Visualisation of graphs

Upward planar drawings

Flow methods

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Upward planar drawings – motivation

- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchie
  - . . .

- Would be nice to have general direction preserved in drawing.

**PERT diagram**

**Petri net**

**Phylogenetic network**
Upward planar drawings – definition

**Definition.**
A directed graph \( G = (V, E) \) is **upward planar** when it admits a drawing \( \Gamma \) (vertices = points, edges = simple curves) that is
- planar and
- where each edge is drawn as an upward, y-monotone curve.
Upward planarity – necessary conditions

- For a digraph $G$ to be upward planar, it has to be:
  - planar
  - acyclic
  - bimodal

- ... but these conditions are not sufficient.
Upward planarity – characterisation

**Theorem 1.** [Kelly 1987, Di Battista, Tamassia, 1988, see GD Ch. 6]

For a digraph $G$ the following statements are equivalent:

1. $G$ is upward planar.
2. $G$ admits an upward planar straight-line drawing.
3. $G$ is the spanning subgraph of a planar $st$-digraph.

Additionally:
- Embedded such that $s$ and $t$ are on the outerface $f_0$.
- or:
- Edge $(s, t)$ exists.

- no crossings
- acyclic digraph with a single source $s$ and single sink $t$
Theorem 1. [Kelly 1987, Di Battista, Tamassia, 1988, see GD Ch. 6]
For a digraph $G$ the following statements are equivalent:
1. $G$ is upward planar.
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3. $G$ is the spanning subgraph of a planar $st$-digraph.

Proof.

$(2) \Rightarrow (1)$ By definition. $(1) \Leftrightarrow (3)$ Example:

$(3) \Rightarrow (2)$ Triangulate & construct drawing:

Case 2:
Upward planarity – complexity

**Theorem.** [Garg, Tamassia, 1995]
For a *planar acyclic* digraph it is in general NP-hard to decide whether it is upward planar.

**Theorem 2.** [Bertolazzi et al., 1994]
For a *combinatorially embedded* planar digraph it can be tested in $O(n^2)$ time whether it is upward planar.

**Corollary.**
For a *triconnected* planar digraph it can be tested in $O(n^2)$ time whether it is upward planar.

**Theorem.** [Hutton, Libow, 1996]
For a *single-source* acyclic digraph it can be tested in $O(n)$ time whether it is upward planar.
The problem

Fixed embedding upward planarity testing.
Let $G = (V, E)$ be a plane digraph with the embedding given by the set of faces $F$ and the outer face $f_0$. Test whether $G$ is upward planar (wrt to $F$, $f_0$).

Idea.

- Find property that any upward planar drawing of $G$ satisfies.
- Formalise property.
- Find algorithm to test property.
Angles, local sources & sinks

Definitions.
■ A vertex $v$ is a **local source** wrt to a face $f$ if $v$ has two outgoing edges on $\partial f$.
■ A vertex $v$ is a **local sink** wrt to a face $f$ if $v$ has two incoming edges on $\partial f$.
■ An angle $\alpha$ is **large** when $\alpha > \pi$ and **small** otherwise.
■ $L(v) = \#$ large angles at $v$
■ $L(f) = \#$ large angles in $f$
■ $S(v) \& S(f)$ for $\#$ small angles
■ $A(f) = \#$ local sources wrt to $f$
  $= \#$ local sinks wrt to $f$

**Lemma 1.**
$L(f) + S(f) = 2A(f)$
Assignment problem

- Vertex $v$ is a global source for $f_1$ and $f_2$.
- Has $v$ a large angle in $f_1$ or $f_2$?
Angle relations

\[ L(f) - S(f) = \begin{cases} 
-2, & f \neq f_0 \\
+2, & f = f_0 
\end{cases} \]

\[ L(f) \geq 1 \]

Split \( f \) with edge from a large angle at a “low” sink \( u \) to

- sink \( v \) with small/large angle:

\[
L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1)
= -2
\]

Proof by induction.

- \( L(f) = 0 \)
  \[ \Rightarrow S(f) = 2 \]
Angle relations

Lemma 2.

\[ L(f) - S(f) = \begin{cases} 
-2, & f \neq f_0 \\
+2, & f = f_0 
\end{cases} \]

Proof by induction.

- \( L(f) = 0 \) \Rightarrow S(f) = 2

\[ L(f) \geq 1 \]

Split \( f \) with edge from a large angle at a “low” sink \( u \) to

- source \( v \) with small/large angle:

\[ L(f) - S(f) = L(f_1) + L(f_2) + 2 - (S(f_1) + S(f_2)) = -2 \]
Angle relations

**Lemma 2.**

\[ L(f) - S(f) = \begin{cases} 
-2, & f \neq f_0 \\
+2, & f = f_0 
\end{cases} \]

**Proof** by induction.

- \( L(f) = 0 \) \( \implies S(f) = 2 \)

\( L(f) \geq 1 \)

Split \( f \) with **edge** from a large angle at a “low” sink \( u \) to

- vertex \( v \) that is neither source nor sink:

Graph showing a graph with a large angle at sink \( u \), an edge pointing from \( u \) to vertex \( v \), and two regions \( f_1 \) and \( f_2 \).

\[
L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1) = -2
\]

- Otherwise “high” source \( u \) exists.

Graph showing a graph with a large angle at source \( u \), an edge pointing from source \( u \) to vertex \( v \), and two regions \( f_1 \) and \( f_2 \).
Proof.
Observation and from Lemma 1: \( L(f) + S(f) = 2A(f) \)
and from Lemma 2: \( L(f) - S(f) = \pm 2. \)

Lemma 3.
In every upward planar drawing of \( G \) holds that
\( \text{for each vertex } v \in V: L(v) = \begin{cases} 
0 & v \text{ inner vertex,} \\
1 & v \text{ source/sink;}
\end{cases} \)
\( \text{for each face } f: L(f) = \begin{cases} 
A(f) - 1 & f \neq f_0, \\
A(f) + 1 & f = f_0.
\end{cases} \)
Assignment of large angles to faces

Let $S$ and $T$ be the sets of sources and sinks, respectively.

**Definition.**

A **consistent assignment** $\Phi: S \cup T \to F$ is a mapping where

$\Phi: v \mapsto$ incident face, where $v$ forms large angle

such that

$$|\Phi^{-1}(f)| = L(f) = \begin{cases} 
A(f) - 1 & \text{if } f \neq f_0, \\
A(f) + 1 & \text{if } f = f_0.
\end{cases}$$
Example of angle to face assignment

- Global sources & sinks

\[ A(f) \] # sources/sinks of \( f \)

Assignment

\[ \Phi : S \cup T \rightarrow F \]
**Theorem 3.**
Let $G = (V, E)$ be an acyclic plane digraph with embedding given by $F, f_0$. Then $G$ is upward planar (respecting $F, f_0$) if and only if $G$ is bimodal and there exists consistent assignment $\Phi$.

**Proof.**
$\Rightarrow$: As constructed before.
$\Leftarrow$: Idea:
- Construct planar st-digraph that is supergraph of $G$.
- Apply equivalence from Theorem 1.
Let $f$ be a face. Consider the clockwise angle sequence $\sigma_f$ of L/S on local sources and sinks of $f$.

- **Goal:** Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle L, S, S \rangle$ at vertices $x, y, z$:
  - $x$ source $\Rightarrow$ insert edge $(z, x)$
Refinement algorithm – $\Phi, F, f_0 \rightarrow \text{st-digraph}$

Let $f$ be a face. Consider the clockwise angle sequence $\sigma_f$ of L/S on local sources and sinks of $f$.

- **Goal:** Add edges to break large angles (sources and sinks).

- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle L, S, S \rangle$ at vertices $x, y, z$:
  - $x$ source $\Rightarrow$ insert edge $(z, x)$
  - $x$ sink $\Rightarrow$ insert edge $(x, z)$.

- Refine outer face $f_0$.

- Refine all faces. $\Rightarrow$ $G$ is contained in a planar st-digraph.

- Planarity, acyclicity, bimodality are invariants under construction.
Refinement example
Refinement example
Refinement example
Result upward planarity test

Theorem 2. [Bertolazzi et al., 1994]
For a combinatorially embedded planar digraph $G$ it can be tested in $O(n^2)$ time whether it is upward planar.

Proof.
- Test for bimodality.
- Test for a consistent assignment $\Phi$ (via flow network).
- If $G$ bimodal and $\Phi$ exists, refine $G$ to plane st-digraph $H$.
- Draw $H$ upward planar.
- Deleted edges added in refinement step.
Finding a consistent assignment

Idea.
Flow \((v, f) = 1\) from global source/sink \(v\) to the incident face \(f\) its large angle gets assigned to.

Flow network.
\[ N_{F,f_0}(G) = ((W, E'); \ell; u; d) \]
- \(W = \{v \in V \mid v \text{ source or sink}\} \cup F\)
- \(E' = \{(v, f) \mid v \text{ incident to } f\}\)
- \(\ell(e) = 0 \quad \forall e \in E'\)
- \(u(e) = 1 \quad \forall e \in E'\)
- \(d(p) = \begin{cases} 1 & \forall p \in W \cap V \\ -(A(p) - 1) & \forall p \in F \setminus \{f_0\} \\ -(A(p) + 1) & p = f_0 \end{cases} \)
Discussion

- There exist fixed-parameter tractable algorithms to test upward planarity of general digraphs with the parameter being the number of triconnected components. [Healy, Lynch 2005, Didimo et al. 2009]

- Finding assignment in Theorem 2 can be sped up to $O(n + r^{1.5})$ where $r = \#$ sources/sinks. [Abbasi, Healy, Rextin 2010]

- Many related concepts have been studied: quasi-planarity, upward drawings of mixed graphs, upward planarity on cyclinder/torus, ...
Literature

- [GD Ch. 6] for detailed explanation

Original papers referenced:

- [Kelly '87] Fundamentals of Planar Ordered Sets
- [Di Battista, Tamassia '88] Algorithms for Plane Representations of Acyclic Digraphs
- [Garg, Tamassia '95] On the Computational Complexity of Upward and Rectilinear Planarity Testing
- [Hutton, Lubiw '96] Upward Planar Drawing of Single-Source Acyclic Digraphs
- [Bertolazzi, Di Battista, Mannino, Tamassia '94] Upward Drawings of Triconnected Digraphs
- [Healy, Lynch '05] Building Blocks of Upward Planar Digraphs
- [Didimo, Giardano, Liotta '09] Upward Spirality and Upward Planarity Testing
- [Abbasi, Healy, Rextin '10] Improving the running time of embedded upward planarity testing