Visualization of Graphs

Lecture 5:
Orthogonal Layouts

Part I:
Topology – Shape – Metric

Alexander Wolff
Orthogonal Layout – Applications

ER diagram in OGDF
Orthogonal Layout – Applications

Organigram of HS Limburg

Circuit diagram by Jeff Atwood

ER diagram in OGDF

UML diagram by Oracle
Orthogonal Layout – Definition

Observations.
- Edges lie on grid ⇒ bends lie on grid points
- Max degree of each vertex is at most 4
- Otherwise

Planarization.
- Fix embedding
- Crossings become vertices

Aesthetic criteria.
- Number of bends
- Length of edges
- Width, height, area
- Monotonicity of edges
- ...

Definition.
A drawing $\Gamma$ of a graph $G = (V, E)$ is called orthogonal if
- vertices are drawn as points on a grid,
- each edge is represented as a sequence of alternating horizontal and vertical segments, and
- pairs of edges are disjoint or cross orthogonally.
Orthogonal Layout – Definition

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- Length of edges
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- ...
Topology – Shape – Metrics

Three-step approach:

\[ V = \{v_1, v_2, v_3, v_4\} \]

\[ E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\} \]

combinatorial embedding/planarization

reduce crossings

bend minimization

orthogonal representation

planar orthogonal drawing

area minimization

[Tamassia 1987]
Visualization of Graphs

Lecture 5: Orthogonal Layouts

Part II: Orthogonal Representation

Alexander Wolff
Orthogonal Representation

**Idea.**
Describe orthogonal drawing combinatorially.

**Definitions.**
Let $G = (V, E)$ be a plane graph with faces $F$ and outer face $f_0$.

- Let $e$ be an edge with the face $f$ to the right.
  An **edge description** of $e$ wrt $f$ is a triple $(e, \delta, \alpha)$ where
  - $\delta \in \{0, 1\}^*$ (where $0 =$ right bend, $1 =$ left bend)
  - $\alpha$ is angle $\in \{\pi/2, \pi, 3\pi/2, 2\pi\}$ between $e$ and next edge $e'$

- A **face representation** $H(f)$ of $f$ is a clockwise ordered sequence of edge descriptions $(e_1, \delta_1, \alpha_1), (e_2, \delta_2, \alpha_2), \ldots, (e_{\deg(f)}, \delta_{\deg(f)}, \alpha_{\deg(f)})$.

- An **orthogonal representation** $H(G)$ of $G$ is defined as

$$H(G) = \{H(f) \mid f \in F\}.$$
Orthogonal Representation – Example

\[ H(f_0) = ((e_1, 11, \frac{\pi}{2}), (e_5, 111, \frac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \frac{\pi}{2})) \]

\[ H(f_1) = ((e_1, 00, \frac{3\pi}{2}), (e_2, \emptyset, \frac{\pi}{2}), (e_6, 00, \pi)) \]

\[ H(f_2) = ((e_5, 000, \frac{\pi}{2}), (e_6, 11, \frac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \frac{\pi}{2})) \]

Concrete coordinates are not fixed yet!
Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to $F, f_0$.

(H2) For each edge $\{u, v\}$ shared by faces $f$ and $g$ with $((u, v), \delta_1, \alpha_1) \in H(f)$ and $((v, u), \delta_2, \alpha_2) \in H(g)$, the sequence $\delta_1$ is like $\delta_2$, but reversed and inverted.

(H3) Let $|\delta|_0$ (resp. $|\delta|_1$) be the number of zeros (resp. ones) in $\delta$, and let $r = (e, \delta, \alpha)$.
Let $C(r) := |\delta|_0 - |\delta|_1 + 2 - \alpha/\pi$. For each face $f$, it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each vertex $v$, the sum of incident angles is $2\pi$. 

$$C(e_3) = 0 - 0 + 2 - 2 = 0$$
$$C(e_4) = 0 - 0 + 2 - 1 = 1$$
$$C(e_5) = 3 - 0 + 2 - 1 = 4$$
$$C(e_6) = 0 - 2 + 2 - 1 = -1$$
Visualization of Graphs

Lecture 5: Orthogonal Layouts

Part III: Bend Minimization

Alexander Wolff
Reminder: $s$-$t$-Flow Networks

**Flow network** $(G = (V, E); S, T; u)$ with
- directed graph $G = (V, E)$
- sources $S \subseteq V$, sinks $T \subseteq V$
- edge capacity $u: E \rightarrow \mathbb{R}^+ \cup \{\infty\}$

A function $X: E \rightarrow \mathbb{R}^+_0$ is called **$S$–$T$ flow** if:

\[
0 \leq X(i, j) \leq u(i, j) \quad \forall (i, j) \in E
\]

\[
\sum_{(i,j)\in E} X(i, j) - \sum_{(j,i)\in E} X(j, i) = 0 \quad \forall i \in V \setminus (S \cup T)
\]

A **maximum $S$–$T$ flow** is an $S$–$T$ flow where $\sum_{(i,j)\in E, i\in S} X(i, j)$ is maximized.
Reminder: \textit{s-t-Flow Networks}

Flow network \((G = (V, E); s, t, u)\) with

- directed graph \(G = (V, E)\)
- \textit{source} \(s \in V\), \textit{sink} \(t \in V\)
- edge \textit{capacity} \(u: E \rightarrow \mathbb{R}^+_0 \cup \{\infty\}\)

A function \(X: E \rightarrow \mathbb{R}_0^+\) is called \textit{s-t flow} if:

\[
0 \leq X(i, j) \leq u(i, j) \quad \forall (i, j) \in E
\]

\[
\sum_{(i, j) \in E} X(i, j) - \sum_{(j, i) \in E} X(j, i) = 0 \quad \forall i \in V \setminus \{s, t\}
\]

A \textbf{maximum} \textit{s-t flow} is an \textit{s-t flow} where \(\sum_{(s, j) \in E} X(s, j)\) is maximized.
General Flow Network

Flow network \((G = (V, E); b; \ell; u)\) with

- directed graph \(G = (V, E)\)
- node production/consumption \(b: V \rightarrow \mathbb{R}\) with \(\sum_{i \in V} b(i) = 0\)
- edge lower bound \(\ell: E \rightarrow \mathbb{R}^+_0\)
- edge capacity \(u: E \rightarrow \mathbb{R}^+_0 \cup \{\infty\}\)

A function \(X: E \rightarrow \mathbb{R}^+_0\) is called valid flow, if:

\[
\ell(i, j) \leq X(i, j) \leq u(i, j) \quad \forall (i, j) \in E
\]

\[
\sum_{(i, j) \in E} X(i, j) - \sum_{(j, i) \in E} X(j, i) = b(i) \quad \forall i \in V
\]

- Cost function \(\text{cost}: E \rightarrow \mathbb{R}^+_0\) and \(\text{cost}(X) := \sum_{(i, j) \in E} \text{cost}(i, j) \cdot X(i, j)\)

A minimum cost flow is a valid flow where \(\text{cost}(X)\) is minimized.
General Flow Network – Algorithms

<table>
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<tr>
<th>Polynomial Algorithms</th>
<th>Year</th>
<th>Running Time</th>
</tr>
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<tbody>
<tr>
<td># Due to</td>
<td></td>
<td></td>
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<tr>
<td>1 Edmonds and Karp</td>
<td>1972</td>
<td>$O(n + m \log U S(n, m, nC))$</td>
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<tr>
<td>2 Rock</td>
<td>1980</td>
<td>$O(n + m \log U S(n, m, nC))$</td>
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<tr>
<td>3 Rock</td>
<td>1980</td>
<td>$O(n \log C M(n, m, U))$</td>
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<tr>
<td>4 Bland and Jensen</td>
<td>1985</td>
<td>$O(m \log C M(n, m, U))$</td>
</tr>
<tr>
<td>5 Goldberg and Tarjan</td>
<td>1987</td>
<td>$O((m^2/n) \log (nC))$</td>
</tr>
<tr>
<td>6 Goldberg and Tarjan</td>
<td>1988</td>
<td>$O(nm \log n \log (nC))$</td>
</tr>
<tr>
<td>7 Ahuja, Goldberg, Orlin and Tarjan</td>
<td>1988</td>
<td>$O(nm \log U \log (nC))$</td>
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<table>
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<th>Strongly Polynomial Algorithms</th>
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<td># Due to</td>
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<tr>
<td>1 Tardos</td>
<td>1985</td>
<td>$O(m^4)$</td>
</tr>
<tr>
<td>2 Orlin</td>
<td>1984</td>
<td>$O((n + m)^2 \log n S(n, m))$</td>
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<tr>
<td>3 Fujishige</td>
<td>1986</td>
<td>$O((n + m)^2 \log n S(n, m))$</td>
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<tr>
<td>4 Galil and Tardos</td>
<td>1986</td>
<td>$O(n^2 \log n S(n, m))$</td>
</tr>
<tr>
<td>5 Goldberg and Tarjan</td>
<td>1987</td>
<td>$O(nm^2 \log n \log(n^2/m))$</td>
</tr>
<tr>
<td>6 Goldberg and Tarjan</td>
<td>1988</td>
<td>$O(nm^2 \log^2 n)$</td>
</tr>
<tr>
<td>7 Orlin (this paper)</td>
<td>1988</td>
<td>$O(n + m^2 \log n S(n, m))$</td>
</tr>
</tbody>
</table>

**Theorem.** 
[Orlin 1991]
The minimum cost flow problem can be solved in $O(n^2 \log^2 n + m^2 \log n)$ time.

**Theorem.** 
[Cornelsen & Karrenbauer 2011]
The minimum cost flow problem for planar graphs with bounded costs and face sizes can be solved in $O(n^{3/2})$ time.
Topology – Shape – Metrics

Three-step approach:

\[ V = \{v_1, v_2, v_3, v_4\} \]
\[ E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\} \]

reduce crossings

combinatorial embedding/planarization

planear orthogonal drawing

bend minimization

orthogonal representation

area minimization

[Tamassia 1987]
Bend Minimization with Given Embedding

**Geometric bend minimization.**

Given:  
- Plane graph $G = (V, E)$ with maximum degree 4  
- Combinatorial embedding $F$ and outer face $f_0$

Find:  
Orthogonal drawing with minimum number of bends that preserves the embedding.

Compare with the following variation.

**Combinatorial bend minimization.**

Given:  
- Plane graph $G = (V, E)$ with maximum degree 4  
- Combinatorial embedding $F$ and outer face $f_0$

Find:  
**Orthogonal representation** $H(G)$ with minimum number of bends that preserves the embedding.
Combinatorial Bend Minimization

**Combinatorial bend minimization.**

**Given:**
- Plane graph $G = (V, E)$ with maximum degree 4
- Combinatorial embedding $F$ and outer face $f_0$

**Find:** **Orthogonal representation** $H(G)$ with minimum number of bends that preserves the embedding

**Idea.**

Formulate as a network flow problem:
- a unit of flow $= \angle \frac{\pi}{2}$
- vertices $\rightarrow$ faces ($\# \angle \frac{\pi}{2}$ per face)
- faces $\rightarrow$ neighbouring faces ($\#$ bends toward the neighbour)
Flow Network for Bend Minimization

Define flow network \( N(G) = ((V \cup F, E); b; \ell; u; \text{cost}) \):

\[
E = \{(v, f)_{ee'} \in V \times F \mid v \text{ between edges } e, e' \text{ of } \partial f \} \cup \{(f, g)_e \in F \times F \mid f, g \text{ have common edge } e\}
\]

(H1) \( H(G) \) corresponds to \( F, f_0 \).

(H2) For each edge \( \{u, v\} \) shared by faces \( f \) and \( g \), sequence \( \delta_1 \) is reversed and inverted \( \delta_2 \).

(H3) For each face \( f \) it holds that:

\[
\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ 4 & \text{otherwise.} \end{cases}
\]

(H4) For each vertex \( v \) the sum of incident angles is \( 2\pi \).

Directed multigraph!
Flow Network for Bend Minimization

Define flow network \( N(G) = ((V \cup F, E); b; \ell; u; {\text{cost}}) \):

- \( E = \{(v, f)_{ee'} \in V \times F \mid v \text{ between edges } e, e' \text{ of } \partial f \} \cup \{(f, g)_e \in F \times F \mid f, g \text{ have common edge } e \} \)
- \( b(v) = 4 \quad \forall v \in V \)
- \( b(f) = -2 \deg_G(f) + \begin{cases} -4 & \text{if } f = f_0, \\ +4 & \text{otherwise} \end{cases} \)

\[ \Rightarrow \sum_w b(w) = 0 \quad \text{(Euler)} \]

We model only the number of bends.
Why is it enough?
Flow Network for Bend Minimization

Define flow network \( N(G) = ((V ∪ F, E); b; ℓ; u; cost): \)

- \( E = \{(v, f)_{ee'} ∈ V × F \mid v \text{ between edges } e, e' \text{ of } ∂f\} \cup \{(f, g)_{e} ∈ F × F \mid f, g \text{ have common edge } e\} \)

- \( b(v) = 4 \quad ∀v ∈ V \)

- \( b(f) = -2 \deg_{G}(f) + \begin{cases} -4 & \text{if } f = f_0, \\ +4 & \text{otherwise} \end{cases} \Rightarrow \sum_{w} b(w) = 0 \) (Euler)

\( \forall (v, f) ∈ E, v ∈ V, f ∈ F \quad ℓ(v, f) := 1 \leq X(v, f) ≤ 4 =: u(v, f) \)

\( \forall (f, g) ∈ E, f, g ∈ F \quad ℓ(f, g) := 0 \leq X(f, g) ≤ ∞ =: u(f, g) \)

\( \text{We model only the number of bends. Why is it enough?} \)

Exercise!
Flow Network Example

- **Legend**
  - \( V \)  
  - \( F \)  
  - \( \ell/u/cost \)
  - \( V \times F \supseteq 1/4/0 \)
  - \( F \times F \supseteq 0/\infty/1 \)
  - \( 4 = b\)-value
  - \( 3 \) flow

- **Cost:** 1
- **One Bend** (outward)
Flow Network Example

Legend

- $V$ - source
- $F$ - sink

$V \times F \supseteq \ell/u/cost$

$F \times F \supseteq 1/4/0$

$4 = b\text{-value}$

flow
Bend Minimization – Result

**Theorem.** [Tamassia ’87]
A plane graph \((G, F, f_0)\) has a valid orthogonal representation \(H(G)\) with \(k\) bends. ⇔
The flow network \(N(G)\) has a valid flow \(X\) with cost \(k\).

**Proof.**

\(-\) Given valid flow \(X\) in \(N(G)\) with cost \(k\).

Construct orthogonal representation \(H(G)\) with \(k\) bends.

- Transform from flow to orthogonal description.
- Show properties (H1)–(H4).
  - (H1) \(H(G)\) matches \(F, f_0\)
  - (H2) Bend order inverted and reversed on opposite sides
  - (H3) Angle sum of \(f = \pm 4\)
  - (H4) Total angle at each vertex = 2\(\pi\)

\(\Rightarrow\)

- Exercise.

(H1) \(H(G)\) corresponds to \(F, f_0\).
(H2) For each edge \(\{u, v\}\) shared by faces \(f\) and \(g\), sequence \(\delta_1\) is reversed and inverted \(\delta_2\).
(H3) For each face \(f\) it holds that:
\[
\sum_{r \in H(f)} C(r) = \begin{cases} 
-4 & \text{if } f = f_0 \\ +4 & \text{otherwise.}
\end{cases}
\]
(H4) For each vertex \(v\) the sum of incident angles is 2\(\pi\).
Bend Minimization – Result

**Theorem.** [Tamassia '87]
A plane graph \((G, F, f_0)\) has a valid orthogonal representation \(H(G)\) with \(k\) bends. ⇔
The flow network \(N(G)\) has a valid flow \(X\) with cost \(k\).

**Proof.**
⇒ Given an orthogonal representation \(H(G)\) with \(k\) bends.
Construct valid flow \(X\) in \(N(G)\) with cost \(k\).

- Define flow \(X : E \rightarrow \mathbb{R}_0^+\).
- Show that \(X\) is a valid flow and has cost \(k\).

\[(N1) \quad X(v_f) = 1/2/3/4 \quad \checkmark\]
\[(N2) \quad X(fg) = |\delta_{fg}|_0, (e, \delta_{fg}, x) \text{ describes } e^* = fg \text{ from } f \quad \checkmark\]
\[(N3) \quad \text{capacities, deficit/demand coverage} \quad \checkmark\]
\[(N4) \quad \text{cost} = k \quad \checkmark\]
Theorem. [Garg & Tamassia 1996]
The min-cost flow problem for planar graphs with bounded costs and vertex degrees can be solved in $O(n^{7/4} \sqrt{\log n})$ time.

Theorem. [Cornelsen & Karrenbauer 2011]
The min-cost flow problem for planar graphs with bounded costs and face sizes can be solved in $O(n^{3/2})$ time.

Theorem. [Garg & Tamassia 2001]
Bend minimization without given combinatorial embedding is NP-hard.
Visualization of Graphs

Lecture 5:
Orthogonal Layouts

Part IV:
Area Minimization

Alexander Wolff
Topology – Shape – Metrics

Three-step approach:

\[ V = \{v_1, v_2, v_3, v_4\} \]
\[ E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\} \]

reduce crossings

combinatorial embedding/planarization

area minimization

planar orthogonal drawing

orthogonal representation

bend minimization

[Tamassia 1987]
Compaction

**Compaction problem.**
Given:  
- Plane graph $G = (V, E)$ with maximum degree 4
- Orthogonal representation $H(G)$

Find: Compact orthogonal layout of $G$ that realizes $H(G)$

**Special case.**
All faces are rectangles.

→ Guarantees possible  
- minimum total edge length
- minimum area

**Properties.**
- bends only on the outer face
- opposite sides of a face have the same length

**Idea.**
- Formulate flow network for horizontal/vertical compaction
Flow Network for Edge Length Assignment

Definition.
Flow Network $N_{\text{hor}} = ((W_{\text{hor}}, E_{\text{hor}}); b; \ell; u; \text{cost})$

- $W_{\text{hor}} = F \setminus \{f_0\} \cup \{s, t\}$
- $E_{\text{hor}} = \{(f, g) \mid f, g \text{ share a horizontal segment and } f \text{ lies below } g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in E_{\text{hor}}$
- $u(a) = \infty \quad \forall a \in E_{\text{hor}}$
- $\text{cost}(a) = 1 \quad \forall a \in E_{\text{hor}}$
- $b(f) = 0 \quad \forall f \in W_{\text{hor}}$
Flow Network for Edge Length Assignment

Definition.
Flow Network $N_{\text{ver}} = ((W_{\text{ver}}, E_{\text{ver}}); b; \ell; u; \text{cost})$

- $W_{\text{ver}} = F \setminus \{f_0\} \cup \{s, t\}$
- $E_{\text{ver}} = \{(f, g) \mid f, g \text{ share a vertical segment and } f \text{ lies to the left of } g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in E_{\text{ver}}$
- $u(a) = \infty \quad \forall a \in E_{\text{ver}}$
- $\text{cost}(a) = 1 \quad \forall a \in E_{\text{ver}}$
- $b(f) = 0 \quad \forall f \in W_{\text{ver}}$
What values of the drawing do the following quantities represent?

- $|X_{\text{hor}}(t, s)|$ and $|X_{\text{ver}}(t, s)|$? width and height of drawing
- $\sum_{e \in E_{\text{hor}}} X_{\text{hor}}(e) + \sum_{e \in E_{\text{ver}}} X_{\text{ver}}(e)$ total edge length

**Theorem.**
A valid flow for $N_{\text{hor}}$ and $N_{\text{ver}}$ exists $\iff$ corresponding edge lengths induce an orthogonal drawing.
Refinement of \((G, H) - \text{Inner Face}\)

$$\text{corner}(e)$$

$$\text{next}(e)$$

$$\text{extend}(e')$$

$$\text{project}(e')$$

$$\text{front}(e')$$

- Dummy vertices for bends
- $$\text{turn}(e) = \begin{cases} 
  1 & \text{left turn} \\
  0 & \text{no turn} \\
  -1 & \text{right turn} 
\end{cases}$$
Refinement of \((G, H)\) – Inner Face

\[
\text{corner}(e) = \begin{cases} 
1 & \text{left turn} \\
0 & \text{no turn} \\
-1 & \text{right turn}
\end{cases}
\]

\[
\text{turn}(e) = \begin{cases} 
1 & \text{left turn} \\
0 & \text{no turn} \\
-1 & \text{right turn}
\end{cases}
\]

- Dummy vertices for bends
Refinement of $(G, H)$ – Outer Face
Refinement of \((G, H)\) – Outer Face
Refinement of \((G, H)\) – Outer Face
Refinement of \((G, H)\) – Outer Face

Area minimized? \[\text{No!}\]

But we get bound \(O((n + b)^2)\) on the area.

\textbf{Theorem.} [Patrignani 2001]
Compaction for given orthogonal representation is NP-hard in general.
Visualization of Graphs

Lecture 5:
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Part V:
NP-Hardness

Alexander Wolff
Boundary, **belt**, and “piston” gadget

\[(w \times h)\text{-rectangle}\]
Boundary, belt, and “piston” gadget
Boundary, belt, and “piston” gadget
Clause gadgets

Example:

\[ C_1 = x_2 \lor \bar{x}_4 \]
\[ C_2 = x_1 \lor x_2 \lor \bar{x}_3 \]
\[ C_3 = x_5 \]
\[ C_4 = x_4 \lor \bar{x}_5 \]

Insert \((2n-1)\)-chain through each clause
Complete reduction

Pick
\[ K = (9n + 2) \cdot (9m + 7) \]

Then:
\[(G, H)\text{ has an area } K\text{ drawing }\]
\[\iff\]
\[\Phi\text{ satisfiable}\]
Literature

- [GD Ch. 5] for detailed explanation

- [Tamassia 1987] “On embedding a graph in the grid with the minimum number of bends”
  Original paper on flow for bend minimization.

  NP-hardness proof for orthogonal representation of planar max-degree-4 graphs.

- [Evans, Fleszar, Kindermann, Saeedi, Shin, Wolff 2022]
  “Minimum rectilinear polygons for given angle sequences”
  NP-hardness proof for compaction of cycles.