Visualization of Graphs

Lecture 6:
Upward Planar Drawings

Part I:
Characterization

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Upward Planar Drawings – Motivation

- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchy
  - . . .

- Would be nice to have general direction preserved in drawing.
A directed graph $G = (V, E)$ is **upward planar** when it admits a drawing $\Gamma$ that is
- planar and
- where each edge is drawn as an upward, y-monotone curve.
Upward Planarity – Necessary Conditions

- For a digraph $G$ to be upward planar, it has to be:
  - planar
  - acyclic
  - bimodal

- ...but these conditions are not sufficient.
Upward Planarity – Characterization

**Theorem 1.** [Kelly 1987, Di Battista & Tamassia 1988]

For a digraph $G$ the following statements are equivalent:

1. $G$ is upward planar.
2. $G$ admits an upward planar straight-line drawing.
3. $G$ is the spanning subgraph of a planar $st$-digraph.

Additionally:
- Embedded such that $s$ and $t$ are on the outerface $f_0$.
- Acyclic digraph with a single source $s$ and single sink $t$

or:
- No crossings

$$(s, t)$$ exists.
Upward Planarity – Characterization

**Theorem 1.** [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph $G$ the following statements are equivalent:
1. $G$ is upward planar.
2. $G$ admits an upward planar straight-line drawing.
3. $G$ is the spanning subgraph of a planar $st$-digraph.

**Proof.**

(2) ⇒ (1) By definition. (1) ⇔ (3) Example:
(3) ⇒ (2) Triangulate & construct drawing:

**Claim.**
Can draw in prespecified triangle.
Induction on $n$. 

Case 1: chord

Case 2: no chord
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Part II:
Assignment Problem

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Upward Planarity – Complexity

**Theorem.** [Garg, Tamassia, 1995]
For a planar acyclic digraph it is in general NP-hard to decide whether it is upward planar.

**Theorem.** [Hutton, Libow, 1996]
For a single-source acyclic digraph it can be tested in $O(n)$ time whether it is upward planar.

**Corollary.**
For a triconnected planar digraph it can be tested in $O(n^2)$ time whether it is upward planar.

**Theorem.** [Bertolazzi et al., 1994]
For a combinatorially embedded planar digraph it can be tested in $O(n^2)$ time whether it is upward planar.
The Problem

**Fixed Embedding Upward Planarity Testing.**
Let $G = (V, E)$ be a plane digraph with set of faces $F$ and outer face $f_0$.
Test whether $G$ is upward planar (wrt to $F$, $f_0$).

**Idea.**

- Find property that any upward planar drawing of $G$ satisfies.
- Formalise property.
- Find algorithm to test property.
Angles, Local Sources & Sinks

Definitions.

- A vertex \( v \) is a **local source** wrt to a face \( f \) if \( v \) has two outgoing edges on \( \partial f \).
- A vertex \( v \) is a **local sink** wrt to a face \( f \) if \( v \) has two incoming edges on \( \partial f \).
- An angle \( \alpha \) at a local source / sink is **large** when \( \alpha > \pi \) and **small** otherwise.
- \( L(v) \) = \# large angles at \( v \)
- \( L(f) \) = \# large angles in \( f \)
- \( S(v) \) & \( S(f) \) for \# small angles
- \( A(f) \) = \# local sources wrt to \( f \)
  = \# local sinks wrt to \( f \)

**Lemma 1.**

\[
L(f) + S(f) = 2A(f)
\]
Assignment Problem

- Vertex $v$ is a **global source** at faces $f_1$ and $f_2$.
- Does $v$ have a **large** angle in $f_1$ or $f_2$?

![Diagram](attachment:image.png)
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Part III:
Angle Relations

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Angle Relations

**Lemma 2.**

\[
L(f) - S(f) = \begin{cases} 
-2, & f \neq f_0 \\
+2, & f = f_0 
\end{cases}
\]

- \(L(f) \geq 1\)

Split \(f\) with edge from a large angle at a “low” sink \(u\) to 

- sink \(v\) with small/large angle:

\[
L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1) = -2
\]

**Proof** by induction.

- \(L(f) = 0\) \(\Rightarrow S(f) = 2\)
Angle Relations

**Lemma 2.**

\[
L(f) - S(f) = \begin{cases} 
-2, & f \neq f_0 \\
+2, & f = f_0 
\end{cases}
\]

**Proof** by induction.

- \( L(f) = 0 \) \( \Rightarrow S(f) = 2 \)

- \( L(f) \geq 1 \)

Split \( f \) with edge from a large angle at a “low” sink \( u \) to

- sink \( v \) with small/large angle:

\[
L(f) - S(f) = \left( L(f_1) + L(f_2) + 2 \right) - \left( S(f_1) + S(f_2) \right) \\
= -2
\]
Angle Relations

Lemma 2.

\[ L(f) - S(f) = \begin{cases} 
-2, & f \neq f_0 \\
+2, & f = f_0 
\end{cases} \]

Proof by induction.

- \( L(f) = 0 \) \implies S(f) = 2

\( L(f) \geq 1 \)

Split \( f \) with edge from a large angle at a “low” sink \( u \) to source \( v \) with small/large angle:

\[ L(f) = \begin{cases} 
-2, & f \neq f_0 \\
+2, & f = f_0 
\end{cases} \]
**Angle Relations**

**Lemma 2.**
\[ L(f) - S(f) = \begin{cases} 
-2, & f \neq f_0 \\
+2, & f = f_0 
\end{cases} \]

**Proof** by induction.
- **Step 1:** \( L(f) = 0 \) \Rightarrow \( S(f) = 2 \)

- **Step 2:** \( L(f) \geq 1 \)

Split \( f \) with edge from a large angle at a “low” sink \( u \) to source \( v \) with small/large angle:

- \( L(f) - S(f) = L(f_1) + L(f_2) + 2 - (S(f_1) + S(f_2)) = -2 \)
Angle Relations

**Lemma 2.**

\[
L(f) - S(f) = \begin{cases} 
-2, & f \neq f_0 \\
+2, & f = f_0 
\end{cases}
\]

**Proof** by induction.

- \( L(f) = 0 \) \( \Rightarrow S(f) = 2 \)

\(-L(f) \geq 1\)

Split \( f \) with edge from a large angle at a “low” sink \( u \) to

- vertex \( v \) that is neither source nor sink:

\[
L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1) = -2
\]

- Otherwise “high” source \( u \) exists.
Number of Large Angles

Lemma 3.
In every upward planar drawing of $G$ holds that
- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex}, \\ 1 & v \text{ source / sink}; \end{cases}$
- for each face $f$: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$

Proof. Lemma 1: $L(f) + S(f) = 2A(f)$
Lemma 2: $L(f) - S(f) = \pm 2$.
$\Rightarrow 2L(f) = 2A(f) \pm 2.$
Assignment of Large Angles to Faces

Let $S$ and $T$ be the sets of sources and sinks, respectively.

**Definition.**
A consistent assignment $\Phi: S \cup T \rightarrow F$ is a mapping where

$$\Phi: v \mapsto \text{incident face, where } v \text{ forms large angle}$$

such that

$$|\Phi^{-1}(f)| = L(f) = \begin{cases} A(f) - 1 & \text{if } f \neq f_0, \\ A(f) + 1 & \text{if } f = f_0. \end{cases}$$
Example of Angle to Face Assignment

- \( A(f) \) # sources / sinks of \( f \)
- \( L(f) \) # large angles of \( f \)

assignment \( \Phi : S \cup T \rightarrow F \)

- \( A(f) = 1 \)
- \( L(f) = 2 \)
- \( A(f) = 3 \)
- \( L(f) = 1 \)
- \( A(f) = 1 \)
- \( L(f) = 0 \)
- \( A(f) = 2 \)
- \( L(f) = 1 \)
- \( A(f) = 2 \)
- \( L(f) = 1 \)
- \( A(f) = 1 \)
- \( L(f) = 0 \)
- \( A(f) = 2 \)
- \( L(f) = 0 \)

- \( A(f) = 3 \)
- \( L(f) = 0 \)
- \( A(f) = 1 \)
- \( L(f) = 4 \)
- \( A(f) = 1 \)
- \( L(f) = 1 \)
- \( A(f) = 1 \)
- \( L(f) = 1 \)
- \( A(f) = 1 \)
- \( L(f) = 1 \)
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Part IV:
Refinement Algorithm

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Theorem 3.
Let $G = (V, E)$ be an acyclic plane digraph with embedding given by $F, f_0$.
Then $G$ is upward planar (respecting $F, f_0$) if and only if $G$ is bimodal and there exists consistent assignment $\Phi$.

Proof.
⇒: As constructed before.
⇐: Idea:
- Construct planar st-digraph that is supergraph of $G$.
- Apply equivalence from Theorem 1.
Refinement Algorithm – $\Phi, F, f_0 \rightarrow \text{st-digraph}$

Let $f$ be a face. Consider the clockwise angle sequence $\sigma_f$ of $L/S$ on local sources and sinks of $f$.

- **Goal:** Add edges to break large angles (sources and sinks).
- **For** $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle L, S, S \rangle$ at vertices $x, y, z$:
  - $x$ source $\Rightarrow$ insert edge $(z, x)$

For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle L, S, S \rangle$ at vertices $x, y, z$:
- $x$ source $\Rightarrow$ insert edge $(z, x)$

![Diagram showing the process of refining a digraph with sources and sinks]
Refinement Algorithm – $\Phi, F, f_0 \rightarrow$ st-digraph

Let $f$ be a face. Consider the clockwise angle sequence $\sigma_f$ of $L/S$ on local sources and sinks of $f$.

- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle L, S, S \rangle$ at vertices $x, y, z$:
  - $x$ source $\Rightarrow$ insert edge $(z, x)$
  - $x$ sink $\Rightarrow$ insert edge $(x, z)$.

Goal: Add edges to break large angles (sources and sinks).

For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle L, S, S \rangle$ at vertices $x, y, z$:

- $x$ source $\Rightarrow$ insert edge $(z, x)$
- $x$ sink $\Rightarrow$ insert edge $(x, z)$.
Refinement Algorithm – \( \Phi, F, f_0 \rightarrow \text{st-digraph} \)

Let \( f \) be a face. Consider the clockwise angle sequence \( \sigma_f \) of \( L/S \) on local sources and sinks of \( f \).

- Goal: Add edges to break large angles (sources and sinks).
- For \( f \neq f_0 \) with \( |\sigma_f| \geq 2 \) containing \( \langle L, S, S \rangle \) at vertices \( x, y, z \):
  - \( x \text{ source} \Rightarrow \text{insert edge } (z, x) \)
  - \( x \text{ sink} \Rightarrow \text{insert edge } (x, z) \).
- Refine outer face \( f_0 \).

- Refine all faces. \( \Rightarrow G \) is contained in a planar st-digraph.
- Planarity, acyclicity, bimodality are invariants under construction.
Refinement Example
Refinement Example
Refinement Example
Result Upward Planarity Test

Theorem 2. [Bertolazzi et al., 1994]
For a combinatorially embedded planar digraph $G$ it can be tested in $O(n^2)$ time whether it is upward planar.

Proof.
- Test for bimodality.
- Test for a consistent assignment $\Phi$ (via flow network).
- If $G$ bimodal and $\Phi$ exists, refine $G$ to plane st-digraph $H$.
- Draw $H$ upward planar.
- Deleted edges added in refinement step.
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Part V:
Finding a Consistent Assignment

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Finding a Consistent Assignment

Idea.
Flow \((v, f) = 1\) from global source / sink \(v\) to the incident face \(f\) its large angle gets assigned to.

Flow network.
\(N_{F, f_0}(G) = ((W, E'); b; \ell; u)\)
- \(W = \{v \in V \mid v \text{ source or sink}\} \cup F\)
- \(E' = \{(v, f) \mid v \text{ incident to } f\}\)
- \(\ell(e) = 0 \forall e \in E'\)
- \(u(e) = 1 \forall e \in E'\)
- \(b(w) = \begin{cases} 1 & \forall w \in W \cap V \\ -(A(w) - 1) & \forall w \in F \setminus \{f_0\} \\ -(A(w) + 1) & w = f_0 \end{cases} \)

Example.

![Flow network diagram](image)
Discussion

- There exist fixed-parameter tractable algorithms to test upward planarity of general digraphs with the parameter being the number of triconnected components. [Healy, Lynch 2005, Didimo et al. 2009]

- Finding assignment in Theorem 2 can be sped up to $O(n + r^{1.5})$ where $r = \# \text{ sources} / \text{ sinks}$. [Abbasi, Healy, Rextin 2010]

- Many related concepts have been studied: quasi-planarity, upward drawings of mixed graphs, upward planarity on cylinder/torus, ...
Literature

- [GD Ch. 6] for detailed explanation

Organized papers referenced:

- [Kelly ’87] Fundamentals of Planar Ordered Sets
- [Di Battista, Tamassia ’88] Algorithms for Plane Representations of Acyclic Digraphs
- [Hutton, Lubiw ’96] Upward Planar Drawing of Single-Source Acyclic Digraphs
- [Bertolazzi, Di Battista, Mannino, Tamassia ’94] Upward Drawings of Triconnected Digraphs
- [Healy, Lynch ’05] Building Blocks of Upward Planar Digraphs
- [Didimo, Giardano, Liotta ’09] Upward Spirality and Upward Planarity Testing
- [Abbasi, Healy, Rextin ’10] Improving the running time of embedded upward planarity testing