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Lehrstuhl für
INFORMATIK I
Effiziente Algorithmen und
wissensbasierte Systeme

Institut für Informatik

Advanced Algorithms

Winter term 2019/20

Lecture 11. Alternative Parameterization: Tree Decomposition
Source: **PA §7.2, 7.3.1**

(slides by Thomas van Dijk & Alexander Wolff)

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Independent Set

INDEPENDENT SET

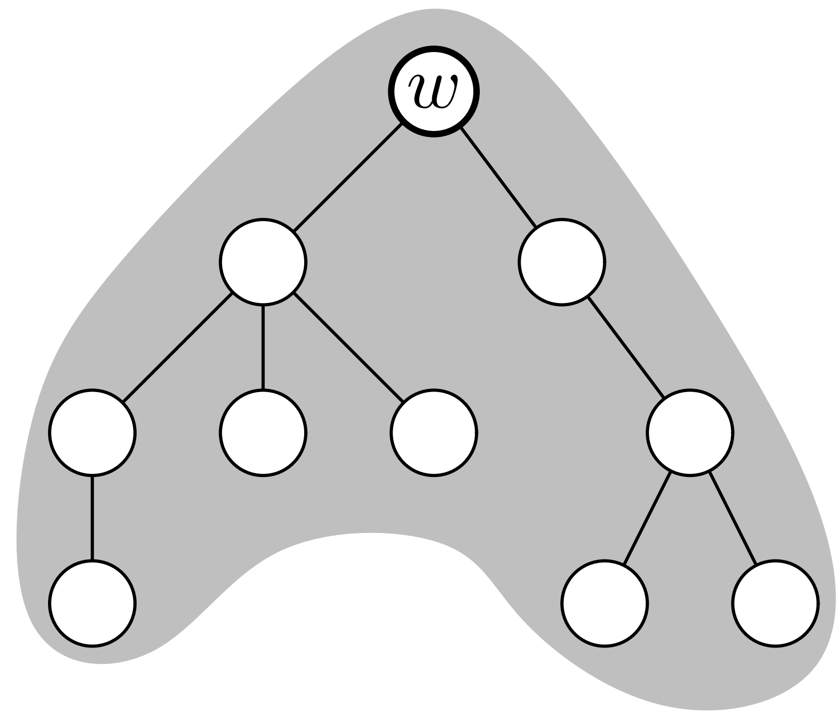
Given:	graph G, weight function $\omega : V \to \mathbb{N}$
Question:	What is the maximum weight of a set $S \subseteq V$ where no pair in S forms an edge in G?

Thm: Independent Set is NP-complete.

Thm: Independent Set can be solved in linear time on trees.

Independent Sets in Trees

$A(w) =$ solution



Choose an arbitrary root w.

Let $T(v) :=$ subtree rooted at v

Let $A(v) :=$ maximum weight of an independent set S in $T(v)$

Let $B(v) :=$ maximum weight of an independent set S in $T(v)$ *where* $v \notin S$

When v is a leaf: $A(v) = \omega(v)$ and $B(v) = 0$

When v has children $x_1, \ldots, x_r$:

$$A(v) = \max\{ \sum_{i=1}^{r} A(x_i), \ \ \omega(v) + \sum_{i=1}^{r} B(x_i) \}$$

$$B(v) = \sum_{i=1}^{r} A(x_i)$$

Algo: Compute $A(\cdot)$ and $B(\cdot)$ bottom-up

s, t-series parallel graphs

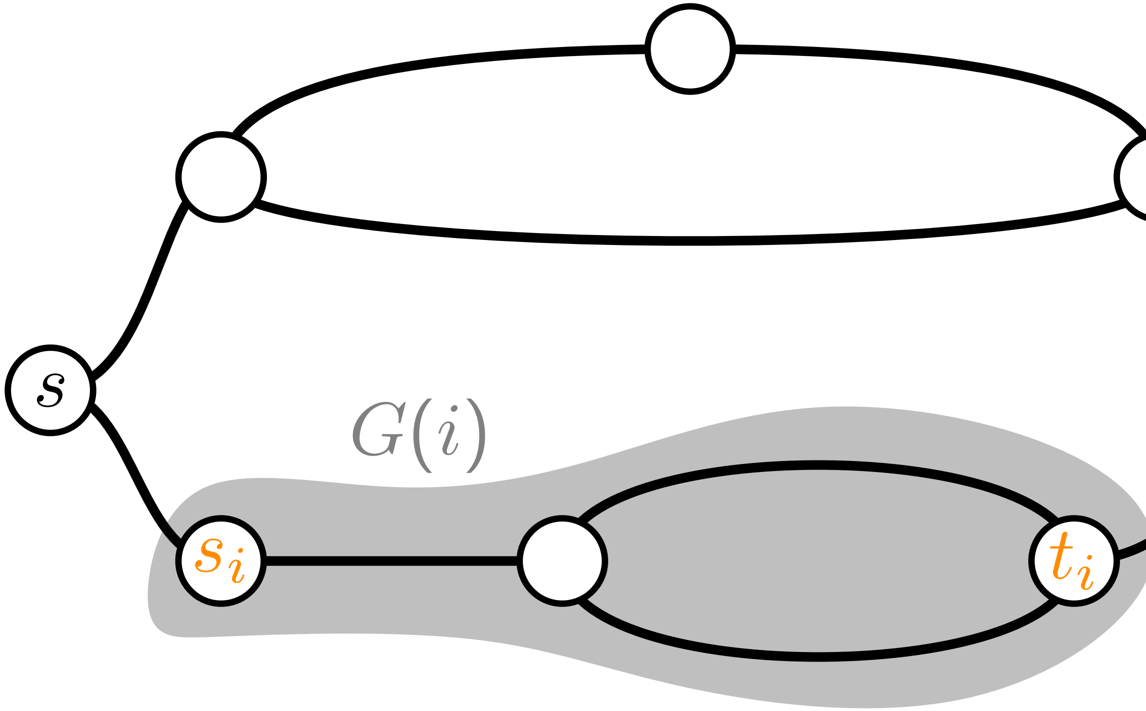
Def.: A graph $G = (V, E)$ is *2-terminal* when it contains two special vertices s and t

Def.: A 2-terminal graph G is *series parallel* when:

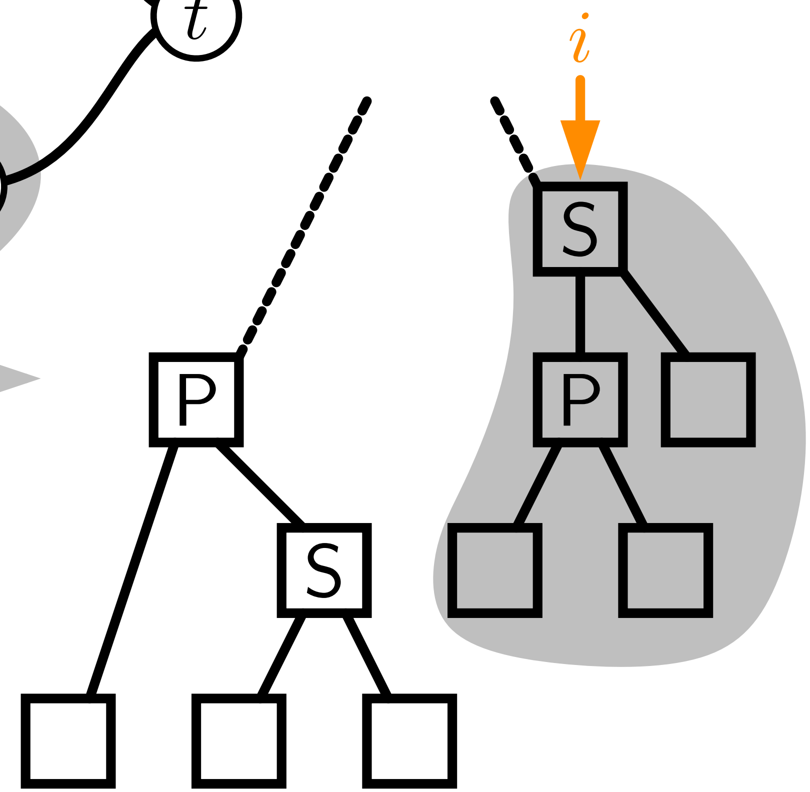
- G is a single edge (s, t)
- G is a *series composition* of two series parallel graphs
- G is a *parallel composition* of two series parallel graphs

recursive definition:
series parallel graphs have a natural tree-structure

SP-tree



Let i be a node in an SP-tree.
$G(i) :=$ graph represented by the subtree rooted at i



Independent Set on SP-trees

Dynamic program on SP-tree indexed by $G(i)$

$AA(i) :=$ maximum weight independent set S in $G(i)$ where $s_i \in S$ and $t_i \in S$

$BA(i) :=$ maximum weight independent set S in $G(i)$ where $s_i \notin S$ and $t_i \in S$

$AB(i)$ and $BB(i)$ def. similarly

other cases omitted... (easy exercise)

$O(1)$ time per SP-node

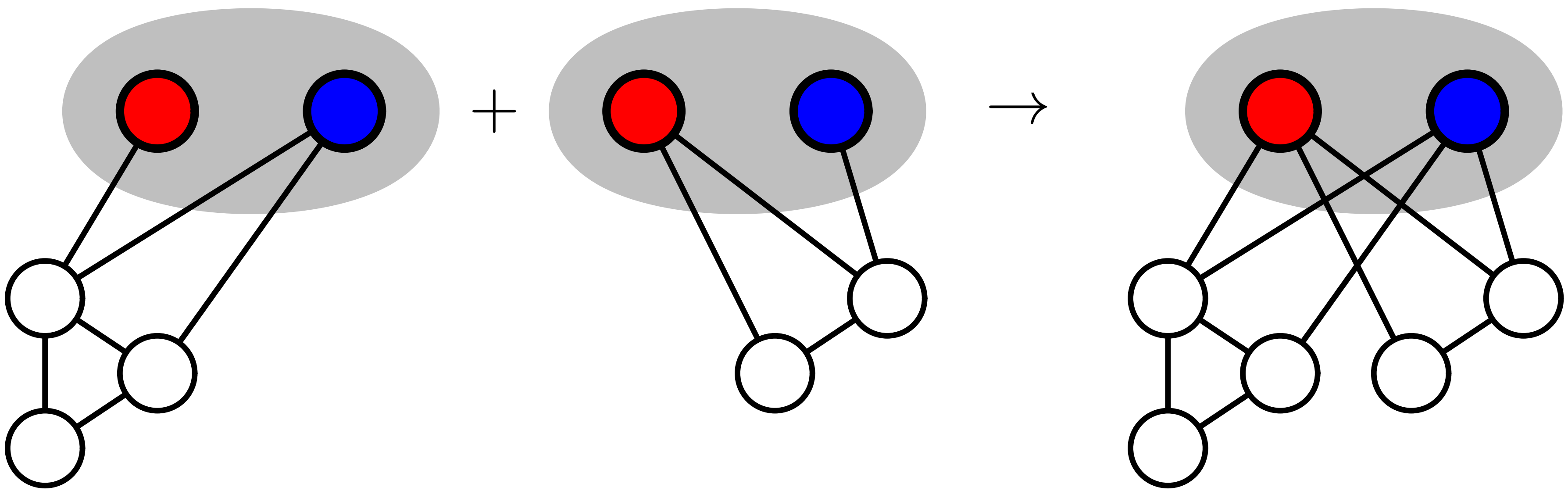
Thm: $\textsc{Independent Set}$ on series parallel graphs with a given SP-tree can be solved in $O(n)$ time.

Generalization?

Many ways to generalize the concept of having a "tree structure"

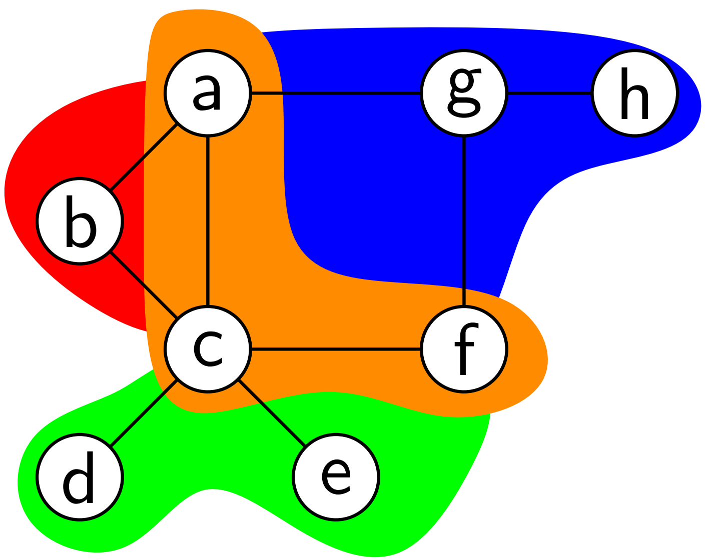
Ex.: k-terminal graph $G = (V, E, T)$, $|T| = k$

Example Operation: "gluing"

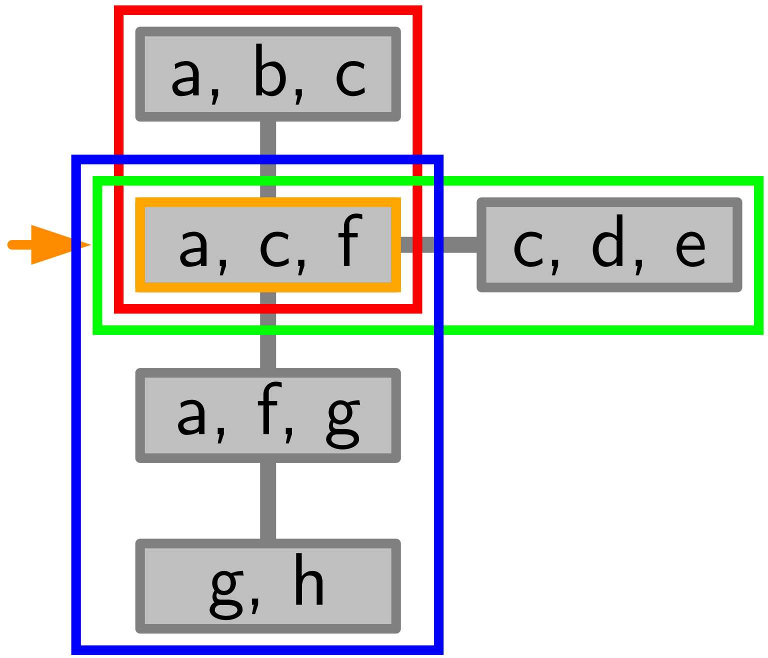


Example: Tree Decomposition

Graph $G = (V, E)$:



Tree Decomposition:



Tree Decompostion (formal)

Def. A *tree decomposition* of a graph $G = (V, E)$ is:

- a tuple $D = (X, T)$
- $T = (P, F)$ is a tree
- $X = \{X_p \mid p \in P\}$ is a set family of subsets of V (one for each node in P)
- $\bigcup\limits_{p \in P} X_p = V$
- $\forall \{u, v\} \in E \; \exists p \in P$ where $u, v \in X_p$
- $\forall v \in V$: $\{p \in P \mid v \in X_p\}$ is connected in T

Treewidth (formal)

- a tuple $D = (X, T)$
- $T = (P, F)$ is a tree

Def. Width (tree decomposition): $\max_{p \in P} |X_p| - 1$,
i.e., cardinality of the largest bag -1

Def. Treewidth $\text{tw}(G)$ is the minimum width of a tree decomposition of G

Obs. $\text{tw}(G) < n$

Question: Which graphs have treewidth 0? $E = \varnothing$

Exercise: Trees have treewidth 1

Exercise: Series parallel graphs have treewidth 2

Thm: There is a tree decomposition of width $\text{tw}(G)$ where $|P|$ is polynomial in n, i.e., the tree has polynomial size in n

Parameterized Problems

See **PA** §**13.3**

Given: Instance of size n and parameter k

Def. Problem is FPT when solvable in $O(f(k) \cdot poly(n))$ time.
$O(f(\text{tw}(G)) \cdot poly(n))$ time.

Ex.:

k-Vertex Cover	**FPT**
k-Independent Set	likely not FPT, $W[1]$-**comp.**
k-Dominating Set	likely not FPT, $W[2]$-**comp.**
k-Coloring	**NP-comp.** $k \geq 3$
Independent Set (treewidth)	**FPT**
List Coloring (treewidth)	$W[1]$-**comp.**
Channel Assignment (treewidth)	**NP-comp.** $k \geq 3$

Computing Treewidth

TREEWIDTH
Given: Graph $G = (V, E)$, number k
Question: $\text{tw}(G) \leq k$?

Thm: TREEWIDTH is NP-complete

k-TREEWIDTH
Given: graph $G = (V, E)$
Parameter: number k
Question: $\text{tw}(G) \leq k$?

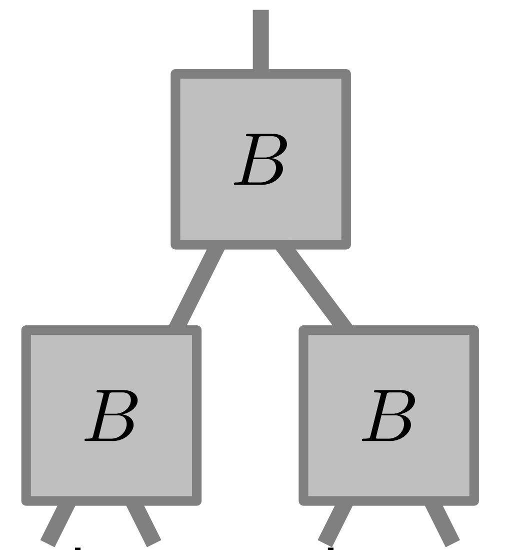
Thm: k-TREEWIDTH is FPT See PA §7.6.

How can we make "fixed-treewidth-tractable" algorithms?

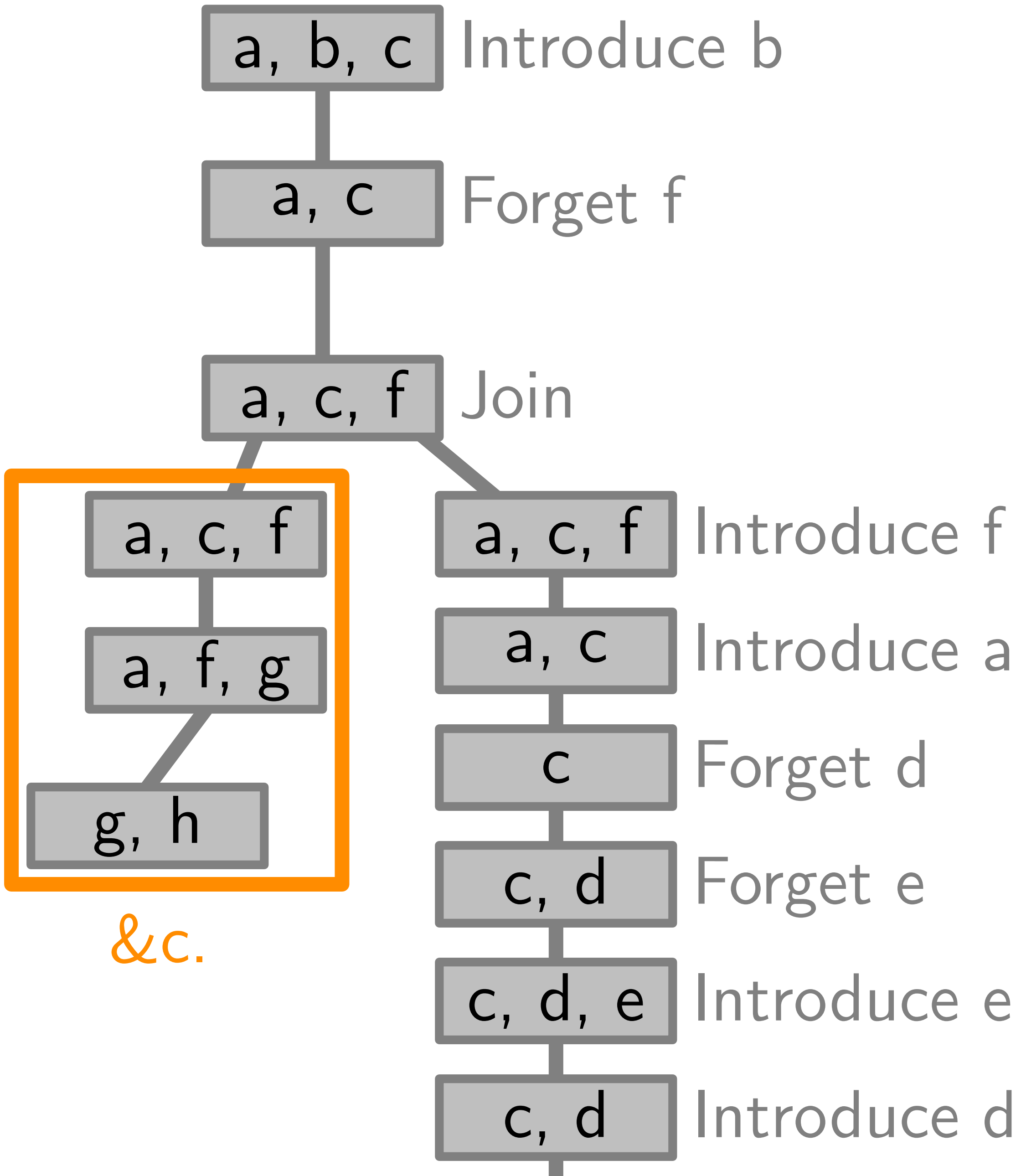
item #1: nice tree decompositoins

In a *nice* tree decomp., one bag is marked as the root and there are only 4 types of bags:

- **Leaf:** the bag is a leaf and contains only one vertex
- **Introduce:**
  The bag has exactly one child and contains the child's vertices and exactly one new vertex.
- **Forget:**
  The bag has exactly one child and contains one vertex fewer than the child.
- **Join:**
  The bag has exactly two children and these three nodes have exactly the same vertices



item #1: nice tree decompositoins



item #2: DP on nice Tree Decomp.

Thm: k-TREEWIDTH is FPT

Thm: Tree decompositions $\rightarrow$ *nice* in polynomial time.

Cor: For FPT-Algorithms it suffices to use nice tree decomp.

Strategy: Build a recurrence for each type of bag, and use dynamic programming.

Indep.Set on Nice Tree Decomp.

Let $G(i) :=$ Graph induced by the vertices in the subtree at i

For bag i and $S \subseteq X_i$, let:
$R(i, S) :=$ maximum weight of an indep. set I in $G(i)$
with $I \cap X_i = S$

Algo.: Compute $R(i, S)$ for all i and corresponding S

Runtime: ?

Thm: The independent set problem is FPT parameterized by treewidth.