

Advanced Algorithms

Winter term 2019/20

Lecture 13. Linear Time Planarity Testing via PQ-trees

(based on slides of Ignaz Rutter)

An Efficient Certifying Planarity Test

Thm

Given an n -vertex m -edge graph $G = (V, E)$, testing whether G is planar can be done in $O(n + m)$ time.

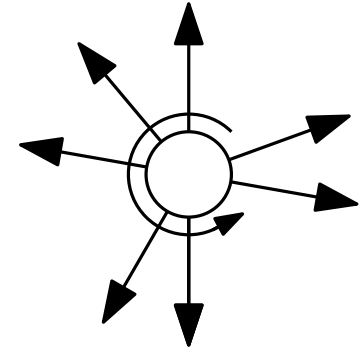
When G is planar, a planar embedding can be produced in the same time. Otherwise, a K_5 or $K_{3,3}$ minor can be found in the same time.

we will skip the details of the certifying part

Edge Ordering and Embeddings

Embeddings are encoded as an edge ordering for each vertex.

~> **Rotation-System**



Does a given rotation system describe a planar embedding?

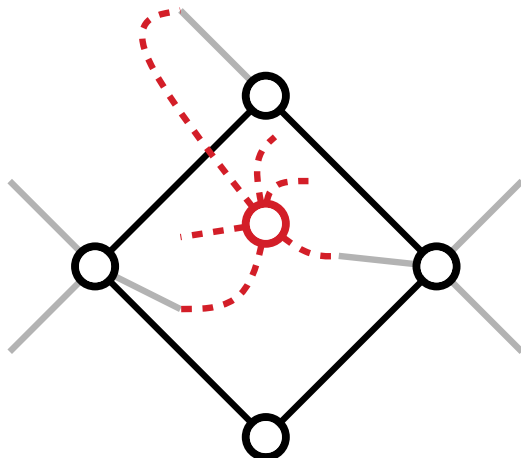
How can you test this?
(hint: Euler's formula)

Planarity Testing, 1st idea

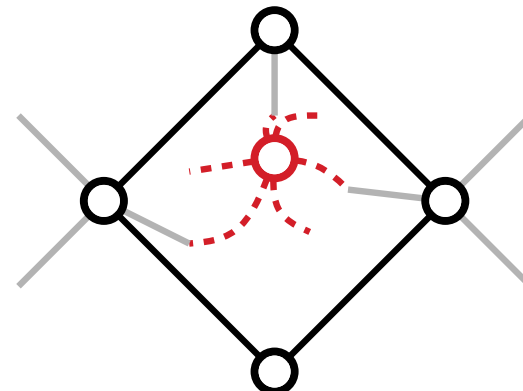
Idea: Iteratively add nodes \rightsquigarrow three types of edges:

- embedded 
- half-embedded 
- absent 

Objective: Save all possible partial embeddings (i.e., positions of the embedded and half-embedded edges).



No planar full embedding possible



Full embedding possible.

\rightsquigarrow Exponential runtime & space



Planarity Testing, refined

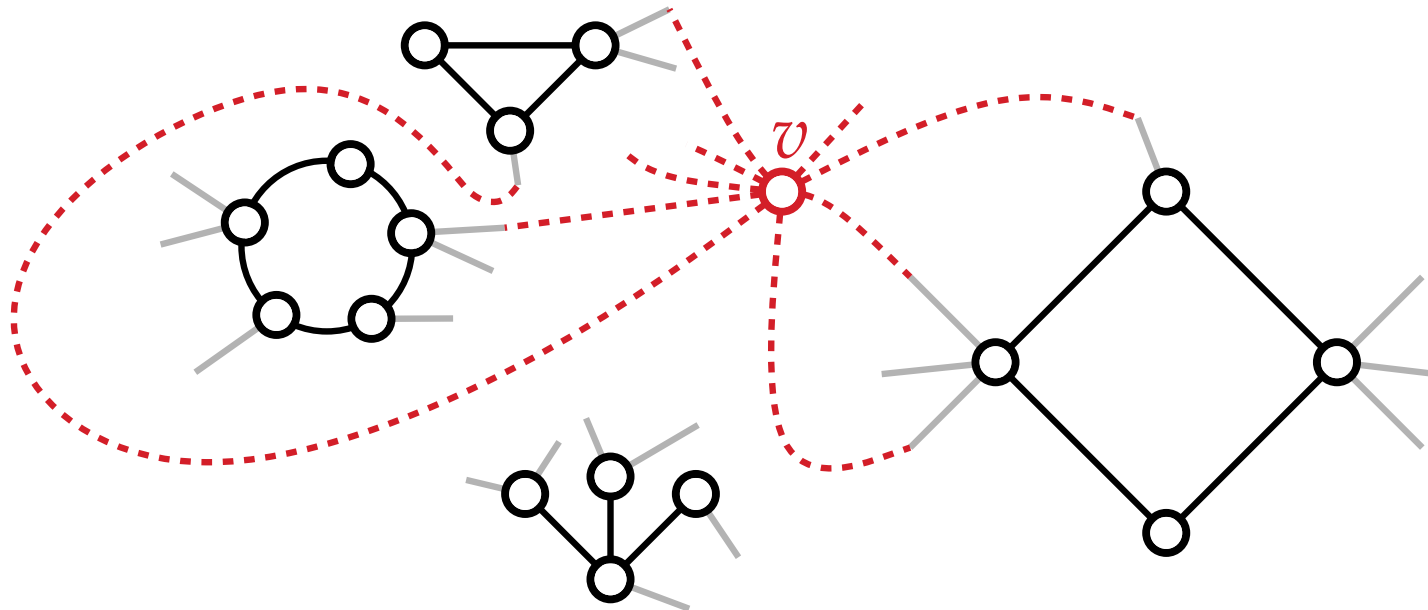
Idea to reduce options for insertions:

Force insertions always on the outerface.

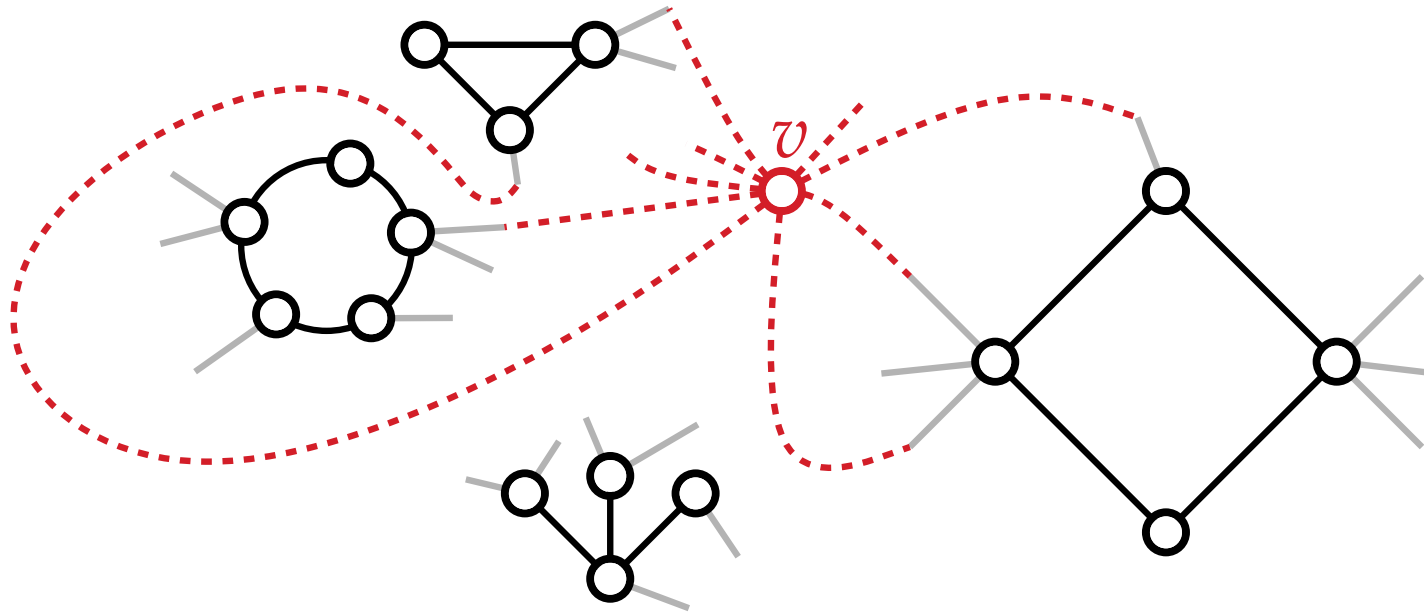
Is this possible?

What do we get from it?

Process vertices bottom-up via a DFS spanning tree.
All halfedges must be embedded on the same (outer) face.



Planarity Testing, Vertex Insertion



1. **Restriction:** Half-embedded edges incident to v that belong to the same component must be consecutive.
2. **Combination:** Components and half-embedded edges hanging from v can be ordered arbitrarily.

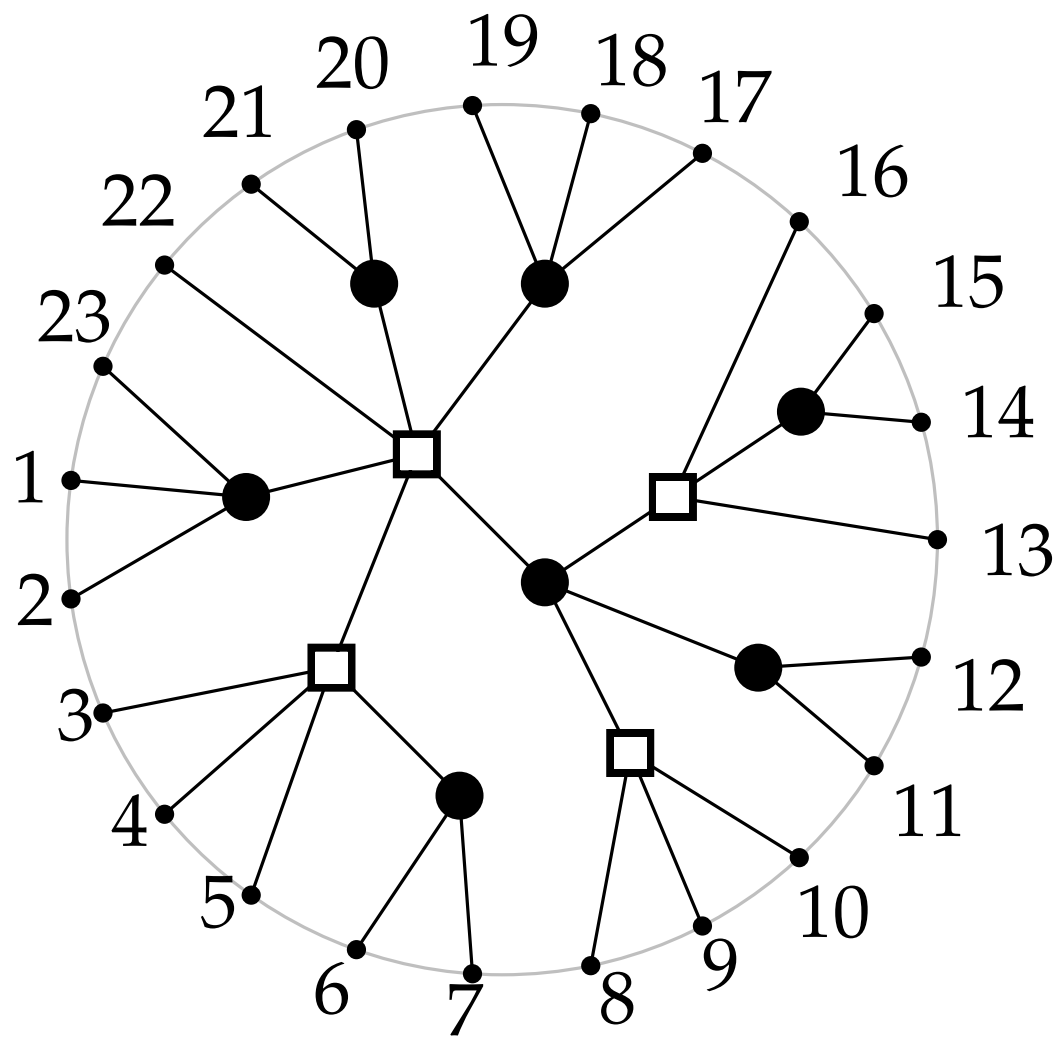
Still too many options to store all of them. 😞

Solution: data structure to compactly represent such constrained orders.

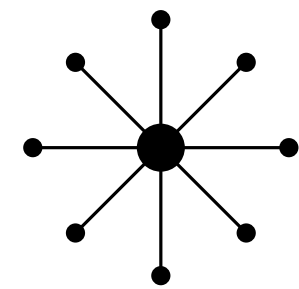


PQ-Tree

PQ-Tree



- P-node
permute children freely
- Q-node
only "flipping" allowed

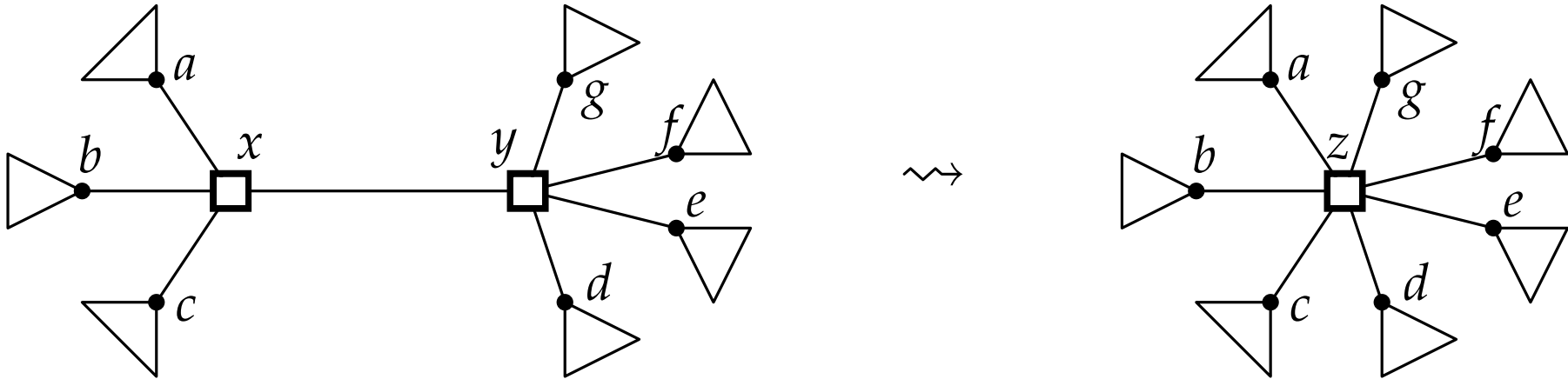


Single P-node \rightsquigarrow all possible circular orderings of its leaves.

PQ-tree represents all circular orderings of its leaves by these operations

Order-Preserving Contraction, Null-Tree

Order-preserving contraction

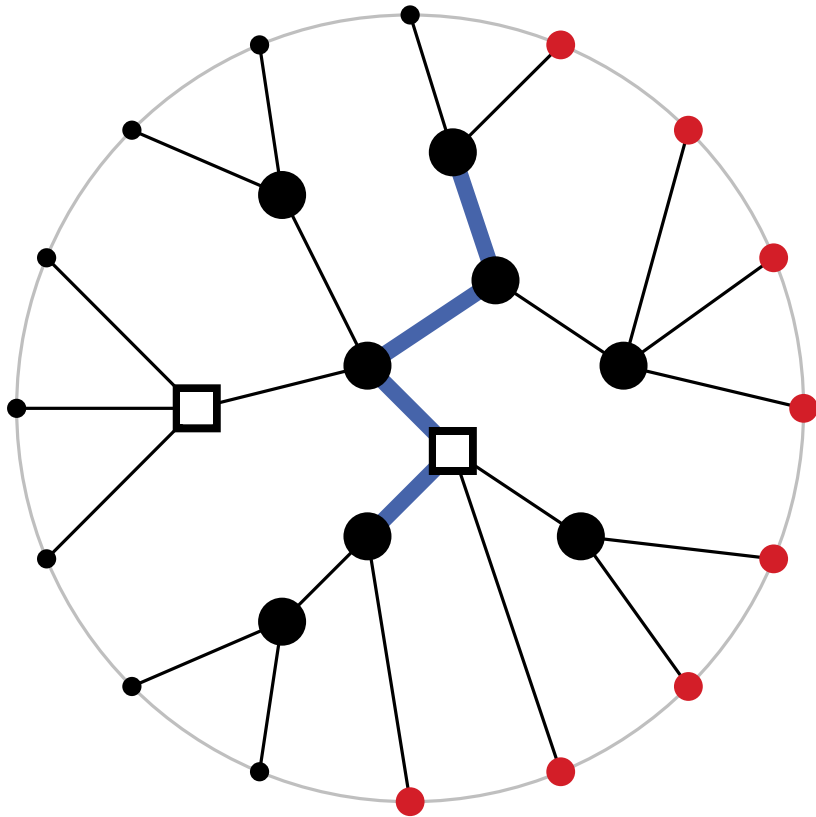


NOTE this restricts the representable orderings!

Null-tree: represents the empty set of permutations

NOTE Null-tree \neq empty tree (represents permutations of the empty set)

Consecutivity of Subsets



Find new PQ-tree, representing exactly the orderings which :

- are admitted by the current tree and
- have the **red leaves** consecutive

No rearrangement with the red leaves consecutive

\Rightarrow

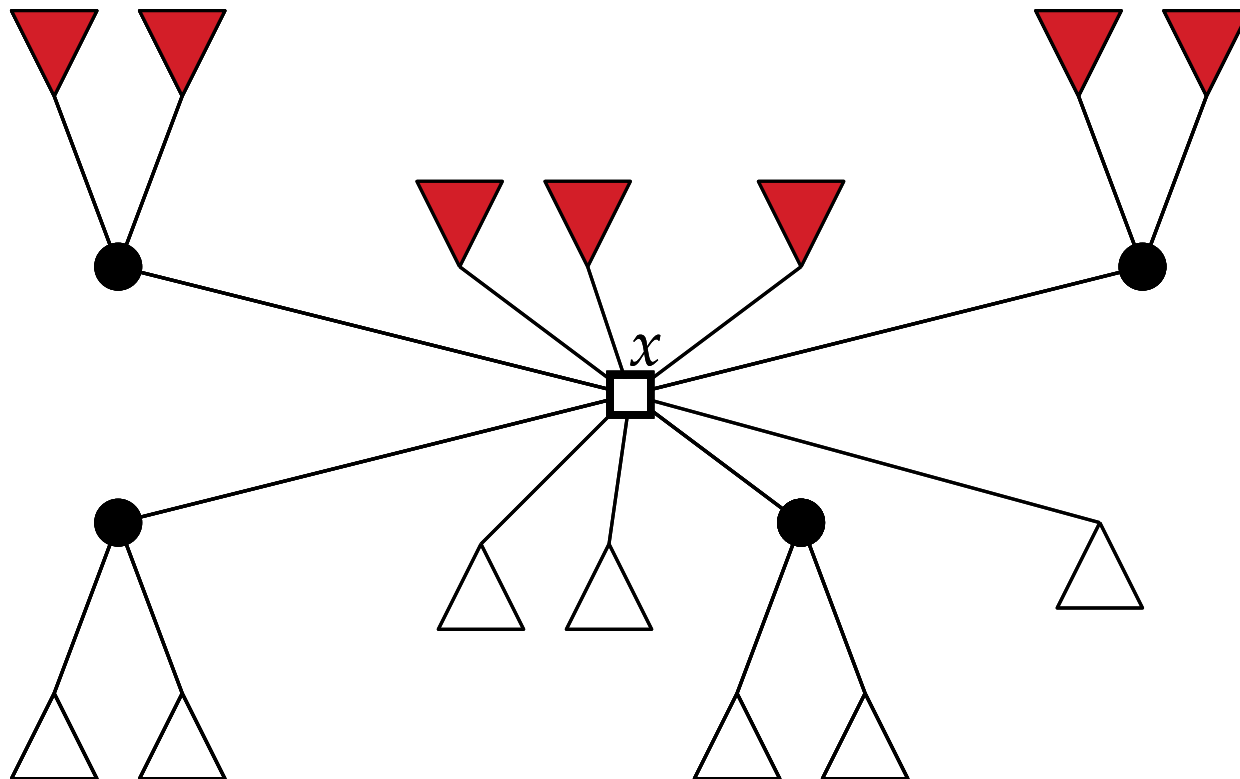
The result is a Null-tree

An edge is **partial** when both subtrees have both red and black leaves \rightsquigarrow partial edges allow forbidden orderings

Lemma: If an arrangement of the PQ-tree has the red leaves consecutive, the partial edges form a path.

Updating PQ-Trees

1. Find the path of partial edges and arrange the tree to split red and black leaves.
2. Split partial path (*terminal path*), insert a new Q-node x
3. Contract edges from x to other Q-nodes, dissolve inner nodes of degree 2.



Correctness

Thm: The update algorithm is correct.

Proof:

- For any ordering represented before with the red leaves consecutive, the operation is the same and succeeds.
- Each ordering of the resulting PQ-tree is represented by the original. ■

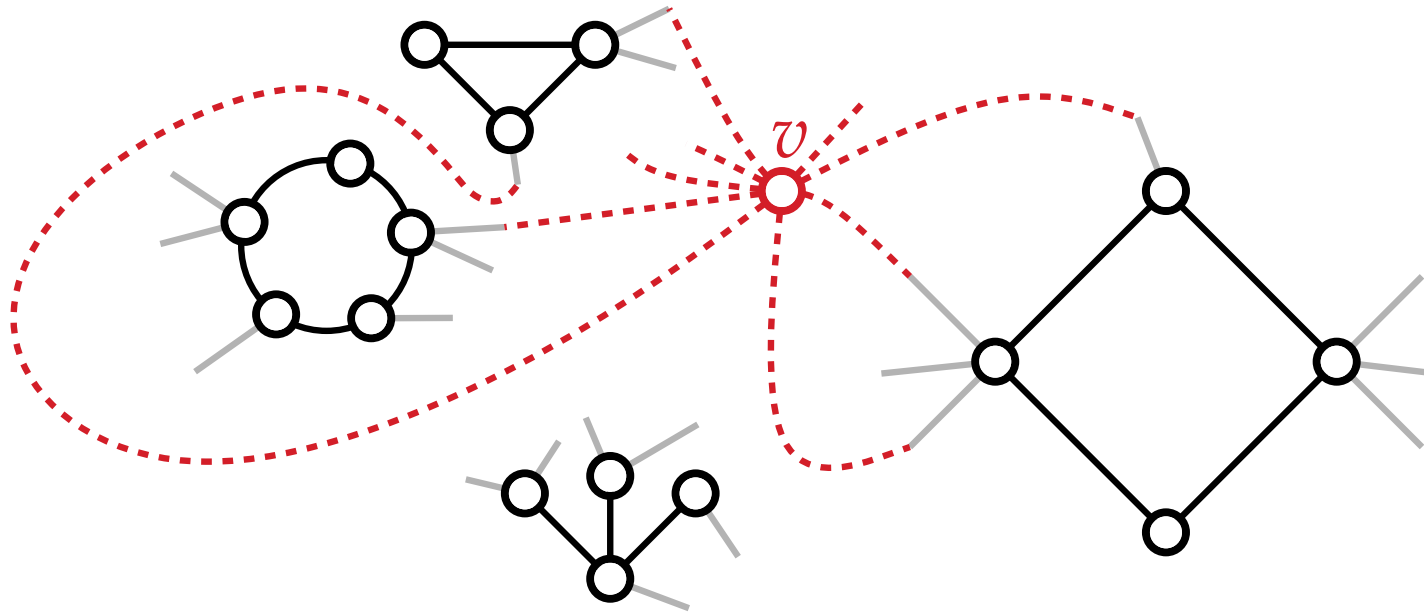
Obs.: Update can be implemented in polynomial time.

What runtime can we easily guarantee?

A linear time implementation needs more ideas ...

Back to Planarity Testing

Planarity Testing, vertex insertion



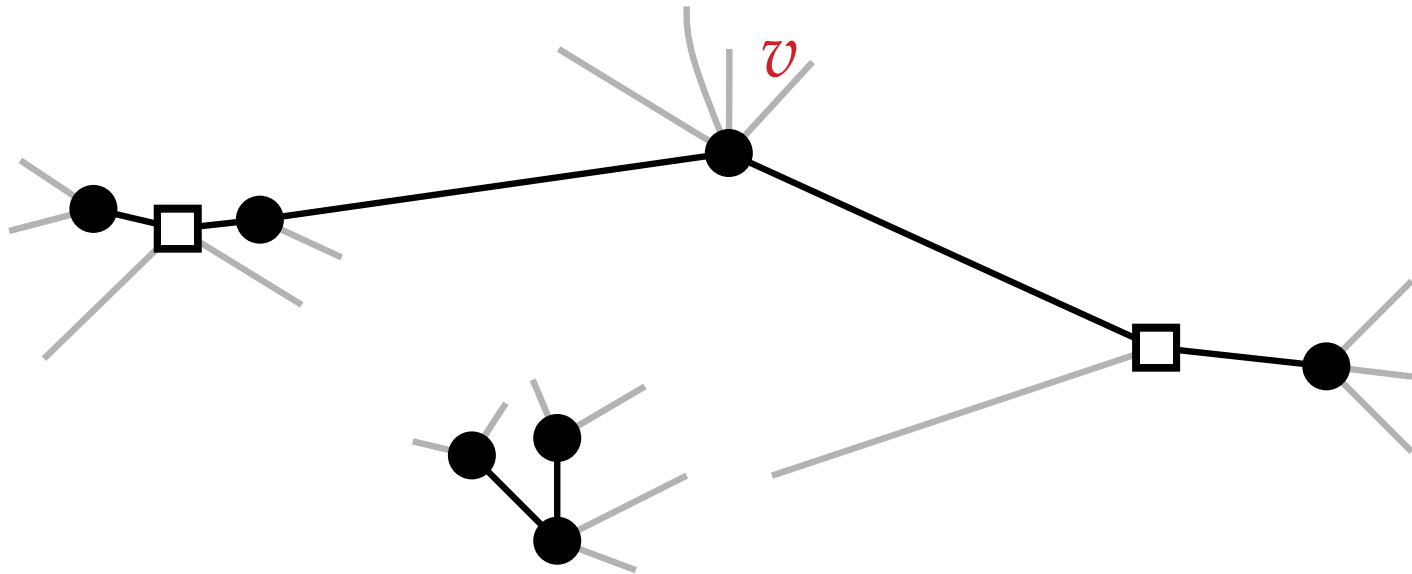
1. **Restriction:** Half-embedded edges incident to v that belong to the same component must be consecutive.
2. **Combination:** Components and half-embedded edges hanging from v can be ordered arbitrarily.

Explicitly representing all options is too expensive.

↪ use PQ-trees!



Planarity Testing, vertex ins. (PQ-trees)



One PQ-tree for each component

- **Restriction:** for each component C , make Cv edges consecutive in C 's PQ-tree
- **Combination:** merge PQ-trees of v 's components into one, and hang new half-embedded edges.
 \rightsquigarrow single PQ-tree for the resulting component

Cost per vertex: one consecutivity constraint $O(\deg v)$

Planarity Testing

Graph is planar if and only if all reduction steps succeed.

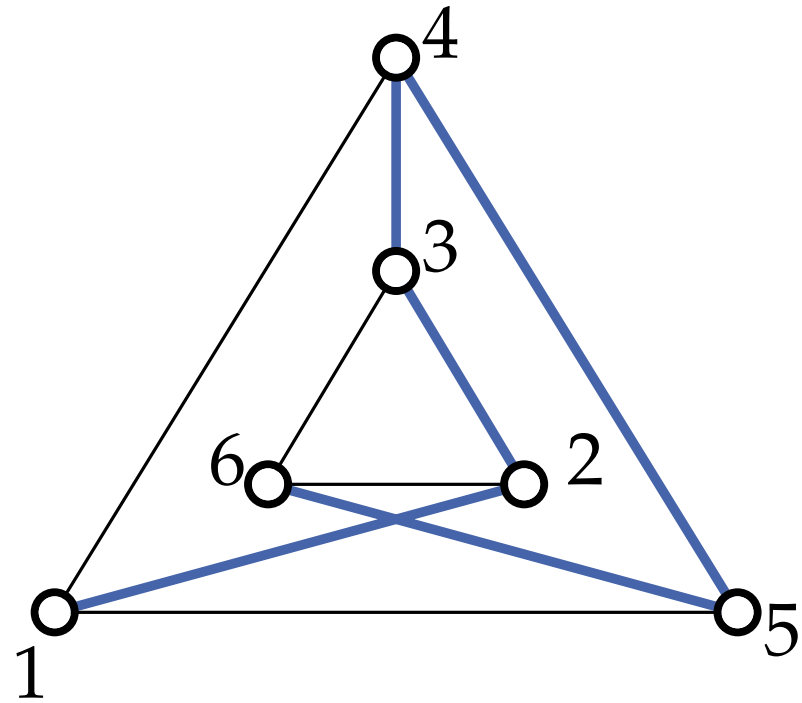
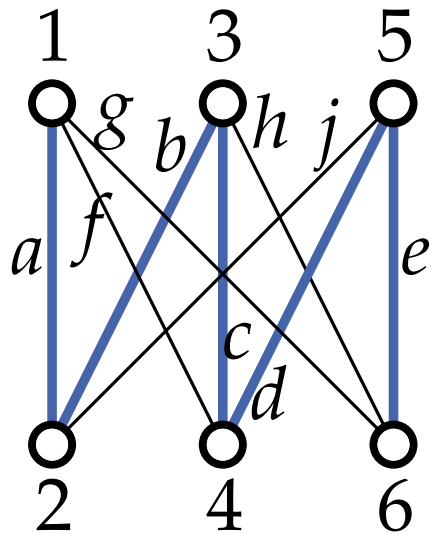
Embedding can be recovered by undoing steps.
Select/expand orders within the PQ-tree.

Warning on linear runtime: finding a terminal path efficiently is tricky, involves rooting each PQ-tree.
When merging, need to root consistently.

What about the sequence to process vertices?

- (s, t) -ordering (two-connected graphs)
(see *graph visualization* lecture) [Lempel, Even, Cederbaum '67]
- Depth-first search [Shih, Hsu '99, Boyer, Myrvold '04]

Examples

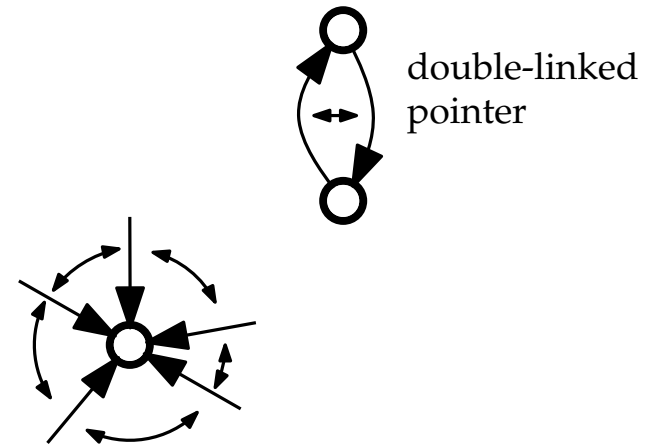


Linear-time PQ-Tree construction

Linear-time PQ-tree construction

Linear time implementation details

- choose root
- store each edge in both directions
- store incoming edges at each node in double-linked list



- mark each edge regarding orientation to root
- P-nodes have pointer to parent

***NOTE*:** Parent of a Q-node is “expensive” to determine, but this means we do not need to keep track of it when editing Q-nodes.

Computing the Terminal Path

Lemma: Terminal path can be found in $O(\overset{\text{length of terminal path}}{p} + \overset{\text{number of consecutive elements}}{k})$ time.

Classify the nodes of the PQ-tree:

- a leaf is **full** when it is a **red** element
- an inner node is **full** when all but one neighbor is full.
- a non-full node is **partial** at least one neighbor is full.

$\rightsquigarrow O(k)$ time until all nodes are labelled.

Key idea: Partial nodes must belong to the terminal path

- extend potential path from each partial node to parent
- stop extending when another path is hit, and join
- leads to a tree with at most one degree 3 node (o.w. reject)
- highest node found which is either partial or meeting of two extensions, is the high point of the terminal path.

How do we find the parent nodes quickly?

where is the parent-edge with respect to full child-edges?

Update Step

Split terminal path, new nodes will receive full neighbors.

Single Split, for each node of the terminal path:

- detach full neighbors F , and delete incident edges to neighbors on the path $O(1)$
- make a copy, hang F from it. $O(\#full\ neighbors)$

$$O(p + k)$$

Create central Q-node, $O(p)$ time

Each contraction, $O(1)$ time

NOTE if terminal path contains $q \geq 2$ Q-nodes, then number of Q-nodes decreases by $q - 1$, i.e., cost of processing this terminal path saves $q - 1$ for us for later.

Runtime analysis

X ground set

$\mathcal{U} = \{U_1, \dots, U_\ell\}$ collection of subsets of X .

Thm: PQ-tree representing all orderings where U_1, \dots, U_ℓ are consecutive can be computed in $O(|X| + |U_1| + \dots + |U_\ell|)$ time (amortized analysis).

Proof: Consider potential function

$\phi(\mathcal{U}, i) = 2u_i + |Q_i| + \sum_{x \in P_i} (\deg(x) - 1)$, where:

- $u_i = \sum_{j>i} |U_j|$
- $Q_i =$ Q-nodes in PQ-tree T_i after processing U_1, \dots, U_i .
- $P_i =$ P-nodes in T_i

$\phi(\mathcal{U}, 0) = \Theta(|X| + \sum_i |U_i|)$ (budget to be used)

Inductively show budget $\phi(\mathcal{U}, i - 1) - \text{cost}(U_i) \geq \phi(\mathcal{U}, i)$

need: ϕ stay to stay ≥ 0

\rightsquigarrow claimed runtime.