Visualization of Graphs

Lecture 8: Hierarchical Layouts: Sugiyama Framework

Part I: The Framework

Tim Hegemann
Hierarchical Drawings – Motivation
Hierarchical Drawing

Problem Statement.
- **Input:** digraph $G = (V, E)$
- **Output:** drawing of $G$ that “closely” reproduces the hierarchical properties of $G$

Desirable Properties.
- vertices occur on (few) horizontal lines
- edges directed upwards
- edge crossings minimized
- edges as short as possible
- vertices evenly spaced

Criteria can be contradictory!
Hierarchical Drawing – Applications

yEd Gallery: Java profiler JProfiler using yFiles

Source: "Design Considerations for Optimizing Storyline Visualizations" Tanahashi et al.

Source: Visualization that won the Graph Drawing Contest 2016. Klawitter & Mchedlidze

Star Wars (Original Trilogy)

Source: Visualization that won the Graph Drawing Contest 2016. Klawitter & Mchedlidze
Classical Approach – Sugiyama Framework

[Sugiyama, Tagawa, Toda '81]
Visualization of Graphs

Lecture 7:
Hierarchical Layouts:
Sugiyama Framework

Part II:
Cycle Breaking

Jonathan Klawitter
Step 1: Cycle breaking

Input → Cycle Breaking → Leveling → Crossing Minimization → Vertex Positioning → Edge Drawing
Step 1: Cycle breaking

Approach.
- Find minimum set $E^*$ of edges which are not upwards.
- Remove $E^*$ and insert reversed edges.

Problem **Minimum Feedback Arc Set (FAS).**
- Input: directed graph $G = (V, E)$
- Output: min. set $E^* \subseteq E$, so that $G - E^* + E_r^*$ acyclic

...NP-hard 😞
Heuristic 1
[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$$E' \leftarrow \emptyset$$

foreach $v \in V$ do

if $|N\rightarrow(v)| \geq |N\leftarrow(v)|$ then

$$E' \leftarrow E' \cup N\rightarrow(v)$$

else

$$E' \leftarrow E' \cup N\leftarrow(v)$$

remove $v$ and $N(v)$ from $G$.

return $(V, E')$

- $G' = (V, E')$ is a DAG
- $E \setminus E'$ is a feedback set

$N\rightarrow(v) := \{(v, u) | (v, u) \in E\}$

$N\leftarrow(v) := \{(u, v) | (u, v) \in E\}$

$N(v) := N\rightarrow(v) \cup N\leftarrow(v)$

- Time: $O(n + m)$
- Quality guarantee: $|E'| \geq |E|/2$
Heuristic 2

[Eades, Lin, Smyth '93]

\[ E' \leftarrow \emptyset \]

\[ \text{while } V \neq \emptyset \text{ do} \]

\[ \text{while in } V \text{ exists a sink } v \text{ do} \]

\[ E' \leftarrow E' \cup N^\leftarrow (v) \]

\[ \text{remove } v \text{ and } N^\leftarrow (v) \]

Remove all isolated vertices from \( V \)

\[ \text{while in } V \text{ exists a source } v \text{ do} \]

\[ E' \leftarrow E' \cup N^\rightarrow (v) \]

\[ \text{remove } v \text{ and } N^\rightarrow (v) \]

\[ \text{if } V \neq \emptyset \text{ then} \]

\[ \text{let } v \in V \text{ such that } |N^\rightarrow (v)| - |N^\leftarrow (v)| \text{ maximal} \]

\[ E' \leftarrow E' \cup N^\rightarrow (v) \]

\[ \text{remove } v \text{ and } N(v) \]

- **Time:** \( O(n + m) \)
- **Quality guarantee:** \( |E'| \geq |E|/2 + |V|/6 \)
Visualization of Graphs

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Part III: Leveling

Jonathan Klawitter
Step 2: Leveling

Input → Cycle Breaking → Leveling

Crossing Minimization → Vertex Positioning → Edge Drawing
Step 2: Leveling

Problem.
- Input: acyclic digraph $G = (V, E)$
- Output: Mapping $y : V \rightarrow \{1, \ldots, n\}$, so that for every $uv \in E$, $y(u) < y(v)$.

Objective is to minimize . . .
- number of layers, i.e. $|y(V)|$
- length of the longest edge, i.e. $\max_{uv \in E} y(v) - y(u)$
- width, i.e. $\max\{|L_i| \mid 1 \leq i \leq h\}$
- total edge length, i.e. number of dummy vertices
Min Number of Layers

Algorithm.

- for each source \( q \)
  set \( y(q) := 1 \)

- for each non-source \( v \)
  set \( y(v) := \max \{ y(u) \mid uv \in E \} + 1 \)

Observation.

- \( y(v) \) is length of the longest path from a source to \( v \) plus 1.
  ...which is optimal!

- Can be implemented in linear time with recursive algorithm.
Example
Total Edge Length – ILP

Can be formulated as an integer linear program:

\[
\begin{align*}
\min & \quad \sum_{(u,v) \in E} (y(v) - y(u)) \\
\text{subject to} & \quad y(v) - y(u) \geq 1 \quad \forall (u,v) \in E \\
& \quad y(v) \geq 1 \quad \forall v \in V \\
& \quad y(v) \in \mathbb{Z} \quad \forall v \in V
\end{align*}
\]

One can show that:

- Constraint-matrix is totally unimodular
  - \( \Rightarrow \) Solution of the relaxed linear program is integer
- The total edge length can be minimized in polynomial time
Width

Drawings can be very wide.
**Narrower Layer Assignment**

**Problem: Leveling With a Given Width.**

- **Input:** acyclic, digraph $G = (V, E)$, width $W > 0$
- **Output:** Partition the vertex set into a minimum number of layers such that each layer contains at most $W$ elements.

**Problem: Precedence-Constrained Multi-Processor Scheduling**

- **Input:** $n$ jobs with unit (1) processing time, $W$ identical machines, and a partial ordering $< \text{on the jobs}$.
- **Output:** Schedule respecting $<$ and having minimum processing time.
- **NP-hard, $(2 - \frac{1}{W})$-Approx., no $(\frac{4}{3} - \varepsilon)$-Approx. ($W \geq 3$).
Approximating PCMPS

- jobs stored in a list \( L \)
  (in any order, e.g., topologically sorted)

- for each time \( t = 1, 2, \ldots \) schedule \( \leq W \) available jobs

- a job in \( L \) is available when all its predecessors have been scheduled

- as long as there are free machines and available jobs, take the first available job and assign it to a free machine
Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)

Number of Machines is $W = 2$.

Output: Schedule

<table>
<thead>
<tr>
<th>$M_1$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>A</th>
<th>C</th>
<th>E</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_2$</td>
<td>–</td>
<td>3</td>
<td>–</td>
<td>–</td>
<td>7</td>
<td>9</td>
<td>B</td>
<td>D</td>
<td>F</td>
<td>–</td>
</tr>
<tr>
<td>$t$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
</tbody>
</table>

Question: Good approximation factor?
Approximating PCMPS - Analysis for $W = 2$

Precedence graph $G_<$

```
1  2  3  4  5  6  7  8  9  10
A  B  C  D  E  F  G
```

Schedule

```
<table>
<thead>
<tr>
<th>M_1</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>A</th>
<th>C</th>
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<tr>
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<td>3</td>
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<td>9</td>
<td>B</td>
<td>D</td>
<td>F</td>
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<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
</tbody>
</table>
```

„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell := \text{Number of layers of } G_<$

**Goal:** measure the quality of our algorithm using the lower bounds

\[ \text{ALG} \leq \left\lfloor \frac{n + \ell}{2} \right\rfloor \approx \left\lfloor n/2 \right\rfloor + \ell/2 \leq 3/2 \cdot \text{OPT} \]

**Bound.**

insertion of pauses (−) in the schedule (except the last) maps to layers of $G_<$

\[ \leq (2 - 1/W) \cdot \text{OPT} \text{ in general case} \]
Visualization of Graphs

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Part IV:
Crossing Minimisation

Jonathan Klawitter
Step 3: Crossing Minimization

Input → Cycle Breaking → Leveling

Crossing Minimization → Vertex Positioning → Edge Drawing
Step 3: Crossing Minimization

Problem.
- **Input:** Graph $G$, layering $y: V \rightarrow \{1, \ldots, n\}$
- **Output:** (Re-)ordering of vertices in each layer so that the number of crossings in minimized.

- NP-hard, even for 2 layers  
  [Garey & Johnson '83]
- hardly any approaches optimize over multiple layers :(
Iterative Crossing Reduction – Idea

**Observation.**
The number of crossings only depends on permutations of adjacent layers.

- Add dummy-vertices for edges connecting “far” layers.
- Consider adjacent layers \((L_1, L_2), (L_2, L_3), \ldots\) bottom-to-top.
- Minimize crossings by permuting \(L_{i+1}\) while keeping \(L_i\) fixed.
Iterative Crossing Reduction – Algorithm

(1) choose a random permutation of $L_1$

(2) iteratively consider adjacent layers $L_i$ and $L_{i+1}$

(3) minimize crossings by permuting $L_{i+1}$ and keeping $L_i$ fixed

(4) repeat steps (2)–(3) in the reverse order (starting from $L_h$)

(5) repeat steps (2)–(4) until no further improvement is achieved

(6) repeat steps (1)–(5) with different starting permutations

*one-sided crossing minimization*
One-Sided Crossing Minimization

**Problem.**

- **Input:** bipartite graph $G = (L_1 \cup L_2, E)$, permutation $\pi_1$ on $L_1$
- **Output:** permutation $\pi_2$ of $L_2$ minimizing the number of edge crossings.

One-sided crossing minimization is NP-hard.

[Eades & Whitesides ’94]

**Algorithms.**

- barycenter heuristic
- median heuristic
- Greedy-Switch
- ILP
- . . .
Barycenter Heuristic
[Sugiyama et al. ’81]

- Intuition: few intersections occur when vertices are close to their neighbors

- The barycentre of $u$ is the mean $x$-coordinate of the neighbours of $u$ in layer $L_1$ \([x_1 \equiv \pi_1]\)

\[
x_2(u) := \text{bary}(u) := \frac{1}{\deg(u)} \sum_{v \in N(u)} x_1(v)
\]

- Vertices with the same barycentre are offset by a small $\delta$.

- linear runtime

- relatively good results

- optimal if no crossings are required

- $O(\sqrt{n})$-approximation factor

Exercise!
Median Heuristic
[Eades & Wormald '94]

\[ \{v_1, \ldots, v_k\} := N(u) \text{ with } \pi_1(v_1) < \pi_1(v_2) < \cdots < \pi_1(v_k) \]

\[ x_2(u) := \text{med}(u) := \begin{cases} 0 & \text{when } N(u) = \emptyset \\ \pi_1(v_{\lceil k/2 \rceil}) & \text{otherwise} \end{cases} \]

Move vertices \( u \) und \( v \) by small \( \delta \), when \( x_2(u) = x_2(v) \)

- Linear runtime
- Relatively good results
- Optimal if no crossings are required
- 3-Approximation factor

\[ 2k(k + 1) + k^2 \text{ vs. } (k + 1)^2 \]

Proof in [GD Ch 11]
Greedy-Switch Heuristic

- Iteratively swap adjacent nodes as long as crossings decrease
- Runtime $O(L_2)$ per iteration; at most $|L_2|$ iterations
- Suitable as post-processing for other heuristics

Worst case?

$\approx k^2/4$

$\approx 2k$
Integer Linear Program

[Jünger & Mutzel, '97]

- Constant $c_{ij} := \#$ crossings between edges incident to $v_i$ or $v_j$ when $\pi_2(v_i) < \pi_2(v_j)$

- Variable $x_{ij}$ for each $1 \leq i < j \leq n_2 := |L_2|

\[
x_{ij} = \begin{cases} 
1 & \text{when } \pi_2(v_i) < \pi_2(v_j) \\
0 & \text{otherwise}
\end{cases}
\]

- The number of crossings of a permutations $\pi_2$

\[
\text{cross}(\pi_2) = \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} (c_{ij} - c_{ji})x_{ij} + \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} c_{ji}
\]

constant
Integer Linear Program

- Minimize the number of crossings:

\[
\text{minimize } \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} (c_{ij} - c_{ji})x_{ij}
\]

- Transitivity constraints:

\[
0 \leq x_{ij} + x_{jk} - x_{ik} \leq 1 \quad \text{for } 1 \leq i < j < k \leq n_2
\]

i.e., if \( x_{ij} = 1 \) and \( x_{jk} = 1 \), then \( x_{ik} = 1 \)

Properties.

- Branch-and-cut technique for DAGs of limited size
- Useful for graphs of small to medium size
- Finds optimal solution
- Solution in polynomial time is not guaranteed
Iterations on Example
Iterations on Example
Iterations on Example
Iterations on Example
Iterations on Example
Iterations on Example
Iterations on Example
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Part V:
Vertex Positioning & Drawing Edges

Jonathan Klawitter
Step 4: Vertex Positioning

Input → Cycle Breaking → Leveling → Crossing Minimization → Vertex Positioning → Edge Drawing
Step 4: Vertex Positioning

Goal.
Paths should be close to straight, vertices evenly spaced

- **Exact:** Quadratic Program (QP)
- **Heuristic:** Iterative approach
Quadratic Program

Consider the path $p_e = (v_1, \ldots, v_k)$ of an edge $e = v_1v_k$ with dummy vertices: $v_2, \ldots, v_{k-1}$

$x$-coordinate of $v_i$ according to the line $v_1v_k$ (with equal spacing):

$$x(v_i) = x(v_1) + \frac{i - 1}{k - 1}(x(v_k) - x(v_1))$$

Define the deviation from the line

$$\text{dev}(p_e) := \sum_{i=2}^{k-1} \left( x(v_i) - \overline{x(v_i)} \right)^2$$

Objective function: $\min \sum_{e \in E} \text{dev}(p_e)$

Constraints for all vertices $v, w$ in the same layer with $w$ right of $v$:

$$x(w) - x(v) \geq \rho(w, v)$$

QP is time-expensive

width can be exponential
Iterative Heuristic

- Compute an initial layout
- Apply the following steps as long as improvements can be made:
  1. Vertex positioning,
  2. edge straightening,
  3. Compactifying the layout width.
Example
Step 5: Drawing Edges

Input → Cycle Breaking → Leveling

Input → Cycle Breaking → Leveling

Crossing Minimization → Vertex Positioning → Edge Drawing
Step 5: Drawing Edges

Possibility.
Substitute polylines by Bézier curves
Example
Example
Example
Classical Approach – Sugiyama Framework
[Sugiyama, Tagawa, Toda '81]

- Flexible framework to draw directed graphs
- Sequential optimization of various criteria
- Modelling gives NP-hard problems, which can still be solved quite well

Input → Cycle breaking → Leveling

Crossing minimization → Vertex positioning → Edge drawing
Literature

Detailed explanations of steps and proofs in

- [GD Ch. 11] and [DG Ch. 5]

based on

- [Sugiyama, Tagawa, Toda '81] Methods for visual understanding of hierarchical system structures

and refined with results from

- [Berger, Shor '90] Approximation algorithms for the maximum acyclic subgraph problem
- [Eades, Lin, Smith '93] A fast and effective heuristic for the feedback arc set problem
- [Garey, Johnson '83] Crossing number is NP-complete
- [Eades, Whiteside '94] Drawing graphs in two layers
- [Eades, Wormland '94] Edge crossings in drawings of bipartite graphs
- [Jünger, Mutzel '97] 2-Layer Straightline Crossing Minimization: Performance of Exact and Heuristic Algorithms