Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG
Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems
Institute for Informatics

Advanced Algorithms

Winter term 2019/20

Lecture 13. Linear Time Planarity Testing via PQ-trees

(based on slides of Ignaz Rutter)

Steven Chaplick

Chair for Computer Science I

An Efficient Certifying Planarity Test

Thm
Given an n-vertex m-edge graph $G = (V, E)$, testing whether G is planar can be done in $O(n + m)$ time.

When G is planar, a planar embedding can be produced in the same time. Otherwise, a K_5 or $K_{3,3}$ minor can be found in the same time.

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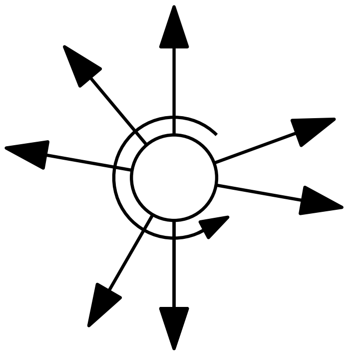
we will skip the details of the certifying part

Edge Ordering and Embeddings

Embeddings are encoded as an edge ordering for each vertex.



$\rightsquigarrow$ Rotation-System

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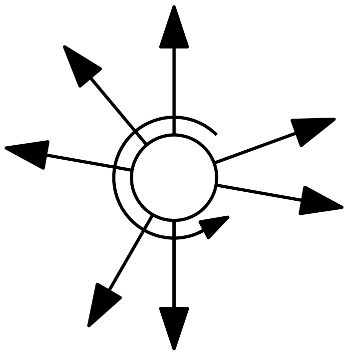

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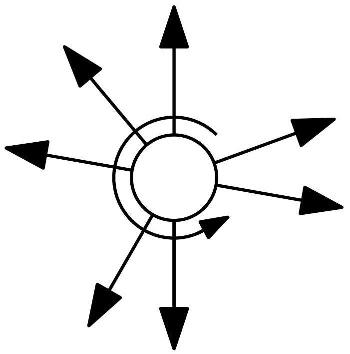

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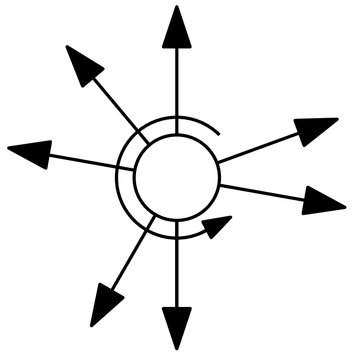

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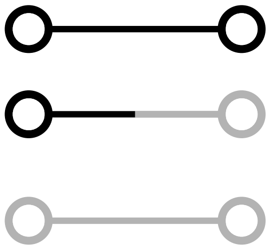
How can you test this?
(hint: Euler's formula)

Planarity Testing, 1st idea

Idea: Iteratively add nodes $\rightsquigarrow$ three types of edges:



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- absent

Objective: Save all possible partial embeddings (i.e., positions of the embedded and half-embedded edges).

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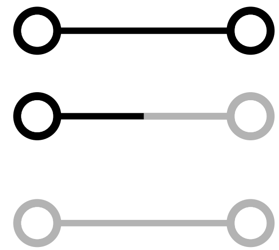

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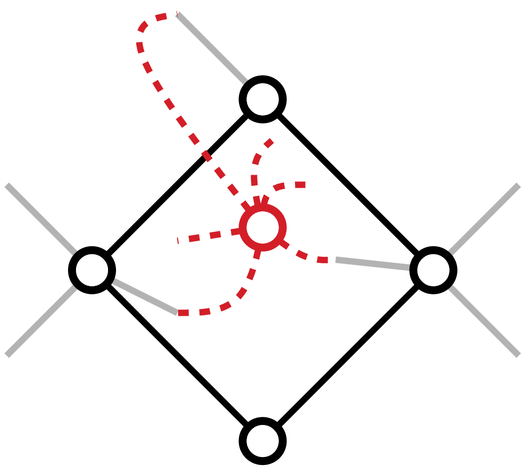
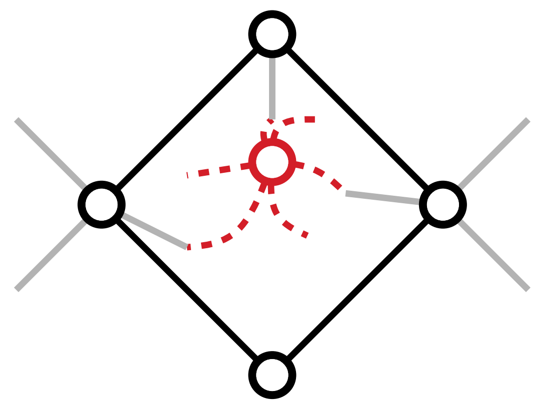


No planar full embedding possible



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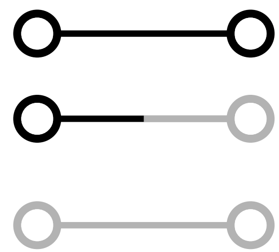

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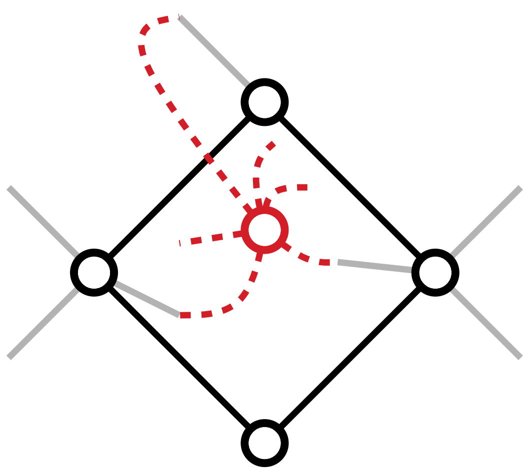

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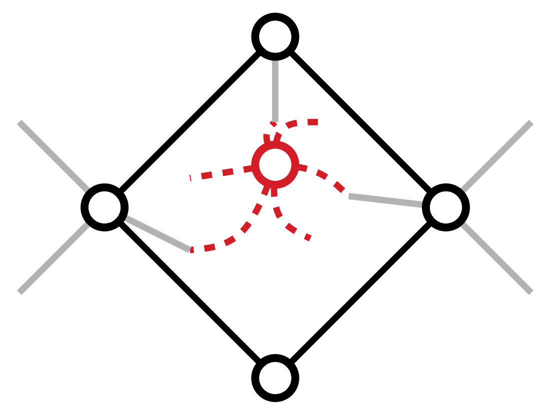


Full embedding possible.

$\rightsquigarrow$ Exponential runtime & space 

Planarity Testing, refined

Idea to reduce options for insertions:

Force insertions always on the outerface.

Is this possible?

What do we get from it?

Planarity Testing, refined

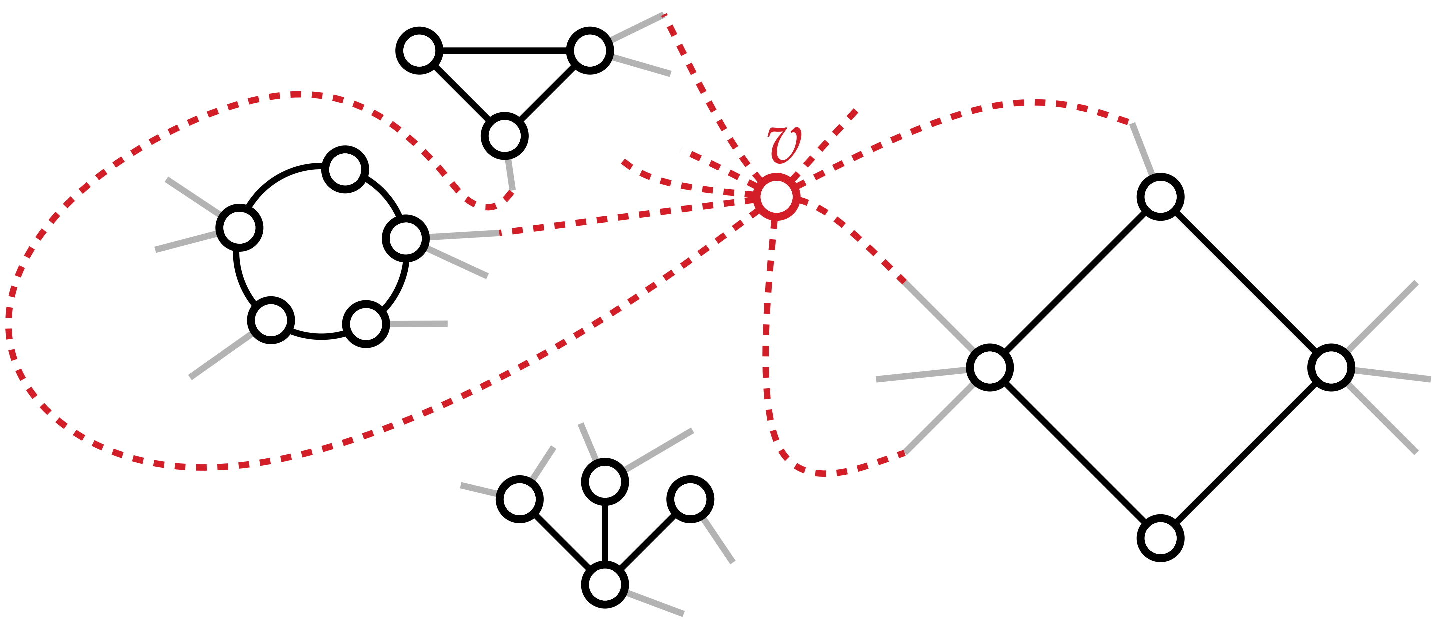
Idea to reduce options for insertions:

Force insertions always on the outerface.

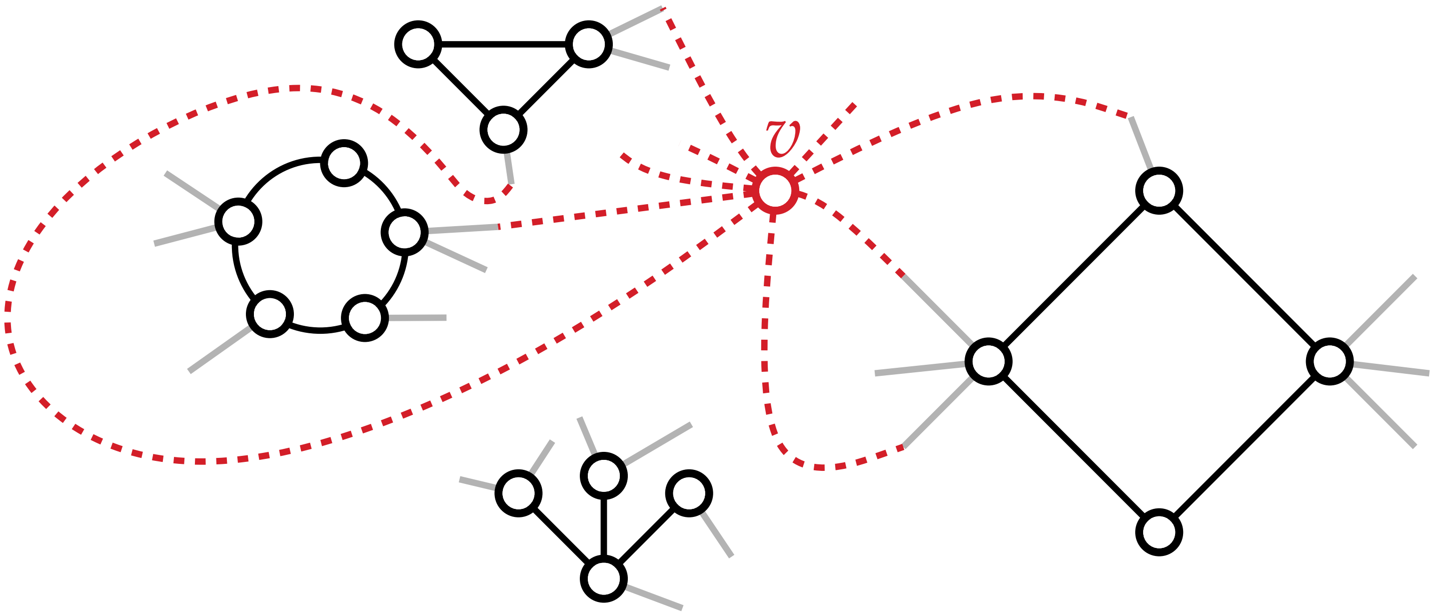
Is this possible?

What do we get from it?

Process vertices bottom-up via a DFS spanning tree.
All halfedges must be embedded on the same (outer) face.



Planarity Testing, Vertex Insertion



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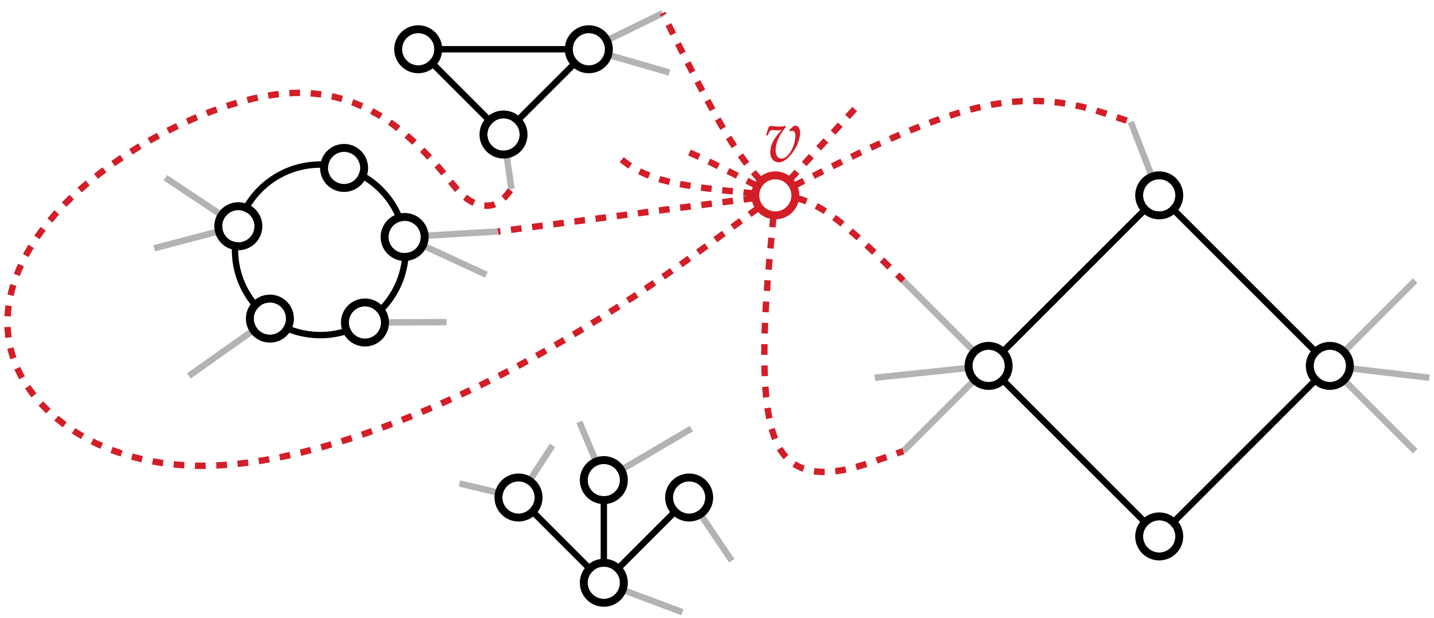

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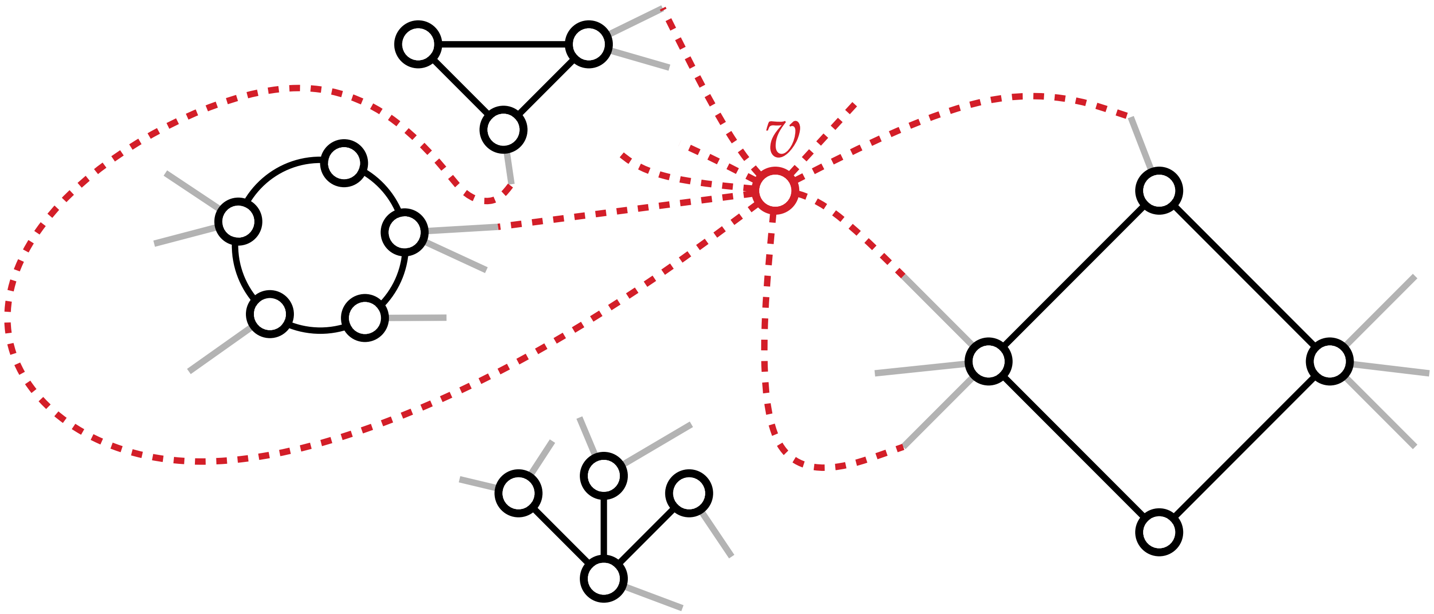

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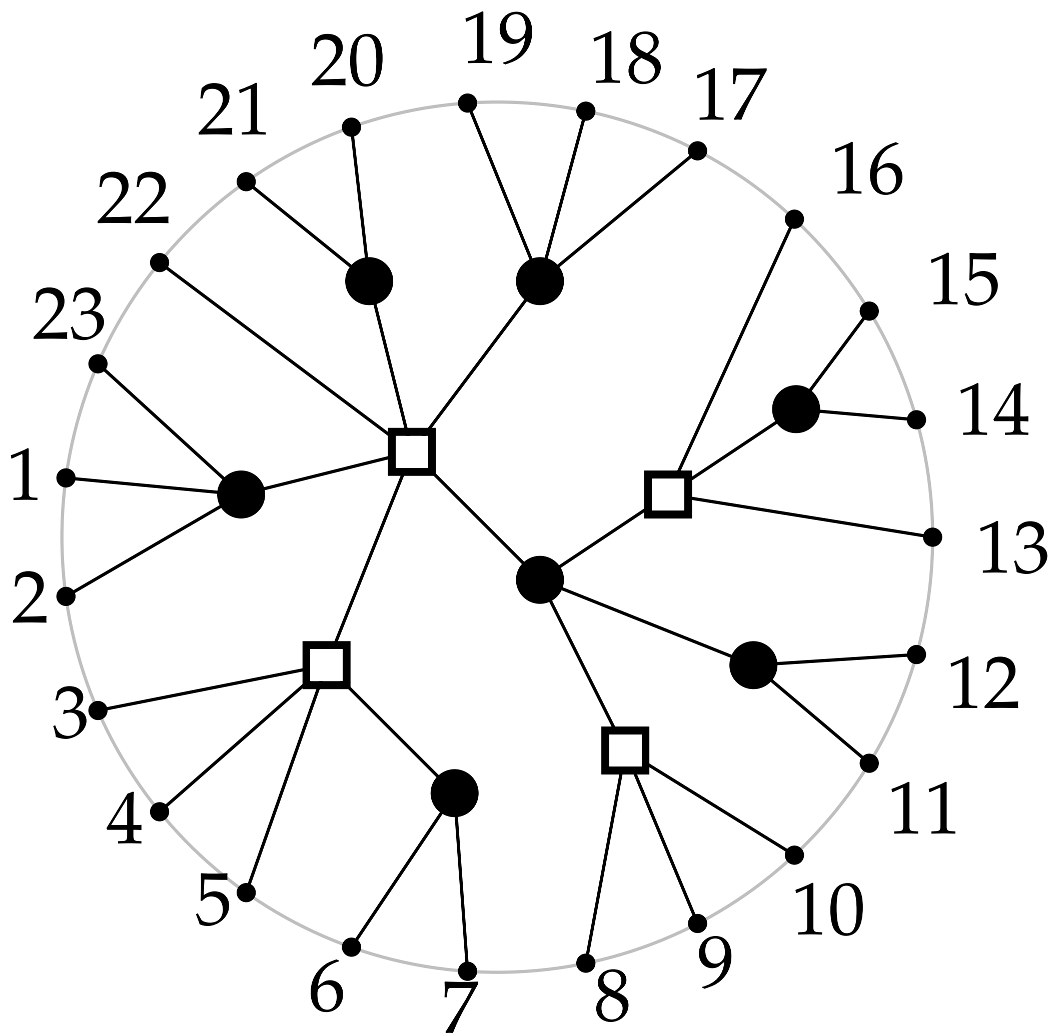
Solution: data structure to compactly represent such constrained orders.

PQ-Tree

PQ-Tree



PQ-tree represents all circular orderings of its leaves by these operations

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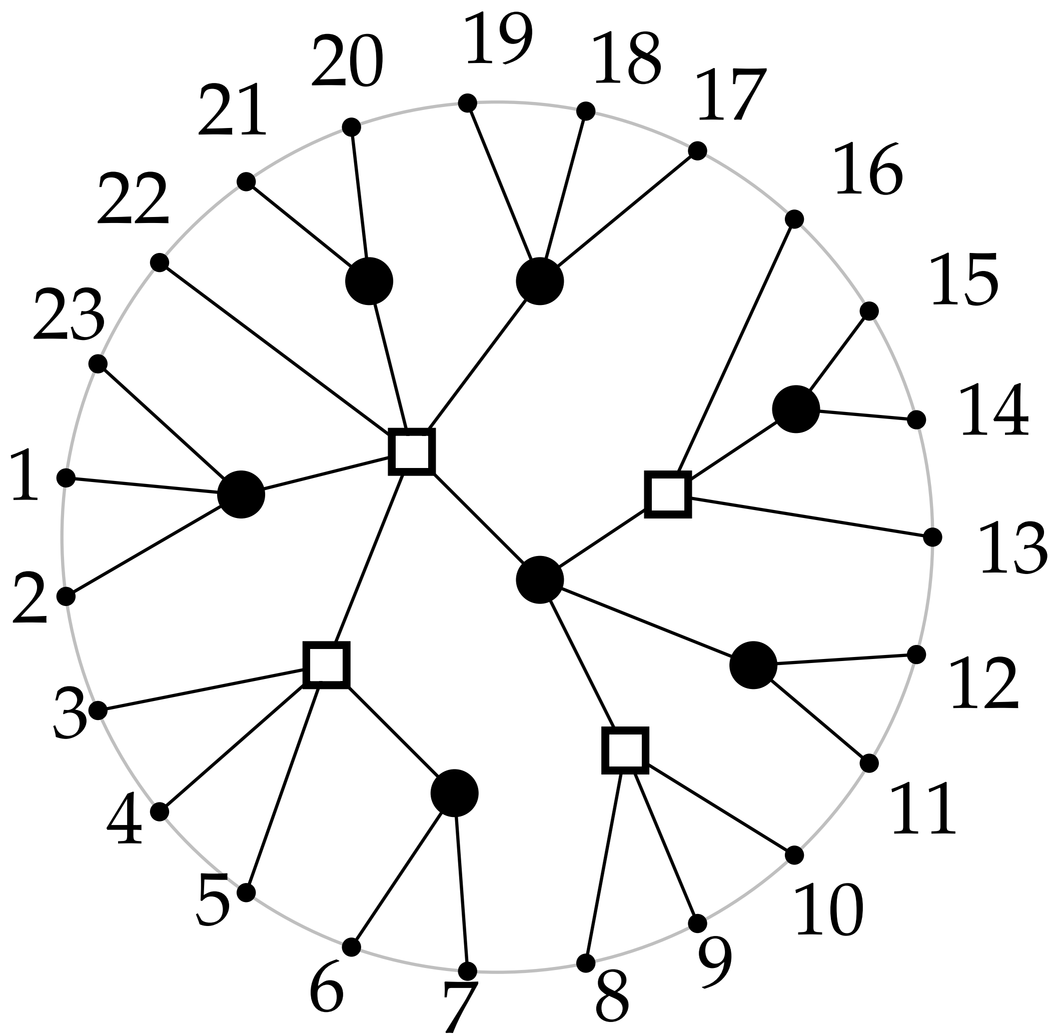
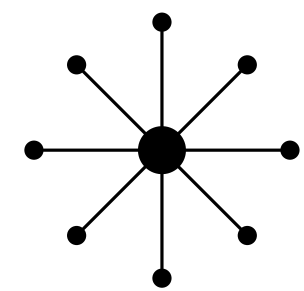


● P-node
permute children freely

□ Q-node
only "flipping" allowed

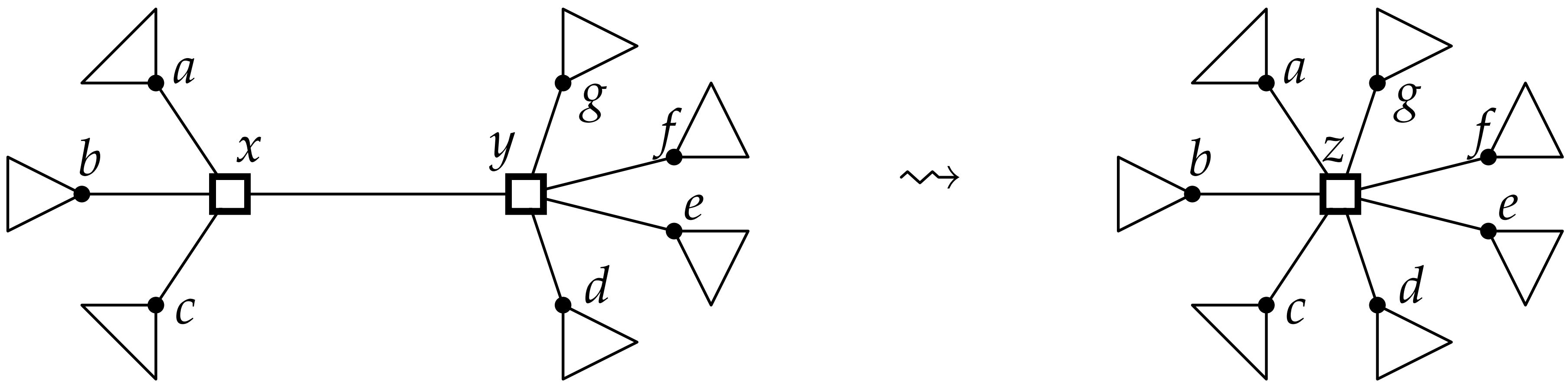


Single P-node ⇝ all possible circular orderings of its leaves.

PQ-tree represents all circular orderings of its leaves by these operations

Order-Preserving Contraction, Null-Tree

Order-preserving contraction

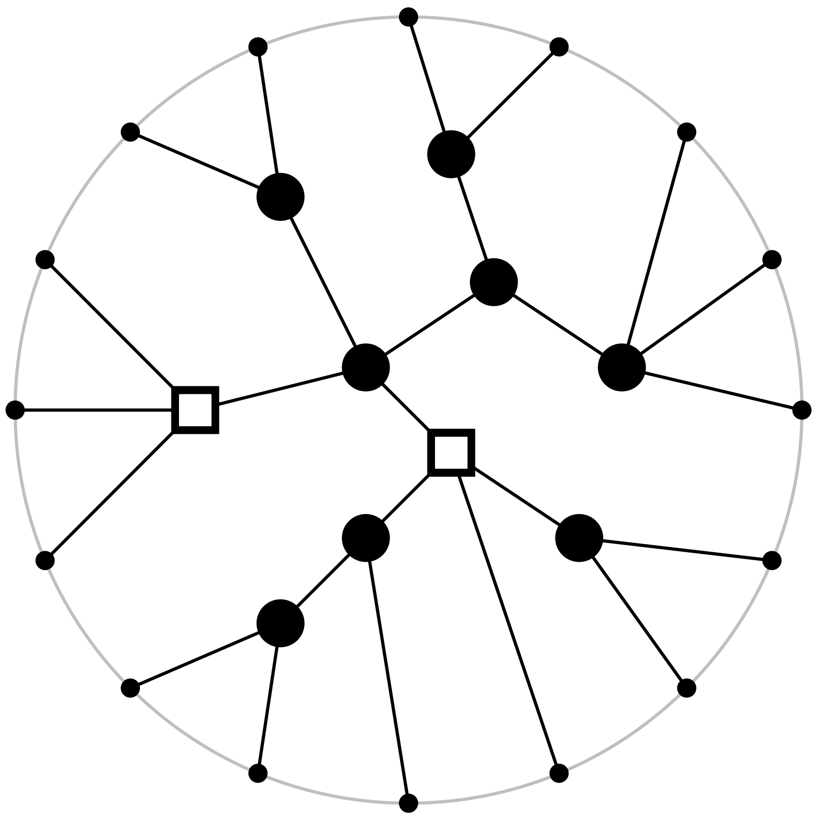


NOTE this restricts the representable orderings!

Null-tree: represents the empty set of permutations

NOTE Null-tree$\neq$ empty tree (represents permutations of the empty set)

Consecutivity of Subsets



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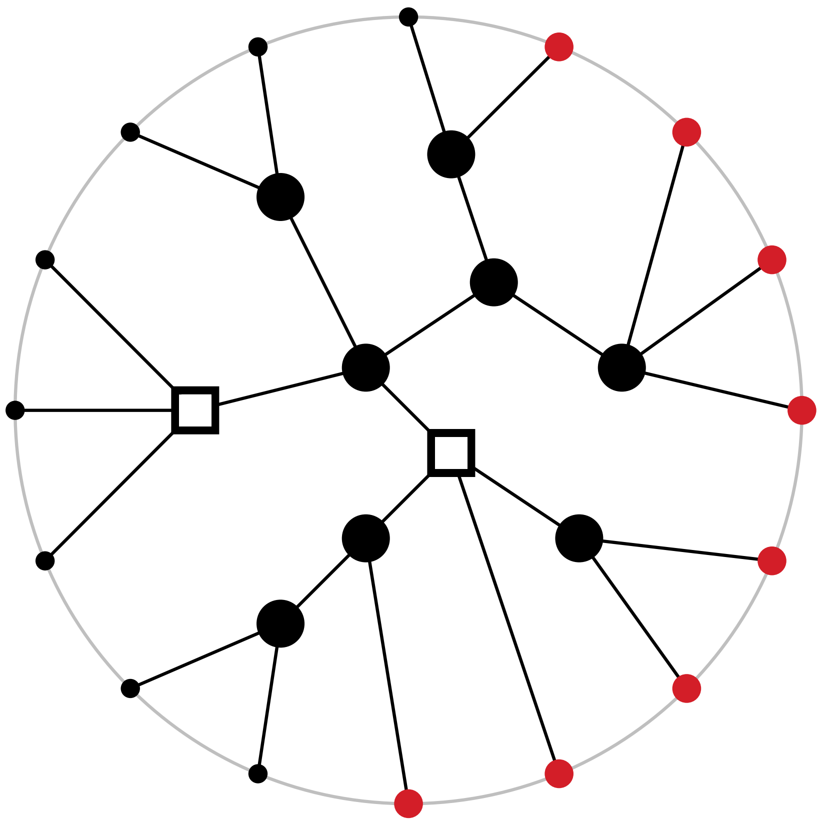

Find new PQ-tree, representing exactly the orderings which :

- are admitted by the current tree and
- have the red leaves consecutive

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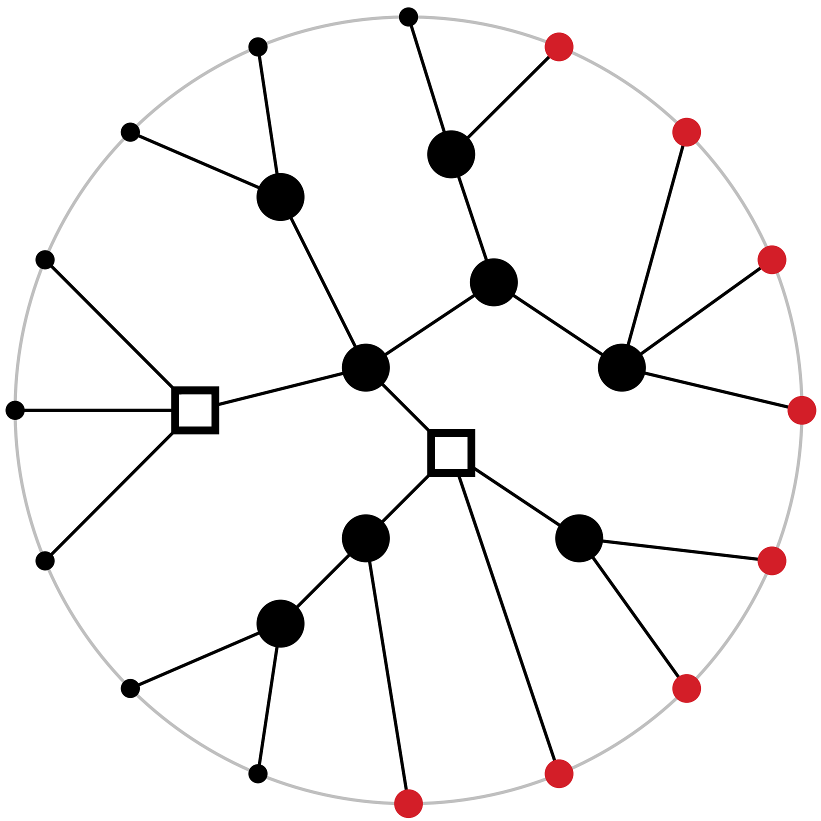

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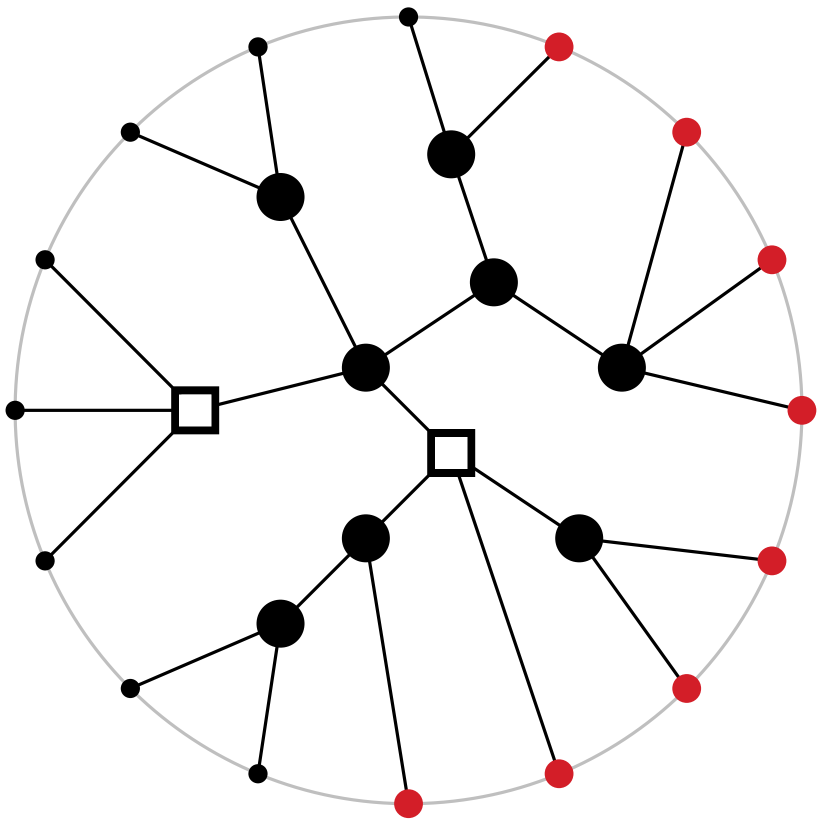

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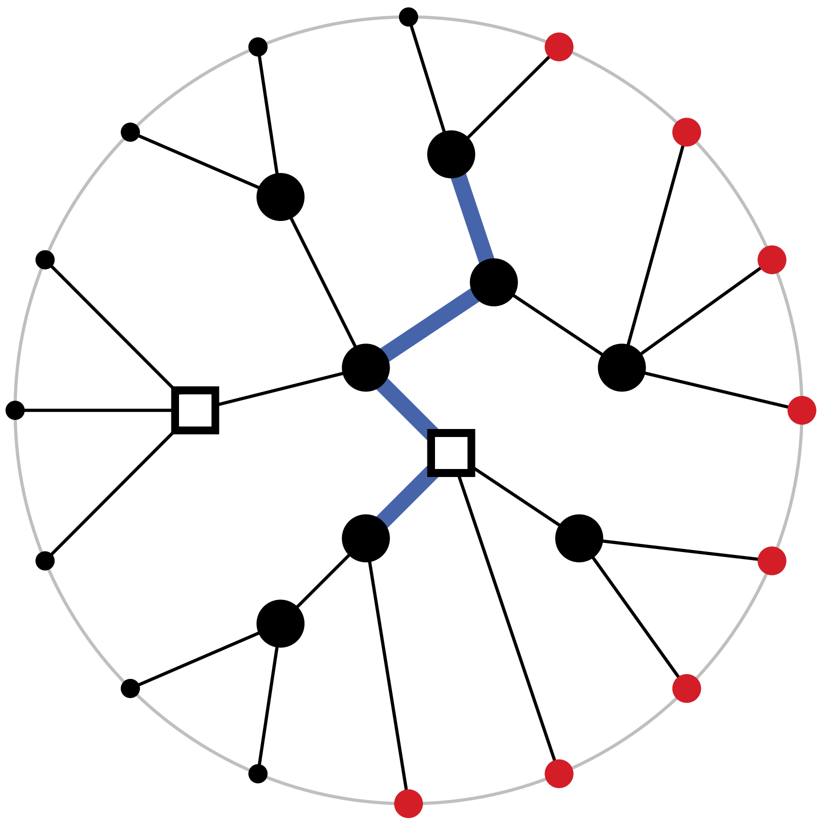

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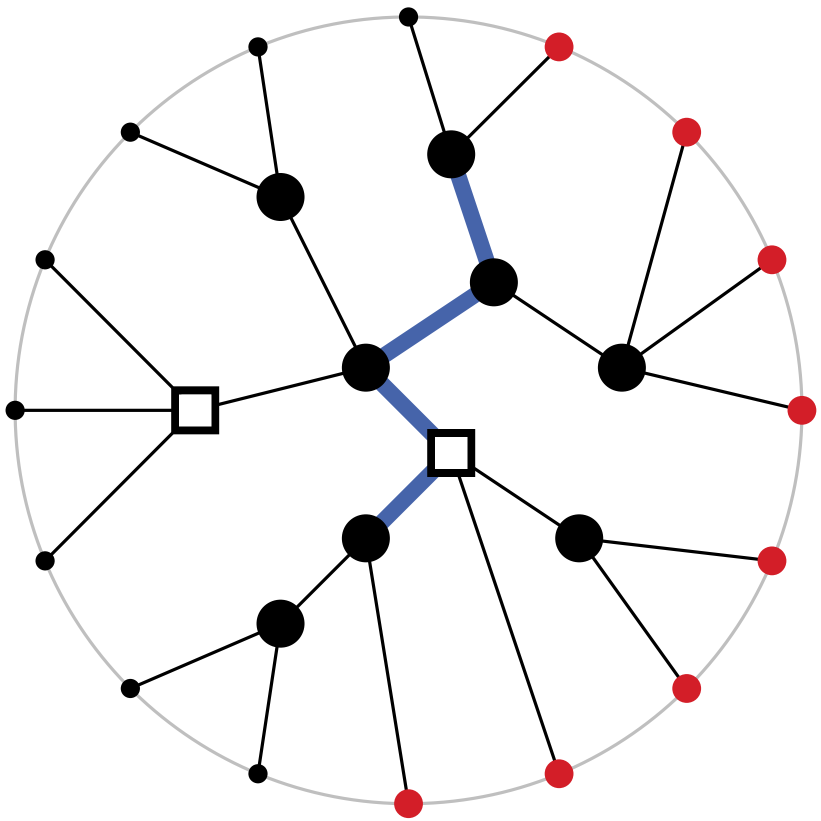

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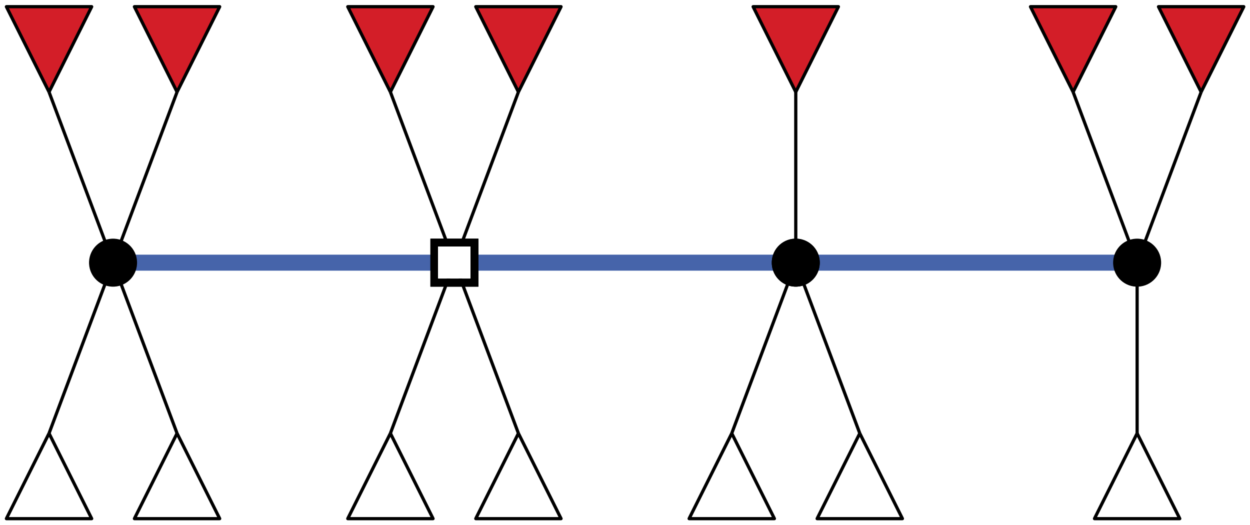
Lemma: If an arrangement of the PQ-tree has the red leaves consecutive, the partial edges form a path.

Updating PQ-Trees

1. Find the path of partial edges and arrange the tree to split red and black leaves.



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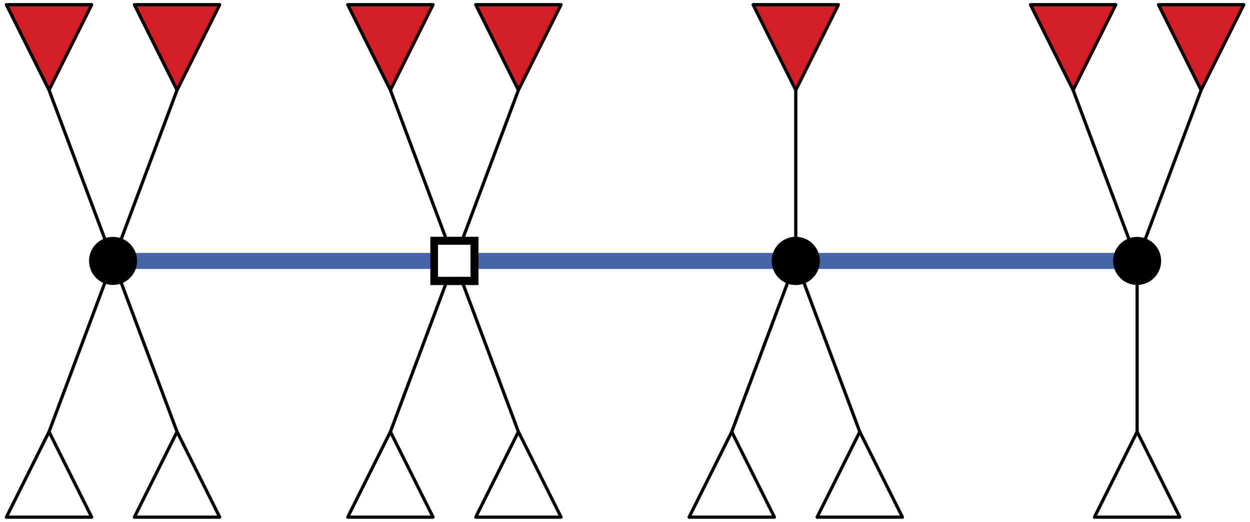

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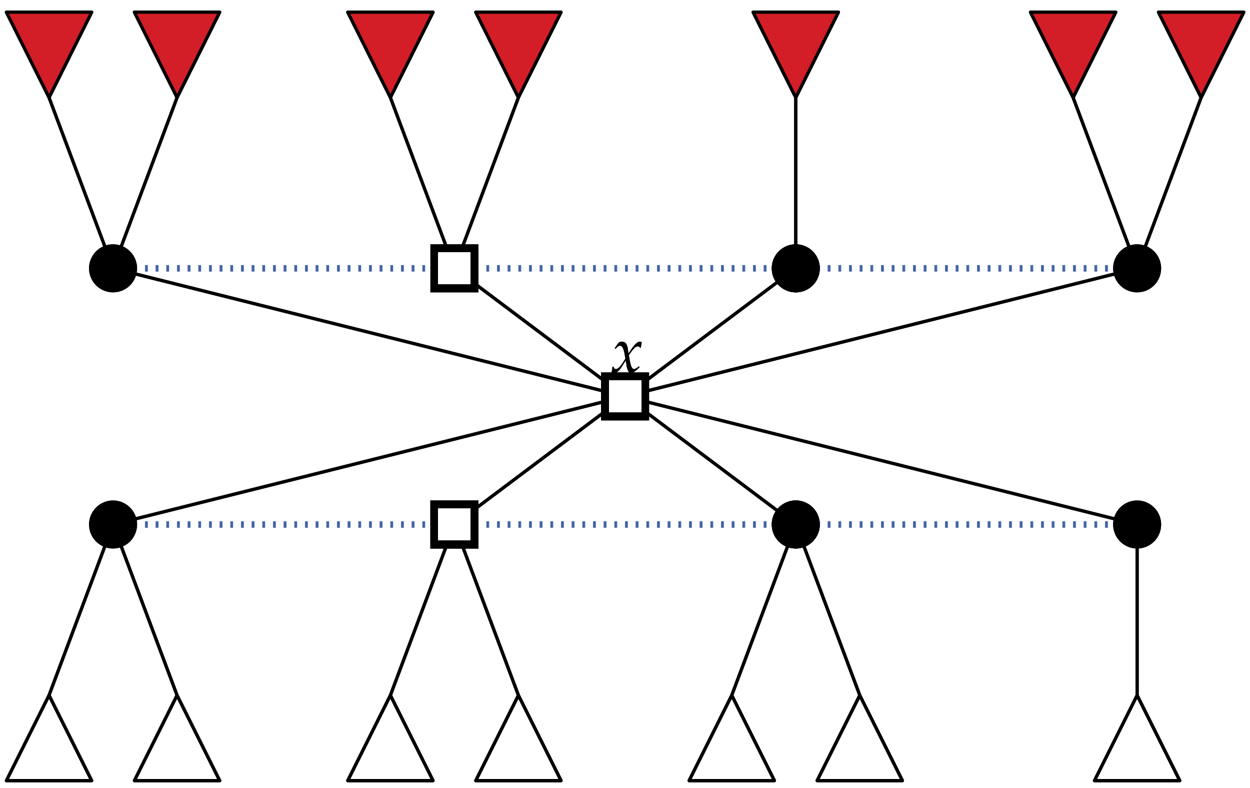

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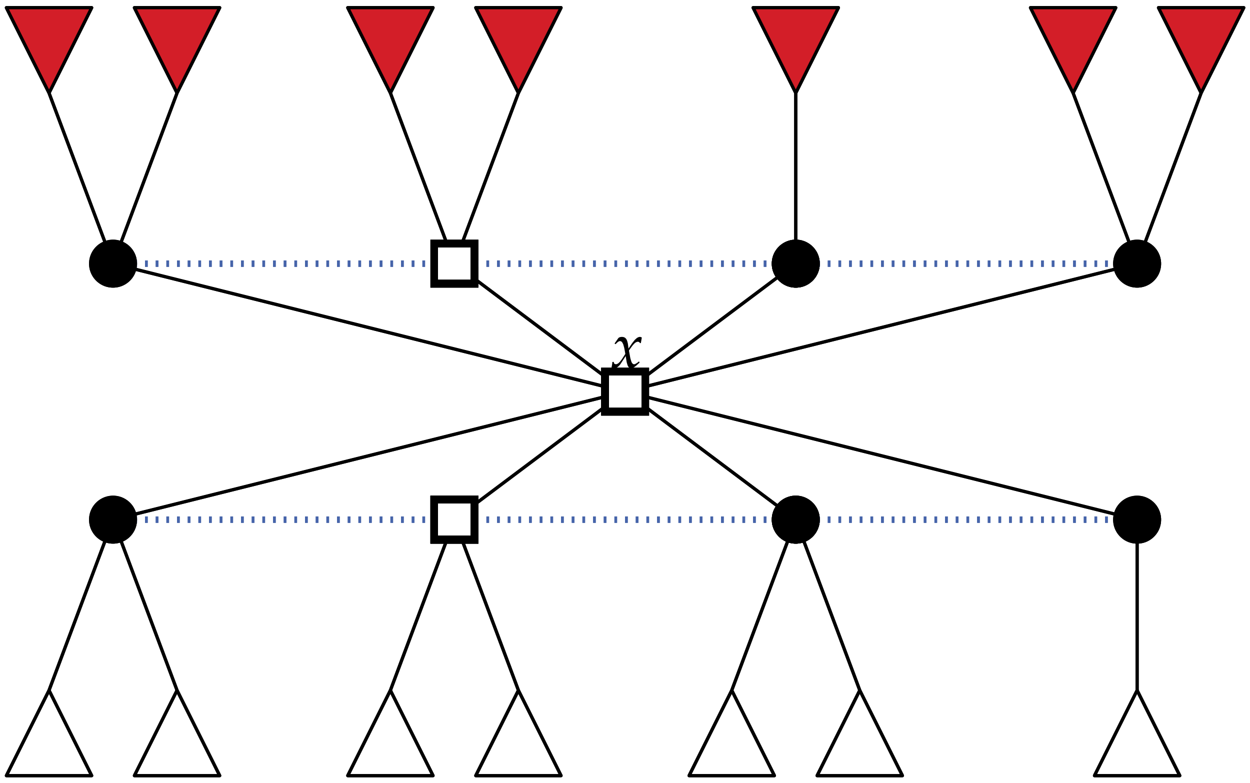

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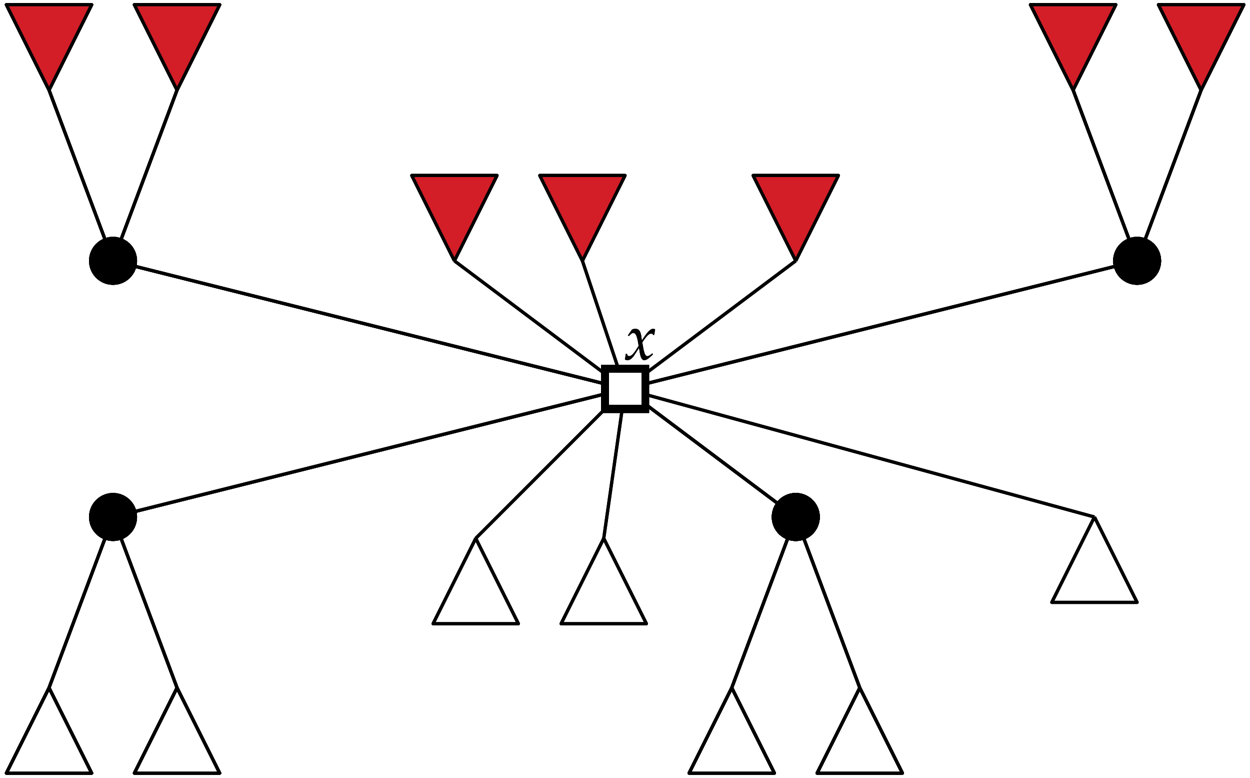

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Thm: The update algorithm is correct.

Proof:

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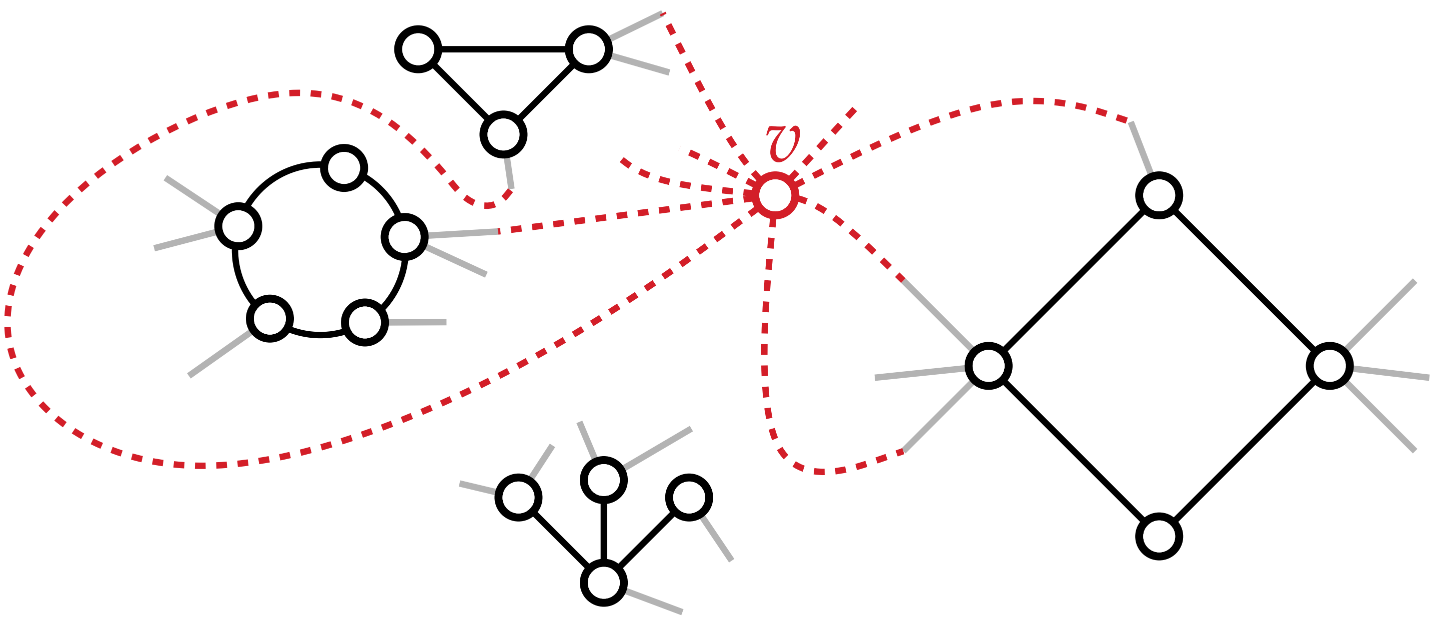
A linear time implementation needs more ideas . . .

Back to Planarity Testing

Planarity Testing, vertex insertion



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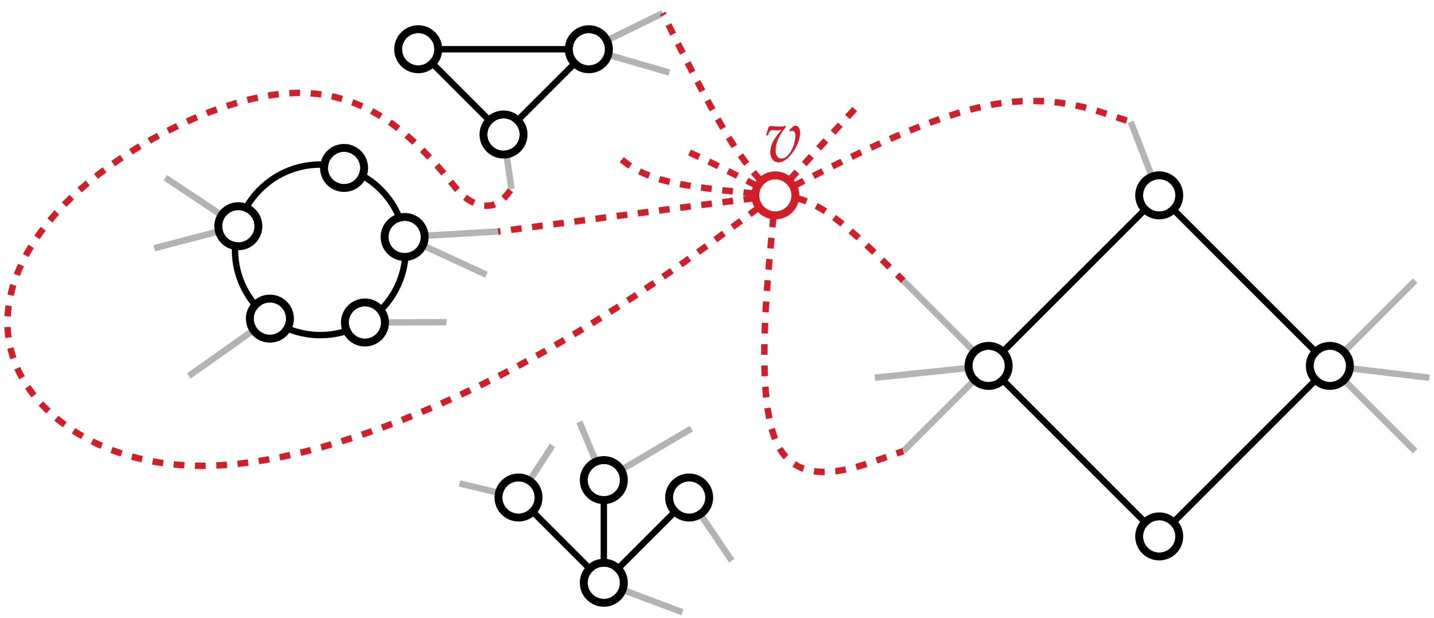

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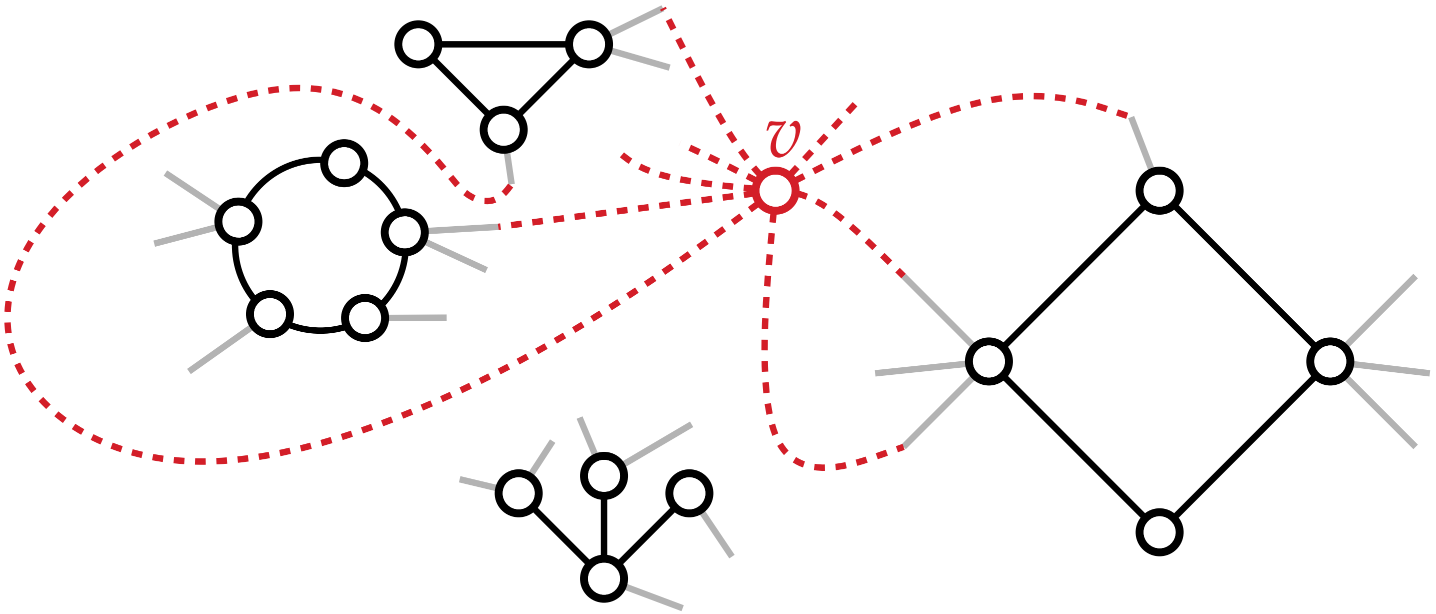
⇝ use PQ-trees!

Planarity Testing, vertex ins. (PQ-trees)



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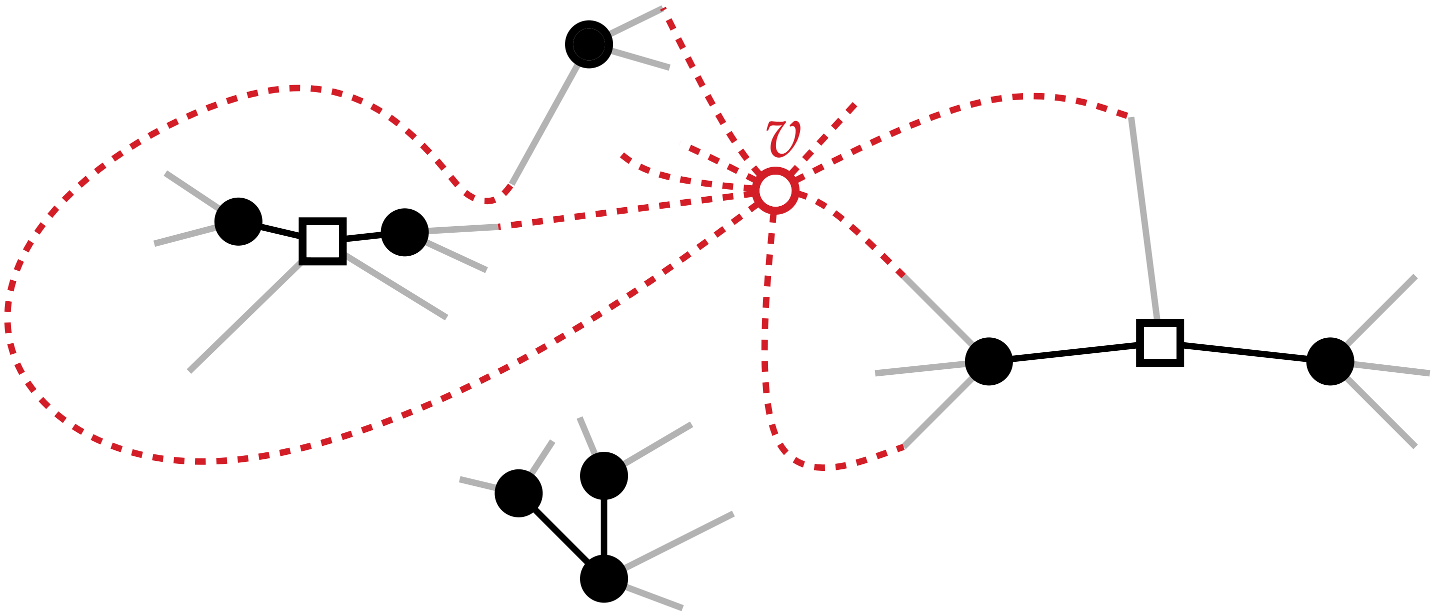
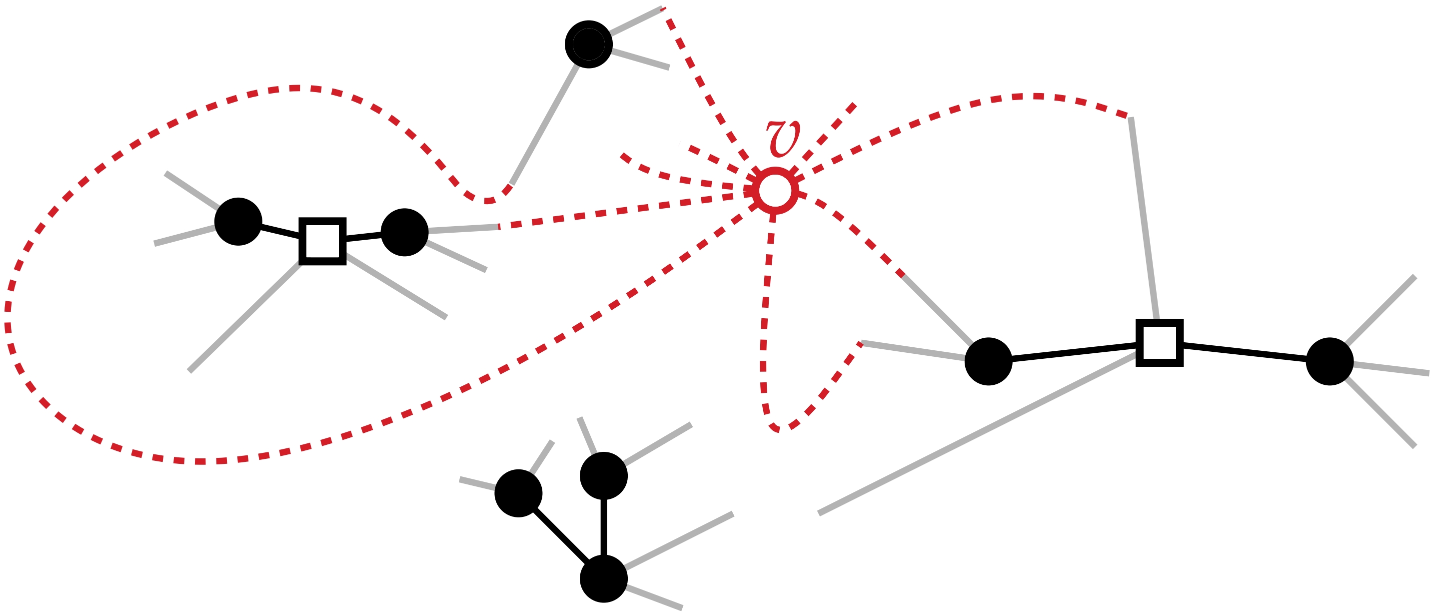


One PQ-tree for each component

Planarity Testing, vertex ins. (PQ-trees)



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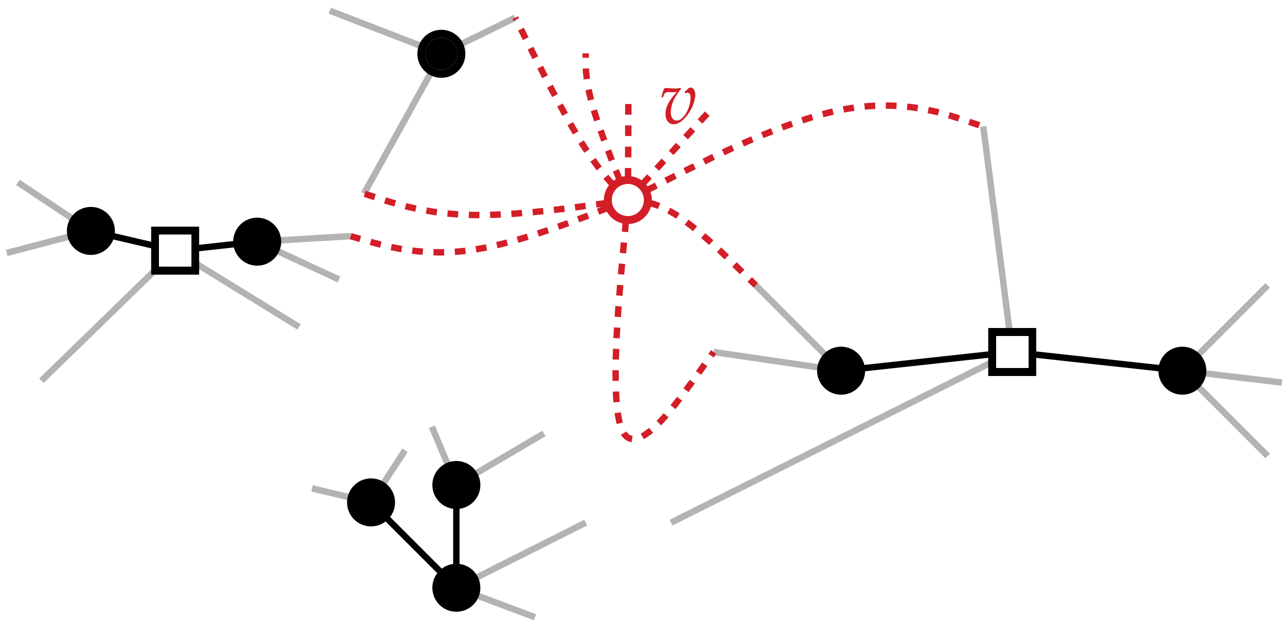

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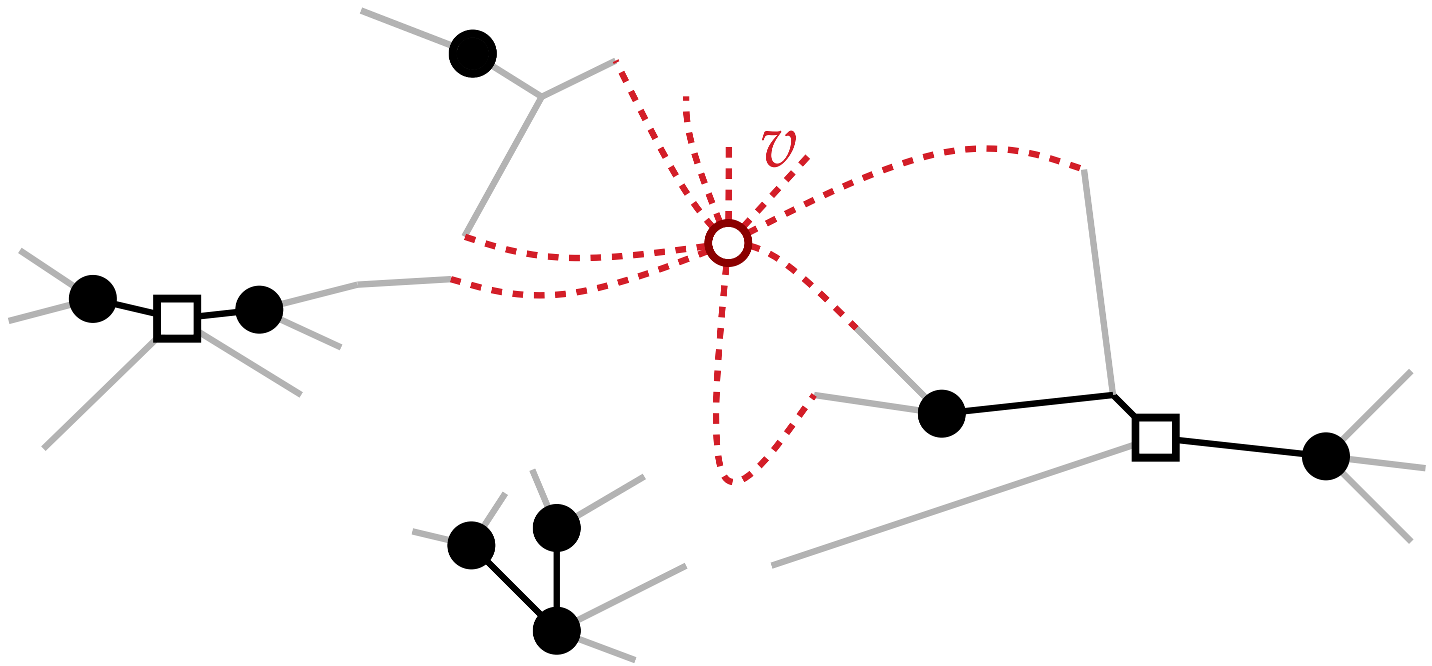

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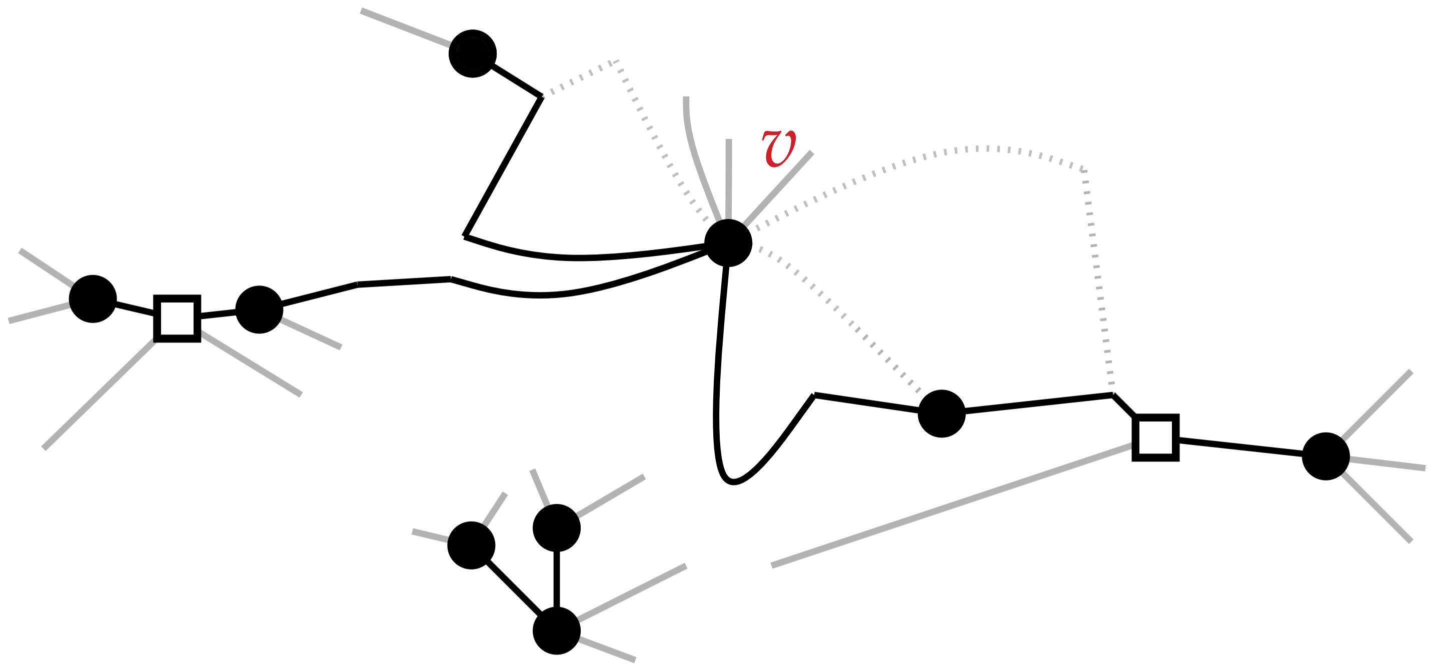

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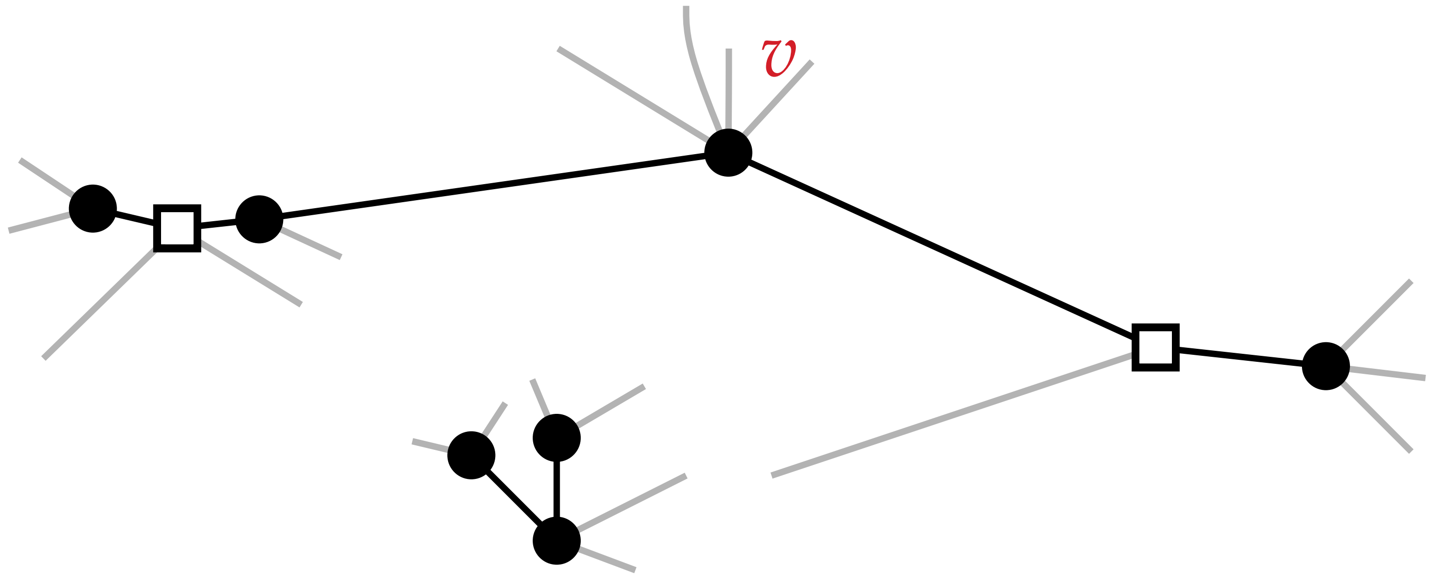

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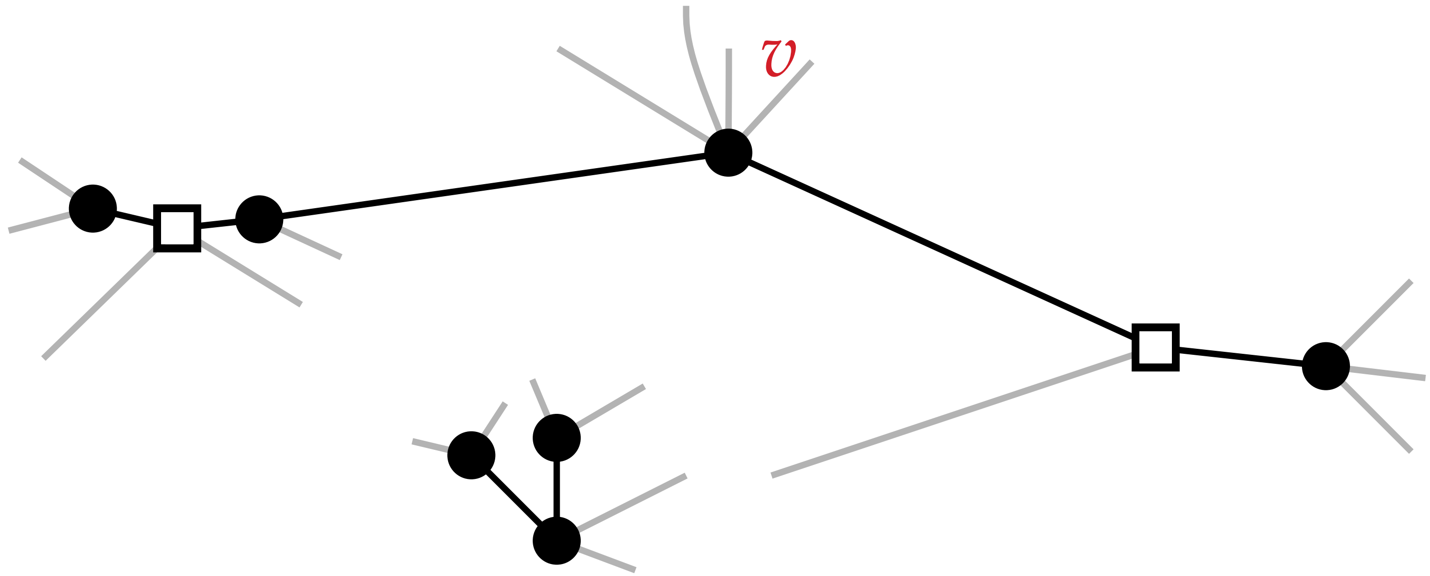

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Cost per vertex: one consecutivity constraint $O(\deg v)$

Planarity Testing

Graph is planar if and only if all reduction steps succeed.

Embedding can be recovered by undoing steps.
Select/expand orders within the PQ-tree.

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Warning on linear runtime: finding a terminal path efficiently is tricky, involves rooting each PQ-tree. When merging, need to root consistently.

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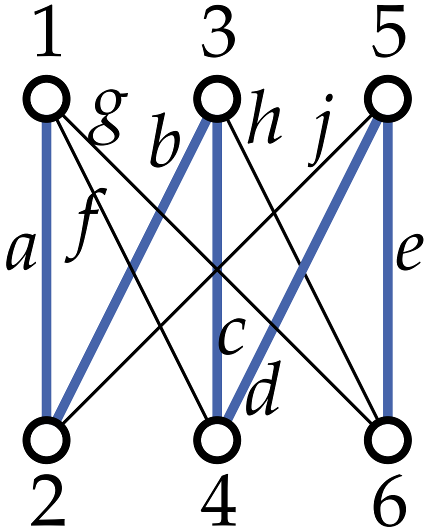
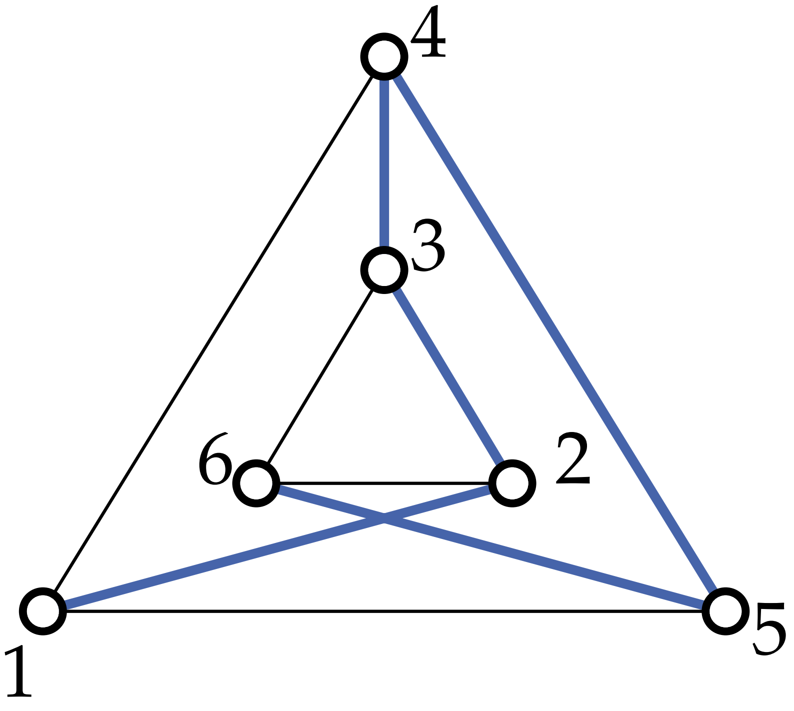
What about the sequence to process vertices?

- (s, t)-ordering (two-connected graphs) (see *graph visualization* lecture) [Lempel, Even, Cederbaum '67]
- Depth-first search [Shih, Hsu '99, Boyer, Myrvold '04]

Examples

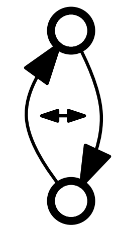




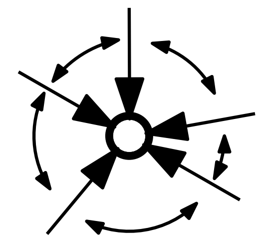
Linear-time PQ-Tree construction

Linear-time PQ-tree construction

Linear time implementation details



- choose root
- store each edge in both directions
- store incoming edges at each node in double-linked list



- mark each edge regarding orientation to root
- P-nodes have pointer to parent

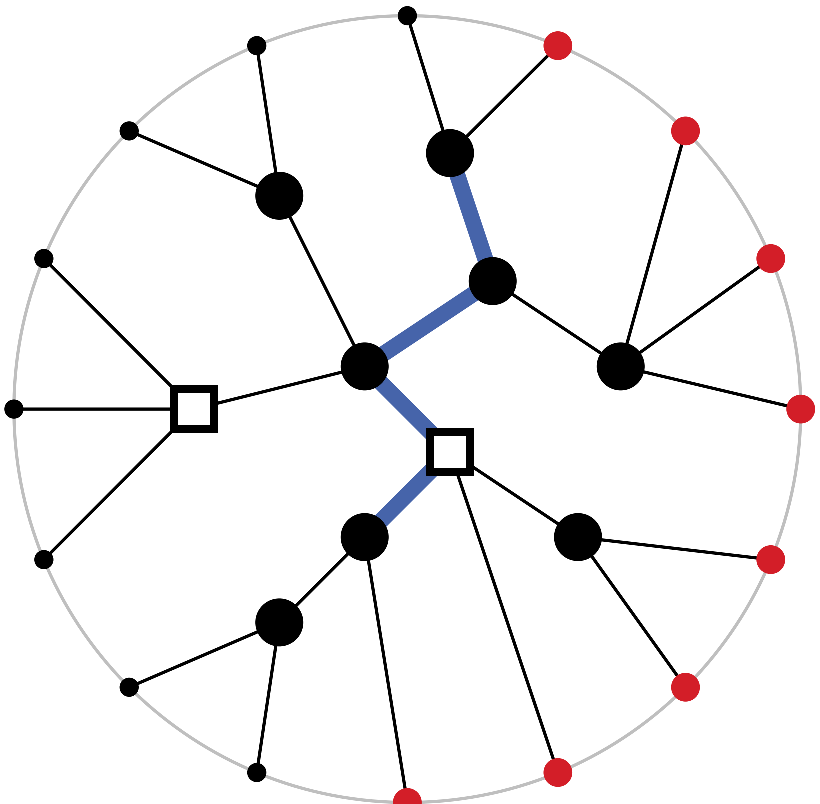
NOTE: Parent of a Q-node is “expensive” to determine, but this means we do not need to keep track of it when editing Q-nodes.

Computing the Terminal Path

length of terminal path

number of consecutive elements

Lemma: Terminal path can be found in $O(p + k)$ time.



Computing the Terminal Path

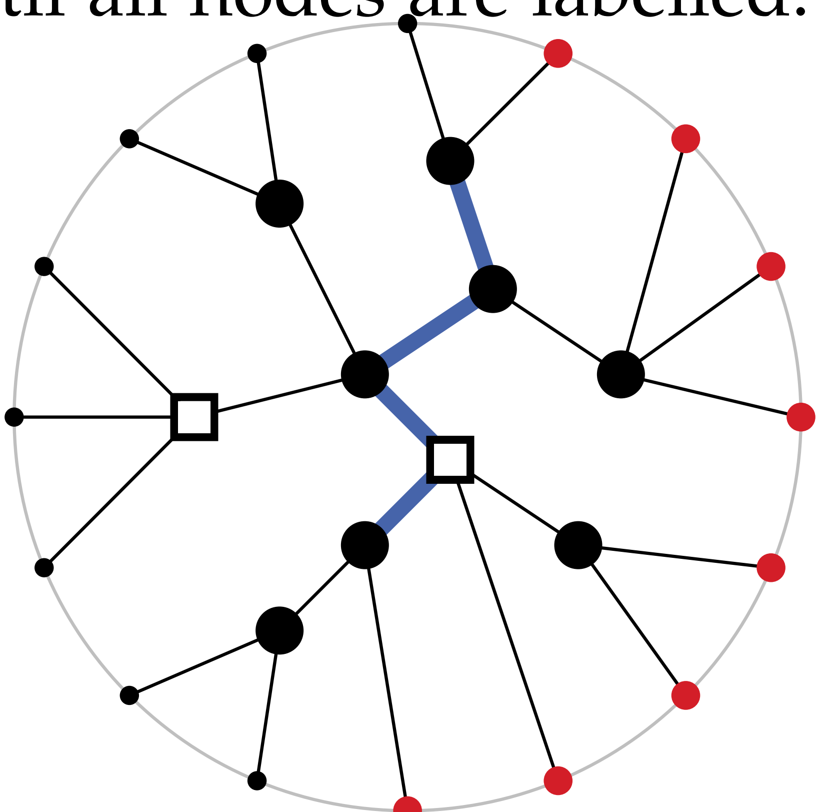
length of terminal path — number of consecutive elements

Lemma: Terminal path can be found in $O(p + k)$ time.

Classify the nodes of the PQ-tree:

- a leaf is full when it is a red element
- an inner node is full when all but one neighbor is full.
- a non-full node is partial at least one neighbor is full.

$\rightsquigarrow$ $O(k)$ time until all nodes are labelled.



Computing the Terminal Path

length of terminal path — number of consecutive elements

Lemma: Terminal path can be found in $O(p + k)$ time.

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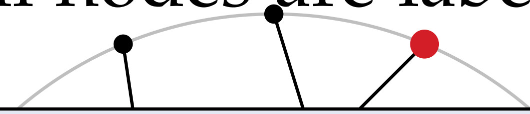
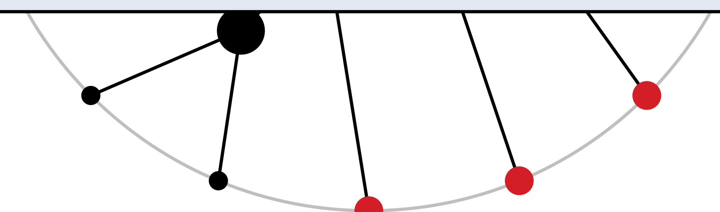


Key idea: Partial nodes must belong to the terminal path

- extend potential path from each partial node to parent
- stop extending when another path is hit, and join
- leads to a tree with at most one degree 3 node (o.w. reject)
- highest node found which is either partial or meeting of two extensions, is the high point of the terminal path.



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where is the parent-edge with respect to full child-edges?

Update Step

Split terminal path, new nodes will receive full neighbors.

Single Split, for each node of the terminal path:

- detach full neighbors F, and delete incident edges to neighbors on the path — $O(1)$
- make a copy, hang F from it. — $O(\#\text{full neighbors})$

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Create central Q-node, $O(p)$ time

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$O(p+k)$

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NOTE if terminal path contains $q \geq 2$ Q-nodes, then number of Q-nodes decreases by $q-1$, i.e., cost of processing this terminal path saves $q-1$ for us for later.

Runtime analysis

X ground set
$\mathcal{U} = \{U_1, \ldots, U_\ell\}$ collection of subsets of X.

Thm: PQ-tree representing all orderings where $U_1, \ldots, U_\ell$ are consecutive can be computed in $O(|X| + |U_1| + \cdots + |U_\ell|)$ time (amortized analysis).

Proof: Consider potential function
$\phi(\mathcal{U}, i) = 2u_i + |Q_i| + \sum_{x \in P_i}(\deg(x) - 1)$, where:

- $u_i = \sum_{j>i} |U_j|$
- Q_i = Q-nodes in PQ-tree T_i after processing $U_1, \ldots, U_i$.
- P_i = P-nodes in T_i

$\phi(\mathcal{U}, 0) = \Theta(|X| + \sum_i |U_i|)$ (budget to be used)
Inductively show budget $\phi(\mathcal{U}, i-1) -$ cost$(U_i) \geq \phi(\mathcal{U}, i)$
need: ϕ stay to stay ≥ 0
$\leadsto$ claimed runtime.