Visualization of Graphs

Lecture 11: The Crossing Lemma and Its Applications

Part I: Definition and Hanani–Tutte

Alexander Wolff
Crossing Number and Topological Graphs

For a graph $G$, the **crossing number** $\text{cr}(G)$ is the smallest number of edge crossings in a drawing of $G$ (in the plane).

In a crossing-minimal drawing of $G$

- no edge is self-intersecting,
- edges with common endpoints do not intersect,
- two edges intersect at most once,
- and, w.l.o.g., at most two edges intersect at the same point.

Such a drawing is called a **topological drawing** of $G$.

**Example.**
$\text{cr}(K_{3,3}) = 1$

![Diagram showing an example of a crossing-minimal drawing and an illustration of how crossings can be reduced.](image)

- # crossings reduced, so terminates
Theorem. [Hanani '43, Tutte '70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

Proof sketch.
Hanani showed that every drawing of $K_5$ and $K_{3,3}$ must have a pair of edges that crosses an odd number of times.
Every non-planar graph has $K_5$ or $K_{3,3}$ as a minor, so there are two paths that cross an odd number of times.
Hence, there must be two edges on these paths that cross an odd number of times. □
Hanani–Tutte Theorem

**Theorem.** [Hanani '43, Tutte '70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

The odd crossing number $ocr(G)$ of $G$ is the smallest number of pairs of edges that cross oddly in a drawing of $G$.

**Corollary.** $ocr(G') = 0 \Rightarrow cr(G') = 0$

Is $ocr(G) = cr(G)$? **No!**

**Theorem.** [Pelsmajer, Schaefer & Štefankovič '08, Tóth '08]
There is a graph $G$ with $ocr(G') < cr(G') \leq 10$

**Theorem.** [Pach & Tóth '00]
If $\Gamma$ is a drawing of $G$ and $E_0$ is the set of edges with only even numbers of crossings in $\Gamma$, then $G$ can be drawn such that no edge in $E_0$ is involved in any crossings.
Hanani–Tutte Theorem

**Theorem.** [Hanani ’43, Tutte ’70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

The odd crossing number $\text{ocr}(G)$ of $G$ is the smallest number of pairs of edges that cross oddly in a drawing of $G$.

**Corollary.** $\text{ocr}(G') = 0 \Rightarrow \text{cr}(G') = 0$

Is $\text{ocr}(G) = \text{cr}(G)$? No!

**Theorem.** [Pelsmajer, Schaefer & Štefankovič ’08, Tóth ’08]
There is a graph $G$ with $\text{ocr}(G') < \text{cr}(G') \leq 10$

**Theorem.** [Pelsmajer, Schaefer & Štefankovič ’08] [Pach & Tóth ’00]
If $\Gamma$ is a drawing of $G$ and $E_0$ is the set of edges with only even numbers of crossings in $\Gamma$, then $G$ can be drawn such that no edge in $E_0$ is involved in any crossings and no new pairs of edges cross.
Hanani–Tutte Theorem

Theorem. [Hanani ’43, Tutte ’70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

The odd crossing number $\text{ocr}(G)$ of $G$ is the smallest number of pairs of edges that cross oddly in a drawing of $G$.

Corollary.
$\text{ocr}(G') = 0 \Rightarrow \text{cr}(G') = 0$

Is $\text{ocr}(G) = \text{cr}(G)$? No!

Theorem. [Pelsmajer, Schaefer & Štefankovič ’08, Tóth ’08]
There is a graph $G$ with $\text{ocr}(G') < \text{cr}(G') \leq 10$

The pairwise crossing number $\text{pcr}(G)$ of $G$ is the smallest number of pairs of edges that cross in a drawing of $G$.

By definition $\text{ocr}(G) \leq \text{pcr}(G) \leq \text{cr}(G)$

Is $\text{pcr}(G) = \text{cr}(G)$? Open!
Visualization of Graphs

Lecture 11:
The Crossing Lemma
and its Applications

Part II:
Computation & Variations

Alexander Wolff
Computing the Crossing Number

- Computing $\text{cr}(G)$ is NP-hard.  
  ... even if $G$ is a planar graph plus one edge!  
  \[ \text{[Garey & Johnson '83]} \]
  \[ \text{[Cabello & Mohar '08]} \]

- $\text{cr}(G)$ often unknown, only conjectures exist
  - for $K_n$ it is only known for up to $\sim 12$ vertices
  \[ \text{[Garey & Johnson '83]} \]

- In practice, $\text{cr}(G)$ is often not computed directly but rather drawings of $G$ are optimized with
  - force-based methods,
  - multidimensional scaling,
  - heuristics, ...

- $\text{cr}(G)$ is a measure of how far $G$ is from being planar.

- Planarization, where we replace crossings with dummy vertices, also uses only heuristics.

For exact computations, check out http://crossings.uos.de!
Other Crossing Numbers

- Schaefer [Schae20] offers a huge survey on different crossings numbers (and more precise definitions)
- One-sided crossing minimization . . .
- Fixed Linear Crossing Number
- In book embeddings
- Crossings of edge bundles
- On other surfaces, such as donuts
- Weighted crossings
- Crossing minimization is NP-hard for most variants.
Rectilinear Crossing Number

**Definition.**
For a graph $G$, the *rectilinear (straight-line) crossing number* $\overline{cr}(G)$ is the smallest number of crossings in a straight-line drawing of $G$.

**Separation.**
$cr(K_8) = 18$, but $\overline{cr}(K_8) = 19$.

Even more . . .

**Lemma 1.** [Bienstock, Dean '93]
For $k \geq 4$, there exists a graph $G_k$ with $cr(G_k) = 4$ and $\overline{cr}(G_k) \geq k$.

- Each straight-line drawing of $G_1$ has at least one crossing of the following types:
- From $G_1$ to $G_k$ do
Visualization of Graphs

Lecture 11:
The Crossing Lemma
and its Applications

Part III:
First Bounds

Alexander Wolff
Bounds for Complete Graphs

**Theorem.** \[\text{Conjecture.} \quad n \leq 12.\]

\[
\text{cr}(K_n) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor = \frac{3}{8} \binom{n}{4} + O(n^3)
\]

**Theorem.** \[\text{Conjecture.} \quad (\text{Zarankiewicz '54, Urbaník '55})\]

\[
\text{cr}(K_{m,n}) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor
\]

- **Sylvester's four-point problem**
- **Turán's brick factory problem (1944)**

*Pál Turán*

*1910 – 1976*

*Budapest, Hungary*
Bounds for Complete Graphs

**Theorem.** [Guy ’60]

\[\text{cr}(K_n) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor = \frac{3}{8} \binom{n}{4} + O(n^3)\]

Bound is tight for \( n \leq 12 \).

**Theorem.** [Zarankiewicz ’54, Urbaník ’55]

\[\text{cr}(K_{m,n}) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor\]

**Theorem.** [Lovász et al. ’04, Aichholzer et al. ’06]

\[
\left(\frac{3}{8} + \varepsilon\right) \binom{n}{4} + O(n^3) < \text{cr}(K_n) < 0.3807 \binom{n}{4} + O(n^3)
\]

Exact numbers are known for \( n \leq 27 \).

Check out [http://www.ist.tugraz.at/staff/aichholzer/crossings.html](http://www.ist.tugraz.at/staff/aichholzer/crossings.html)!
First Lower Bounds on $\text{cr}(G)$

**Lemma 2.**
For a graph $G$ with $n$ vertices and $m$ edges,

$$\text{cr}(G) \geq m - 3n + 6.$$ 

**Proof.**
- Consider a drawing of $G$ with $\text{cr}(G)$ crossings.
- Obtain a graph $H$ by turning crossings into dummy vertices.
- $H$ has $n + \text{cr}(G)$ vertices and $m + 2\text{cr}(G)$ edges.
- $H$ is planar, so

$$m + 2\text{cr}(G) \leq 3(n + \text{cr}(G)) - 6.$$
First Lower Bounds on $\text{cr}(G)$

**Lemma 3.**
For a non-planar graph $G$ with $n$ vertices and $m$ edges,

\[
\text{cr}(G) \geq r \cdot \left( \frac{\lfloor m/r \rfloor}{2} \right) \in \Omega \left( \frac{m^2}{n} \right)
\]

where $r \leq 3n - 6$ is the maximum number of edges in a planar subgraph of $G$.

**Proof sketch.**
- Take $\left\lfloor \frac{m}{r} \right\rfloor$ edge-disjoint subgraphs of $G$ with $r$ edges.
- In the best case, they are all planar.
- For every $i < j$, any edge of $G_j$ induces at least one crossing with $G_i$.
  (If not, swap edges to reduce $\text{cr}(G_i)$.)
Visualization of Graphs

Lecture 11: The Crossing Lemma and its Applications

Part IV: The Crossing Lemma

Alexander Wolff
The Crossing Lemma

- 1973 Erdős and Guy conjectured that \( \text{cr}(G) \in \Omega\left(\frac{m^3}{n^2}\right) \).

- In 1982 Leighton and, independently, Ajtai, Chávtal, Newborn, and Szemerédi showed that
  \[ \text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}. \]

- Bound is asymptotically tight.

- Result stayed hardly known until Székely demonstrated its usefulness (in 1997).

- We go through the proof from “THE BOOK” by Chazelle, Sharir, and Welzl.

- Factor \( \frac{1}{64} \) was later (with intermediate steps) improved to \( \frac{1}{29} \) by Ackerman in 2013.

Consider this bound for graphs with \( \Theta(n) \) and \( \Theta(n^2) \) many edges.
The Crossing Lemma

**Crossing Lemma.**
For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4n$, 
$$\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}.$$  

**Proof.**
- Consider a crossing-minimal drawing of $G$.
- Let $p$ be a number in $(0, 1]$.
- Keep every vertex of $G$ independently with probability $p$.
- $G_p = \text{remaining graph (with drawing } \Gamma_p).$
- Let $n_p, m_p, X_p$ be the random variables counting the numbers of vertices / edges / crossings of $\Gamma_p$, resp.
- By Lemma 2, $\text{cr}(G_p) - m_p + 3n_p \geq 6$.  
  $\Rightarrow \mathbb{E}(X_p - m_p + 3n_p) \geq 0.$
- $\mathbb{E}(n_p) = pn$ and $\mathbb{E}(m_p) = p^2m$
- $\mathbb{E}(X_p) = p^4\text{cr}(G)$
- $0 \leq \mathbb{E}(X_p) - \mathbb{E}(m_p) + 3\mathbb{E}(n_p)$
  $= p^4\text{cr}(G) - p^2m + 3pn$
- $\text{cr}(G) \geq \frac{p^2m - 3pn}{p^4} = \frac{m}{p^2} - \frac{3n}{p^3}$
- Set $p = \frac{4n}{m}$.
- $\text{cr}(G) \geq \frac{m^3}{16n^2} - \frac{3m^3}{64n^2} = \frac{1}{64} \frac{m^3}{n^2}$  \(\square\)
Visualization of Graphs

Lecture 11:
The Crossing Lemma
and its Applications

Part V:
Applications

Alexander Wolff
Application 1: Point–Line Incidences

- For a set \( P \subset \mathbb{R}^2 \) of points and a set \( \mathcal{L} \) of lines, let \( I(P, \mathcal{L}) \) = number of point–line incidences in \( (P, \mathcal{L}) \).
- Define \( I(n, k) = \max_{|P|=n, |\mathcal{L}|=k} I(P, \mathcal{L}) \).
- For example: \( I(4, 4) = 9 \)

\[ \Rightarrow I(P, \mathcal{L}) = 10 \]

**Theorem 1.**

[Szemerédi, Trotter ’83, Székely ’97]

\[ I(n, k) \leq c(n^{2/3}k^{2/3} + n + k) \]

**Proof.**

- \#(points on \( \ell \)) \(-\) 1 = \#(edges on \( \ell \))
- \( \Rightarrow I(n, k) - k \leq m \) (sum up over \( \mathcal{L} \))
- Crossing Lemma: \( \frac{1}{164} \frac{m^3}{n^2} \leq \text{cr}(G) \)
- \( \Rightarrow \exists c: c(I(n, k) - k)^3/n^2 \leq \text{cr}(G) \leq k^2 \)
- If \( m < 4n \), then \( I(n, k) - k \leq 4n \).  □
Application 2: Unit Distances

For a set $P \subset \mathbb{R}^2$ of points, define

- $U(P) = \text{number of pairs in } P \text{ at unit distance and}$
- $U(n) = \max_{|P|=n} U(P)$.

**Theorem 2.**

[Spencer, Szemerédi, Trotter '84, Székely '97]

$U(n) < 6.7n^{4/3}$

**Proof.**
Application 2: Unit Distances

For a set $P \subset \mathbb{R}^2$ of points, define

- $U(P)$ = number of pairs in $P$ at unit distance and
- $U(n) = \max_{|P|=n} U(P)$.

**Theorem 2.**

[Spencer, Szemerédi, Trotter '84, Székely '97]

$U(n) < 6.7n^{4/3}$

**Proof.**

- $U(P) \leq c'm$
- $cr(G) \leq 2n^2$
- $c \frac{U(P)^3}{n^2} \leq cr(G) \leq 2n^2$
Application 3: Expected Number of Crossings in a Matching

Given set of \( n \) points (in general position, \( n \) even) – what is the average number of crossings in a perfect matching?

Point set spans drawing \( \Gamma \) of \( K_n \).

We will analyze the number of crossings in a random perfect matching in \( \Gamma \! 

Number of crossings in \( \Gamma \) \( \geq \) \( c_r(K_n) > \frac{3}{8} \binom{n}{4} \)

Number of edges in \( K_n \): \( \binom{n}{2} \)

Number of potential crossings (all pairs of edges): \( \text{pot}(K_n) = \binom{\binom{n}{2}}{2} \approx 3 \binom{n}{4} \)

Pick two random edges \( e_1 \) and \( e_2 \).

\( \Pr[e_1 \text{ and } e_2 \text{ cross}] \geq \frac{c_r(K_n)}{\text{pot}(K_n)} > \frac{1}{8} \).

Pick random perfect matching \( M \); it has \( n/2 \) edges, so \( \binom{n/2}{2} = \frac{1}{8} n(n-2) \) pairs of edges.

By linearity of expectation, the expected number of crossings in \( M \) is \( > \frac{1}{8} \binom{n/2}{2} = \frac{1}{64} n(n-2) \). □
Literature

- [Aigner, Ziegler] Proofs from THE BOOK [https://doi.org/10.1007/978-3-662-57265-8]
- [Schaefer ’20] The Graph Crossing Number and its Variants: A Survey
- Terrence Tao’s blog post “The crossing number inequality” from 2007
- [Garey, Johnson ’83] Crossing number is NP-complete
- [Bienstock, Dean ’93] Bounds for rectilinear crossing numbers
- [Székely ’97] Crossing Numbers and Hard Erdős Problems in Discrete Geometry
- Documentary/Biography “N Is a Number: A Portrait of Paul Erdős”
- Exact computations of crossing numbers: http://crossings.uos.de