus, alg. picks only $a \Rightarrow$ cost $3$.

How can we generalize this to larger $W$?