Visualization of Graphs

Lecture 11:
The Crossing Lemma
and its Applications

Part I:
Definition and Hanani–Tutte

Jonathan Klawitter
Crossing Number and Topological Graphs

For a graph $G$, the **crossing number** $cr(G)$ is the smallest number of crossings in a drawing of $G$ (in the plane).
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$\text{cr}(K_{3,3}) = 1$
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![Diagram of a topological drawing with examples of edge intersections and self-intersections]
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# crossings reduced, so terminates
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**Theorem.** [Hanani’43, Tutte’70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.
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**Proof Sketch.**
Hanani showed that every drawing of $K_5$ and $K_{3,3}$ must have a pair of edges that crosses an odd number of times.
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Hence, there must be two edges on these paths that cross an odd number of times.
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**Theorem.** [Pelsmajer, Schaefer & Štefankovič ’08, Tóth ’08]
There is a graph $G$ with $\text{ocr}(G) < \text{cr}(G) \leq 10$
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Is $\text{pcr}(G) = \text{cr}(G)$? Open!
Visualization of Graphs

Lecture 11:
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and its Applications

Part II:
Computation & Variations

Jonathan Klawitter
Computing the Crossing Number

- Computing $cr(G)$ is NP-hard. [Garey & Johnson '83]

... even if $G$ is a planar graph plus one edge! [Cabello & Mohar '08]
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- Planarization, where we replace crossings with dummy vertices, also uses only heuristics
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- Crossing minimization is NP-hard for most of the variants
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Rectilinear Crossing Number

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**Separation.**
$cr(K_8) = 18$, but $\overline{cr}(K_8) = 19$. 
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**Lemma 1.** [Bienstock, Dean '93]
For $k \geq 4$, there exists a graph $G_k$ with $cr(G_k) = 4$ and $\overline{cr}(G_k) \geq k$. 
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![Graph $G_1$](image)
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- From $G_1$ to $G_k$ do
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Part III:
First Bounds

Jonathan Klawitter
Bounds for Complete Graphs

**Theorem.** [Guy '60]

\[
\text{cr}(K_n) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor = \frac{3}{8} \binom{n}{4} + O(n^3)
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Theorem. [Guy ‘60]

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Sylvester’s four-point problem
Bounds for Complete Graphs

<table>
<thead>
<tr>
<th>Theorem.</th>
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<tr>
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Sylvester’s four-point problem
Bounds for Complete Graphs

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Bound is sharp for \( n \leq 12 \).
Bounds for Complete Graphs

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Sylvester’s four-point problem
Bounds for Complete Graphs

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Turán’s brick factory problem (1944)

Sylvester’s four-point problem

Pál Turán  
*1910 – 1976  
Budapest, Hungary

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 Bounds for Complete Graphs

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**Theorem.** [Lovász et al. '04, Aichholzer et al. '06]

\[
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Sylvester's four-point problem
Bounds for Complete Graphs

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Sylvester’s four-point problem
Bounds for Complete Graphs

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Check out http://www.ist.tugraz.at/staff/aichholzer/crossings.html!

Sylvester's four-point problem
First Lower Bounds on $\text{cr}(G)$

**Lemma 2.**
For a graph $G$ with $n$ vertices and $m$ edges,

$$\text{cr}(G) \geq m - 3n + 6.$$
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- Obtain a graph $H$ by turning crossings into dummy vertices.
First Lower Bounds on $\text{cr}(G)$

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- Obtain a graph $H$ by turning crossings into dummy vertices.
- $H$ has $n + cr(G)$ vertices and $m + 2cr(G)$ edges.
- $H$ is planar, so

$$m + 2cr(G) \leq 3(n + cr(G)) - 6.$$
First Lower Bounds on $\text{cr}(G)$

**Lemma 3.**
For a graph $G$ with $n$ vertices and $m$ edges,

$$\text{cr}(G) \geq r \left( \frac{\lfloor m/r \rfloor}{2} \right) \in \Omega \left( \frac{m^2}{n} \right)$$

where $r \leq 3n - 6$ is the maximum number of edges in a planar subgraph of $G$. 
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- Take $\lfloor m/r \rfloor$ edge-disjoint subgraphs of $G$ with $r$ edges.
- In the best case, they are all planar.
First Lower Bounds on \( \text{cr}(G) \)

**Lemma 3.**
For a graph \( G \) with \( n \) vertices and \( m \) edges,

\[
\text{cr}(G) \geq r \left( \left\lfloor \frac{m}{r} \right\rfloor \right)^2 \in \Omega \left( \frac{m^2}{n} \right)
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where \( r \leq 3n - 6 \) is the maximum number of edges in a planar subgraph of \( G \).

**Proof.**
- Take \( \left\lfloor \frac{m}{r} \right\rfloor \) edge-disjoint subgraphs of \( G \) with \( r \) edges.
- In the best case, they are all planar.
- For each pair \( G_i, G_j \), any edge of \( G_j \) induces at least one crossings with \( G_i \).
  (If not, swap edges to reduce \( \text{cr}(G_i) \).)
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Consider this bound for graphs with \( \Theta(n) \) and \( \Theta(n^2) \) many edges.
Visualization of Graphs

Lecture 11:
The Crossing Lemma and its Applications

Part IV:
The Crossing Lemma

Jonathan Klawitter
The Crossing Lemma

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The Crossing Lemma

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- We look at a proof “from THE BOOK” by Chazelle, Sharir and Welz.

- Factor $\frac{1}{64}$ was later (with intermediate steps) improved to $\frac{1}{29}$ by Ackerman in 2013.
The Crossing Lemma

**Crossing Lemma.**
For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4n$, 
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- Consider a minimal embedding of $G$.
- Let $p$ be a number in $(0, 1)$.
- Keep every vertex of $G$ independently with probability $p$. 
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**Proof.**
- Consider a minimal embedding of $G$.
- Let $p$ be a number in $(0, 1)$.
- Keep every vertex of $G$ independently with probability $p$.
- Let $G_p$ be the remaining graph.
- Let $n_p, m_p, X_p$ be the random variables counting the number of vertices/edges/crossings of $G_p$.
- By Lem 2, $\mathbb{E}(X_p - m_p + 3n_p) \geq 0$.
- $\mathbb{E}(n_p) = pn$ and $\mathbb{E}(m_p) = p^2 m$
- $\mathbb{E}(X_p) = p^4 cr(G)$
- $0 \leq \mathbb{E}(X_p) - \mathbb{E}(m_p) + 3\mathbb{E}(n_p)$
  $= p^4 cr(G) - p^2 m + 3pn$
- $cr(G) \geq \frac{p^2 m - 3pn}{p^4} = \frac{m}{p^2} - \frac{3n}{p^3}$
The Crossing Lemma

Crossing Lemma.
For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4n$,
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\[
\begin{align*}
\mathbb{E}(n_p) &= pn \quad \text{and} \quad \mathbb{E}(m_p) = p^2 m \\
\mathbb{E}(X_p) &= p^4 \text{cr}(G) \\
0 &\leq \mathbb{E}(X_p) - \mathbb{E}(m_p) + 3\mathbb{E}(n_p) \\
&= p^4 \text{cr}(G) - p^2 m + 3pn \\
\text{cr}(G) &\geq \frac{p^2 m - 3pn}{p^4} = \frac{m}{p^2} - \frac{3n}{p^3} \\
\text{Set } p &= \frac{4n}{m}. \\
\text{cr}(G) &\geq \frac{m^3}{16n^2} - \frac{3m^3}{64n^2}
\end{align*}
\]
The Crossing Lemma

Crossing Lemma.
For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4n$, 
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Visualization of Graphs

Lecture 11:
The Crossing Lemma
and its Applications

Part V:
Applications

Jonathan Klawitter
Application 1: Point-Line Incidences

- For points $P \subset \mathbb{R}^2$ and lines $\mathcal{L}$,
  \[ I(P, \mathcal{L}) = \text{number of point-line incidences in } (P, \mathcal{L}). \]
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![Diagram](image.png)
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\Rightarrow I(P, \mathcal{L}) = 10
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  ![Diagram](image)

  ⇒ $I(P, \mathcal{L}) = 10$

- Define $I(n, k) = \max_{|P|=n, |\mathcal{L}|=k} I(P, \mathcal{L})$. 

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- For example: $I(4, 4) =$
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\[ \begin{array}{c|c|c|c}
3 & 3 & 2 & 2 \\
\hline
8 & & & \\
\end{array} \]
**Application 1: Point-Line Incidences**

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  \[ 3 \quad 8 \quad 9 \]
Application 1: Point-Line Incidences

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**Theorem 1.** [Szemerédi, Trotter '83, Székely '97]

$I(n, k) \leq 2.7n^{2/3}k^{2/3} + 6n + 2k$. 
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\[ \Rightarrow I(P, \mathcal{L}) = 10 \]

**Theorem 1.**
[Szemerédi, Trotter '83, Székely '97]
\[ I(n, k) \leq c(n^{2/3}k^{2/3} + n + k). \]
Application 1: Point-Line Incidences

- For points $P \subset \mathbb{R}^2$ and lines $L$, $I(P, L)$ is the number of point-line incidences in $(P, L)$.

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Proof.
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Proof.

Example incidence count:
$\Rightarrow I(P, \mathcal{L}) = 10$
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&[\text{Szemerédi, Trotter '83, Székely '97}] \\
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\[\text{Proof.} \]
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- $\#$ points on $l = 1 + \#$ edges on $l$
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- \( c'(I(n, k) - k)^3 / n^2 \leq \text{cr}(G) \leq k^2 \)
- if \( m \not\geq 4n \), then \( I(n, k) - k \leq 4n \)
Application 2: Unit Distances

For points $P \subset \mathbb{R}^2$ define

- $U(P) =$ number of pairs in $P$ at unit distance and
- $U(n) = \max_{|P|=n} U(P)$. 
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[Spencer, Szemerédi, Trotter '84, Székely '97]

$U(n) < 6.7n^{4/3}$
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**Proof.**

- $U(P) \leq c'm$
- $\text{cr}(G) \leq 2n^2$
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- $c\frac{U(P)^3}{n^2} \leq \text{cr}(G) \leq 2n^2$
Application 3: Max. Num. of Crossings in Matchings

Given point set of $n$ points
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Number of potential crossings (all pairs of edges): $\text{pot}(K_n) =$
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Number of potential crossings (all pairs of edges): $\text{pot}(K_n) = \binom{\binom{n}{2}}{2} \approx 3(\binom{n}{4})$
Application 3: Max. Num. of Crossings in Matchings

Given point set of $n$ points
What is the max. number of crossings in any matching?

Point set spans drawing $\Gamma$ of $K_n$
We will analyze the number of crossings in a random matching in $\Gamma$!

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6 crossings
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Number of edges in $K_n$: $\binom{n}{2}$

Number of *potential crossings* (all pairs of edges): $\text{pot}(K_n) = \binom{\binom{n}{2}}{2} = 3 \binom{n}{4}$

Pick two random edges $e_1, e_2$

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By linearity of expectations, exp. number of crossings in $M$ is $> \frac{1}{8}\binom{n/2}{2}$
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Pick two random edges $e_1, e_2$

Pr[$e_1$ and $e_2$ cross] $\geq \overline{cr}(K_n)/pot(K_n) > \frac{1}{8}$

Fix matching $M$; it has $\leq n/2$ edges, so $\binom{n/2}{2} = \frac{1}{8}n(n-2)$ pairs of edges

By linearity of expectations, exp. number of crossings in $M$ is $> \frac{1}{8}\binom{n/2}{2} = \frac{1}{64}n(n-2)$
Literature

- [Aigner, Ziegler] Proofs from THE BOOK
- [Schaefer '20] The Graph Crossing Number and its Variants: A Survey
- Terrence Tao blog post “The crossing number inequality” from 2007
- [Garey, Johnson '83] Crossing number is NP-complete
- [Bienstock, Dean '93] Bounds for rectilinear crossing numbers
- [Székely '97] Crossing Numbers and Hard Erdös Problems in Discrete Geometry
- Documentary/Biography “N Is a Number: A Portrait of Paul Erdös”
- Exact computations of crossing numbers: http://crossings.uos.de