Visualization of Graphs

Lecture 3:
Straight-Line Drawings of Planar Graphs I:
Canonical Ordering and the Shift Method

Part I:
Planar Straight-Line Drawings

Alexander Wolff
Planar Graphs

\( G \) is **planar**: it can be drawn in such a way that no edges cross each other.

**planar embedding**: Clockwise orientation of adjacent vertices around each vertex.

A planar graph can have many planar embeddings.

A planar embedding can have many planar drawings!

**faces**: Connected region of the plane bounded by edges

**Euler’s polyhedra formula.**

\[
\text{#faces} - \text{#edges} + \text{#vertices} = \text{#conn.comp.} + 1
\]

\[ f - m + n = c + 1 \]

**Proof.** By induction on \( m \):

\( m = 0 \Rightarrow f = 1 \) and \( c = n \)
\[ \Rightarrow 1 - 0 + n = n + 1 \checkmark \]

\( m \geq 1 \Rightarrow \text{remove some edge } e \Rightarrow m \to m - 1 \)

\[ e \Rightarrow c \to c + 1 \]
Properties of Planar Graphs

Euler’s polyhedra formula.
\[ f - m + n = c + 1 \]

Theorem. \( G \) simple planar graph with \( n \geq 3 \) vtc.
1. \( m \leq 3n - 6 \)
2. \( f \leq 2n - 4 \)
3. There is a vertex of degree at most 5.

Proof. 1. Every edge incident to \( \leq 2 \) faces
   Every face incident to \( \geq 3 \) edges
   \( \Rightarrow 3f \leq 2m \)
   \( \Rightarrow 6 \leq 3c + 3 \leq 3f - 3m + 3n \leq 2m - 3m + 3n = 3n - m \)
   \( \Rightarrow m \leq 3n - 6 \)
2. \( 3f \leq 2m \leq 6n - 12 \) \( \Rightarrow f \leq 2n - 4 \)
3. \( \sum_{v \in V} \deg(v) = 2m \leq 6n - 12 \)
   \( \Rightarrow \min_{v \in V} \deg(v) \leq \text{average degree}(G) = 1/n \sum_{v \in V} \deg(v) < 6 \)

Handshaking lemma.
\( \sum_{v \in V} \deg(v) = 2|E| \)
Triangulations

A **plane (inner) triangulation** is a plane graph where every (inner) face is a triangle.

A **maximal planar graph** is a planar graph where adding any edge would violate planarity.

**Observation.**
A maximal plane graph is a plane triangulation.

**Lemma.**
A plane triangulation is at least 3-connected and thus has a unique planar embedding.

We focus on plane triangulations:

**Lemma.**
Every plane graph is subgraph of a plane triangulation.
Triangulations

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Motivation

- Why planar and straight-line?

[Bennett, Ryall, Spaltzeholz and Gooch ’07]

The Aesthetics of Graph Visualization

3.2. Edge Placement Heuristics

By far the most agreed-upon edge placement heuristic is to **minimize the number of edge crossings** in a graph [BMRW98, Har98, DH96, Pur02, TR05, TBB88]. The importance of avoiding edge crossings has also been extensively validated in terms of user preference and performance (see Section 4). Similarly, based on perceptual principles, it is beneficial to **minimize the number of edge bends** within a graph [Pur02, TR05, TBB88]. Edge bends make edges more difficult to follow because an edge with a sharp bend is more likely to be perceived as two separate objects. This leads to the heuristic of **keeping edge bends uniform** with respect to the bend’s position on the edge and its angle [TR05]. If an edge must be bent to satisfy other aesthetic criteria, the angle of the bend should be as little as possible, and the bend placement should evenly divide the edge.

**Drawing conventions**
- No crossings $\Rightarrow$ planar
- No bends $\Rightarrow$ straight-line

**Drawing aesthetics**
- Area
Towards Straight-Line Drawings

Theorem. [Kuratowski 1930]

$G$ planar $\iff$ neither $K_5$ nor $K_{3,3}$ minor of $G$

Theorem. [Hopcroft & Tarjan 1974]

Let $G$ be a graph with $n$ vertices. There is an $O(n)$-time algorithm to test whether $G$ is planar. Also computes a planar embedding in $O(n)$ time.

Theorem. [Wagner 1936, Fáry 1948, Stein 1951]

Every planar graph has a planar drawing where the edges are straight-line segments.
Towards Straight-Line Drawings

### Theorem. [Kuratowski 1930]

\[ G \text{ planar} \iff \text{neither } K_5 \text{ nor } K_{3,3} \text{ minor of } G \]

### Theorem. [Hopcroft & Tarjan 1974]

Let \( G \) be a graph with \( n \) vertices. There is an \( O(n) \)-time algorithm to test whether \( G \) is planar.

Also computes a planar embedding in \( O(n) \) time.

### Theorem. [Wagner 1936, Fáry 1948, Stein 1951]

Every planar graph has a planar drawing where the edges are straight-line segments.

The algorithms implied by these theorems produce drawings whose area is **not** bounded by any polynomial in \( n \).
Planar straight-line drawings

**Theorem.** [De Fraysseix, Pach, Pollack '90]
Every \( n \)-vertex planar graph has a planar straight-line drawing of size \( (2n - 4) \times (n - 2) \).

**Idea.**
- Start with single edge \((v_1, v_2)\). Let this be \( G_2 \).
- To obtain \( G_{i+1} \), add \( v_{i+1} \) to \( G_i \) so that neighbours of \( v_{i+1} \) are on the outer face of \( G_i \).
- Neighbours of \( v_{i+1} \) in \( G_i \) have to form path of length at least two.

**Theorem.** [Schnyder '90]
Every \( n \)-vertex planar graph has a planar straight-line drawing of size \( (n - 2) \times (n - 2) \).
Visualization of Graphs

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Part II:
Canonical Order

Alexander Wolff
Definition.
Let $G = (V, E)$ be a triangulated plane graph on $n \geq 3$ vertices. An ordering $\pi = (v_1, v_2, \ldots, v_n)$ of $V$ is called a **canonical order** if the following conditions hold for each $k \in \{3, 4, \ldots, n\}$:

(C1) Vertices $\{v_1, \ldots v_k\}$ induce a biconnected internally triangulated graph; call it $G_k$.

(C2) Edge $(v_1, v_2)$ belongs to the outer face of $G_k$.

(C3) If $k < n$ then vertex $v_{k+1}$ lies in the outer face of $G_k$, and the neighbors of $v_{k+1}$ form a path on the boundary of $G_k$. 
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Canonical Order – Example

(C1) Vertices \( \{v_1, \ldots v_k\} \) induce a biconnected internally triangulated graph; call it \( G_k \).

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\[
\begin{align*}
\text{chord:} & \quad \text{edge joining two nonadjacent vertices in a cycle}
\end{align*}
\]
Canonical Order – Example

(C1) Vertices $\{v_1, \ldots, v_k\}$ induce a biconnected internally triangulated graph; call it $G_k$.

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(C3) If \(k < n\) then vertex \(v_{k+1}\) lies in the outer face of \(G_k\), and the neighbors of \(v_{k+1}\) form a path on the boundary of \(G_k\).
Canonical Order – Existence

**Lemma.**
Every triangulated plane graph has a canonical order.

**Base Case:**
Let $G_n = G$, and let $v_1, v_2, v_n$ be the vertices of the outer face of $G_n$. Conditions (C1)–(C3) hold.

**Induction hypothesis:**
Vertices $v_{n-1}, \ldots, v_{k+1}$ have been chosen such that conditions (C1)–(C3) hold for $k + 1 \leq i \leq n$.

**Induction step:** Consider $G_k$. We search for $v_k$.

(C1) $G_k$ biconnected and internally triangulated

(C2) $(v_1, v_2)$ on outer face of $G_k$

(C3) $k < n \Rightarrow v_{k+1}$ in outer face of $G_k$, neighbors of $v_{k+1}$ form path on boundary of $G_k$

**Have to show:**
1. $v_k$ not incident to chord is sufficient
2. Such $v_k$ exists
Canonical Order – Existence

**Claim 1.**
If \( v_k \) is not incident to a chord, then \( G_{k-1} \) is biconnected.

**Claim 2.**
There exists a vertex in \( G_k \) that is not incident to a chord as choice for \( v_k \).

Contradiction to neighbors of \( v_k \) forming a path on \( \partial G_{k-1} \)!

Not triangulated!

\( G_k \) was not biconnected!

\( v_1 \) \( v_2 \)

This completes the proof of the lemma. \( \square \)
**Canonical Order – Implementation**

**CanonicalOrder**\((G = (V, E), (v_1, v_2, v_n))\)

```plaintext
forall v ∈ V do
  chords(v) ← 0; out(v) ← false; mark(v) ← false
mark(v_1), mark(v_2), out(v_1), out(v_2), out(v_n) ← true
for k = n downto 3 do
  choose v such that mark(v) = false, out(v) = true, and chords(v) = 0
  \(v_k ← v; mark(v) ← true\)
  // Let \(\partial G_{k-1}\) be \(w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2\).
  Let \(w_p, \ldots, w_q\) be the unmarked neighbors of \(v_k\).
  for i = p to q do
    out\(w_i\) ← true
    update chords\(w_i\)
    and for its neighbours
```

- **chord\(v)\)**: \# chords adjacent to \(v\)
- **out\(v\)** = true iff \(v\) is currently outer vertex
- **mark\(v\)** = true iff \(v\) has received its number

**Lemma.**
Algorithm CanonicalOrder computes a canonical order of a plane graph in \(O(n)\) time.
Visualization of Graphs

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Part III:
The Shift Method

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Shift Method – Idea

**Drawing invariants:**

$G_{k−1}$ is drawn such that

- $v_1$ is at $(0, 0)$, $v_2$ is at $(2k − 6, 0)$,
- boundary of $G_{k−1}$ (minus edge $(v_1, v_2)$) is drawn $x$-monotone,
- each edge of the boundary of $G_{k−1}$ (minus edge $(v_1, v_2)$) is drawn with slopes $±1$.

What could be the solution?
Shift Method – Idea

**Drawing invariants:**

$G_{k-1}$ is drawn such that

- $v_1$ is at $(0, 0)$, $v_2$ is at $(2k - 6, 0)$,
- boundary of $G_{k-1}$ (minus edge $(v_1, v_2)$) is drawn $x$-monotone,
- each edge of the boundary of $G_{k-1}$ (minus edge $(v_1, v_2)$) is drawn with slopes $\pm 1$.

Yes, because $w_p$ and $w_q$ have even Manhattan distance $\Delta x + \Delta y$. 

Will $v_k$ lie on the grid?
Shift Method – Example
Shift Method – Example

$L(10)$
Shift Method – Example

$L(16)$
Shift Method – Example
Shift Method – Planarity

**Observations.**
- Each internal vertex is **covered** exactly once.
- Covering relation defines a tree in \( G \)
- and a forest in \( G_i, 1 \leq i \leq n - 1 \).

**Lemma.**

Let \( 0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_t \in \mathbb{N} \), such that \( \delta_q - \delta_p \geq 2 \) and even. If we shift \( L(w_i) \) by \( \delta_i \) to the right, then we get a planar straight-line drawing.

Proof by induction:
If \( G_{k-1} \) is drawn planar and straight-line, then so is \( G_k \).
Shift Method – Pseudocode

Let $v_1, \ldots, v_n$ be a canonical order of $G$.

for $i = 1$ to $3$ do
  $L(v_i) \leftarrow \{v_i\}$
  $P(v_1) \leftarrow (0, 0); P(v_2) \leftarrow (2, 0), P(v_3) \leftarrow (1, 1)$

for $i = 4$ to $n$ do
  Let $\partial G_{i-1}$ be $v_1 = w_1, w_2, \ldots, w_{t-1}, w_t = v_2$.
  Let $w_p, \ldots, w_q$ be the neighbors of $v_i$.
  foreach $v \in \bigcup_{j=p+1}^{q-1} L(w_j)$ do /* $O(n^2)$ in total */
    $x(v) \leftarrow x(v) + 1$
  endforeach
  foreach $v \in \bigcup_{j=q}^{t} L(w_j)$ do /* $O(n^2)$ in total */
    $x(v) \leftarrow x(v) + 2$
  endforeach
  $P(v_i) \leftarrow$ intersection of slope-$\pm 1$ diagonals through $P(w_p)$ and $P(w_q)$
  $L(v_i) \leftarrow \bigcup_{j=p+1}^{q-1} L(w_j) \cup \{v_i\}$

Running Time?
Shift Method – Linear-Time Implementation

Idea 1.
To compute $x(v_k)$ & $y(v_k)$, we only need $y(w_p)$ and $y(w_q)$ and $x(w_q) - x(w_p)$

Idea 2.
Instead of storing explicit x-coordinates, we store x-distances.

(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$
(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$
Idea 1.
To compute \( x(v_k) & y(v_k) \), we only need \( y(w_p) \) and \( y(w_q) \) and \( x(w_q) - x(w_p) \)

Idea 2.
Instead of storing explicit x-coordinates, we store x-distances.
After an x-distance is computed for each \( v_k \), use preorder traversal to compute all x-coordinates.

\[
\begin{align*}
(1) \quad x(v_k) &= \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p)) \\
(2) \quad y(v_k) &= \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p)) \\
(3) \quad x(v_k) - x(w_p) &= \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))
\end{align*}
\]
Shift Method – Linear-Time Implementation

Relative x-distance tree.
For each vertex $v$ store
- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$

Calculations.
- $\Delta_x(w_{p+1})++$, $\Delta_x(w_q)++$
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \ldots + \Delta_x(w_q)$
- $\Delta_x(v_k)$ by (3)
- $y(v_k)$ by (2)
- $\Delta_x(w_q) = \Delta_x(w_p, w_q) - \Delta_x(v_k)$
- $\Delta_x(w_{p+1}) = \Delta_x(w_{p+1}) - \Delta_x(v_k)$

(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$
(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$
(3) $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$
Literature

- [PGD Ch. 4.2] for detailed explanation of shift method
- [de Fraysseix, Pach, Pollack 1990] “How to draw a planar graph on a grid”
  - original paper on shift method