Visualization of Graphs

Lecture 11: The Crossing Lemma and Its Applications

Part I: Definition and Hanani–Tutte

Alexander Wolff
Crossing Number and Topological Graphs

For a graph $G$, the **crossing number** $\text{cr}(G)$ is the smallest number of edge crossings in a drawing of $G$ (in the plane).
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**Example.**

$\text{cr}(K_{3,3}) = 9$?
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$cr(K_{3,3}) = 5$?
Crossing Number and Topological Graphs

For a graph $G$, the **crossing number** $\text{cr}(G)$ is the smallest number of edge crossings in a drawing of $G$ (in the plane).

**Example.**

$\text{cr}(K_{3,3}) = 1$
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**Example.**

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[Diagram showing edge crossings reduced, so terminates]
Hanani–Tutte Theorem

**Theorem.** [Hanani ’43, Tutte ’70]

A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.
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Hanani showed that every drawing of $K_5$ and $K_{3,3}$ must have a pair of edges that crosses an odd number of times.
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Hence, there must be two edges on these paths that cross an odd number of times. □
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Theorem. \[\text{[Hanani '43, Tutte '70]}\]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

The odd crossing number $\text{ocr}(G)$ of $G$ is the smallest number of pairs of edges that cross oddly in a drawing of $G$. 
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**Theorem.** [Pelsmajer, Schaefer & Štefankovič ’08, Tóth ’08]
There is a graph $G'$ with $\text{ocr}(G') < \text{cr}(G') \leq 10$
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If $\Gamma$ is a drawing of $G$ and $E_0$ is the set of edges with only even numbers of crossings in $\Gamma$, then $G$ can be drawn such that no edge in $E_0$ is involved in any crossings.
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By definition $\text{ocr}(G) \leq \text{pcr}(G) \leq \text{cr}(G)$
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Is $pcr(G) = cr(G)$? Open!
Visualization of Graphs

Lecture 11:
The Crossing Lemma
and its Applications

Part II:
Computation & Variations

Alexander Wolff
Computing the Crossing Number

- Computing $\text{cr}(G)$ is NP-hard. [Garey & Johnson '83]

... even if $G$ is a planar graph plus one edge! [Cabello & Mohar '08]
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For exact computations, check out http://crossings.uos.de!
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- Planarization, where we replace crossings with dummy vertices, also uses only heuristics.

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Other Crossing Numbers

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- Crossing minimization is NP-hard for most variants.
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**Separation.**
$cr(K_8) = 18$, but $\overline{cr}(K_8) = 19$. 
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**Lemma 1.** [Bienstock, Dean ’93] For $k \geq 4$, there exists a graph $G_k$ with $cr(G_k) = 4$ and $\overline{cr}(G_k) \geq k$. 

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- From $G_1$ to $G_k$ do
Bounds for Complete Graphs

**Theorem.** \([\text{Guy '60}]\)

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Bounds for Complete Graphs

Theorem. \[cr(K_n) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor = \frac{3}{8} \binom{n}{4} + O(n^3)\] [Guy ’60]
Bounds for Complete Graphs

Theorem. [Guy '60]

\[ cr(K_n) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor = \frac{3}{8} \binom{n}{4} + O(n^3) \]
Bounds for Complete Graphs

**Theorem.** [Guy '60]

\[
\text{cr}(K_n) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor = \frac{3}{8} \binom{n}{4} + O(n^3)
\]
Bounds for Complete Graphs

**Theorem.** \[ \text{cr}(K_n) \leq \frac{1}{4} \begin{bmatrix} n \\ 2 \end{bmatrix} \begin{bmatrix} n - 1 \\ 2 \end{bmatrix} \begin{bmatrix} n - 2 \\ 2 \end{bmatrix} \begin{bmatrix} n - 3 \\ 2 \end{bmatrix} = \frac{3}{8} \binom{n}{4} + O(n^3) \]

[Guy ’60]

Sylvester’s four-point problem
Bounds for Complete Graphs

**Theorem.** \[ \text{cr}(K_n) \leq \frac{1}{4} \left( \binom{n}{2} \binom{n-1}{2} \binom{n-2}{2} \binom{n-3}{2} \right) = \frac{3}{8} \binom{n}{4} + O(n^3) \]

[Guy '60]

**Conjecture.** Sylvester's four-point problem
Bounds for Complete Graphs

**Theorem.** [Guy '60]

\[
\text{cr}(K_n) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor = \frac{3}{8} \binom{n}{4} + O(n^3)
\]

Bound is tight for \( n \leq 12 \).

**Conjecture.**

Sylvester’s four-point problem
Bounds for Complete Graphs

Theorem. \[ \mathrm{cr}(K_n) \leq \frac{1}{4} \left[ \frac{n}{2} \right] \left[ \frac{n-1}{2} \right] \left[ \frac{n-2}{2} \right] \left[ \frac{n-3}{2} \right] = \frac{3}{8} \binom{n}{4} + O(n^3) \]

Bound is tight for \( n \leq 12 \).

Conjecture.

Theorem. \[ \mathrm{cr}(K_{m,n}) \leq \frac{1}{4} \left[ \frac{n}{2} \right] \left[ \frac{n-1}{2} \right] \left[ \frac{m}{2} \right] \left[ \frac{m-1}{2} \right] \]

Sylvester’s four-point problem
Bounds for Complete Graphs

**Theorem.** [Guy ‘60]

\[
\text{cr}(K_n) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor = \frac{3}{8} \binom{n}{4} + O(n^3)
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Bound is tight for \( n \leq 12 \).

**Theorem.** [Zarankiewicz ’54, Urbaník ’55]

\[
\text{cr}(K_{m,n}) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor
\]

Turán’s brick factory problem (1944)

Pál Turán
*1910 – 1976
Budapest, Hungary

Sylvester’s four-point problem
Bounds for Complete Graphs

**Theorem.** Conjecture. [Guy ’60]

\[
\text{cr}(K_n) \leq \frac{1}{4} \left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n-1}{2} \right\rceil \left\lceil \frac{n-2}{2} \right\rceil \left\lceil \frac{n-3}{2} \right\rceil = \frac{3}{8} \binom{n}{4} + O(n^3)
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\]

Turán’s brick factory problem (1944)

Sylvester’s four-point problem
Bounds for Complete Graphs

**Theorem.** \[ \text{Conjecture.} \] [Guy ’60]

\[
\text{cr}(K_n) \leq 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n - 1}{2} \right\rfloor \left\lfloor \frac{n - 2}{2} \right\rfloor \left\lfloor \frac{n - 3}{2} \right\rfloor = \frac{3}{8} \binom{n}{4} + O(n^3)
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Bound is tight for \( n \leq 12 \).

**Theorem.** \[ \text{Conjecture.} \] [Zarankiewicz ’54, Urbaník ’55]

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\text{cr}(K_{m,n}) \leq 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n - 1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m - 1}{2} \right\rfloor
\]

**Theorem.** [Lovász et al. ’04, Aichholzer et al. ’06]

\[
\left( \frac{3}{8} + \varepsilon \right) \binom{n}{4} + O(n^3) < \text{cr}(K_n) < 0.3807 \binom{n}{4} + O(n^3)
\]

Sylvester’s four-point problem
Bounds for Complete Graphs

**Theorem.** [Guy ’60]

\[
\text{cr}(K_n) \leq \frac{1}{4} \left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n-1}{2} \right\rceil \left\lceil \frac{n-2}{2} \right\rceil \left\lceil \frac{n-3}{2} \right\rceil = \frac{3}{8} \binom{n}{4} + O(n^3)
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Exact numbers are known for \( n \leq 27 \).
Bounds for Complete Graphs

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Exact numbers are known for \( n \leq 27 \).

Check out [http://www.ist.tugraz.at/staff/aichholzer/crossings.html](http://www.ist.tugraz.at/staff/aichholzer/crossings.html)!
First Lower Bounds on $\text{cr}(G)$

**Lemma 2.**
For a graph $G$ with $n$ vertices and $m$ edges,

$$\text{cr}(G) \geq m - 3n + 6.$$
First Lower Bounds on $\text{cr}(G)$

Lemma 2.
For a graph $G$ with $n$ vertices and $m$ edges,

$$\text{cr}(G) \geq m - 3n + 6.$$ 

Proof.

- Consider a drawing of $G$ with $\text{cr}(G)$ crossings.
First Lower Bounds on $\text{cr}(G)$

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- Obtain a graph $H$ by turning crossings into dummy vertices.

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**Proof.**
- Consider a drawing of $G$ with $\text{cr}(G)$ crossings.
- Obtain a graph $H$ by turning crossings into dummy vertices.
- $H$ has $n + \text{cr}(G)$ vertices and $m + 2\text{cr}(G)$ edges.
First Lower Bounds on $\text{cr}(G)$

**Lemma 2.**
For a graph $G$ with $n$ vertices and $m$ edges,

$$\text{cr}(G) \geq m - 3n + 6.$$  

**Proof.**
- Consider a drawing of $G$ with $\text{cr}(G)$ crossings.
- Obtain a graph $H$ by turning crossings into dummy vertices.
- $H$ has $n + \text{cr}(G)$ vertices and $m + 2\text{cr}(G)$ edges.
- $H$ is planar, so

$$m + 2\text{cr}(G) \leq 3(n + \text{cr}(G)) - 6. \quad \square$$

---

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$$m + 2\text{cr}(G) \leq 3(n + \text{cr}(G)) - 6.$$  

Consider this bound for graphs with $\Theta(n)$ and $\Theta(n^2)$ many edges.
First Lower Bounds on \( cr(G) \)

**Lemma 3.**

For a non-planar graph \( G \) with \( n \) vertices and \( m \) edges,

\[
\text{cr}(G) \geq r \cdot \left( \frac{\lfloor m/r \rfloor}{2} \right) \in \Omega \left( \frac{m^2}{n} \right)
\]

where \( r \leq 3n - 6 \) is the maximum number of edges in a planar subgraph of \( G \).
First Lower Bounds on \( \text{cr}(G) \)

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**Proof sketch.**
- Take \( \lfloor m/r \rfloor \) edge-disjoint subgraphs of \( G \) with \( r \) edges.
First Lower Bounds on $\text{cr}(G)$

Lemma 3.
For a non-planar graph $G$ with $n$ vertices and $m$ edges,

$$\text{cr}(G) \geq r \cdot \left( \left\lfloor \frac{m}{r} \right\rfloor \right)^2 \in \Omega \left( \frac{m^2}{n} \right)$$

where $r \leq 3n - 6$ is the maximum number of edges in a planar subgraph of $G$.

Proof sketch.
- Take $\left\lfloor \frac{m}{r} \right\rfloor$ edge-disjoint subgraphs of $G$ with $r$ edges.
- In the best case, they are all planar.
First Lower Bounds on $\text{cr}(G)$

**Lemma 3.**
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- Take $\left\lfloor \frac{m}{r} \right\rfloor$ edge-disjoint subgraphs of $G$ with $r$ edges.
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- For every $i < j$, any edge of $G_j$ induces at least one crossing with $G_i$.
  (If not, swap edges to reduce $\text{cr}(G_i)$.)
First Lower Bounds on \( \text{cr}(G) \)

**Lemma 3.**
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**Proof sketch.**
- Take \( \lfloor m/r \rfloor \) edge-disjoint subgraphs of \( G \) with \( r \) edges.
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- For every \( i < j \), any edge of \( G_j \) induces at least one crossing with \( G_i \).
  (If not, swap edges to reduce \( \text{cr}(G_i) \).)
Visualization of Graphs

Lecture 11:
The Crossing Lemma
and its Applications

Part IV:
The Crossing Lemma

Alexander Wolff
1973 Erdős and Guy conjectured that $\text{cr}(G) \in \Omega(m^3/n^2)$. 
The Crossing Lemma

- 1973 Erdős and Guy conjectured that \( \text{cr}(G) \in \Omega(m^3/n^2) \).

- In 1982 Leighton and, independently, Ajtai, Chávtal, Newborn, and Szemerédi showed that

\[
\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}.
\]
The Crossing Lemma

- 1973 Erdős and Guy conjectured that \( \text{cr}(G') \in \Omega(m^3/n^2) \).

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Consider this bound for graphs with \( \Theta(n) \) and \( \Theta(n^2) \) many edges.
The Crossing Lemma

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\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}.
\]

- Bound is asymptotically tight.

**Consider this bound for graphs with \( \Theta(n) \) and \( \Theta(n^2) \) many edges.**
The Crossing Lemma

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- Bound is asymptotically tight.

- Result stayed hardly known until Székely demonstrated its usefulness (in 1997).

Consider this bound for graphs with \( \Theta(n) \) and \( \Theta(n^2) \) many edges.
The Crossing Lemma

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- We go through the proof from “THE BOOK” by Chazelle, Sharir, and Welzl.
The Crossing Lemma

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- Bound is asymptotically tight.

- Result stayed hardly known until Székely demonstrated its usefulness (in 1997).

- We go through the proof from “THE BOOK” by Chazelle, Sharir, and Welzl.

- Factor $\frac{1}{64}$ was later (with intermediate steps) improved to $\frac{1}{29}$ by Ackerman in 2013.

Consider this bound for graphs with $\Theta(n)$ and $\Theta(n^2)$ many edges.
The Crossing Lemma

**Crossing Lemma.**
For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4n$,
$$\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}.$$
The Crossing Lemma

Proof.
- Consider a crossing-minimal drawing of $G$.

**Crossing Lemma.**
For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4n$, 

$$\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}.$$
The Crossing Lemma

Crossing Lemma. For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4n$,
\[
\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}.
\]

Proof.

- Consider a crossing-minimal drawing of $G$.
- Let $p$ be a number in $(0, 1]$. 
The Crossing Lemma

**Crossing Lemma.**
For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4n$, $cr(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}$.

**Proof.**
- Consider a crossing-minimal drawing of $G$.
- Let $p$ be a number in $(0, 1]$.
- Keep every vertex of $G$ independently with probability $p$. 
The Crossing Lemma

Proof.

- Consider a crossing-minimal drawing of $G$.
- Let $p$ be a number in $(0, 1]$.
- Keep every vertex of $G$ independently with probability $p$.
- $G_p = \text{remaining graph (with drawing } \Gamma_p)$. 
- Let $n_p, m_p, X_p$ be the random variables counting the numbers of vertices / edges / crossings of $\Gamma_p$, resp.

**Crossing Lemma.**

For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4n$,

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The Crossing Lemma

**Crossing Lemma.**
For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4n$,
$$\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}. $$

**Proof.**
- Consider a crossing-minimal drawing of $G$.
- Let $p$ be a number in $(0, 1]$.
- Keep every vertex of $G$ independently with probability $p$.
- $G_p =$ remaining graph (with drawing $\Gamma_p$).
- Let $n_p, m_p, X_p$ be the random variables counting the numbers of vertices / edges / crossings of $\Gamma_p$, resp.
- By Lemma 2, $\text{cr}(G_p) - m_p + 3n_p \geq 6.$
The Crossing Lemma

**Crossing Lemma.** For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4n$,  \[
\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}.
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- Consider a crossing-minimal drawing of $G$.
- Let $p$ be a number in $(0, 1]$.
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- $G_p = \text{remaining graph (with drawing } \Gamma_p\text{)}$.
- Let $n_p, m_p, X_p$ be the random variables counting the numbers of vertices / edges / crossings of $\Gamma_p$, resp.
- By Lemma 2, $\text{cr}(G_p) - m_p + 3n_p \geq 6$.
  \[\Rightarrow E(X_p - m_p + 3n_p) \geq 0.\]
The Crossing Lemma

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For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4n$,
\[
\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}.
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- Consider a crossing-minimal drawing of $G$.
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The Crossing Lemma

**Crossing Lemma.** For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4n$, 
$$\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}.$$ 

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- Consider a crossing-minimal drawing of $G$.
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$\mathbb{E}(n_p) = pn$ and $\mathbb{E}(m_p) = p^2 m$
The Crossing Lemma

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- \[\Rightarrow \mathbb{E}(X_p - m_p + 3n_p) \geq 0.\]
- \[\mathbb{E}(n_p) = pn \text{ and } \mathbb{E}(m_p) = p^2m\]
- \[\mathbb{E}(X_p) = p^4\text{cr}(G)\]
The Crossing Lemma

**Crossing Lemma.**
For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4n,$
\[ \text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}. \]

**Proof.**
- Consider a crossing-minimal drawing of $G$.
- Let $p$ be a number in $(0, 1]$.
- Keep every vertex of $G$ independently with probability $p$.
- $G_p$ = remaining graph (with drawing $\Gamma_p$).
- Let $n_p, m_p, X_p$ be the random variables counting the numbers of vertices / edges / crossings of $\Gamma_p$, resp.
- By Lemma 2, $\text{cr}(G_p) - m_p + 3n_p \geq 6$.
  \[ \Rightarrow \mathbb{E}(X_p - m_p + 3n_p) \geq 0. \]

- $\mathbb{E}(n_p) = pn$ and $\mathbb{E}(m_p) = p^2m$
- $\mathbb{E}(X_p) = p^4\text{cr}(G)$
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Proof.

- Consider a crossing-minimal drawing of $G$.
- Let $p$ be a number in $(0, 1]$.
- Keep every vertex of $G$ independently with probability $p$.
- $G_p = \text{remaining graph (with drawing } \Gamma_p \text{)}$.
- Let $n_p, m_p, X_p$ be the random variables counting the numbers of vertices / edges / crossings of $\Gamma_p$, resp.
- By Lemma 2, $\text{cr}(G_p) - m_p + 3n_p \geq 6$.
  \[ \Rightarrow \mathbb{E}(X_p - m_p + 3n_p) \geq 0. \]

\[ \begin{align*}
\mathbb{E}(n_p) &= pn \text{ and } \mathbb{E}(m_p) = p^2m \\
\mathbb{E}(X_p) &= p^4 \text{cr}(G) \\
0 &\leq \mathbb{E}(X_p) - \mathbb{E}(m_p) + 3\mathbb{E}(n_p) \\
&= p^4 \text{cr}(G) - p^2m + 3pn
\end{align*} \]
The Crossing Lemma

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For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4n$, 
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- Let $G_p = \text{remaining graph (with drawing } \Gamma_p)$.
- Let $n_p, m_p, X_p$ be the random variables counting the numbers of vertices / edges / crossings of $\Gamma_p$, resp.
- By Lemma 2, $\text{cr}(G_p) - m_p + 3n_p \geq 6$.
  \[ \Rightarrow E(X_p - m_p + 3n_p) \geq 0. \]
- $E(n_p) = pn$ and $E(m_p) = p^2m$
- $E(X_p) = p^4\text{cr}(G)$
- $0 \leq E(X_p) - E(m_p) + 3E(n_p)$
  \[ = p^4\text{cr}(G) - p^2m + 3pn \]
- $\text{cr}(G) \geq \frac{p^2m - 3pn}{p^4}$
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- Let $n_p, m_p, X_p$ be the random variables counting the numbers of vertices / edges / crossings of $\Gamma_p$, resp.
- By Lemma 2, $\text{cr}(G_p) = m_p + 3n_p \geq 6$.

$$\Rightarrow \mathbb{E}(X_p - m_p + 3n_p) \geq 0.$$  

- $\mathbb{E}(n_p) = pn$ and $\mathbb{E}(m_p) = p^2m$
- $\mathbb{E}(X_p) = p^4\text{cr}(G)$
- $0 \leq \mathbb{E}(X_p) - \mathbb{E}(m_p) + 3\mathbb{E}(n_p) = p^4\text{cr}(G) - p^2m + 3pn$
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  $$= p^4\text{cr}(G) - p^2m + 3pn$$
- $\text{cr}(G) \geq \frac{p^2m - 3pn}{p^4} = \frac{m}{p^2} - \frac{3n}{p^3}$
- Set $p = \frac{4n}{m}$.  

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  \[ \Rightarrow \mathbb{E}(X_p - m_p + 3n_p) \geq 0. \]
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- $\mathbb{E}(X_p) = p^4 \text{cr}(G)$
- $0 \leq \mathbb{E}(X_p) - \mathbb{E}(m_p) + 3\mathbb{E}(n_p)$
  \[ = p^4 \text{cr}(G') - p^2 m + 3pn \]
- $\text{cr}(G) \geq \frac{p^2 m - 3pn}{p^4} = \frac{m}{p^2} - \frac{3n}{p^3}$
- Set $p = \frac{4n}{m}$.
- \[ \text{cr}(G') \geq \frac{m}{p^2} - \frac{3n}{p^3} \]

\[
\boxed{\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}.}
\]
The Crossing Lemma

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- By Lemma 2, $\text{cr}(G_p) - m_p + 3n_p \geq 6.$ \[\Rightarrow \mathbb{E}(X_p - m_p + 3n_p) \geq 0.\]

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- $\mathbb{E}(X_p) = p^4\text{cr}(G)$.
- $0 \leq \mathbb{E}(X_p) - \mathbb{E}(m_p) + 3\mathbb{E}(n_p) = p^4\text{cr}(G) - p^2m + 3pn$.
- $\text{cr}(G) \geq \frac{p^2m - 3pn}{p^4} = \frac{m}{p^2} - \frac{3n}{p^3}$.
- Set $p = \frac{4n}{m}$.
- $\text{cr}(G) \geq \frac{m^3}{16n^2} - \frac{3m^3}{64n^2}.$
The Crossing Lemma

**The Crossing Lemma.**
For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4n$, 

$$\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}.$$  

**Proof.**

- Consider a crossing-minimal drawing of $G$.
- Let $p$ be a number in $(0, 1]$.
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\[ 0 \leq \mathbb{E}(X_p) - \mathbb{E}(m_p) + 3\mathbb{E}(n_p) = p^4 \text{cr}(G) - p^2 m + 3pn \]

\[ \text{cr}(G) \geq \frac{p^2 m - 3pn}{p^4} = \frac{m}{p^2} - \frac{3n}{p^3} \]

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\[ \square \]
Visualization of Graphs

Lecture 11:
The Crossing Lemma
and its Applications

Part V:
Applications

Alexander Wolff
Application 1: Point–Line Incidences

- For a set $P \subset \mathbb{R}^2$ of points and a set $\mathcal{L}$ of lines, let $I(P, \mathcal{L}) = \text{number of point–line incidences in } (P, \mathcal{L})$. 
Application 1: Point–Line Incidences

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\[ P \quad \mathcal{L} \]
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\[ \Rightarrow I(P, \mathcal{L}) = \]
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\[
\Rightarrow I(P, \mathcal{L}) = \]

\[\begin{array}{llll}
1 & 2 & 3 & \\
P & & & \\
\mathcal{L} & & & \\
\end{array}\]
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$$\Rightarrow I(P, \mathcal{L}) = 10$$
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\[ \Rightarrow I(P, \mathcal{L}) = 10 \]

- Define $I(n, k) = \max_{|P|=n, |\mathcal{L}|=k} I(P, \mathcal{L})$. 

\[ \mathcal{L} \]

\[ P \]

\[ 3 \]

\[ 3 \]

\[ 2 \]

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For example: $I(4, 4) =$
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\[
\begin{array}{c}
\mathcal{L} \\
\quad 3 \\
\quad 2 \\
\quad 3 \\
P \\
\quad 2 \\
\quad 3 \\
\end{array}
\]

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\[
\begin{array}{ccc}
& & 3 \\
3 & & 8 \\
& & 9 \\
\end{array}
\]
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Theorem 1. ([Szemerédi, Trotter ’83, Székely ’97] $I(n, k) \leq 2.7n^{2/3}k^{2/3} + 6n + 2k$.)
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- For example: $I(4, 4) = 9$

Theorem 1.
[Szemerédi, Trotter ’83, Székely ’97]
\[I(n, k) \leq c(n^{2/3}k^{2/3} + n + k)\]
Application 1: Point–Line Incidences

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**Theorem 1.**
[Szemerédi, Trotter '83, Székely '97]
\[ I(n, k) \leq c\left(n^{2/3}k^{2/3} + n + k\right). \]

**Proof.**
Application 1: Point–Line Incidences

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Proof.
\[ \text{cr}(G) \leq k^2 \]
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- For example: $I(4, 4) = 9$

\[\begin{align*}
\text{Theorem 1.} & & \text{[Szemerédi, Trotter '83, Székely '97]} \\
& & I(n, k) \leq c(n^{2/3}k^{2/3} + n + k).
\end{align*}\]

\[\begin{align*}
\text{Proof.} & & \text{cr}(G) \leq k^2 \\
& & \#\text{(points on } \ell) - 1 = \#\text{(edges on } \ell)
\end{align*}\]
Application 1: Point–Line Incidences

- For a set $P \subset \mathbb{R}^2$ of points and a set $\mathcal{L}$ of lines, let $I(P, \mathcal{L}) = \text{number of point–line incidences in } (P, \mathcal{L})$.

- Define $I(n, k) = \max_{|P| = n, |\mathcal{L}| = k} I(P, \mathcal{L})$.

- For example: $I(4, 4) = 9$

Theorem 1. [Szemerédi, Trotter '83, Székely '97]

$I(n, k) \leq c\left(\frac{n^{2/3}k^{2/3}}{3} + n + k\right)$.

Proof.

- $\#(\text{points on } \ell) - 1 = \#(\text{edges on } \ell)$

- $\Rightarrow I(n, k) - k \leq m$ (sum up over $\mathcal{L}$)
Application 1: Point–Line Incidences

- For a set $P \subset \mathbb{R}^2$ of points and a set $\mathcal{L}$ of lines, let $I(P, \mathcal{L}) = \text{number of point–line incidences in } (P, \mathcal{L})$.

- Define $I(n, k) = \max_{|P|=n, |\mathcal{L}|=k} I(P, \mathcal{L})$.

- For example: $I(4, 4) = 9$

$\Rightarrow I(P, \mathcal{L}) = 10$

**Theorem 1.**

[Szemerédi, Trotter ’83, Székely ’97] $I(n, k) \leq c(n^{2/3}k^{2/3} + n + k)$.

**Proof.**

- $\#(\text{points on } \ell) - 1 = \#(\text{edges on } \ell)$
- $\Rightarrow I(n, k) - k \leq m \quad \text{(sum up over } \mathcal{L})$
- Crossing Lemma: $\frac{1}{64} \frac{m^3}{n^2} \leq \text{cr}(G)$
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For a set \( P \subset \mathbb{R}^2 \) of points and a set \( \mathcal{L} \) of lines, let \( I(P, \mathcal{L}) = \) number of point–line incidences in \((P, \mathcal{L})\).

Define \( I(n, k) = \max_{|P|=n, |\mathcal{L}|=k} I(P, \mathcal{L}) \).

For example: \( I(4, 4) = 9 \)

\[ \Rightarrow I(P, \mathcal{L}) = 10 \]

\[ \Rightarrow I(n, k) - k \leq m \quad \text{(sum up over } \mathcal{L} \text{)} \]

Crossing Lemma: \( \frac{1}{64} \frac{m^3}{n^2} \leq cr(G) \)

\[ \Rightarrow \exists c: c(I(n, k) - k)^3/n^2 \leq cr(G) \]

Theorem 1.

[Szemerédi, Trotter '83, Székely '97]
\[ I(n, k) \leq c\left(\frac{n^{2/3}k^{2/3}}{} + n + k\right) \]

Proof.

\[ cr(G) \leq k^2 \]

\[ \#(\text{points on } \ell) - 1 = \#(\text{edges on } \ell) \]

\[ \Rightarrow I(n, k) - k \leq m \quad \text{(sum up over } \mathcal{L} \text{)} \]
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- For a set $P \subset \mathbb{R}^2$ of points and a set $\mathcal{L}$ of lines, let $I(P, \mathcal{L}) = \text{number of point–line incidences in } (P, \mathcal{L})$.

\[ \Rightarrow I(P, \mathcal{L}) = 10 \]

- Define $I(n, k) = \max_{|P|=n, |\mathcal{L}|=k} I(P, \mathcal{L})$.

- For example: $I(4, 4) = 9$

\[ I(n, k) \leq c(n^{2/3}k^{2/3} + n + k). \]

Proof.

- $(\text{points on } \ell) - 1 = (\text{edges on } \ell)$

\[ \Rightarrow I(n, k) - k \leq m \quad (\text{sum up over } \mathcal{L}) \]

- Crossing Lemma: $\frac{1}{64} \frac{m^3}{n^2} \leq \text{cr}(G)$

\[ \Rightarrow \exists c: c(I(n, k) - k)^3/n^2 \leq \text{cr}(G) \leq k^2 \]
Application 1: Point–Line Incidences

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- Define \( I(n, k) = \max_{|P|=n, |\mathcal{L}|=k} I(P, \mathcal{L}) \).

- For example: \( I(4, 4) = 9 \)

**Theorem 1.**
[Szemerédi, Trotter ’83, Székely ’97]
\[
I(n, k) \leq c(n^{2/3}k^{2/3} + n + k).
\]

**Proof.**

- \#(points on \( \ell \)) − 1 = #(edges on \( \ell \))

- \( I(n, k) − k \leq m \) (sum up over \( \mathcal{L} \))

- Crossing Lemma: \( \frac{1}{64} \frac{m^3}{n^2} \leq \text{cr}(G) \)

- \( \exists c: c(I(n, k) − k)^3/n^2 \leq \text{cr}(G) \leq k^2 \)

- If \( m < 4n \),
Application 1: Point–Line Incidences

- For a set $P \subset \mathbb{R}^2$ of points and a set $\mathcal{L}$ of lines, let $I(P, \mathcal{L}) =$ number of point–line incidences in $(P, \mathcal{L})$.

- Define $I(n, k) = \max_{|P|=n, |\mathcal{L}|=k} I(P, \mathcal{L})$.

- For example: $I(4, 4) = 9$

- **Theorem 1.**

  [Szemerédi, Trotter ’83, Székely ’97]

  $I(n, k) \leq c(n^{2/3}k^{2/3} + n + k)$.

**Proof.**

- $(\text{points on } \ell) - 1 = (\text{edges on } \ell)$

  $\Rightarrow I(n, k) - k \leq m$ \hspace{1cm} (sum up over $\mathcal{L}$)

- Crossing Lemma: $\frac{1}{64} \frac{m^3}{n^2} \leq \text{cr}(G)$

  $\Rightarrow \exists c: c(I(n, k) - k)^{3/n^2} \leq \text{cr}(G) \leq k^2$

  If $m < 4n$, then $I(n, k) - k \leq 4n$. \hspace{1cm} $\square$
Application 2: Unit Distances

For a set $P \subset \mathbb{R}^2$ of points, define

- $U(P)$ = number of pairs in $P$ at unit distance and
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![Diagram showing the relationship between points and unit distances](image)
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\[ \square \]
Literature

- [Aigner, Ziegler] Proofs from THE BOOK [https://doi.org/10.1007/978-3-662-57265-8]
- [Schaefer ’20] The Graph Crossing Number and its Variants: A Survey
- Terence Tao’s blog post “The crossing number inequality” from 2007
- [Garey, Johnson ’83] Crossing number is NP-complete
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- [Székely ’97] Crossing Numbers and Hard Erdős Problems in Discrete Geometry
- Documentary/Biography “N Is a Number: A Portrait of Paul Erdős”
- Exact computations of crossing numbers: http://crossings.uos.de