Visualization of Graphs

Lecture 7:
Contact Representations of Planar Graphs:
Triangle Contacts and Rectangular Duals

Part I:
Geometric Representations

Alexander Wolff
In an **intersection representation** of a graph,
– each vertex is represented by a set
– such that two sets intersect ⇔
  the corresponding vertices are adjacent.

For a collection $S$ of sets,
the **intersection graph** $G(S)$ of $S$
has vertex set $S$ and edge set
\[ \{S, S'\} : S, S' \in S, S \neq S', \text{ and } S \cap S' \neq \emptyset \].
Contact Representation of Graphs

Let $G$ be a graph.

Let $S$ be a family of geometric objects (e.g., disks).

Represent each vertex $v$ by a geometric object $S(v) \in S$.

In an $S$-contact representation of $G$, $S(u)$ and $S(v)$ touch iff $uv \in E$.

$G$ is planar

[Koebe 1936]

disks

polygons

A contact representation is an intersection representation with interior-disjoint sets.
Contact Representation of Planar Graphs

Is the intersection graph of a contact representation always planar?
■ No, not even for connected object types.

Some object types are used to represent special classes of planar graphs:
- bipartite graphs
- max. triangle-free graphs
- planar triangulations
General Approach

How to compute a contact representation of a given graph $G$?

- Consider only inner triangulations (or maximal bipartite graphs, etc.)
  - Triangulate by adding vertices, not by adding edges

- Describe contact representation combinatorially.
  - Which objects touch each other in which way?

- Compute combinatorical description.

- Show that combinatorical description can be used to construct drawing.
This Lecture

Representation with right-triangles and corner contact:
- Use Schnyder realizer to describe contacts between triangles.
- Use canonical order to compute drawing.

Representation with dissection of a rectangle, called **rectangular dual**:
- Find a description similar to a Schnyder realizer for rectangles.
- Construct drawing via st-digraphs, duals, and topological sorting.
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Part II:
Triangle Contact Representations

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Triangle Corner Contact Representation

**Idea.**
Use canonical order and Schnyder realizer to find coordinates for triangles.

**Observation.**
- Can set base of triangle at height equal to position in canonical order.
- Triangle tip is precisely at base of triangle corresponding to cover neighbor.
- Outgoing edges in Schnyder forest indicate corner contacts.
Triangle Contact Representation Example
Triangle Contact Representation Example
Triangle Contact Representation Example
T-shape Contact Representation
T-shape Contact Representation
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Part III:
Rectangular Duals

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Cartograms

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A **rectangular dual** of a graph $G$ is a contact representation with axis-aligned rectangles s.t.

- no four rectangles share a point, and
- the union of all rectangles is a rectangle

**Theorem.** [Koźmiński, Kinnen ’85]

A graph $G$ has a rectangular dual $R$ if and only if $G$ is a PTP graph.
Regular Edge Labeling

Properly Triangulated Planar Graph $G$

Rectangular Dual $R$

Diagram showing the relationships between $G$ and $R$, with vertices and edges labeled.
Regular Edge Labeling

Properly Triangulated Planar Graph $G$

Rectangular Dual $\mathcal{R}$
Regular Edge Labeling

Properly Triangulated Planar Graph $G$

Rectangular Dual $\mathcal{R}$
Regular Edge Labeling

Properly Triangulated Planar Graph $G$

Regular Edge Labeling

Rectangular Dual $\mathcal{R}$

for every inner vertex
Regular Edge Labeling

Properly Triangulated Planar Graph $G$

Rectangular Dual $\mathcal{R}$

for every inner vertex

for four outer vertices
Regular Edge Labeling

Properly Triangulated Planar Graph $G$

Regular Edge Labeling

Rectangular Dual $R$

[Kant, He '94]: In linear time

for every inner vertex

for four outer vertices
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Part IV:
Computing a REL

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Theorem.
Let $G$ be a PTP graph. There exists a labeling $v_1 = v_S, v_2 = v_W, v_3, \ldots, v_n = v_N$ of the vertices of $G$ such that for every $4 \leq k \leq n$:

- The subgraph $G_{k-1}$ induced by $v_1, \ldots, v_{k-1}$ is biconnected and the boundary $C_{k-1}$ of $G_{k-1}$ contains the edge $(v_S, v_W)$.
- $v_k$ is in exterior face of $G_{k-1}$, and its neighbors in $G_{k-1}$ form a (at least 2-element) subinterval of the path $C_{k-1} \setminus (v_S, v_W)$.
- If $k \leq n - 2$, then $v_k$ has at least 2 neighbors in $G \setminus G_{k-1}$.
Refined Canonical Order Example
Refined Canonical Order Example
Refined Canonical Order Example
Refined Canonical Order Example
Refined Canonical Order Example
Refined Canonical Order $\rightarrow$ REL

We construct a REL as follows:

- For $i < j$, orient $(v_i, v_j)$ from $v_i$ to $v_j$;
- $v_k$ has incoming edges from $v_{t_1}, \ldots, v_{t_l}$, we say that $v_{t_1}$ is left point of $v_k$ and $v_{t_l}$ is right point of $v_k$.
- Base edge of $v_k$ is $(v_{t_a}, v_k)$, where $t_a < k$ is minimal.
- If $v_{k_1}, \ldots, v_{k_o}$ are higher numbered neighbors of $v_k$, we call $(v_k, v_{k_1})$ left edge and $(v_k, v_{k_o})$ right edge.

**Lemma 1.**
A left edge or right edge cannot be a base edge.

**Proof.** Suppose left edge $(v_k, v_{k_1})$ is base edge of $v_{k_1}$.
Since $G$ triangulated, $(v_{t_1}, v_{k_1}) \in E(G)$.
Contradiction since $k > t_1$. 

![Diagram of REL construction](image_url)
Refined Canonical Order $\rightarrow$ REL

Lemma 2.
An edge is either a left edge, a right edge or a base edge.

Proof.
- Exclusive “or” follows from Lemma 1.
- Let $(v_{t_a}, v_k)$ be base edge of $v_k$.
- $v_{t_a}$ is right point of $v_{t_{a-1}}$.
  - $v_{t_i}$ has at least two higher-numbered neighbors.
  - One of them is $v_k$; the other one is either $v_{t_{i-1}}$ or $v_{t_{i+1}}$.
  - For $1 \leq i < a - 1$, it is $v_{t_{i-1}}$. Thus, $v_{t_i}$ is right point of $v_{t_{i-1}}$.
- Analogously, $v_{t_i}$ is left point of $v_{t_{i+1}}$ for $i \geq a$.
- Edges $(v_{t_i}, v_k)$, $1 \leq i < a - 1$, are right edges.
- Similarly, $(v_{t_i}, v_k)$, for $a + 1 \leq i \leq l$, are left edges.
Refined Canonical Order → REL

**Coloring.**
- Color right (left) edges in red (blue).
- Color a base edge \((v_{ti}, v_k)\) red if \(i = 1\) and blue if \(i = l\) and otherwise arbitrarily.

Let \(T_r\) be the red edges and \(T_b\) the blue edges.

**Lemma 3.**
\(\{T_r, T_b\}\) is a regular edge labeling.

**Proof.**
\[ k_0 \geq 2 \]

\[ k_d = \max\{k_1, \ldots, k_o\} \]
base edges of \(v_{k_2}, \ldots, v_{k_{o-1}}\)

\[ k_1 < k_2 < \ldots < k_d \text{ and } k_d > k_{d+1} > \ldots > k_o \]

\[(v_k, v_{k_i}), 2 \leq i \leq d - 1 \text{ are blue} \]

\[(v_k, v_{k_i}), d + 1 \leq i \leq o - 1 \text{ are red} \]

\[(v_k, v_{kd}) \text{ is either red or blue} \]

\[ \Rightarrow \text{Circular order of outgoing edges at } v_k \text{ correct.} \]
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Part V:
Computing the Coordinates

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[Kant, He '94]:

\[ O(n) \]

PTP \rightarrow REL \rightarrow RD

\[ O(n) \]
From REL to st-Digraphs to Coordinates
From REL to st-Digraphs to Coordinates

WE network $G_{\text{hor}}$
From REL to st-Digraphs to Coordinates

SN network $G_{\text{ver}}$
dual of $G_{\text{ver}}$
From REL to st-Digraphs to Coordinates

SN network $G_{ver}$

dual of $G_{ver}$
From REL to st-Digraphs to Coordinates

dual of \( G_{\text{ver}} \)
compute topological order
From REL to st-Digraphs to Coordinates

dual of $G_{\text{ver}}$
compute topological order
From REL to st-Digraphs to Coordinates
From REL to st-Digraphs to Coordinates
Rectangular Dual Algorithm

For a PTP graph $G = (V, E)$:

- Find a REL $\{T_r, T_b\}$ of $G$;
- Construct a SN network $G_{\text{ver}}$ of $G$ (consists of $T_b$ plus outer edges);
- Construct the dual $G^*_{\text{ver}}$ of $G_{\text{ver}}$ and compute a topological ordering $f_{\text{ver}}$ of $G^*_{\text{ver}}$;
- For each vertex $v \in V$, let $g$ and $h$ be the face on the left and face on the right of $v$. Set $x_1(v) = f_{\text{ver}}(g)$ and $x_2(v) = f_{\text{ver}}(h)$.
- Define $x_1(v_N) = 1, x_1(v_S) = 2$ and $x_2(v_N) = \max f_{\text{ver}} - 1, x_2(v_S) = \max f_{\text{ver}}$.
- Analogously compute $y_1$ and $y_2$ with $G_{\text{hor}}$.
- For each $v \in V$, let $R(v) = [x_1(v), x_2(v)] \times [y_1(v), y_2(v)]$. 
Reading off Coordinates to Get Rectangular Dual

\[
\begin{align*}
x_1(v_N) &= 1, & x_2(v_N) &= 15 \\
x_1(v_S) &= 2, & x_2(v_S) &= 16 \\
x_1(v_W) &= 0, & x_2(v_W) &= 1 \\
x_1(v_E) &= 15, & x_2(v_E) &= 16 \\
x_1(a) &= 1, & x_2(a) &= 3 \\
x_1(b) &= 3, & x_2(b) &= 5 \\
x_1(c) &= 5, & x_2(c) &= 14 \\
x_1(d) &= 14, & x_2(d) &= 15 \\
x_1(e) &= 13, & x_2(e) &= 15 \\
\ldots
\end{align*}
\]

\[
\begin{align*}
y_1(v_W) &= 0, & y_2(v_W) &= 9 \\
y_1(v_E) &= 1, & y_2(v_E) &= 10 \\
y_1(v_N) &= 9, & y_2(v_N) &= 10 \\
y_1(v_S) &= 0, & y_2(v_S) &= 1 \\
y_1(a) &= 1, & y_2(a) &= 2 \\
y_1(b) &= 1, & y_2(b) &= 2 \\
\ldots
\end{align*}
\]
Reading off Coordinates to Get Rectangular Dual

\[ x_1(v_N) = 1, \quad x_2(v_N) = 15 \]
\[ x_1(v_S) = 2, \quad x_2(v_S) = 16 \]
\[ x_1(v_W) = 0, \quad x_2(v_W) = 1 \]
\[ x_1(v_E) = 15, \quad x_2(v_E) = 16 \]
\[ x_1(a) = 1, \quad x_2(a) = 3 \]
\[ x_1(b) = 3, \quad x_2(b) = 5 \]
\[ x_1(c) = 5, \quad x_2(c) = 14 \]
\[ x_1(d) = 14, \quad x_2(d) = 15 \]
\[ x_1(e) = 13, \quad x_2(e) = 15 \]
\[ \ldots \]

\[ y_1(v_W) = 0, \quad y_2(v_W) = 9 \]
\[ y_1(v_E) = 1, \quad y_2(v_E) = 10 \]
\[ y_1(v_N) = 9, \quad y_2(v_N) = 10 \]
\[ y_1(v_S) = 0, \quad y_2(v_S) = 1 \]
\[ y_1(a) = 1, \quad y_2(a) = 2 \]
\[ y_1(b) = 1, \quad y_2(b) = 2 \]
\[ \ldots \]
Correctness of Algorithm (Sketch)

- If edge \((u, v)\) exists, then \(x_2(u) = x_1(v)\)

\[
x_2(u) = f_{\text{ver}}(g) = x_1(v)
\]

- and the vertical segments of their rectangles overlap

\[
y_1(v) = f_{\text{hor}}(a) \leq y_1(u) = f_{\text{hor}}(b)
< y_2(v) = f_{\text{hor}}(c) \leq y_2(u) = f_{\text{hor}}(d)
\]

- If path from \(u\) to \(v\) in red at least two edges long, then \(x_2(u) < x_1(v)\).
- No two boxes overlap.
- For details, see He's paper [He '93].
Rectangular Dual Result

**Theorem.**
Every PTP graph $G$ has a rectangular dual, which can be computed in linear time.

**Proof.**
- Compute a planar embedding of $G$.
- Compute a refined canonical ordering of $G$.
- Traverse the graph and color the edges.
- Construct $G_{\text{ver}}$ and $G_{\text{hor}}$.
- Construct their duals $G^*_{\text{ver}}$ and $G^*_{\text{hor}}$.
- Compute a topological ordering for vertices of $G^*_{\text{ver}}$ and $G^*_{\text{hor}}$.
- Assign coordinates to the rectangles representing vertices.
Discussion

- A layout is area-universal if any assignment of areas to rectangles can be realized by a combinatorially equivalent rectangular layout.

- A rectangular layout is area-universal if and only if it is one-sided. ([Eppstein et al. SIAM J. Comp. 2012])

- Area-universal rectilinear representation: possible for all planar graphs.

- [Alam et al. 2013]: 8 sides (matches the lower bound)
Literature

Construction of triangle contact representations based on
- [de Fraysseix, Ossona de Mendez, Rosenstiehl ’94] On Triangle Contact Graphs

Construction of rectangular dual based on
- [He ’93] On Finding the Rectangular Duals of Planar Triangulated Graphs
- [Kant, He ’94] Two algorithms for finding rectangular duals of planar graphs
and originally from
- [Koźmiński, Kinnen ’85] Rectangular Duals of Planar Graphs