Visualization of Graphs

Lecture 11: Beyond Planarity
Drawing Graphs with Crossings

Part I:
Graph Classes and Drawing Styles

Alexander Wolff

Partially based on slides by Fabrizio Montecchini, Michalis Bekos, and Walter Didimo.
Planar Graphs

Planar graphs admit drawings in the plane without crossings.
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Plane graph is a planar graph with a plane embedding = rotation system.
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Planarity is recognizable in linear time.
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Different drawing styles...
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Different drawing styles...

straight-line drawing
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Different drawing styles...

straight-line drawing  orthogonal drawing
Planar Graphs

Planar graphs admit drawings in the plane without crossings.

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Planarity is recognizable in linear time.

Different drawing styles...

- straight-line drawing
- orthogonal drawing
- grid drawing with bends & 3 slopes
Planar Graphs

Planar graphs admit drawings in the plane without crossings.

Plane graph is a planar graph with a plane embedding = rotation system.

Planarity is recognizable in linear time.

Different drawing styles...

- straight-line drawing
- orthogonal drawing
- grid drawing with bends & 3 slopes
- circular-arc drawing
And Non-Planar Graphs?

We have seen a few drawing styles:
And Non-Planar Graphs?

We have seen a few drawing styles:

force-directed drawing
And Non-Planar Graphs?

We have seen a few drawing styles:

force-directed drawing

hierarchical drawing
And Non-Planar Graphs?

We have seen a few drawing styles:

- force-directed drawing
- hierarchical drawing
- orthogonal layouts (via planarization)
And Non-Planar Graphs?

We have seen a few drawing styles:

- force-directed drawing
- hierarchical drawing
- orthogonal layouts (via planarization)

Maybe not all crossings are equally bad?
And Non-Planar Graphs?

We have seen a few drawing styles:

- force-directed drawing
- hierarchical drawing
- orthogonal layouts (via planarization)

Maybe not all crossings are equally bad?

block crossings
And Non-Planar Graphs?

We have seen a few drawing styles:

- force-directed drawing
- hierarchical drawing
- orthogonal layouts (via planarization)

Maybe not all crossings are equally bad?

Which crossings feel worse?
Eye-Tracking Experiment

Input: A graph drawing and designated path.
Eye-Tracking Experiment

**Input:** A graph drawing and designated path.

[Eades, Hong & Huang 2008]
Eye-Tracking Experiment

**Input:** A graph drawing and designated path.

**Task:** Trace path and count number of edges.

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**Results:**
Eye-Tracking Experiment

**Input:** A graph drawing and designated path.

**Task:** Trace path and count number of edges.

**Results:**
- no crossings
- eye movements smooth and fast
Eye-Tracking Experiment

**Input:** A graph drawing and designated path.

**Task:** Trace path and count number of edges.

**Results:**
- no crossings
- large crossing angles
- eye movements smooth and fast
- eye movements smooth but slightly slower
Eye-Tracking Experiment

**Input:** A graph drawing and designated path.

**Task:** Trace path and count number of edges.

**Results:**
- no crossings: eye movements smooth and fast
- large crossing angles: eye movements smooth but slightly slower
- small crossing angles: eye movements no longer smooth and very slow (back-and-forth movements at crossing points)

[Eades, Hong & Huang 2008]
Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize “bad” crossing configurations.
Some Beyond-Planar Graph Classes

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$k$-planar ($k = 1$)
Some Beyond-Planar Graph Classes

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\[ k \text{-planar (} k = 1 \text{)} \quad k \text{-quasi-planar (} k = 3 \text{)} \]
Some Beyond-Planar Graph Classes

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\(k\)-planar \( (k = 1) \)

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fan-planar
Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize “bad” crossing configurations.

- **$k$-planar** ($k = 1$)
- **$k$-quasi-planar** ($k = 3$)
- **fan-planar**
- **RAC**
Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize “bad” crossing configurations.

- $k$-planar ($k = 1$)
- $k$-quasi-planar ($k = 3$)
- Fan-planar
- Right-angle crossing (RAC)
Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize “bad” crossing configurations.

- **$k$-planar ($k = 1$)**
- **$k$-quasi-planar ($k = 3$)**
- **fan-planar**

**Topological graphs**

- ✓
- ×
- ✓
- ×

**Geometric graphs**

- ✓
- ×

**RAC**

- right-angle crossing
- ✓
- ×
Some Beyond-Planar Graph Classes

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There are many more beyond-planar graph classes. . .
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There are many more beyond-planar graph classes...
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There are many more beyond-planar graph classes. . .

- **fan-planar**
- **RAC** (right-angle crossing)
- **IC** (independent crossing)
- **fan-crossing-free**
- **skewness-\(k\) (\(k = 2\))**

Combinations, . . .
Drawing Styles for Crossings

RAC
right-angle crossing
Drawing Styles for Crossings

RAC
right-angle crossing

orthogonal

slanted orthogonal
Drawing Styles for Crossings

RAC
right-angle crossing

orthogonal

slanted orthogonal

block / bundled crossings

circular layout: 28 individual vs. 12 bundle crossings
Drawing Styles for Crossings

- RAC (right-angle crossing)
  - ✓: valid
  - ✗: invalid

- Orthogonal

- Slanted orthogonal

- Block / bundled crossings
  - Circular layout: 28 individual vs. 12 bundle crossings
Drawing Styles for Crossings

- **RAC**
  - right-angle crossing

- **orthogonal**

- **slanted orthogonal**

- **block / bundled crossings**
  - circular layout: 28 individual vs. 12 bundle crossings

![RAC](image)

![orthogonal](image)

![slanted orthogonal](image)

![block / bundled crossings](image)
Drawing Styles for Crossings

- **RAC** (right-angle crossing)
- **orthogonal**
- **slanted orthogonal**
- **block / bundled crossings**
  - circular layout: 28 individual vs. 12 bundle crossings
- **cased crossings**
Drawing Styles for Crossings

- **RAC** (right-angle crossing)
- **orthogonal**
- **slanted orthogonal**
- **block / bundled crossings**
  - circular layout: 28 individual vs. 12 bundle crossings
- **cased crossings**
- **symmetric partial edge drawing**

[Diagram showing different drawing styles for crossings]
Drawing Styles for Crossings

- RAC (right-angle crossing)
- orthogonal
- slanted orthogonal
- block / bundled crossings (circular layout: 28 individual vs. 12 bundle crossings)
- cased crossings
- symmetric partial edge drawing
- 1/4-SHPED
Geometric Representations

$K_4$

bar visibility representation
Geometric Representations

\[ K_5 \]

Bar 1-visibility representation (B1VR)
Geometric Representations

\[ K_5 \]

bar 1-visibility representation (B1VR)
Geometric Representations

\( K_5 \)

c

d

e

bar 1-visibility representation (B1VR)
Geometric Representations

$K_5$

bar 1-visibility representation (B1VR)
Every 1-planar graph admits a B1VR.

[Brandenburg 2014; Evans et al. 2014; Angelini et al. 2018]
Geometric Representations

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Geometric Representations

- Every 1-planar graph admits a B1VR. [Brandenburg 2014; Evans et al. 2014; Angelini et al. 2018]

- $G$ has at most $6n - 20$ edges. [Bose et al. 1997]
Geometric Representations

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- $G$ has at most $6n - 20$ edges. [Bose et al. 1997]

- Recognition is NP-complete. [Shermer 1996]
Every 1-planar graph admits a B1VR. [Brandenburg 2014; Evans et al. 2014; Angelini et al. 2018]

$G$ has at most $6n - 20$ edges. [Bose et al. 1997]

Recognition is NP-complete. [Shermer 1996]

Recognition becomes polynomial if embedding is fixed. [Biedl et al. 2018]
Visualization of Graphs

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Part II:
Density & Relationships

Alexander Wolff
GD Beyond Planarity: a Taxonomy

- Density
  - Many families
  - Fixed rot. system
  - NP-hard
  - Poly-time

- Recognition
  - NP-hard
  - Poly-time

- Stretchability
  - 1-plan.
  - Maximal 1-plan.
  - Fan-plan.
  - RAC & 1-plan.
  - Fan-plan & k-plan.
  - 2-layer fan & RAC

- Relationships
  - Opt 1-plan.
  - Out 1-plan.
  - Fan-plan.
  - RAC & 1-plan.
  - Fan-plan & k-plan.
  - 2-layer fan & RAC

- Constraints
  - Circ. & layers
  - Simult.
  - Circ. & layers
  - Quas-plan.
  - RAC
  - Book emb.

- Aesthetics
  - Area
  - Edge-complexity
  - Bends
  - Slopes
  - K-plan
  - RAC
  - Out 1-plan.

Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017
GD Beyond Planarity: a Taxonomy

- Density
  - many families
- Recognition
- Stretchability
  - 1-plan.
- Relationships
  - 1-plan.
  - fan-plan. & k-plan.
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  - book emb.

GD Beyond Planarity

- fixed rot. system
- variable embedding
- NP-hard
  - 1-plan.
  - fan-plan.
  - maximal 1-plan.
- poly-time
  - RAC
  - 1-plan.
  - fan-plan.
  - out. 1-plan.
  - out. fan-plan.
  - opt. 1-plan.
  - 2-layer RAC
  - 2-layer fan-plan

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Density of 1-Planar Graphs

**Theorem.** [Ringel 1965, Pach & Tóth 1997]
A 1-planar graph with $n$ vertices has at most $4n - 8$ edges, which is a tight bound.
Density of 1-Planar Graphs

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**Proof sketch.**
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- Let the red edges be those that do not cross.
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- Let the red edges be those that do not cross.
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\[ m_{rb} \leq 3n - 6 \]
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- and a green plane graph $G_g$ with
  \[ m_g \leq 3n - 6 \quad \Rightarrow \quad m \leq m_{rb} + m_g \leq 6n - 12 \]
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Observe that each green edge joins two faces in \( G_{rb} \).
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- Let the **red** edges be those that do not cross.
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m_{rb} \leq 3n - 6
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- and a **green** plane graph \( G_g \) with
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Observe that each **green** edge joins two faces in \( G_{rb} \).

\[
m_g \leq f_{rb}/2
\]
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Observe that each green edge joins two faces in $G_{rb}$.

$$m_g \leq f_{rb}/2 \leq (2n - 4)/2$$
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Observe that each green edge joins two faces in \( G_{rb} \).

\[ m_g \leq f_{rb}/2 \leq (2n - 4)/2 = n - 2 \]
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  \]
Observe that each green edge joins two faces in \( G_{rb} \).
  \[
m_g \leq \frac{f_{rb}}{2} \leq \frac{(2n - 4)}{2} = n - 2
  \]
  \[
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A 1-planar graph with $n$ vertices has at most $4n - 8$ edges, which is a tight bound.

$n = 12, m = 40$
Density of 1-Planar Graphs

Theorem. [Ringel 1965, Pach & Tóth 1997]
A 1-planar graph with $n$ vertices has at most $4n - 8$ edges, which is a tight bound.

A 1-planar graph with $n$ vertices is called **optimal** if it has exactly $4n - 8$ edges.

\[ n = 12, m = 40 \]
Density of 1-Planar Graphs

A 1-planar graph with \( n \) vertices is called **optimal** if it has exactly \( 4n - 8 \) edges.

A 1-planar graph is called **maximal** if adding any edge would result in a non-1-planar graph.

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**Theorem.** [Brandenburg et al. 2013]
There are maximal 1-planar graphs with $n$ vertices and $\frac{45}{17} n - O(1)$ edges.
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**Theorem.** [Brandenburg et al. 2013]
There are **maximal** 1-planar graphs with \( n \) vertices and \( \frac{45}{17}n - O(1) \) edges.

\[ \approx 2.65n \]
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\[ n = 12, m = 40 \]

\[ n = 20, m = 48 \]
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\[ n = 20, m = 48 \]
Density of 1-Planar Graphs

**Theorem.** [Ringel 1965, Pach & Tóth 1997]
A 1-planar graph with \( n \) vertices has at most \( 4n - 8 \) edges, which is a tight bound.

A 1-planar graph with \( n \) vertices is called **optimal** if it has exactly \( 4n - 8 \) edges.

A 1-planar graph is called **maximal** if adding any edge would result in a non-1-planar graph.

**Theorem.** [Brandenburg et al. 2013]
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**Theorem.** [Didimo 2013]
A 1-planar graph with \( n \) vertices that admits a straight-line drawing has at most \( 4n - 9 \) edges.

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\( n = 12, m = 40 \)
Density of $k$-Planar Graphs

**Theorem.**
A $k$-planar graph with $n$ vertices has at most:

\[ k \text{ number of edges} \]
Density of $k$-Planar Graphs

**Theorem.**
A $k$-planar graph with $n$ vertices has at most:
- $k$ number of edges
- 0 Euler’s formula
Density of $k$-Planar Graphs

**Theorem.**
A $k$-planar graph with $n$ vertices has at most:

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<tr>
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Euler’s formula
# Density of $k$-Planar Graphs

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optimal 2-planar
Density of $k$-Planar Graphs

**Theorem.**

A $k$-planar graph with $n$ vertices has at most:

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Euler’s formula

**[Ringel 1965]**

**[Pach and Tóth 1997]**

Planar structure:
Density of $k$-Planar Graphs

**Theorem.**
A $k$-planar graph with $n$ vertices has at most:

- **$k$** number of edges
  - $0$: $3(n - 2)$, Euler's formula
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Planar structure:

Edges per face:
Total:
Density of \( k \)-Planar Graphs

**Theorem.**

A \( k \)-planar graph with \( n \) vertices has at most:

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Optimal 2-planar

Planar structure:

Edges per face:

Total:
Density of $k$-Planar Graphs

**Theorem.**
A $k$-planar graph with $n$ vertices has at most:

- $k$ number of edges
- 0: $3(n - 2)$ (Euler's formula)
- 1: $4(n - 2)$ ([Ringel 1965])
- 2: $5(n - 2)$ ([Pach and Tóth 1997])

Planar structure:

$n - m + f = 2$

$Euler's$ $formula$

$Pach$ $and$ $Tóth$ $1997$

optimal 2-planar

$m = c \cdot f$?
Density of $k$-Planar Graphs

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A $k$-planar graph with $n$ vertices has at most:

- $0$ edges: $3(n - 2)$ (Euler’s formula)
- $1$ edge: $4(n - 2)$ ([Ringel 1965])
- $2$ edges: $5(n - 2)$ ([Pach and Tóth 1997])

Planar structure:

$$n - m + f = 2$$

Edges per face:

$$m = c \cdot f$$

Total:

$$m = \frac{5}{2} f$$
Density of $k$-Planar Graphs

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Euler's formula

Total:

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Density of $k$-Planar Graphs

**Theorem.**
A $k$-planar graph with $n$ vertices has at most:

- $k$  
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  - $2$: $5(n - 2)$

Euler’s formula

- $n - m + f = 2$

Optimal 2-planar

Planar structure:
- $\frac{5}{3}(n - 2)$ edges
- $\frac{2}{3}(n - 2)$ faces

Edges per face: 5 edges

Total:
Density of $k$-Planar Graphs

**Theorem.**

A $k$-planar graph with $n$ vertices has at most:

- $k$ number of edges

<table>
<thead>
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**Planar structure:**

- $\frac{5}{3}(n - 2)$ edges
- $\frac{2}{3}(n - 2)$ faces
- Edges per face: 5 edges
- Total: $5(n - 2)$ edges

$n - m + f = 2$

$m = c \cdot f$ ?
Density of $k$-Planar Graphs

**Theorem.**
A $k$-planar graph with $n$ vertices has at most:

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Planar structure:

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Total: $5(n - 2)$ edges
Density of $k$-Planar Graphs

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---

![optimal 3-planar graph](image)
Density of $k$-Planar Graphs

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A $k$-planar graph with $n$ vertices has at most:

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![optimal 3-planar graph](image-url)
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Optimal 3-planar structure:
- $\frac{3}{2}(n-2)$ edges
- $\frac{1}{2}(n-2)$ faces
- Edges per face: 8 edges
- Total: $5.5(n-2)$ edges
Density of $k$-Planar Graphs

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### GD Beyond Planarity: a Hierarchy

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<tr>
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<tbody>
<tr>
<td>4-planar $6n - 12$</td>
<td>thickness-2 $6n - 12$</td>
</tr>
<tr>
<td>1-bend RAC $\leq 5.5n - 10$</td>
<td>3-planar $5.5n - 11$</td>
</tr>
<tr>
<td>fan-planar $5n - 10$</td>
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<tr>
<td>bipart. fan-planar $\leq 4n - 12$</td>
<td>RAC $4n - 10$</td>
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<tr>
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<td>planar $3n - 6$</td>
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- quasi-planar $6.5n - 20$: $6.5n \pm c$ [Agarwal et al. 1997]
- 4-planar $6n - 12$: $6n \pm c$ [Ackerman 2015]
- 1-bend RAC $\leq 5.5n - 10$: $5.5n \pm c$ [Pach & Tóth 1997], [Bekos et al. 2018]
- fan-planar $5n - 10$: $5n \pm c$ [Kaufmann & Ueckerdt 2014], [Pach & Tóth 1997]
- RAC $4n - 10$: $4n \pm c$ [Didimo et al. 2011], [Bodendiek et al. 1983], [Cheong et al. 2013]
- 1-planar $4n - 8$: $3.5n \pm c$ [Dehkordi et al. 2013], [Auer et al. 2016]
- outer 1-planar $2.5n - 4$: $2.5n \pm c$ [Dehkordi et al. 2013], [Auer et al. 2016]
GD Beyond Planarity: a Hierarchy

- 4-planar $6n - 12$
- thickness-2 $6n - 12$
- 1-bend RAC $\leq 5.5n - 10$
- 3-planar $5.5n - 11$
- fan-planar $5n - 10$
- 2-planar $5n - 10$
- bipart. fan-planar $\leq 4n - 12$

**Quasi-planar** $6.5n - 20$

- 2.5-planar $2.5n - 5$
- outer 1-planar $2.5n - 4$
- outer RAC $2.5n - 4$
- 1-planar $4n - 4$
- RAC $4n - 4$

- bipartite $3n - 7$
- bipart. fan-planar $\leq 4n - 12$
- bipartite RAC $\leq 3.5n - 7$
- outer fan-planar $3n - 5$
- planar $3n - 6$

- outer fan-planar $3n - 5$
- outer RAC $2.5n - 5$

**Outer fan-planar** $3n - 5$

**Planar** $3n - 6$

**Sparse**

**Dense**

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- 4-planar $6n - 12$
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- bipart. fan-planar $\leq 4n - 12$

- 2.5-planar $2.5n - 5$
- outer 1-planar $2.5n - 4$
- outer RAC $2.5n - 4$
- 1-planar $4n - 4$
- RAC $4n - 4$

- bipartite $3n - 7$
- bipart. fan-planar $\leq 4n - 12$
- bipartite RAC $\leq 3.5n - 7$
- outer fan-planar $3n - 5$
- planar $3n - 6$

- outer fan-planar $3n - 5$
- outer RAC $2.5n - 5$

**Outer fan-planar** $3n - 5$

**Planar** $3n - 6$

**Sparse**

**Dense**

- 3-planar $5.5n - 11$
- 4-planar $6n - 12$
- thickness-2 $6n - 12$
- 1-bend RAC $\leq 5.5n - 10$

- quasi-planar $6.5n \pm c$ [Agarwal et al. 1997]
- 4-planar $6n - 12$
- thickness-2 $6n - 12$
- 1-bend RAC $\leq 5.5n - 10$
- 3-planar $5.5n - 11$
- fan-planar $5n - 10$
- 2-planar $5n - 10$
- bipart. fan-planar $\leq 4n - 12$

- 2.5-planar $2.5n - 5$
- outer 1-planar $2.5n - 4$
- outer RAC $2.5n - 4$
- 1-planar $4n - 4$
- RAC $4n - 4$

- bipartite $3n - 7$
- bipart. fan-planar $\leq 4n - 12$
- bipartite RAC $\leq 3.5n - 7$
- outer fan-planar $3n - 5$
- planar $3n - 6$

- outer fan-planar $3n - 5$
- outer RAC $2.5n - 5$

**Outer fan-planar** $3n - 5$

**Planar** $3n - 6$
GD Beyond Planarity: a Hierarchy
GD Beyond Planarity: a Hierarchy

For triconnected graphs, the bound is $3n - 6$.

- **planar** $3n - 6$
- **bipartite RAC** $3n - 7$
- **bipart. 1-planar** $3n - 8$
- **bipart. 2-planar** $3.5n - 7$
- **outer fan-planar** $3n - 5$
- **outer 1-planar** $2.5n - 4$
- **outer RAC** $2.5n - 5$
- **fan-planar** $5n - 10$
- **2-planar** $5n - 10$
- **RAC** $4n - 10$
- **1-planar** $4n - 8$
- **1-bend RAC** $\leq 5.5n - 10$
- **3-planar** $5.5n - 11$
- **thickness-2 RAC** $6n - 12$
- **4-planar** $6n - 12$

- **quasi-planar** $6.5n - 20$
- **6.5n ± c** [Agarwal et al. 1997]
- **6n ± c** [Ackerman 2015]
- **5.5n ± c** [Pach & Tóth 1997], [Bekos et al. 2018]
- **5n ± c** [Kaufmann & Ueckerdt 2014], [Pach & Tóth 1997]
- **4n ± c** [Didimo et al. 2011], [Bodendiek et al. 1983], [Cheong et al. 2013]
- **3.5n ± c** [Dehkordi et al. 2013], [Auer et al. 2016]
- **3n ± c** [Bekos et al. 2017], [Binucci et al. 2015], [Angelini et al. 2018]
- **2.5n ± c** [Dehkordi et al. 2013], [Auer et al. 2016]
# GD Beyond Planarity: a Hierarchy

<table>
<thead>
<tr>
<th>Class</th>
<th>Formula</th>
<th>Bound</th>
<th>Reference</th>
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<tbody>
<tr>
<td><strong>Dense</strong></td>
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<tr>
<td>4-planar</td>
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<td>$6.5n - 20$</td>
<td>[Agarwal et al. 1997]</td>
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<td>[Ackerman 2015]</td>
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<td>1-bend RAC</td>
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<td>$5.5n - 10$</td>
<td>[Pach &amp; Tóth 1997]</td>
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<td>2-planar</td>
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<td>[Pach &amp; Tóth 1997]</td>
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<td>[Bodendiek et al. 1983]</td>
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<td>1-planar</td>
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<td>planar</td>
<td>$3n - 6$</td>
<td>$3n$</td>
<td>[Binucci et al. 2015]</td>
</tr>
<tr>
<td>bipartite 1-planar</td>
<td>$3n - 8$</td>
<td>$3n$</td>
<td>[Angelini et al. 2018]</td>
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<td>outer RAC</td>
<td>$2.5n - 5$</td>
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<td>[Dehkordi et al. 2013]</td>
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<tr>
<td>outer 1-planar</td>
<td>$2.5n - 4$</td>
<td>$2.5n$</td>
<td>[Auer et al. 2016]</td>
</tr>
<tr>
<td><strong>Sparse</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2-planar</td>
<td>$4n - 12$</td>
<td>$3.5n - 7$</td>
<td>[Didimo et al. 2011]</td>
</tr>
<tr>
<td>quasi-planar</td>
<td>$6.5n $</td>
<td>$6.5n$</td>
<td>[Agarwal et al. 1997]</td>
</tr>
<tr>
<td>outer 1-planar</td>
<td>$2.5n - 4$</td>
<td>$2.5n$</td>
<td>[Dehkordi et al. 2013]</td>
</tr>
</tbody>
</table>
GD Beyond Planarity: a Hierarchy

- 4-planar $6n - 10$
- thickness-2 $6.5n - 20$
- quasi-planar $6.5n - 20$

- 3-planar $5.5n - 11$
- 1-planar $4n - 8$
- not 1-planar
- fan-planar $5n - 10$

- planar $3n - 6$
- outer fan-planar $3n - 5$
- bipartite RAC $3n - 7$
- bipartite 1-planar $3n - 8$
- outer RAC $2.5n - 5$
- outer 1-planar $2.5n - 4$

- 1-bend RAC $5.5n + c$
- outer 1-planar

- 2-planar $5n - 10$
- quasi-planar $5n - 20$

- 4-planar $6n - 12$
- thickness-2 $6n - 12$
- quasi-planar $6n - 12$

- outer RAC $5n - 10$
- RAC $4n - 10$
- 1-planar $4n - 10$

- 6-planar $6n - 12$
- fan-planar $5n - 10$
- 1-planar $4n - 8$
- outer RAC $2.5n - 5$

- 3-planar $5.5n - 11$
- fan-planar $5.5n - 11$
- 1-planar $4n - 8$
- outer RAC $2.5n - 5$

- 4-planar $6n - 10$
- thickness-2 $5.5n - 10$
- quasi-planar $6.5n - 20$

- 3-planar $5.5n - 11$
- fan-planar $5.5n - 11$
- 1-planar $4n - 8$
- outer RAC $2.5n - 5$

- 4-planar $6n - 10$
- thickness-2 $5.5n - 10$
- quasi-planar $6.5n - 20$

- 3-planar $5.5n - 11$
- fan-planar $5.5n - 11$
- 1-planar $4n - 8$
- outer RAC $2.5n - 5$

- 4-planar $6n - 10$
- thickness-2 $5.5n - 10$
- quasi-planar $6.5n - 20$

- 3-planar $5.5n - 11$
- fan-planar $5.5n - 11$
- 1-planar $4n - 8$
- outer RAC $2.5n - 5$

- 4-planar $6n - 10$
- thickness-2 $5.5n - 10$
- quasi-planar $6.5n - 20$

- 3-planar $5.5n - 11$
- fan-planar $5.5n - 11$
- 1-planar $4n - 8$
- outer RAC $2.5n - 5$

- 4-planar $6n - 10$
- thickness-2 $5.5n - 10$
- quasi-planar $6.5n - 20$

- 3-planar $5.5n - 11$
- fan-planar $5.5n - 11$
- 1-planar $4n - 8$
- outer RAC $2.5n - 5$

- 4-planar $6n - 10$
- thickness-2 $5.5n - 10$
- quasi-planar $6.5n - 20$

- 3-planar $5.5n - 11$
- fan-planar $5.5n - 11$
- 1-planar $4n - 8$
- outer RAC $2.5n - 5$

- 4-planar $6n - 10$
- thickness-2 $5.5n - 10$
- quasi-planar $6.5n - 20$

- 3-planar $5.5n - 11$
- fan-planar $5.5n - 11$
- 1-planar $4n - 8$
- outer RAC $2.5n - 5$

- 4-planar $6n - 10$
- thickness-2 $5.5n - 10$
- quasi-planar $6.5n - 20$

- 3-planar $5.5n - 11$
- fan-planar $5.5n - 11$
- 1-planar $4n - 8$
- outer RAC $2.5n - 5$

- 4-planar $6n - 10$
- thickness-2 $5.5n - 10$
- quasi-planar $6.5n - 20$

- 3-planar $5.5n - 11$
- fan-planar $5.5n - 11$
- 1-planar $4n - 8$
- outer RAC $2.5n - 5$
GD Beyond Planarity: a Hierarchy

- **Dense**
  - **4-planar** $6n - 12$
  - **thickness-2** $6n - 12$
  - **1-bend RAC** $\leq 5.5n - 10$
  - **3-planar** $5.5n - 11$
  - **fan-planar** $5n - 10$
  - **2-planar** $5n - 10$
  - **bipart. fan-planar** $\leq 4n - 12$
  - **RAC** $4n - 10$
  - **1-planar** $4n - 8$
  - **bipart. 2-planar** $\leq 3.5n - 7$
  - **planar** $3n - 6$
  - **outer fan-planar** $3n - 5$
  - **bipartite RAC** $3n - 7$
  - **bipart. 1-planar** $\leq 3n - 8$
  - **outer RAC** $2.5n - 5$
  - **outer 1-planar** $2.5n - 4$
  - **quasi-planar** $6.5n - 20$

- **Sparse**
  - **bipartite RAC** $3n - 7$

- **Outer**
  - **outer fan-planar** $3n - 5$
  - **bipartite RAC** $3n - 7$
  - **bipart. 1-planar** $\leq 3n - 8$
  - **outer RAC** $2.5n - 5$
  - **outer 1-planar** $2.5n - 4$

- **Thickness**
  - **planar** $3n - 6$
  - **outer fan-planar** $3n - 5$
  - **bipartite RAC** $3n - 7$
  - **bipart. 1-planar** $\leq 3n - 8$
  - **outer RAC** $2.5n - 5$
  - **outer 1-planar** $2.5n - 4$

- **Arcs**
  - **bipart. fan-planar** $\leq 4n - 12$
  - **RAC** $4n - 10$
  - **1-planar** $4n - 8$
  - **bipart. 2-planar** $\leq 3.5n - 7$

- **Planarity Parameters**
  - **6.5n ± c** [Agarwal et al. 1997]
  - **6n ± c** [Ackerman 2015]
  - **5.5n ± c** [Pach & Tóth 1997] [Bekos et al. 2018]
  - **5n ± c** [Kaufmann & Ueckerdt 2014] [Pach & Tóth 1997]
  - **4n ± c** [Didimo et al. 2011] [Bodendiek et al. 1983] [Cheong et al. 2013]
  - **3.5n ± c** [Dehkordi et al. 2013] [Auer et al. 2016]
  - **3n ± c** [Bekos et al. 2017] [Binucci et al. 2015] [Angelini et al. 2018]
  - **2.5n ± c** [Dehkordi et al. 2013] [Auer et al. 2016]
GD Beyond Planarity: a Hierarchy

- **Dense**
  - Fan-planar: $5n - 10$
  - 2-planar: $5n - 10$
  - Bipart. fan-planar: $\leq 4n - 12$
  - RAC: $4n - 10$
  - 1-planar: $4n - 8$

- **Sparse**
  - Planar: $3n - 6$
  - Outer fan-planar: $3n - 5$
  - Bipartite RAC: $3n - 7$
  - Bipart. 1-planar: $\leq 3.5n - 7$

- 1-bend RAC: $2.5n - 5$
- Inner fan-planar: $3n - 5$
- Outer 1-planar: $2.5n - 4$
- Bipartite 1-planar: $\leq 3.5n - 8$
- RAC: $2.5n - 5$
- Bipart. 2-planar: $\leq 3.5n - 7$
- Bipartite RAC: $3n - 7$

- 6.5n ± c
- 5n ± c
- 4n ± c
- 3.5n ± c
- 3n ± c
- 2.5n ± c
- K₄,n - 4 is not 2-planar

- K₇ is not planar

- References:
  - [Agarwal et al. 1997]
  - [Ackerman 2015]
  - [Bekos et al. 2018]
  - [Kaufmann & Ueckerdt 2014]
  - [Pach & Tóth 1997]
  - [Bodendiek et al. 1983]
  - [Didimo et al. 2011]
  - [Cheong et al. 2013]
  - [Dehkordi et al. 2013]
  - [Auer et al. 2016]
  - [Bekos et al. 2017]
  - [Angelini et al. 2018]

- [Angelini et al. 2018]
GD Beyond Planarity: a Hierarchy

- 4-planar, $6n - 12$
- thickness-2, $6n - 12$
- 1-bend RAC, $5.5n - 10$
- 3-planar, $5.5n - 11$
- fan-planar, $5n - 10$
- 2-planar, $5n - 10$
- bipart. fan-planar, $4n - 12$
- RAC, $4n - 10$
- 1-planar, $4n - 8$
- bipart. 2-planar, $3.5n - 7$
- planar, $3n - 6$
- outer fan-planar, $3n - 5$
- bipartite RAC, $3n - 7$
- bipart. 1-planar, $3n - 8$
- outer RAC, $2.5n - 5$
- outer 1-planar, $2.5n - 4$
- quasi-planar, $6.5n - 20$
- 6n ± c
- $5.5n ± c$
- $5n ± c$
- $4n ± c$
- $3.5n ± c$
- $3n ± c$
- $2.5n ± c$

[Agarwal et al. 1997]
[Ackerman 2015]
[Pach & Tóth 1997]
[Bekos et al. 2018]
[Kaufmann & Ueckerdt 2014]
[Pach & Tóth 1997]
[Didimo et al. 2011]
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[Auer et al. 2016]
[Bekos et al. 2017]
[Binucci et al. 2015]
[Angelini et al. 2018]
[Dehkordi et al. 2013]
[Auer et al. 2016]
GD Beyond Planarity: a Hierarchy

Conjecture (Pach et al. 1996):
A simple \( k \)-quasi planar graph with \( n \) vertices has \( O(n) \) edges.

Proved for:
- \( k = 3 \) \[\text{Agarwal et al. 1997}\]
- \( k = 4 \) \[\text{Ackerman 2009}\]
- \( k = 3 \) \[\text{Angelini et al. 2018}\]
- \( k = 4 \) \[\text{Binucci et al. 2015}\]
- \( k = 5 \) \[\text{Dehkordi et al. 2013}\]
- \( k = 6 \) \[\text{Auer et al. 2016}\]

<table>
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<tr>
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<th>Graph Class</th>
<th>Bound</th>
<th>Reference</th>
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<td>( 6.5n - 20 )</td>
<td>[Agarwal et al. 1997]</td>
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<td>[Ackerman 2015]</td>
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<td>5.5n ± c</td>
<td>[Pach &amp; Tóth 1997]</td>
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<tr>
<td></td>
<td>4n ± c</td>
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<tr>
<td></td>
<td>2.5n ± c</td>
<td>[Dehkordi et al. 2013]</td>
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</tr>
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</table>

| Sparse  | planar      | \( 3n - 6 \) | |
|         | outer fan-planar | \( 3n - 5 \) | |
|         | bipartite RAC   | \( 3n - 7 \) | |
|         | bipart. 1-planar | \( \leq 3n - 8 \) | |
|         | outer RAC       | \( 2.5n - 5 \) | |
|         | outer 1-planar  | \( 2.5n - 4 \) | |

| Planar  | RAC       | \( 4n - 10 \) | |
|         | 1-planar  | \( 4n - 8 \) | |
|         | bipart. 2-planar | \( \leq 3.5n - 7 \) | |
|         | fan-planar | \( 5n - 10 \) | |
|         | bipart. fan-planar | \( \leq 4n - 12 \) | |
|         | 2-planar  | \( 5n - 10 \) | |
Crossing Numbers

The $k$-planar crossing number $\text{cr}_{k\text{-pl}}(G)$ of a graph $G$ is the number of crossings required in any $k$-planar drawing of $G$. 
The $k$-planar crossing number $cr_k\text{-pl}(G)$ of a graph $G$ is the number of crossings required in any $k$-planar drawing of $G$.

- $cr_{1\text{-pl}}(G) \leq n - 2$
Crossing Numbers

The \textbf{\textit{k}-planar crossing number} \( \text{cr}_{k\text{-pl}}(G) \) of a graph \( G \) is the number of crossings required in any \( k \)-planar drawing of \( G \).

- \( \text{cr}_{1\text{-pl}}(G) \leq n - 2 \)
- \( \text{cr}(G) = 1 \Rightarrow \text{cr}_{1\text{-pl}}(G) = 1 \)
Crossing Numbers

The $k$-planar crossing number $cr_k(G)$ of a graph $G$ is the number of crossings required in any $k$-planar drawing of $G$.

- $cr_{1-pl}(G) \leq n - 2$
- $cr(G) = 1 \Rightarrow cr_{1-pl}(G) = 1$

**Theorem.** [Chimani, Kindermann, Montecchiani & Valtr 2019]
For every $\ell \geq 7$, there is a 1-planar graph $G$ with $n = 11\ell + 2$ vertices such that $cr(G) = 2$ and $cr_{1-pl}(G) = n - 2$. 
Crossing Numbers

The \textbf{$k$-planar crossing number} $\text{cr}_{\text{k-pl}}(G)$ of a graph $G$ is the number of crossings required in any $k$-planar drawing of $G$.

- $\text{cr}_{\text{1-pl}}(G) \leq n - 2$
- $\text{cr}(G) = 1 \Rightarrow \text{cr}_{\text{1-pl}}(G) = 1$

**Theorem.** [Chimani, Kindermann, Montecchiani & Valtr 2019]
For every $\ell \geq 7$, there is a 1-planar graph $G$ with $n = 11\ell + 2$ vertices such that $\text{cr}(G) = 2$ and $\text{cr}_{\text{1-pl}}(G) = n - 2$. 

\[\text{cr}_{\text{1-pl}}(G) \leq n - 2\]
\[\text{cr}(G) = 1 \Rightarrow \text{cr}_{\text{1-pl}}(G) = 1\]
Crossing Numbers

The \textbf{k-planar crossing number} \( \text{cr}_{k\text{-pl}}(G) \) of a graph \( G \) is the number of crossings required in any \( k \)-planar drawing of \( G \).

\begin{itemize}
  \item \( \text{cr}_{1\text{-pl}}(G) \leq n - 2 \)
  \item \( \text{cr}(G) = 1 \Rightarrow \text{cr}_{1\text{-pl}}(G) = 1 \)
\end{itemize}

\textbf{Theorem.} [Chimani, Kindermann, Montecchiani & Valtr 2019]
For every \( \ell \geq 7 \), there is a 1-planar graph \( G \) with \( n = 11\ell + 2 \) vertices such that \( \text{cr}(G) = 2 \) and \( \text{cr}_{1\text{-pl}}(G) = n - 2 \).
Crossing Numbers

The \textit{k-planar crossing number} \( c_{k-pl}(G) \) of a graph \( G \) is the number of crossings required in any \( k \)-planar drawing of \( G \).

- \( c_{1-pl}(G) \leq n - 2 \)
- \( c(G) = 1 \Rightarrow c_{1-pl}(G) = 1 \)

Theorem. [Chimani, Kindermann, Montecchiani & Valtr 2019]

For every \( \ell \geq 7 \), there is a 1-planar graph \( G \) with \( n = 11\ell + 2 \) vertices such that \( c(G) = 2 \) and \( c_{1-pl}(G) = n - 2 \).
Crossing Numbers

The \textit{$k$-planar crossing number} $\text{cr}_{k\text{-pl}}(G)$ of a graph $G$ is the number of crossings required in any $k$-planar drawing of $G$.

- $\text{cr}_{1\text{-pl}}(G) \leq n - 2$
- $\text{cr}(G) = 1 \Rightarrow \text{cr}_{1\text{-pl}}(G) = 1$

\textbf{Theorem.} [Chimani, Kindermann, Montecchiani & Valtr 2019]
For every $\ell \geq 7$, there is a 1-planar graph $G$ with $n = 11\ell + 2$ vertices such that $\text{cr}(G) = 2$ and $\text{cr}_{1\text{-pl}}(G) = n - 2$.  

\begin{center}
\begin{tikzpicture}
  \draw[blue, thick] (0,0) -- (1,1) -- (2,0) -- (1,-1) -- cycle;
  \draw[blue, thick] (0,0) -- (1,-1) -- (2,0) -- (1,1) -- cycle;
  \draw[blue, thick] (1,1) -- (2,2) -- (1,3) -- cycle;
  \draw[blue, thick] (1,-1) -- (2,-2) -- (1,-3) -- cycle;
  \draw[blue, thick] (0,0) -- (0,2)
\end{tikzpicture}
\end{center}
Crossing Numbers

The \( k \)-planar crossing number \( \text{cr}_{k\text{-pl}}(G) \) of a graph \( G \) is the number of crossings required in any \( k \)-planar drawing of \( G \).

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- \( \text{cr}(G) = 1 \Rightarrow \text{cr}_{1\text{-pl}}(G) = 1 \)

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For every \( \ell \geq 7 \), there is a 1-planar graph \( G \) with \( n = 11\ell + 2 \) vertices such that \( \text{cr}(G) = 2 \) and \( \text{cr}_{1\text{-pl}}(G) = n - 2 \).
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**Crossing ratio**

$\rho_{1\text{-pl}}(n) = (n - 2)/2$
### Crossing Ratios

<table>
<thead>
<tr>
<th>Family</th>
<th>Forbidden Configurations</th>
<th>Lower</th>
<th>Upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$-planar</td>
<td>An edge crossed more than $k$ times</td>
<td>$\Omega(n/k)$</td>
<td>$O(k\sqrt{kn})$</td>
</tr>
<tr>
<td>$k$-quasi-planar</td>
<td>$k$ pairwise crossing edges</td>
<td>$\Omega(n/k^3)$</td>
<td>$f(k)n^2 \log^2 n$</td>
</tr>
<tr>
<td>Fan-planar</td>
<td>Two independent edges crossing a third or two adjacent edges crossing another edge from different &quot;side&quot;</td>
<td>$\Omega(n)$</td>
<td>$O(n^2)$</td>
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<tr>
<td>$(k, l)$-grid-free</td>
<td>Set of $k$ edges such that each edge crosses each edge from a set of $l$ edges.</td>
<td>$\Omega\left(\frac{n}{kl(k+l)}\right)$</td>
<td>$g(k, l)n^2$</td>
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<tr>
<td>$k$-gap-planar</td>
<td>More than $k$ crossings mapped to an edge in an optimal mapping</td>
<td>$\Omega(n/k^3)$</td>
<td>$O(k\sqrt{kn})$</td>
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<tr>
<td>Skewness-$k$</td>
<td>Set of crossings not covered by at most $k$ edges</td>
<td>$\Omega(n/k)$</td>
<td>$O(kn + k^2)$</td>
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<tr>
<td>$k$-apex</td>
<td>Set of crossings not covered by at most $k$ vertices</td>
<td>$\Omega(n/k)$</td>
<td>$O(k^2n^2 + k^4)$</td>
</tr>
<tr>
<td>Planarly connected</td>
<td>Two crossing edges that do not have two of their endpoint connected by a crossing-free edge</td>
<td>$\Omega(n^2)$</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td>$k$-fan-crossing-free</td>
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<td>Straight-line RAC</td>
<td>Two edges crossing at an angle $&lt; \frac{\pi}{2}$</td>
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</tr>
</tbody>
</table>

Table from “Crossing Numbers of Beyond-Planar Graphs Revisited” [van Beusekom, Parada & Speckmann 2021]
Visualization of Graphs

Lecture 11:
Beyond Planarity
Drawing Graphs with Crossings

Part III:
Recognition

Alexander Wolff
GD Beyond Planarity: a Taxonomy

Density
- many families
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$G$ planar $\iff$ neither $K_5$ nor $K_{3,3}$ minor of $G$
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  - fixed rot. system
  - variable embedding
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  - NP-hard
- Stretchability
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  - 2-layer fan & RAC
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Area of Straight-Line RAC Drawings

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Every IC-planar graph has an IC-planar straight-line RAC drawing, and such a drawing can be found in polynomial time.
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Some IC-planar straight-line RAC drawings require exponential area.
RAC Drawings
RAC Drawings
RAC Drawings
RAC Drawings

✓

✓
RAC Drawings

Every graph admits a RAC drawing...
Every graph admits a RAC drawing . . .
    . . . if we use enough bends.
Every graph admits a RAC drawing . . .

. . . if we use enough bends.

How many do we need at most in total or per edge?
3-Bend RAC Drawings

Theorem. [Didimo, Eades & Liotta 2017]
Every graph admits a 3-bend RAC drawing, that is, a RAC drawing where every edge has at most 3 bends.
3-Bend RAC Drawings

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3-Bend RAC Drawings

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Every graph admits a 3-bend RAC drawing, that is, a RAC drawing where every edge has at most 3 bends.
Kite Triangulations

This is a kite:
Kite Triangulations

This is a **kite**:

\[ \begin{array}{c}
\text{This is a kite:} \\
\begin{array}{c}
\text{u and v are opposite} \\
w.r.t. \{z, w\}
\end{array}
\end{array} \]
This is a **kite**:

Let $G'$ be a plane triangulation.

$u$ and $v$ are **opposite** w.r.t. \{z, w\}
Kite Triangulations

This is a **kite**: 

Let \( G' \) be a plane triangulation.

Let \( S \subset E(G') \) s.t. no two edges in \( S \) lie on the same face.

\( u \) and \( v \) are **opposite** w.r.t. \( \{z, w\} \).
Kite Triangulations

This is a **kite**:

Let $G'$ be a plane triangulation.

Let $S \subset E(G')$ s.t. no two edges in $S$ lie on the same face

... and their opposite vertices do not have an edge in $E(G')$. 

$u$ and $v$ are **opposite** w.r.t. $\{z, w\}$
Kite Triangulations

This is a kite:

\[\begin{align*}
  &w \\
  &v \\
  &z \\
  &u
\end{align*}\]

\[\begin{align*}
  &w \\
  &v \\
  &z \\
  &u
\end{align*}\]

\hspace{2cm}

\[u\text{ and } v\text{ are opposite w.r.t. } \{z, w\}\]

Let \(G'\) be a plane triangulation.

Let \(S \subset E(G')\) s.t. no two edges in \(S\) lie on the same face

\[\ldots \text{and their opposite vertices do not have an edge in } E(G').\]

Add set \(T\) of edges connecting opposite vertices.
Kite Triangulations

This is a **kite**:

Let $G'$ be a plane triangulation.

Let $S \subset E(G')$ s.t. no two edges in $S$ lie on the same face

... and their opposite vertices do not have an edge in $E(G')$.

Add set $T$ of edges connecting opposite vertices.

The resulting graph $G$ is a **kite-triangulation**.

$u$ and $v$ are **opposite** w.r.t. $\{z, w\}$
Kite Triangulations

This is a **kite**:

Let $G'$ be a plane triangulation.

Let $S \subset E(G')$ s.t. no two edges in $S$ lie on the same face and their opposite vertices do not have an edge in $E(G')$.

Add set $T$ of edges connecting opposite vertices.

The resulting graph $G$ is a **kite-triangulation**.

**Note:** optimal 1-planar graphs $\subset$ kite-triangulations.
Kite Triangulations

This is a **kite**: $u \quad v$

$w \quad z$

$u$ and $v$ are **opposite** w.r.t. $\{z, w\}$

Let $G'$ be a plane triangulation.

Let $S \subset E(G')$ s.t. no two edges in $S$ lie on the same face

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The resulting graph $G$ is a **kite-triangulation**.

**Theorem.** [Angelini et al. '11]

Every kite-triangulation $G$ on $n$ vertices admits a 1-planar 1-bend RAC drawing.

**Note:** optimal 1-planar graphs $\subset$ kite-triangulations.
Kite Triangulations

This is a **kite**:

\[ \begin{array}{c}
\text{w} \\
\text{v} \\
\text{z} \\
\text{u} \\
\end{array} \]

\[ \begin{array}{c}
\text{w} \\
\text{v} \\
\text{z} \\
\text{u} \\
\end{array} \]

\( u \) and \( v \) are **opposite** w.r.t. \( \{z, w\} \)

Let \( G' \) be a plane triangulation.

Let \( S \subset E(G') \) s.t. no two edges in \( S \) lie on the same face

\( \ldots \) and their opposite vertices do not have an edge in \( E(G') \).

Add set \( T \) of edges connecting opposite vertices.

The resulting graph \( G \) is a **kite-triangulation**.

**Theorem.** [Angelini et al. ’11]
Every kite-triangulation \( G \) on \( n \) vertices admits a 1-planar 1-bend RAC drawing, which can be constructed in \( O(n) \) time.

**Note:** optimal 1-planar graphs \( \subset \) kite-triangulations.
Theorem. [Angelini et al. ’11]
Every kite-triangulation $G$ on $n$ vertices admits a 1-planar 1-bend RAC drawing, which can be constructed in $O(n)$ time.

Proof.

Let $G'$ be a plane triangulation.

Let $S \subset E(G')$ s.t. no two edges in $S$ lie on the same face and their opposite vertices do not have an edge in $E(G')$.

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### Kite Triangulations

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**Proof.**

Let $G'$ be the underlying plane triangulation of $G$.

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The resulting graph $G$ is a **kite-triangulation**.

**Note:** optimal 1-planar graphs $\subset$ kite-triangulations.
Kite Triangulations

This is a kite:

\[ \begin{align*}
    & w \\
    & v \\
    & z \\
    & u
\end{align*} \]

Let \( G' \) be a plane triangulation.

Let \( S \subset E(G') \) s.t. no two edges in \( S \) lie on the same face

\( \ldots \) and their opposite vertices do not have an edge in \( E(G') \).

Add set \( T \) of edges connecting opposite vertices.

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**Proof.**

Let \( G' \) be the underlying plane triangulation of \( G \). Let \( G'' = G' - S \).
Kite Triangulations

This is a kite:

\[
\begin{array}{c}
\begin{array}{c}
\text{w} \\
\text{u} \\
\text{z} \\
\text{v} \\
\text{w}
\end{array}
\end{array}
\]

Let \( G' \) be a plane triangulation.

Let \( S \subset E(G') \) s.t. no two edges in \( S \) lie on the same face

\[
\begin{array}{c}
\begin{array}{c}
\text{w} \\
\text{u} \\
\text{z} \\
\text{v} \\
\text{w}
\end{array}
\end{array}
\]

...and their opposite vertices do not have an edge in \( E(G') \).

Add set \( T \) of edges connecting opposite vertices.

The resulting graph \( G \) is a kite-triangulation.

Note: optimal 1-planar graphs \( \subset \) kite-triangulations.

**Theorem.** [Angelini et al. '11]
Every kite-triangulation \( G \) on \( n \) vertices admits a 1-planar 1-bend RAC drawing, which can be constructed in \( O(n) \) time.

**Proof.**
Let \( G' \) be the underlying plane triangulation of \( G \). Let \( G'' = G' - S \).

Construct straight-line drawing of \( G'' \).
Kite Triangulations

This is a kite:

\begin{center}
\begin{tikzpicture}
  \node (u) at (0,0) {$u$};
  \node (v) at (1,1) {$v$};
  \node (w) at (0,1) {$w$};
  \node (z) at (1,0) {$z$};

  \draw (u) -- (v);
  \draw (u) -- (w);
  \draw (u) -- (z);
  \draw (v) -- (w);
  \draw (v) -- (z);
  \draw (w) -- (z);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
  \node (u) at (2,0) {$u$};
  \node (v) at (3,1) {$v$};
  \node (w) at (2,1) {$w$};
  \node (z) at (3,0) {$z$};

  \draw (u) -- (v);
  \draw (u) -- (w);
  \draw (u) -- (z);
  \draw (v) -- (w);
  \draw (v) -- (z);
  \draw (w) -- (z);
\end{tikzpicture}
\end{center}

\textbf{Theorem.} [Angelini et al. ’11] Every kite-triangulation \(G\) on \(n\) vertices admits a 1-planar 1-bend RAC drawing, which can be constructed in \(O(n)\) time.

\textbf{Proof.}

Let \(G'\) be a plane triangulation. Let \(S \subset E(G')\) s.t. no two edges in \(S\) lie on the same face and their opposite vertices do not have an edge in \(E(G')\).

Add set \(T\) of edges connecting opposite vertices.

The resulting graph \(G\) is a kite-triangulation.

\textbf{Note:} optimal 1-planar graphs \(\subset\) kite-triangulations.
Kite Triangulations

This is a kite:

\[ \begin{tikzpicture}
  \node (u) at (0,0) [circle, fill=black] {u};
  \node (v) at (1,1) [circle, fill=black] {v};
  \node (w) at (1,0) [circle, fill=black] {w};
  \node (z) at (0,1) [circle, fill=black] {z};
  \draw (u) -- (v) -- (w) -- (z) -- (u);
  \draw[blue] (u) -- (w);
\end{tikzpicture} \]

\[ \text{\textbf{u} and \textbf{v} are opposite w.r.t. \{z, w\}} \]

Let \( G' \) be a plane triangulation.

Let \( S \subset E(G') \) s.t. no two edges in \( S \) lie on the same face ... and their opposite vertices do not have an edge in \( E(G') \).

Add set \( T \) of edges connecting opposite vertices.

The resulting graph \( G \) is a \textbf{kite-triangulation}.

**Theorem.** [Angelini et al. '11] Every kite-triangulation \( G \) on \( n \) vertices admits a 1-planar 1-bend RAC drawing, which can be constructed in \( O(n) \) time.

**Proof.** Let \( G'' = G' - S \).

Construct straight-line drawing of \( G'' \).

Fill faces as follows:

\[ \begin{tikzpicture}
  \node (u) at (0,0) [circle, fill=black] {u};
  \node (v) at (1,1) [circle, fill=black] {v};
  \node (w) at (1,0) [circle, fill=black] {w};
  \node (z) at (0,1) [circle, fill=black] {z};
  \draw (u) -- (v) -- (w) -- (z) -- (u);
  \draw[blue] (u) -- (w);
\end{tikzpicture} \]

strictly convex face

\[ \text{Note: optimal 1-planar graphs \subset kite-triangulations.} \]
Kite Triangulations

This is a kite:

Let $G'$ be a plane triangulation.

Let $S \subset E(G')$ s.t. no two edges in $S$ lie on the same face
... and their opposite vertices do not have an edge in $E(G')$.

Add set $T$ of edges connecting opposite vertices.

The resulting graph $G$ is a kite-triangulation.

Note: optimal 1-planar graphs $\subset$ kite-triangulations.

**Theorem.** [Angelini et al. '11]
Every kite-triangulation $G$ on $n$ vertices admits a 1-planar 1-bend RAC drawing, which can be constructed in $O(n)$ time.

**Proof.**
Let $G'' = G' - S$.
Construct straight-line drawing of $G''$.
Fill faces as follows:

strictly convex face
Kite Triangulations

Theorem. [Angelini et al. '11] Every kite-triangulation $G$ on $n$ vertices admits a 1-planar 1-bend RAC drawing, which can be constructed in $O(n)$ time.

Proof.
Let $G'$ be the underlying plane triangulation of $G$. Let $G'' = G' - S$. Construct straight-line drawing of $G''$. Fill faces as follows:

The resulting graph $G$ is a kite-triangulation.

Note: optimal 1-planar graphs $\subset$ kite-triangulations.
Kite Triangulations

This is a kite:

\[
\begin{array}{c}
\text{\textbullet} \text{ w} \\
\text{\textbullet} \text{ v} \\
\text{\textbullet} \text{ z} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\textbullet} \text{ u} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\textbullet} \text{ w} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\textbullet} \text{ v} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\textbullet} \text{ z} \\
\end{array}
\]

\[u \text{ and } v \text{ are opposite w.r.t. } \{z, w\}\]

Let \( G' \) be a plane triangulation.

Let \( S \subset E(G') \) s.t. no two edges in \( S \) lie on the same face

\[
\begin{array}{c}
\text{\textbullet} \text{ u} \\
\text{\textbullet} \text{ v} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\textbullet} \text{ w} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\textbullet} \text{ z} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\textbullet} \text{ z} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\textbullet} \text{ w} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\textbullet} \text{ v} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\textbullet} \text{ u} \\
\end{array}
\]

\[u \text{ and } v \text{ are opposite w.r.t. } \{z, w\}\]

\[
\begin{array}{c}
\text{\textbullet} \text{ w} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\textbullet} \text{ v} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\textbullet} \text{ z} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\textbullet} \text{ u} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\textbullet} \text{ w} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\textbullet} \text{ v} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\textbullet} \text{ z} \\
\end{array}
\]

\[\text{Note: optimal 1-planar graphs } \subset \text{ kite-triangulations.}\]

\[\text{The resulting graph } G \text{ is a kite-triangulation.}\]

\[\text{Theorem. } [\text{Angelini et al. ’11}]\]

Every kite-triangulation \( G \) on \( n \) vertices admits a 1-planar 1-bend RAC drawing, which can be constructed in \( \mathcal{O}(n) \) time.

\[\text{Proof.}\]

Let \( G'' \) be the underlying plane triangulation of \( G \). Let \( G''' = G' - S \).

Construct straight-line drawing of \( G''' \).

Fill faces as follows:

\[\text{strictly convex face}\]

\[\text{strictly convex face}\]
Kite Triangulations

This is a **kite**:

\[ u \quad \begin{array}{c} w \\ v \end{array} \\
\begin{array}{c} z \\ u \end{array} \]

\[ \quad \begin{array}{c} w \\ v \end{array} \\
\begin{array}{c} z \\ u \end{array} \]

**u** and **v** are **opposite** w.r.t. \{**z**, **w**\}

Let **G** be a plane triangulation.

Let **G**' be the underlying plane triangulation of **G**.

Let \( S \subset E(G') \) s.t. no two edges in \( S \) lie on the same face

... and their opposite vertices do not have an edge in \( E(G') \).

Add set \( T \) of edges connecting opposite vertices.

The resulting graph **G** is a **kite-triangulation**.

**Note:** optimal 1-planar graphs \( \subset \) kite-triangulations.

**Theorem.** [Angelini et al. '11]

Every kite-triangulation **G** on **n** vertices admits a 1-planar 1-bend RAC drawing, which can be constructed in \( \mathcal{O}(n) \) time.

**Proof.**

Let **G**' be the underlying plane triangulation of **G**. Let **G**'' = **G**' − **S**.

Construct straight-line drawing of **G**''.

Fill faces as follows:

strictly convex face

otherwise

Note: optimal 1-planar graphs \( \subset \) kite-triangulations.
Kite Triangulations

This is a **kite**:

\[
\begin{array}{c}
\text{\begin{minipage}{0.3\textwidth}
\begin{tikzpicture}
  \node (v1) at (0,0) {$u$};
  \node (v2) at (1,1) {$v$};
  \node (v3) at (1,-1) {$w$};
  \node (v4) at (0,2) {$z$};
  \draw (v1) -- (v2) -- (v3) -- (v4) -- (v1);
\end{tikzpicture}
\end{minipage}}
\end{array}
\]

Let \( G \) be a plane triangulation.

\[
\begin{array}{c}
\text{\begin{minipage}{0.3\textwidth}
\begin{tikzpicture}
  \node (v1) at (0,0) {$u$};
  \node (v2) at (1,1) {$v$};
  \node (v3) at (1,-1) {$w$};
  \node (v4) at (0,2) {$z$};
  \draw (v1) -- (v2) -- (v3) -- (v4) -- (v1);
\end{tikzpicture}
\end{minipage}}
\end{array}
\]

\textbf{Theorem.} [Angelini et al. '11]

Every kite-triangulation \( G \) on \( n \) vertices admits a 1-planar 1-bend RAC drawing, which can be constructed in \( O(n) \) time.

\textbf{Proof.}

Let \( G' \) be the underlying plane triangulation of \( G \). Let \( G'' = G' - S \).

Construct straight-line drawing of \( G'' \).

Fill faces as follows:

![Strictly convex face](strictly-convex-face)

![Otherwise](otherwise)

Add set \( T \) of edges connecting opposite vertices.

The resulting graph \( G \) is a **kite-triangulation**.

\textbf{Note:} optimal 1-planar graphs \( \subset \) kite-triangulations.
Kite Triangulations

This is a **kite**: $u$ and $v$ are **opposite** w.r.t. $\{z, w\}$

Let $G'$ be a plane triangulation.

Let $S \subset E(G')$ s.t. no two edges in $S$ lie on the same face

... and their opposite vertices do not have an edge in $E(G')$.

Add set $T$ of edges connecting opposite vertices.

The resulting graph $G$ is a **kite-triangulation**.

Note: optimal 1-planar graphs $\subset$ kite-triangulations.

**Theorem.** [Angelini et al. ’11]
Every kite-triangulation $G$ on $n$ vertices admits a 1-planar 1-bend RAC drawing, which can be constructed in $\mathcal{O}(n)$ time.

**Proof.**
Let $G'' = G' - S$. Construct straight-line drawing of $G''$. Fill faces as follows:

strictly convex face

otherwise

Add set $T$ of edges connecting opposite vertices.
Kite Triangulations

This is a kite:

\[
\begin{array}{c}
\text{\hspace{2cm} } \\
u & \hspace{2cm} & w \\
\text{\hspace{2cm} } & \text{\hspace{2cm} } \\
z & \hspace{2cm} & v \\
\end{array}
\]

\[u \text{ and } v \text{ are opposite w.r.t. } \{z, w\}\]

Let \(G'\) be a plane triangulation.

Let \(S \subset E(G')\) s.t. no two edges in \(S\) lie on the same face ... and their opposite vertices do not have an edge in \(E(G')\).

Add set \(T\) of edges connecting opposite vertices.

The resulting graph \(G\) is a kite-triangulation.

Note: optimal 1-planar graphs \(\subset\) kite-triangulations.

**Theorem.** [Angelini et al. ’11]
Every kite-triangulation \(G\) on \(n\) vertices admits a 1-planar 1-bend RAC drawing, which can be constructed in \(O(n)\) time.

**Proof.**
Let \(G'' = G' - S\).

Construct straight-line drawing of \(G''\).

Fill faces as follows:

Note: optimal 1-planar graphs \(\subset\) kite-triangulations.
Kite Triangulations

This is a **kite**:

![Kite Diagram]

Let $G'$ be a plane triangulation.

Let $S \subset E(G')$ s.t. no two edges in $S$ lie on the same face and their opposite vertices do not have an edge in $E(G')$.

Add set $T$ of edges connecting opposite vertices.

The resulting graph $G$ is a **kite-triangulation**.

**Theorem.** [Angelini et al. ’11] Every kite-triangulation $G$ on $n$ vertices admits a 1-planar 1-bend RAC drawing, which can be constructed in $O(n)$ time.

**Proof.**


Fill faces as follows:

- **Note:** optimal 1-planar graphs $\subset$ kite-triangulations.
Visualization of Graphs

Lecture 11:
Beyond Planarity
Drawing Graphs with Crossings

Part V:
1-Planar 1-Bend RAC Drawings

Alexander Wolff
1-Planar 1-Bend RAC Drawings

**Theorem.** [Bekos, Didimo, Liotta, Mehrabi & Montecchiani 2017]
Every 1-planar graph $G$ admits a 1-planar 1-bend RAC drawing.
1-Planar 1-Bend RAC Drawings

**Theorem.** [Bekos, Didimo, Liotta, Mehrabi & Montecchiani 2017]
Every 1-planar graph $G$ admits a 1-planar 1-bend RAC drawing.
If a 1-planar embedding of $G$ is given as part of the input,
such a drawing can be computed in linear time.
1-Planar 1-Bend RAC Drawings

**Theorem.** [Bekos, Didimo, Liotta, Mehrabi & Montecchiani 2017]
Every 1-planar graph $G$ admits a 1-planar 1-bend RAC drawing. If a 1-planar embedding of $G$ is given as part of the input, such a drawing can be computed in linear time.

**Observation.**
In a triangulated 1-plane graph (not necessarily simple), each pair of crossing edges of $G$ forms an empty kite,
1-Planar 1-Bend RAC Drawings

**Theorem.** [Bekos, Didimo, Liotta, Mehrabi & Montecchiani 2017] Every 1-planar graph $G$ admits a 1-planar 1-bend RAC drawing. If a 1-planar embedding of $G$ is given as part of the input, such a drawing can be computed in linear time.

**Observation.**
In a triangulated 1-plane graph (not necessarily simple), each pair of crossing edges of $G$ forms an empty kite, except for at most one pair if their crossing point is on the outer face of $G$. 
1-Planar 1-Bend RAC Drawings

**Theorem.** [Bekos, Didimo, Liotta, Mehrabi & Montecchiani 2017]
Every 1-planar graph $G$ admits a 1-planar 1-bend RAC drawing.
If a 1-planar embedding of $G$ is given as part of the input, such a drawing can be computed in linear time.

**Observation.**
In a triangulated 1-plane graph (not necessarily simple), each pair of crossing edges of $G$ forms an empty kite, except for at most one pair if their crossing point is on the outer face of $G$.

**Theorem.** [Chiba, Yamanouchi & Nishizeki 1984]
For every plane graph $G$ with outer face $C_k$ and every convex $k$-gon $P$, there exists a strictly convex planar straight-line drawing of $G$ whose outer face coincides with $P$. Such a drawing can be computed in linear time.
Algorithm Outline

**input**

- $G$
  - simple 1-plane

**output**

- $\Gamma$
  - 1-bend 1-planar RAC drawing of $G$
- $\Gamma^+$
  - 1-bend 1-planar RAC drawing of $G^+$

**augmentation**

- $(the\ embedding\ may\ change)$
- $G^+$
  - triangulated 1-plane (multi-edges)

**recursive procedure**

- $G^*$
  - hierarchical contraction of $G^+$

**removal of dummy elements**

- removal of dummy elements
Algorithm Outline

**input**

\( G \)  
simple 1-plane

\( \Gamma \)  
1-bend 1-planar RAC  
drawing of \( G \)

**output**

\( G^+ \)  
triangulated 1-plane  
(multi-edges)

\( \Gamma^+ \)  
1-bend 1-planar RAC  
drawing of \( G^+ \)

 augmentation  
(the embedding may change)

**recursive procedure**

\( G^* \)  
hierarchical contraction of \( G^+ \)

removal of dummy elements
Algorithm Step 1: Augmentation

$G$: simple 1-plane graph
Algorithm Step 1: Augmentation

1. For each pair of crossing edges add an enclosing 4-cycle.

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$G$: simple 1-plane graph
Algorithm Step 1: Augmentation

1. For each pair of crossing edges add an enclosing 4-cycle.

2. Remove those multiple edges that belong to $G$. $G$: simple 1-plane graph
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2. Remove those multiple edges that belong to $G$.  

$G$: simple 1-plane graph
Algorithm Step 1: Augmentation

1. For each pair of crossing edges add an enclosing 4-cycle.

2. Remove those multiple edges that belong to $G$.

3. Remove one (multiple) edge from each face of degree two (if any).

$G$: simple 1-plane graph
Algorithm Step 1: Augmentation

1. For each pair of crossing edges add an enclosing 4-cycle.

2. Remove those multiple edges that belong to $G$.

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4. Triangulate faces of degree $> 3$ by inserting a star inside them.
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Algorithm Outline

**Input**

$G$

simple 1-plane

**Output**

$\Gamma$

1-bend 1-planar RAC drawing of $G$

$\Gamma^+$

1-bend 1-planar RAC drawing of $G^+$

$G^+$

triangulated 1-plane (multi-edges)

$G^*$

hierarchical contraction of $G^+$

Recursive procedure

Augmentation (the embedding may change)

Removal of dummy elements
Algorithm Step 2: Hierarchical Contractions

$G^+$
triangulated 1-plane
(multi-edges)
Algorithm Step 2: Hierarchical Contractions

$G^+$

triangulated 1-plane
(multi-edges)

- triangular faces
Algorithm Step 2: Hierarchical Contractions

$G^+$
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges
  never crossed
Algorithm Step 2: Hierarchical Contractions

$G^+$
triangulated 1-plane
(multi-edges)
- triangular faces
- multiple edges
  never crossed
- only empty kites
Algorithm Step 2: Hierarchical Contractions

$G^+$
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges
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- only empty kites

⇒
structure of each separation pair
Algoritm Step 2: Hierarchical Contractions

\[ G^+ \]
triangulated 1-plane
(multi-edges)
- triangular faces
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\[ \Rightarrow \]
structure of each separation pair
Algoritm Step 2: Hierarchical Contractions

$G^+$
triangulated 1-plane
(multi-edges)
- triangular faces
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structure of each separation pair

Contract all inner components of each separation pair into a thick edge.
Algorithm Step 2: Hierarchical Contractions

$G^+$ triangulated 1-plane (multi-edges)
- triangular faces
- multiple edges never crossed
- only empty kites

⇒ structure of each separation pair
Contract all inner components of each separation pair into a thick edge.
Algorithm Step 2: Hierarchical Contractions

$G^+$
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges
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structure of each separation pair

Contract all inner components of each separation pair into a **thick edge**.
Algorithm Step 2: Hierarchical Contractions

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triangulated 1-plane
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Contract all inner components of each separation pair into a thick edge.
Algorithm Step 2: Hierarchical Contractions

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triangulated 1-plane (multi-edges)

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Algorithm Step 2: Hierarchical Contractions

$G^+$ triangulated 1-plane (multi-edges)

- triangular faces
- multiple edges never crossed
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Contract all inner components of each separation pair into a thick edge.
Algorithm Step 2: Hierarchical Contractions

$G^*$

hierarchical contraction of $G^+$

$G^+$
Algorithm Step 2: Hierarchical Contractions

$G^*$
Hierarchical contraction of $G^+$

$G^+$
Simple 3-connected triangulated 1-plane graph
Algorithm Outline

**input**

- \( G \)
  - simple 1-plane

**output**

- \( \Gamma \)
  - 1-bend 1-planar RAC drawing of \( G \)

Augmentation (the embedding may change)

- \( G^+ \)
  - triangulated 1-plane (multi-edges)

Hierarchical contraction of \( G^+ \)

Recursive procedure

Recursive procedure

- \( \Gamma^+ \)
  - 1-bend 1-planar RAC drawing of \( G^+ \)

Removal of dummy elements
Algorithm Step 3: Drawing Procedure
Algorithm Step 3: Drawing Procedure

remove crossing edges
Algorithm Step 3: Drawing Procedure

1. Remove crossing edges.
2. Result: 3-connected plane graph.
Algorithm Step 3: Drawing Procedure

remove crossing edges

3-connected plane graph

apply Chiba et al.
Algorithm Step 3: Drawing Procedure

remove crossing edges

3-connected plane graph

apply Chiba et al.

clean faces & prescribed outer face
Algorithm Step 3: Drawing Procedure

1. Remove crossing edges from the graph.
2. Apply Chiba et al. to the 3-connected plane graph.
3. Reinsert the crossing edges.
4. Ensure convex faces and a prescribed outer face.
Algorithm Step 3: Drawing Procedure

1. Remove crossing edges
2. 3-connected plane graph
3. Apply Chiba et al.
4. Convex faces & prescribed outer face
5. Reinsert crossing edges

Diagram:
- Initial graph with crossings
- Step 1: Crossing edges removed
- Step 2: 3-connected plane graph
- Step 3: Applying Chiba et al.
- Step 4: Convex faces and prescribed outer face
- Step 5: Reinserting crossing edges
Algorithm Step 3: Drawing Procedure

1. Remove crossing edges
2. Apply Chiba et al.
3. Convex faces & prescribed outer face
4. Reinsert crossing edges

3-connected plane graph
Algorithm Step 3: Drawing Procedure

1. remove crossing edges
2. apply Chiba et al.
3. convex faces & prescribed outer face
4. reinsert crossing edges
5. partial drawing

3-connected plane graph
Algorithm Step 3: Drawing Procedure
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remove crossing edges
Algorithm Step 3: Drawing Procedure

remove crossing edges
Algorithm Step 3: Drawing Procedure

apply Chiba et al.
Algorithm Step 3: Drawing Procedure

reinsert crossing edges
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remove crossing edges
Algorithm Step 3: Drawing Procedure

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Algorithm Step 3: Drawing Procedure

apply Chiba et al.
Algorithm Step 3: Drawing Procedure

reinsert crossing edges
Algorithm Step 3: Drawing Procedure
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Algorithm Step 3: Drawing Procedure
Algorithm Step 3: Drawing Procedure

$\Gamma^+$: 1-bend 1-planar RAC drawing of $G^+$
Algorithm Outline

**Input**

\[ G \]

simple 1-plane

**Output**

\[ \Gamma \]

1-bend 1-planar RAC drawing of \( G \)

\[ \Gamma^+ \]

1-bend 1-planar RAC drawing of \( G^+ \)

**Algorithm Steps**

1. Start with a simple 1-plane graph \( G \).
2. Perform an augmentation (the embedding may change) to obtain \( G^+ \), which is triangulated 1-plane (multi-edges).
3. Apply a hierarchical contraction procedure to obtain \( G^* \).
4. Finally, remove all dummy elements to obtain \( \Gamma^+ \), the 1-bend 1-planar RAC drawing of \( G^+ \).
Algorithm Step 4: Removal of Dummy Vertices
Algorithm Step 4: Removal of Dummy Vertices

$$\Gamma: \text{1-bend 1-planar RAC drawing of } G$$
Algorithm Step 4: Removal of Dummy Vertices

$G$: simple 1-plane graph

$\Gamma$: 1-bend 1-planar RAC drawing of $G$
GD Beyond Planarity: a Taxonomy

- RAC &. 1-plan.
- fan-plan. & k-plan.
- 2-layer fan & RAC
- k-plan. & k-qu.-plan.

- circ. RAC
- out. 1-plan.
- 2-layer RAC
- 2-layer fan
- book emb.

- RAC
- 2-layer RAC
- k-plan.
- book emb.

- 1-plan.
- fan-plan.
- quas.-plan.

- 1-plan.
- poly-time
- variable embedding
- 1-plan.

- maximal 1-plan.
- RAC
- out. 1-plan.
- opt. 1-plan.
- 2-layer RAC
- 2-layer fan-plan

- area
- edge-complexity
- bends
- slopes
- out. 1-plan.
- k-plan.
- RAC

- Density
- Recognition
- Stretchability
- Relationships
- Constraints
- Eng. & Exper.

- fixed rot. system
- many families
- variable embedding
- 1-plan.
- NP-hard
- poly-time

Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017
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Relationships
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Eng. & Exper.
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A Survey on Graph Drawing Beyond Planarity.

An Annotated Bibliography on 1-Planarity.

Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017
Literature

Books and surveys:
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- [Kobourov, Liotta & Montecchiani '17] An Annotated Bibliography on 1-Planarity
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- [Eades, Huang, Hong '08] Effects of Crossing Angles
- [Brandenburg et al. '13] On the density of maximal 1-planar graphs
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- [Grigoriev and Bodlaender '07] Algorithms for graphs embeddable with few crossings per edge
- [Angelini et al. '11] On the Perspectives Opened by Right Angle Crossing Drawings
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- [Bekos et al. '17] On RAC drawings of 1-planar graphs