Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Lehrstuhl für
INFORMATIK I
Effiziente Algorithmen und
wissensbasierte Systeme

Institut für Informatik

Advanced Algorithms

Winter term 2019/20

Lecture 11. Alternative Parameterization: Tree Decomposition
Source: **PA §7.2, 7.3.1**

(slides by Thomas van Dijk & Alexander Wolff)

Steven Chaplick *Lehrstuhl für Informatik I*

Independent Set

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Given: graph G, weight function $\omega : V \to \mathbb{N}$

Question: What is the maximum weight of a set $S \subseteq V$ where no pair in S forms an edge in G?

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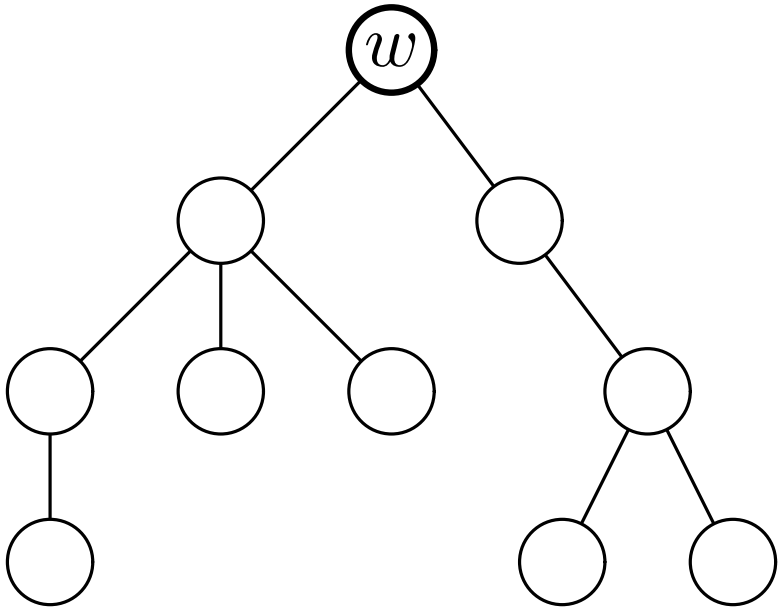
Thm: Independent Set is NP-complete.

Thm: Independent Set can be solved in linear time on trees.

Independent Sets in Trees

Choose an arbitrary root w.



Independent Sets in Trees

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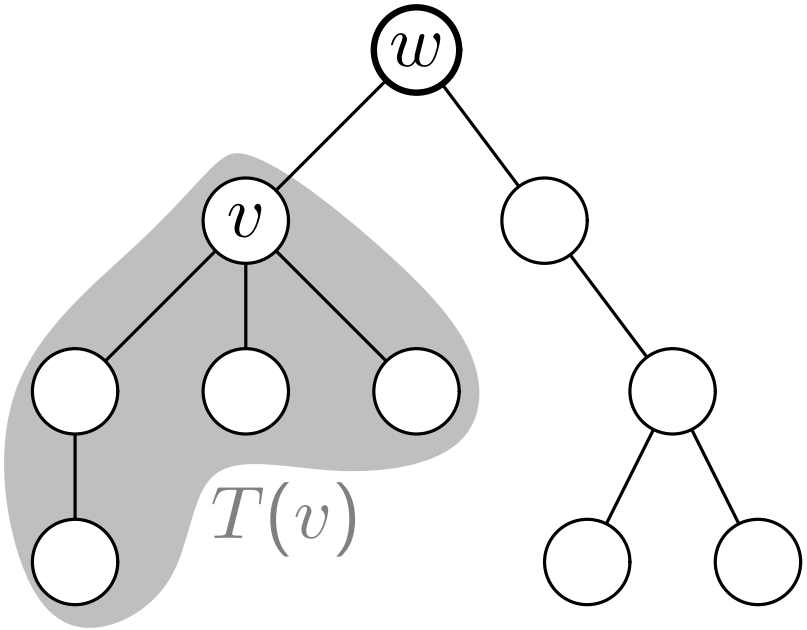
Let $T(v) :=$ subtree rooted at v



Independent Sets in Trees

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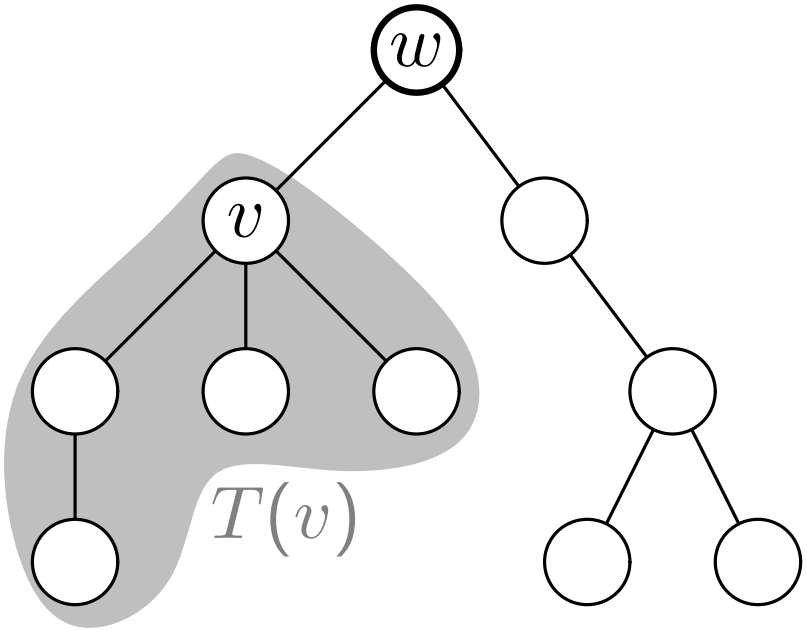
Let $A(v) :=$ maximum weight of an independent set S in $T(v)$



Independent Sets in Trees

$A(w) =$ solution

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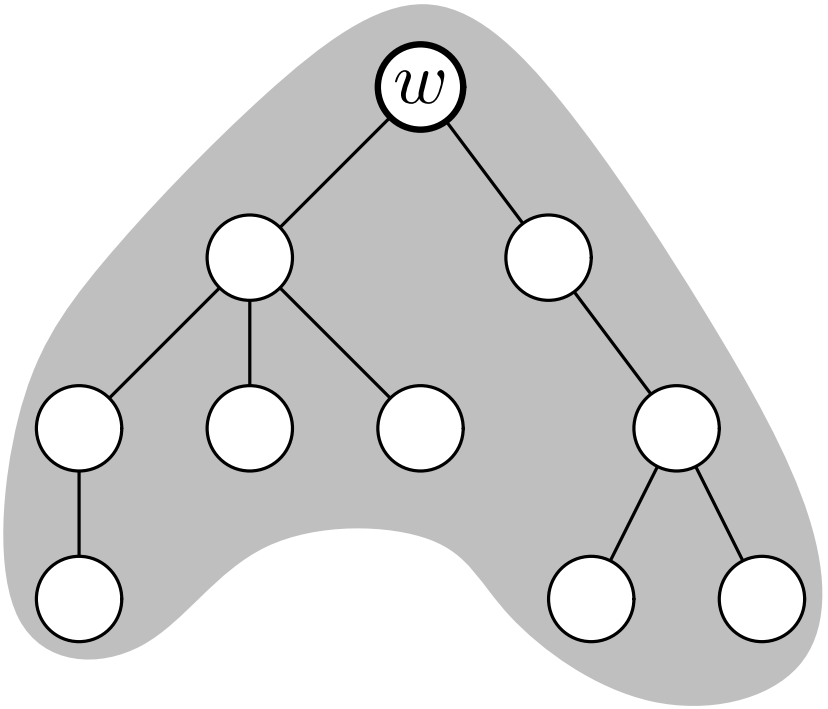

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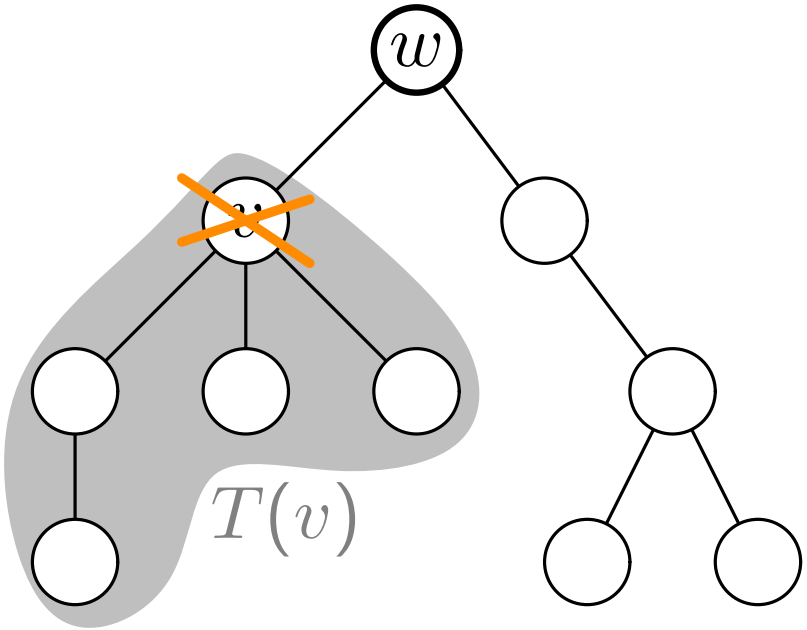

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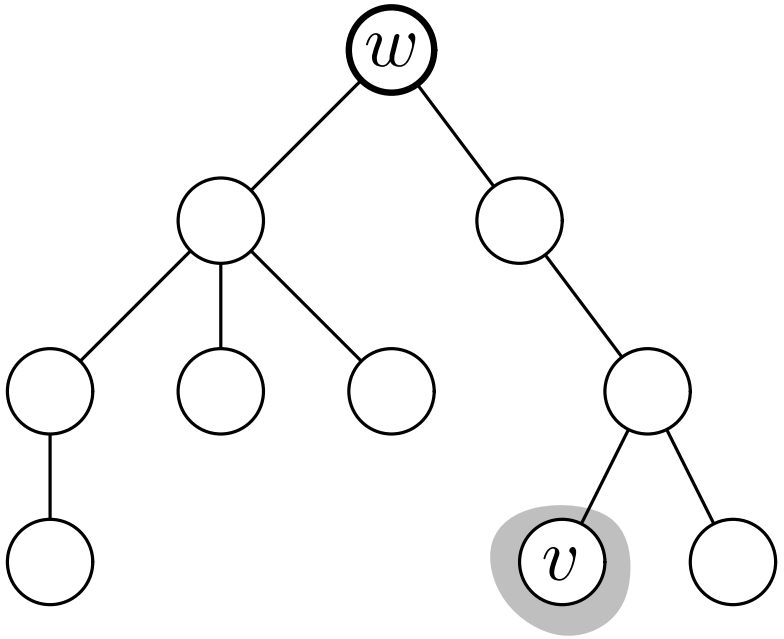
When v is a leaf: $A(v) =$



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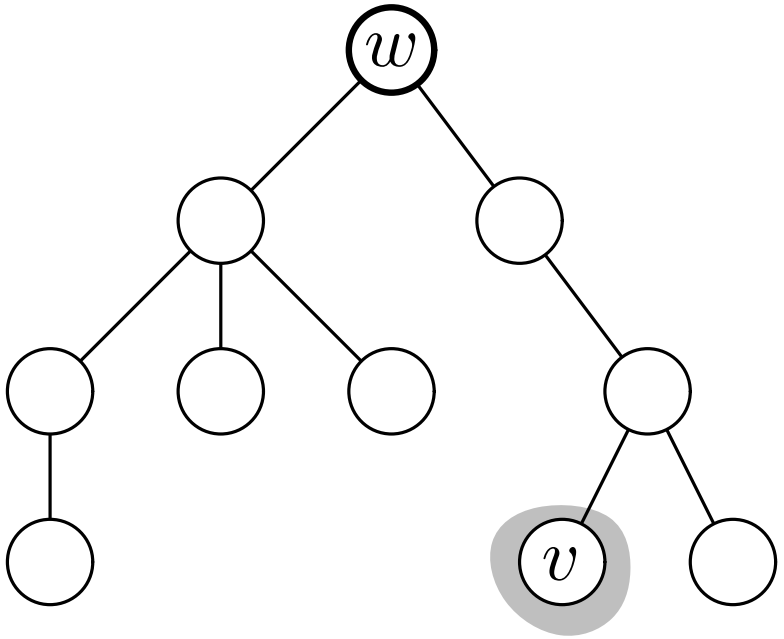

When v is a leaf: $A(v) = \ \omega(v)$ and $B(v) =$

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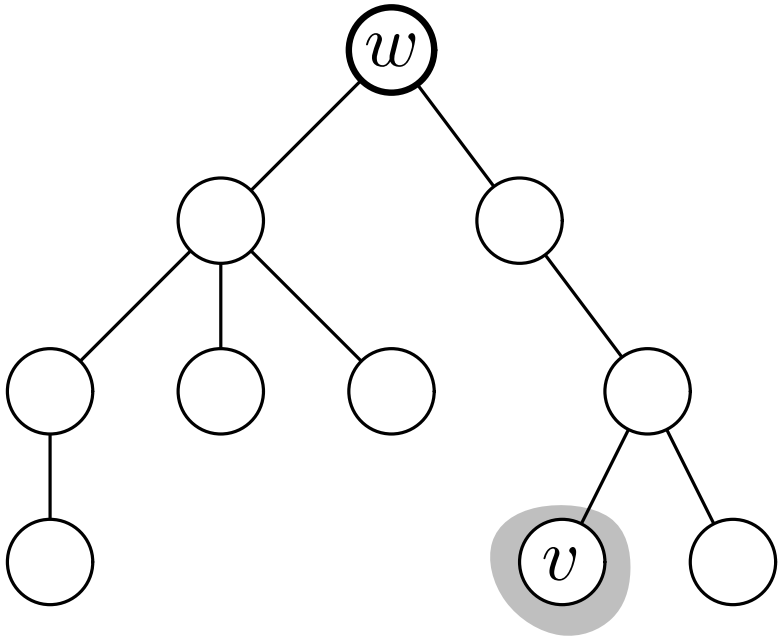

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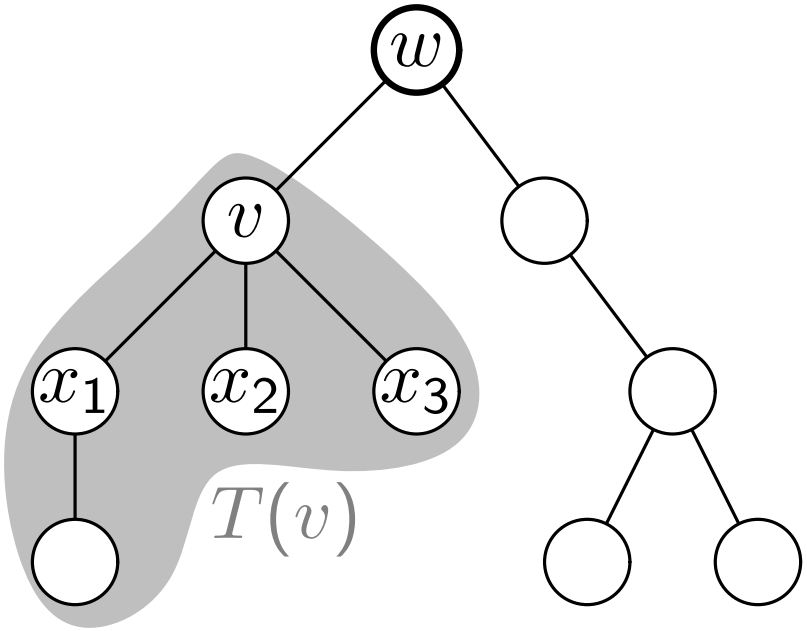

When v is a leaf: $A(v) = \omega(v)$ and $B(v) = 0$

When v has children $x_1, \ldots, x_r$:

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Independent Sets in Trees

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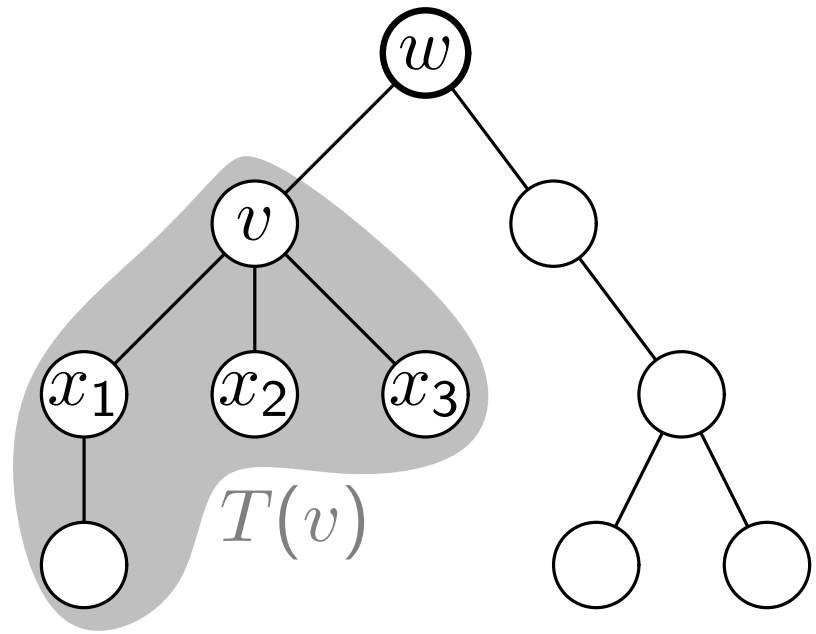

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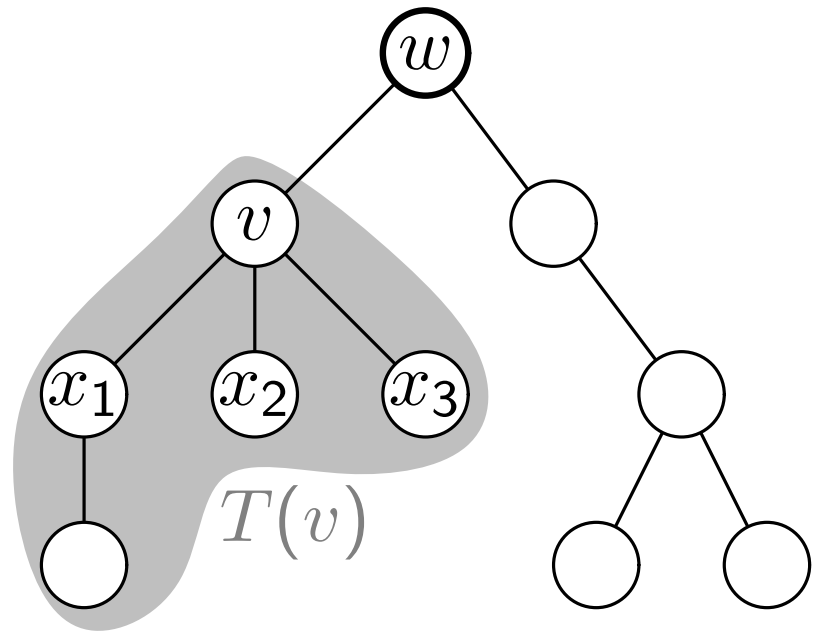

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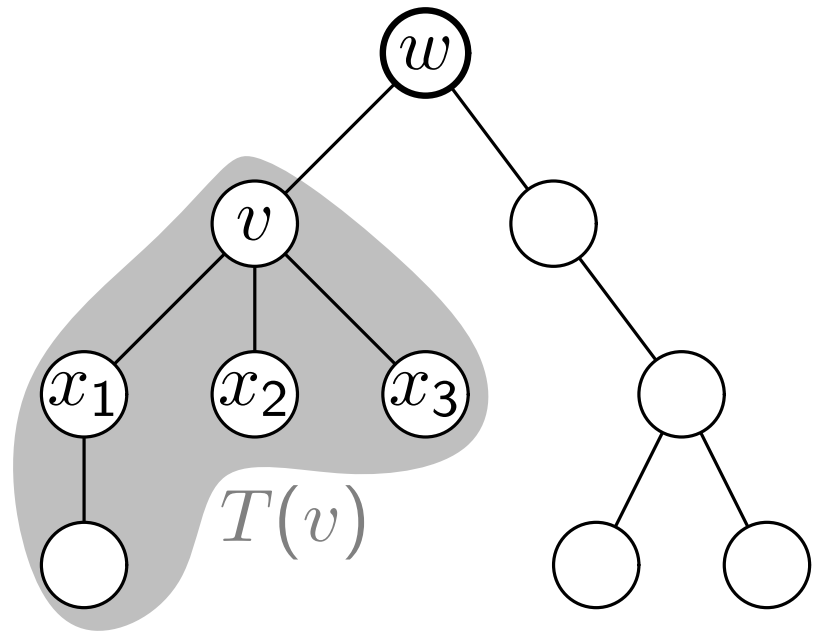

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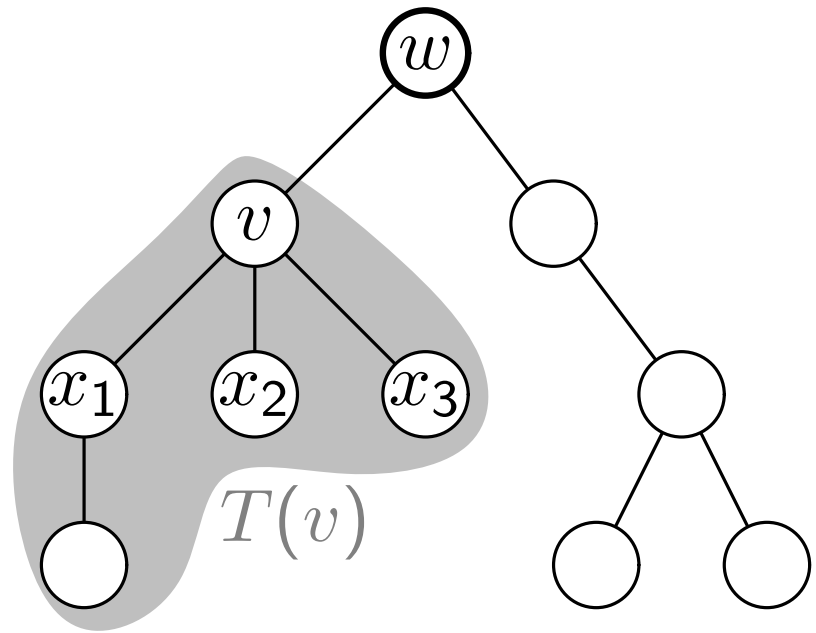

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When v has children $x_1, \ldots, x_r$:

$$A(v) = \max\{ \textstyle\sum_{i=1}^{r} A(x_i), \quad \omega(v) + \sum_{i=1}^{r} B(x_i) \}$$

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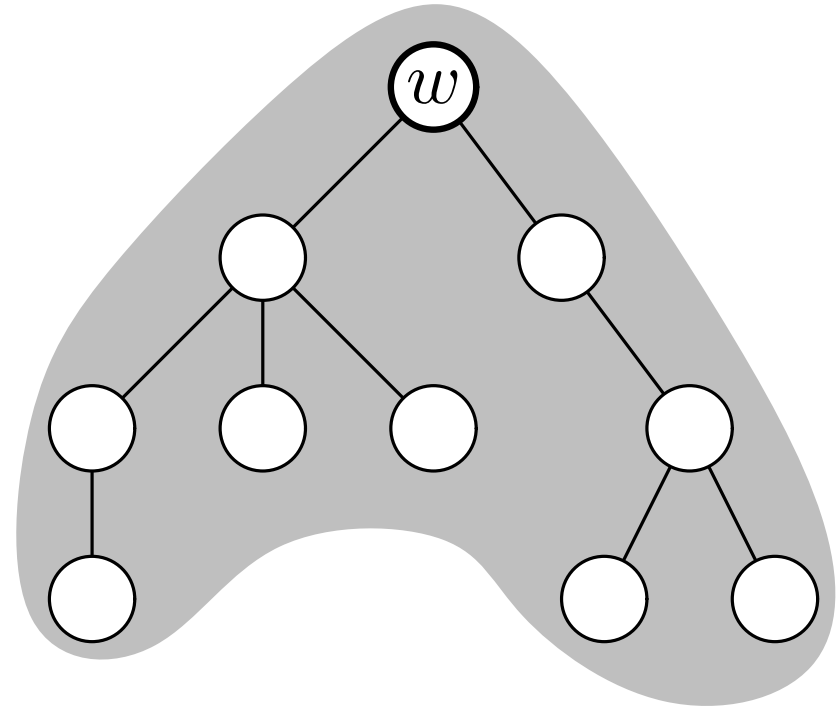
Independent Sets in Trees

$A(w) =$ solution



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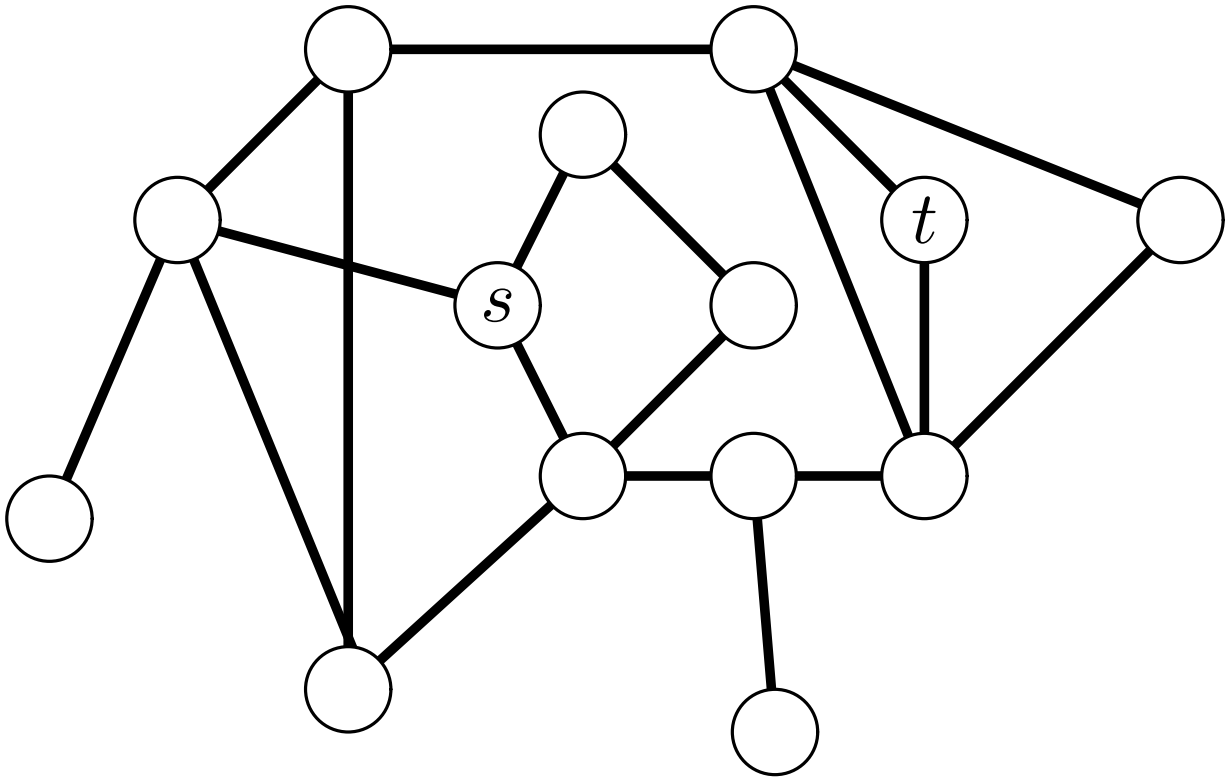
$B(v) = \sum_{i=1}^{r} A(x_i)$ **Algo:** Compute $A(\cdot)$ and $B(\cdot)$ bottom-up

s, t-series parallel graphs

Def.: A graph $G = (V, E)$ is *2-terminal* when it contains two special vertices s and t



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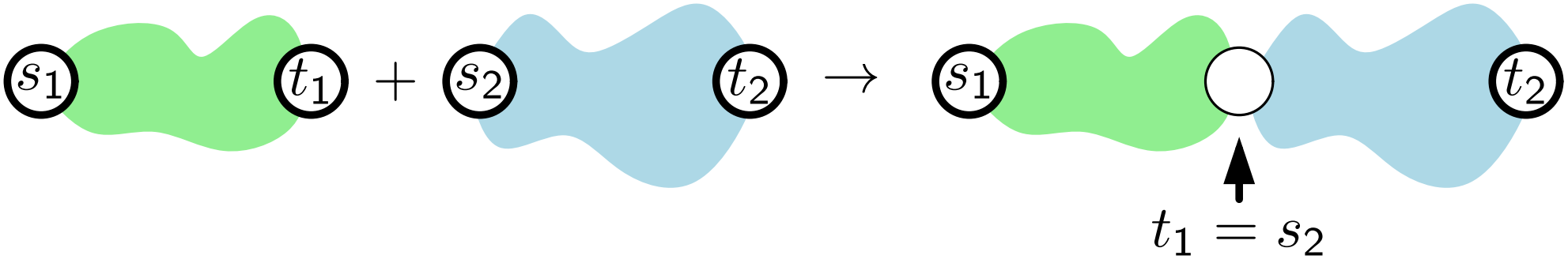
- G is a *series composition* of two series parallel graphs



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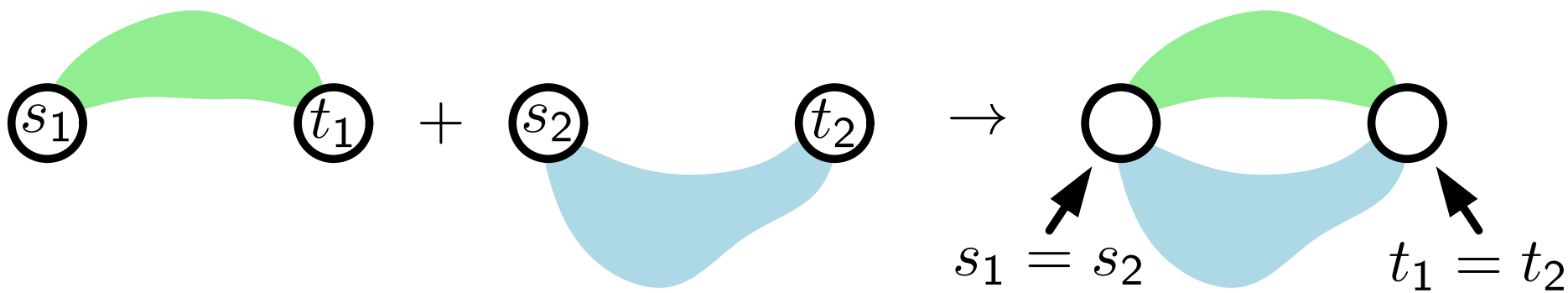

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recursive definition:
series parallel graphs have a natural tree-structure

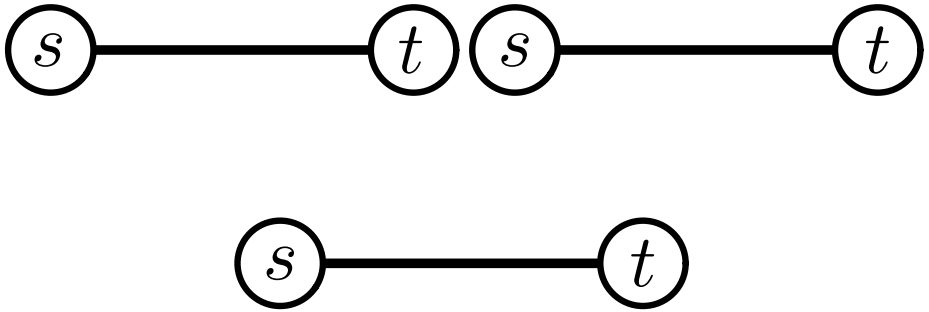
SP-tree

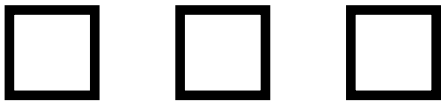
$\square$

SP-tree

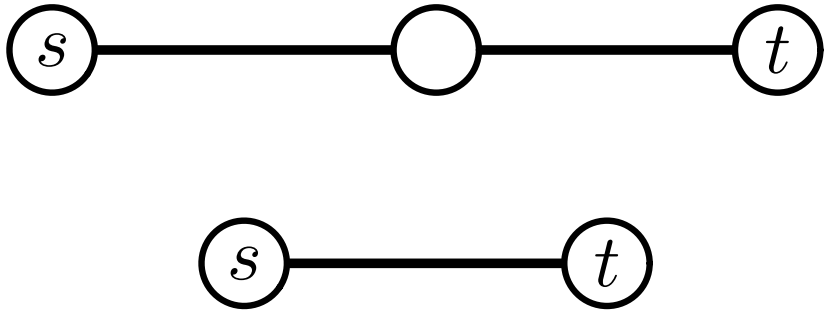
$\square \quad \square$

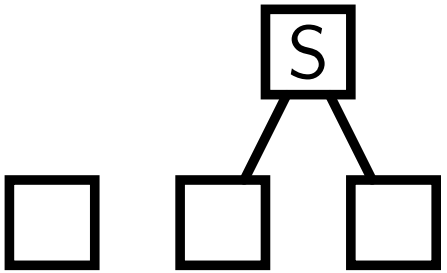
SP-tree





SP-tree





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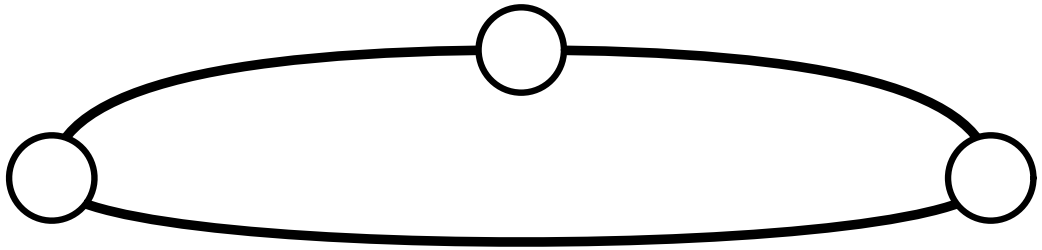

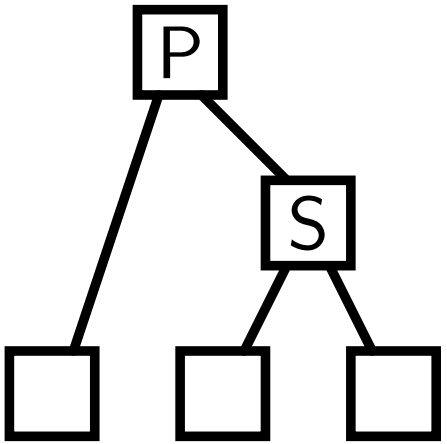

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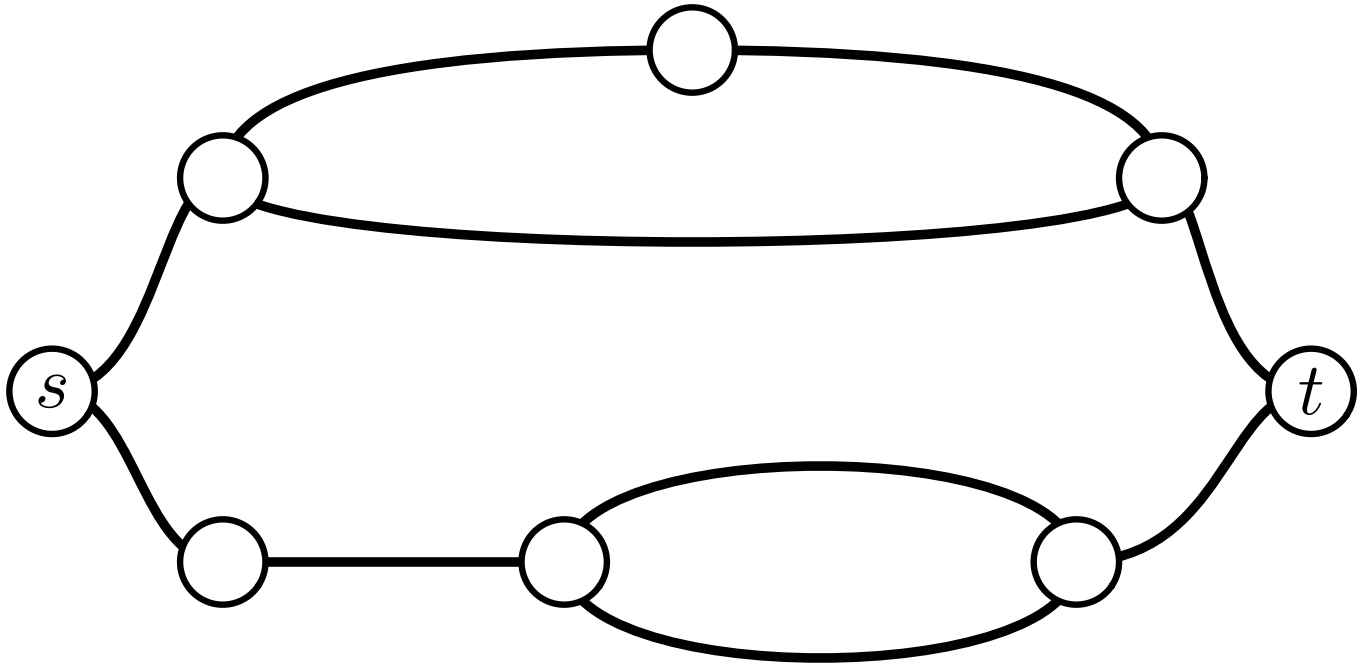

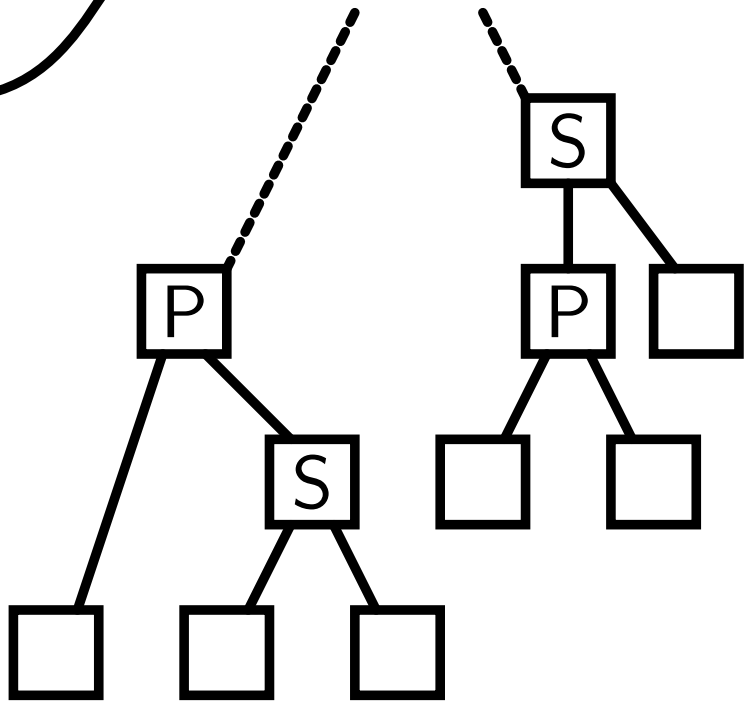

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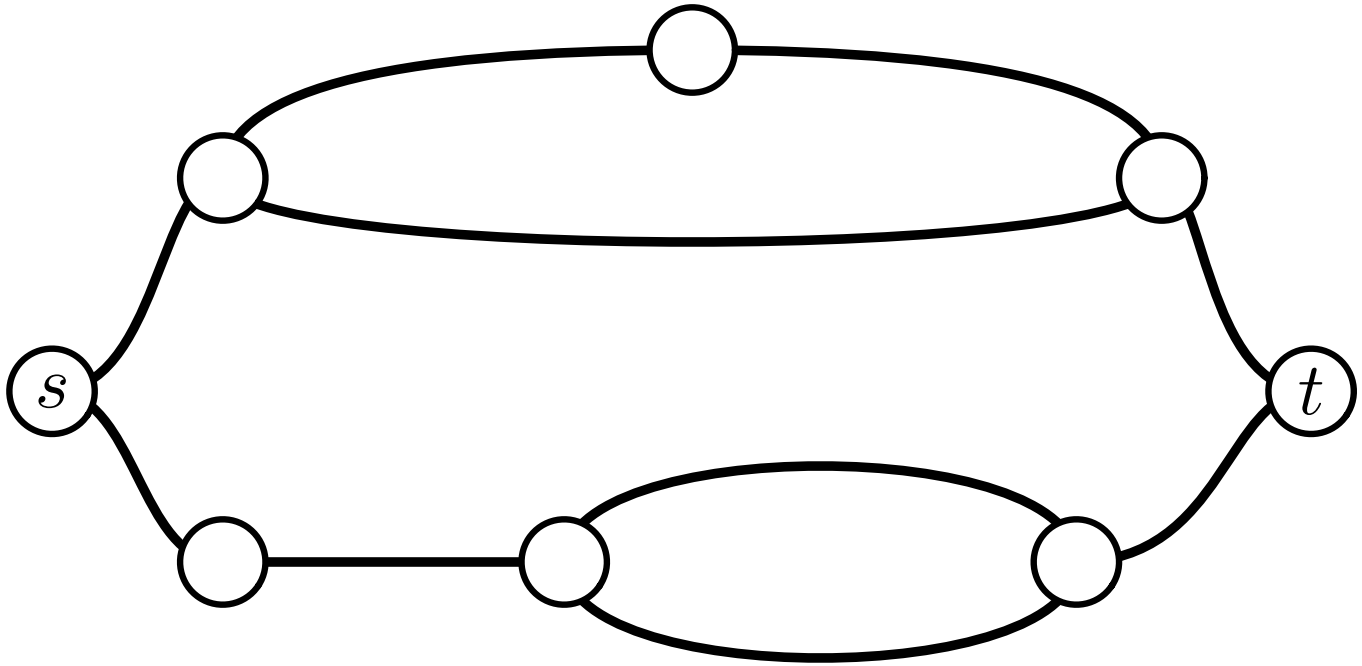
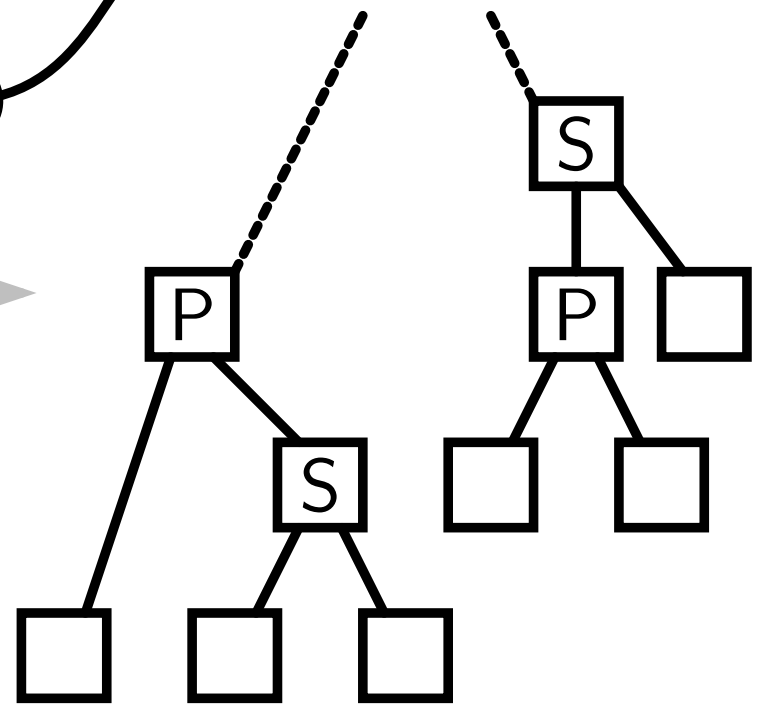


Let i be a node in an SP-tree.
$G(i) :=$ graph represented by the subtree rooted at i



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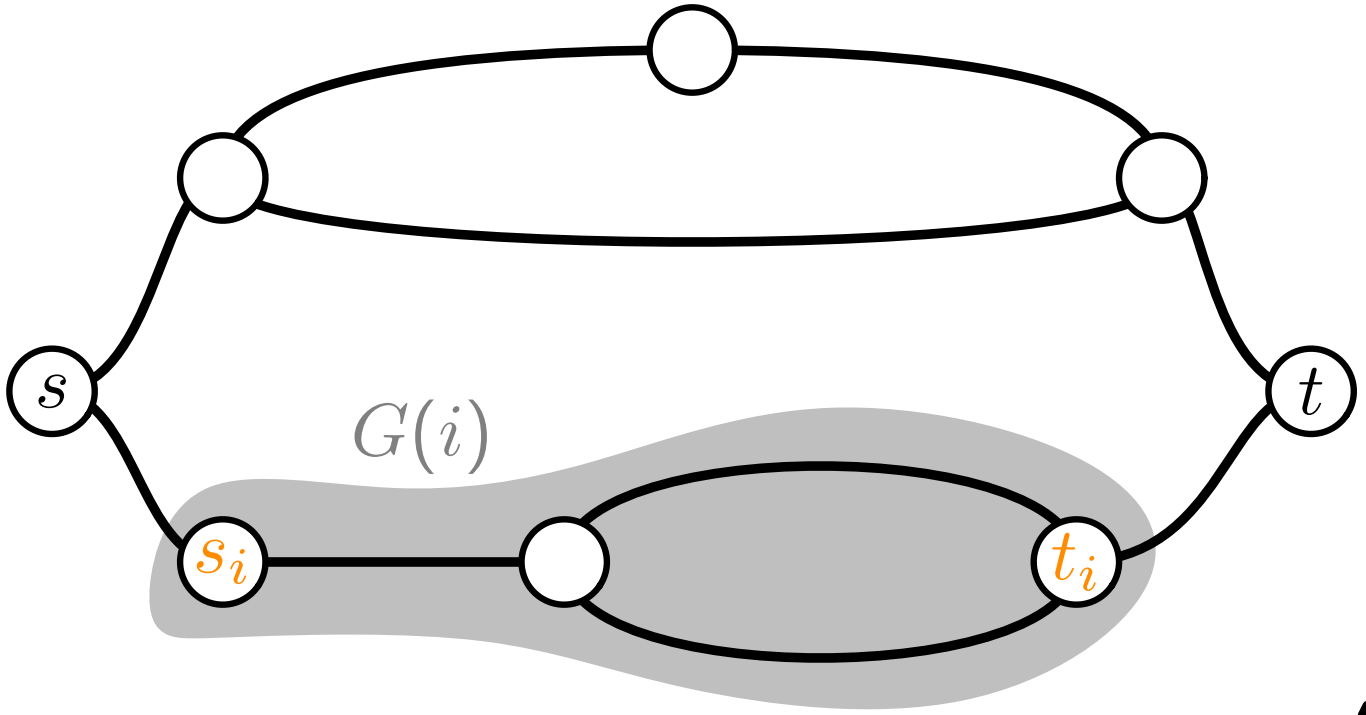

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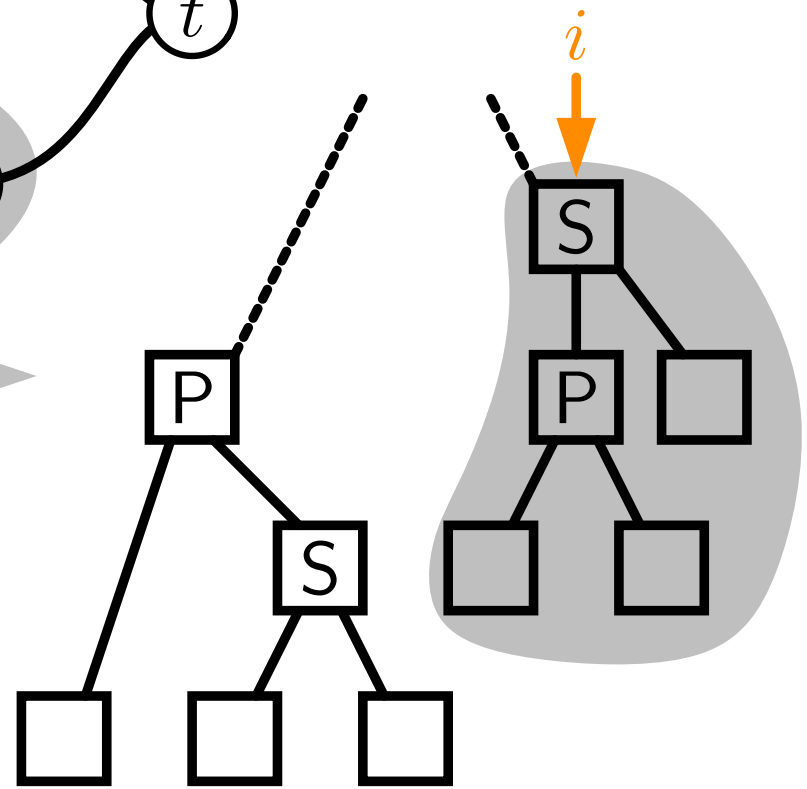
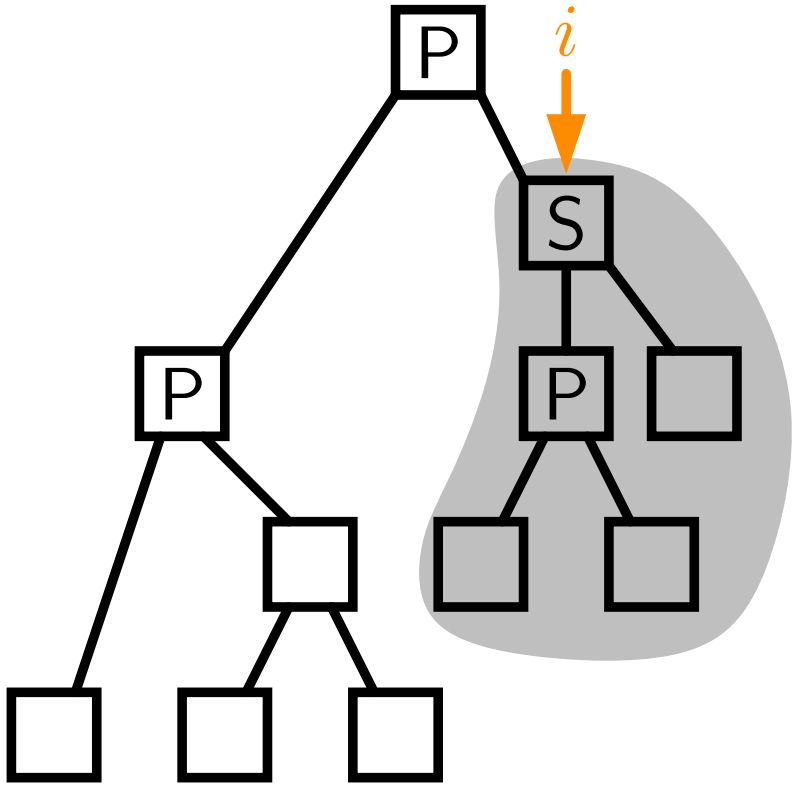


Independent Set on SP-trees

Dynamic program on SP-tree indexed by $G(i)$



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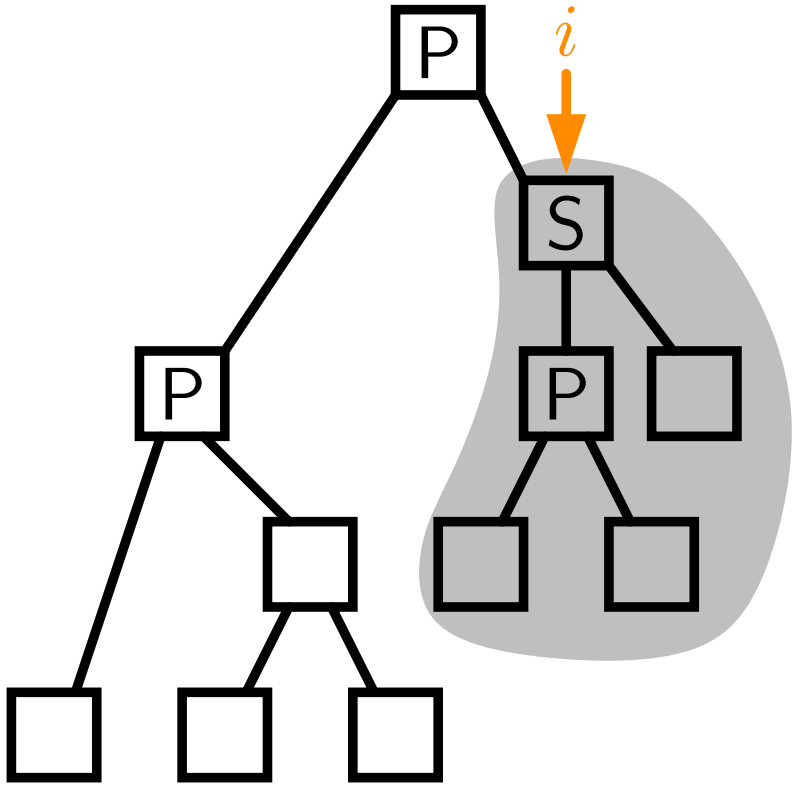

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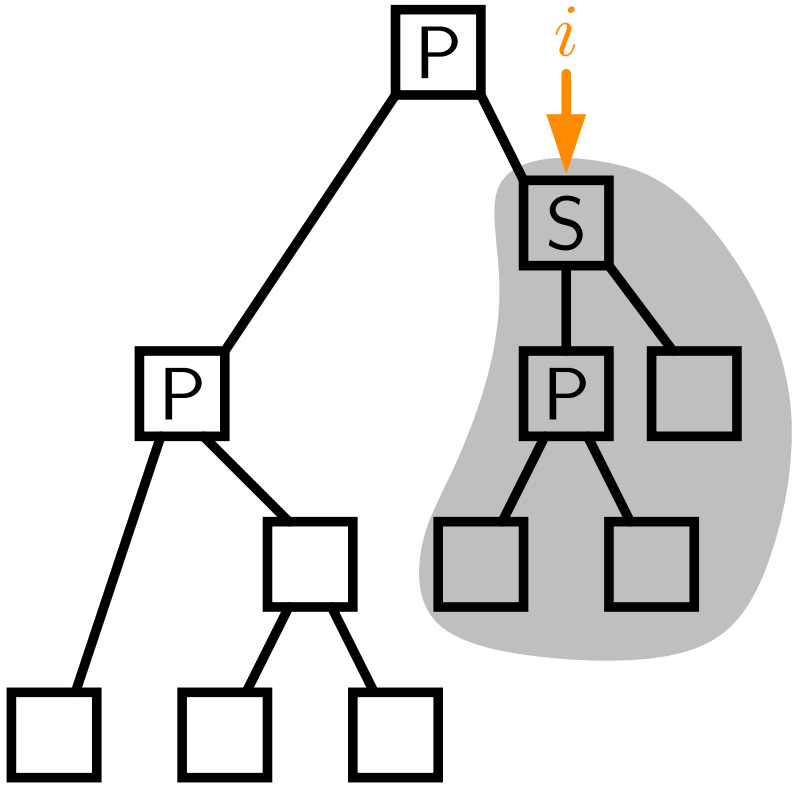

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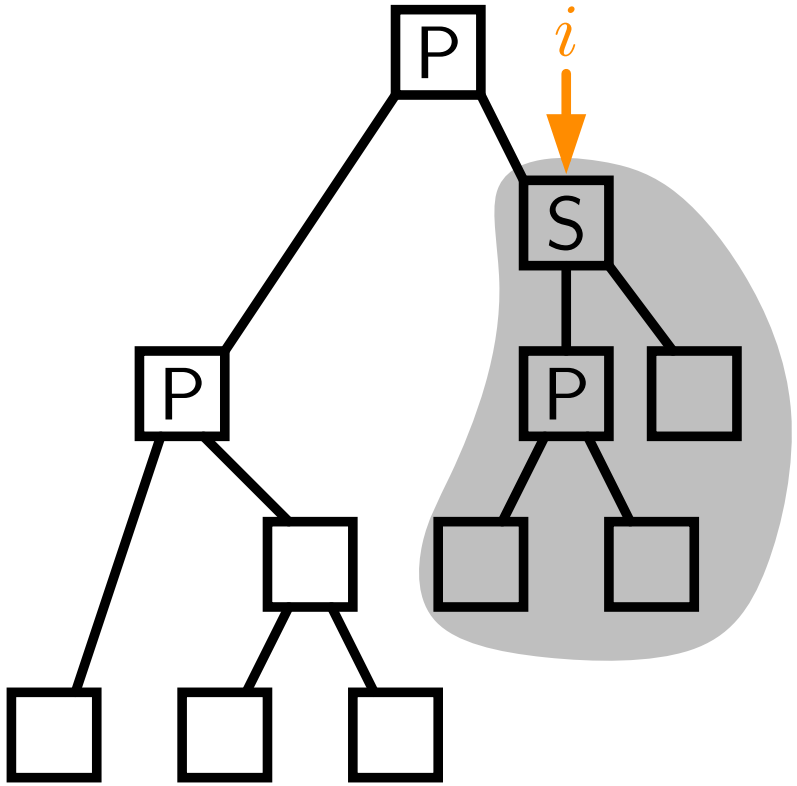
$AB(i)$ and $BB(i)$ def. similarly



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When i is a leaf...

$BB(i) =$

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$$BA(i) = \omega(t_i)$$
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$$\begin{aligned} AB(i) &= \\ BA(i) &= \omega(t_i) \\ BB(i) &= 0 \end{aligned}$$

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Independent Set on SP-trees

Dynamic program on SP-tree indexed by $G(i)$

$AA(i) :=$ maximum weight independent set S in $G(i)$ where $s_i \in S$ and $t_i \in S$

$BA(i) :=$ maximum weight independent set S in $G(i)$ where $s_i \notin S$ and $t_i \in S$

$AB(i)$ and $BB(i)$ def. similarly

When i is a leaf...

$$\begin{aligned} AA(i) &= -\infty \\ AB(i) &= \omega(s_i) \\ BA(i) &= \omega(t_i) \\ BB(i) &= 0 \end{aligned}$$

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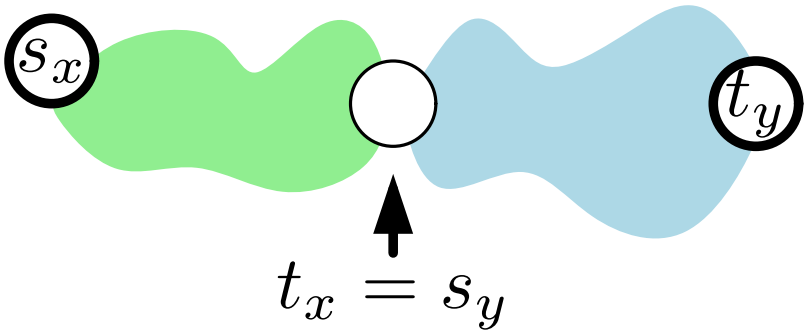
When i is a series composition with children x and y, ...



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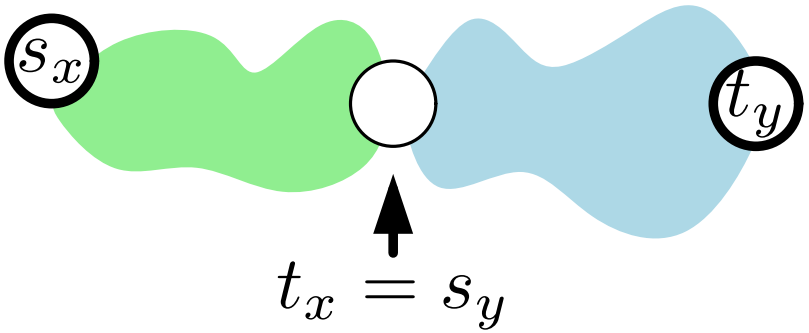

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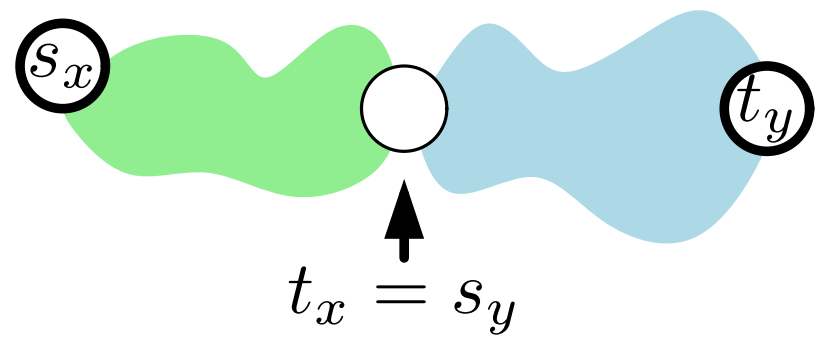

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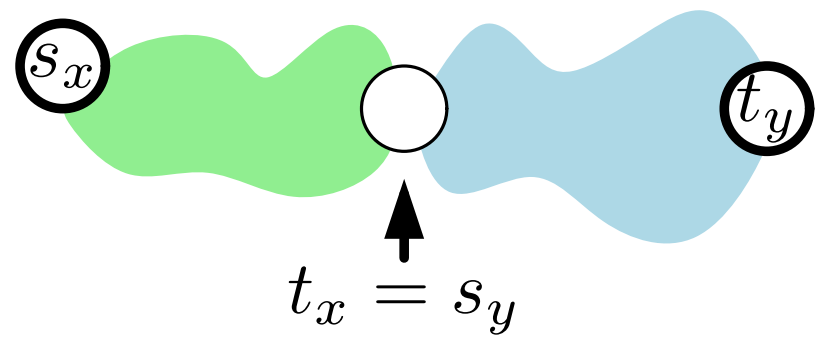

$$AA(i) = \max\{\ AB(x) + BA(y),\ AA(x) + AA(y) - \omega(t_x)\ \}$$

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$O(1)$ time per SP-node

Thm: INDEPENDENT SET on series parallel graphs with a given SP-tree can be solved in $O(n)$ time.

Generalization?

Many ways to generalize the concept of having a "tree structure"

Ex.: k-terminal graph $G = (V, E, T)$, $|T| = k$

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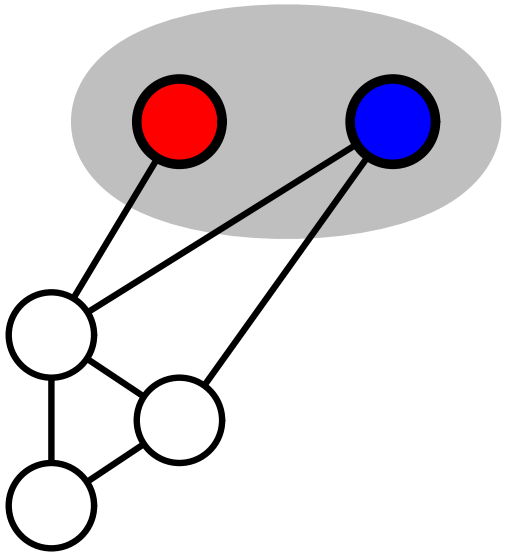

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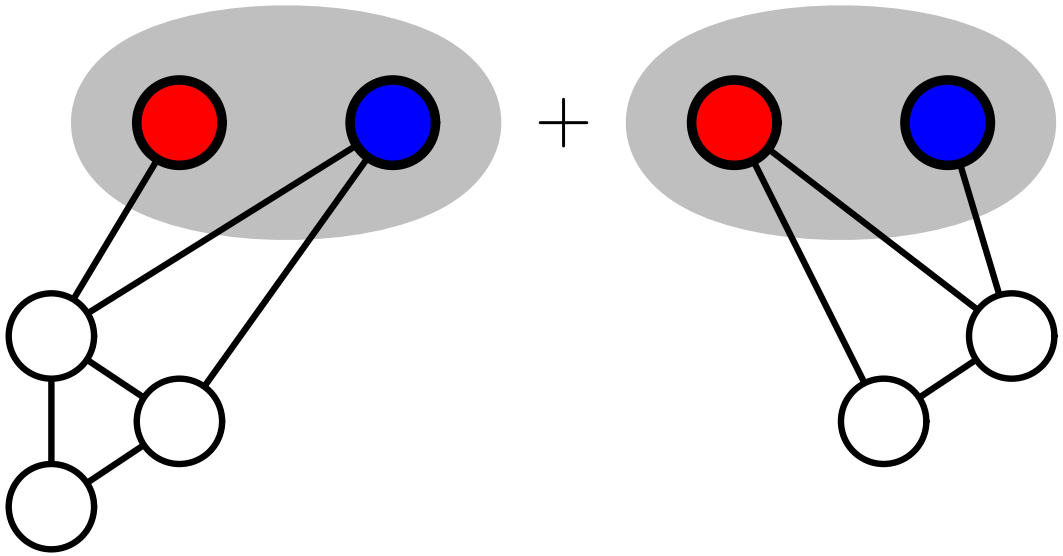

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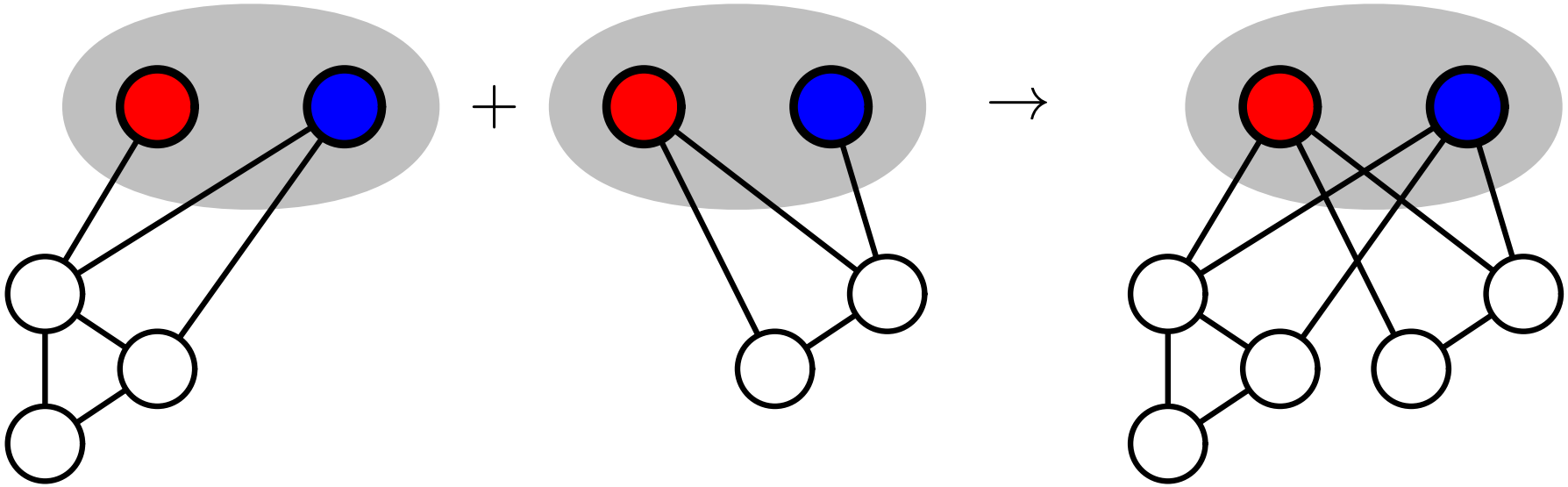
Example Operation: "gluing"

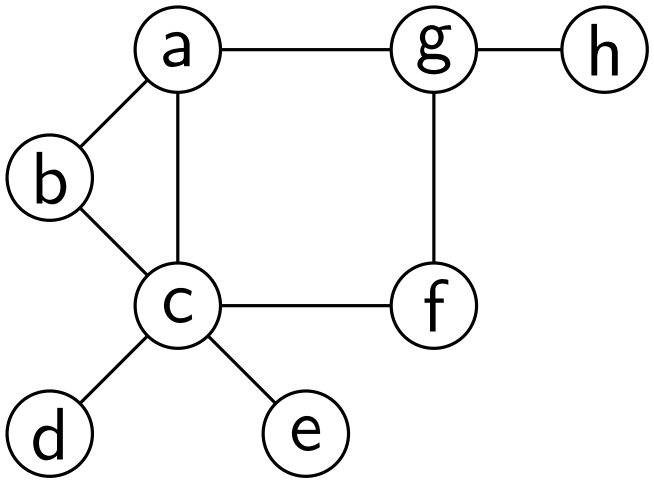


Example: Tree Decomposition

"bags"

Tree Decomposition is a tree where the nodes map to subsets of V so that...

Graph $G = (V, E)$:



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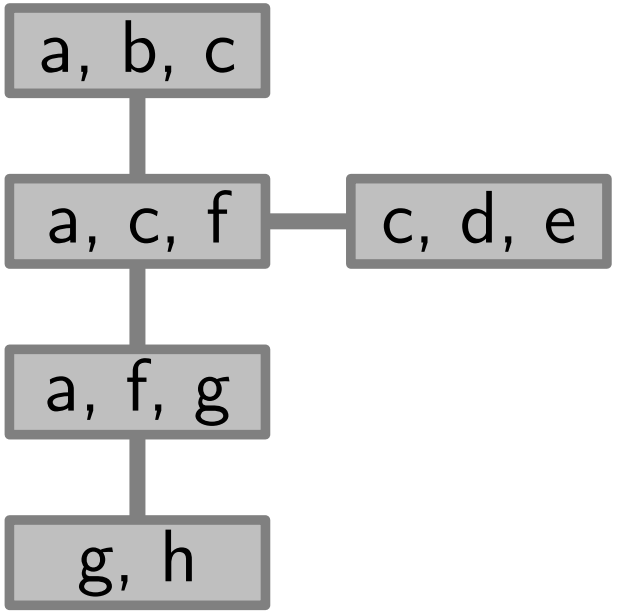

Example: Tree Decomposition

1 each vertex belongs to at least one bag

these bags are connected

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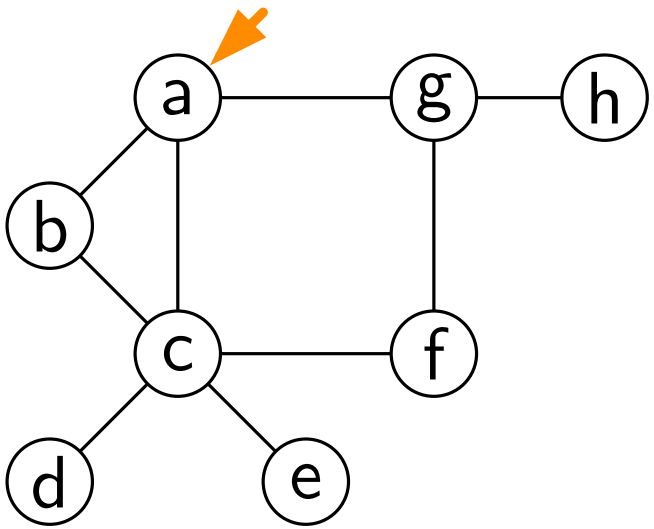

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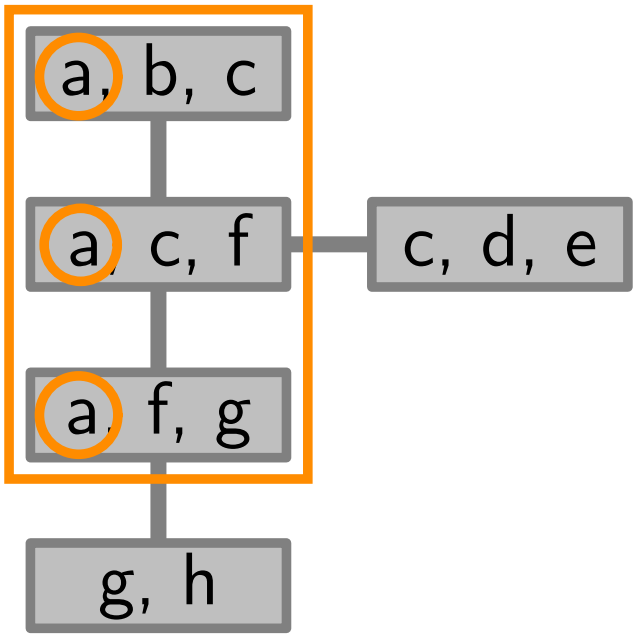

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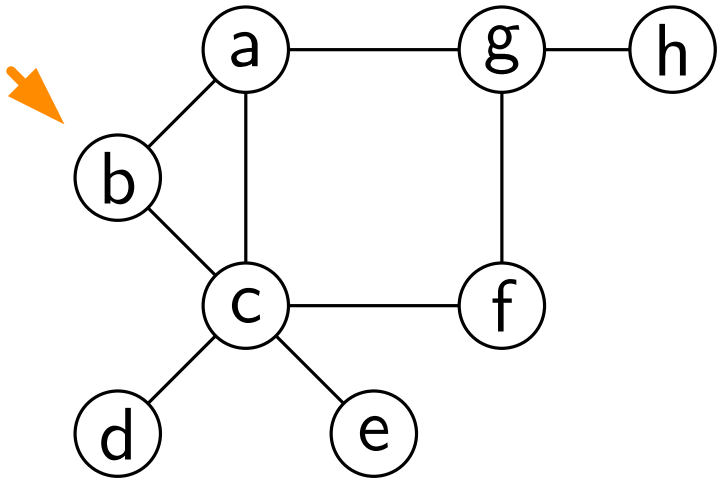

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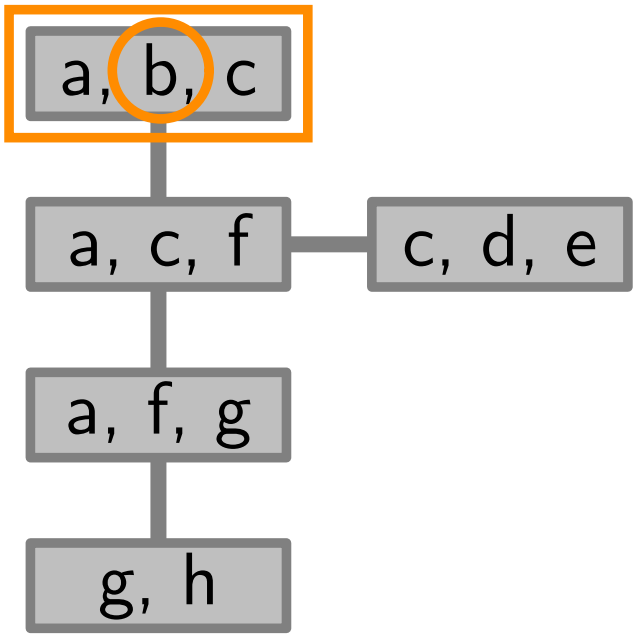

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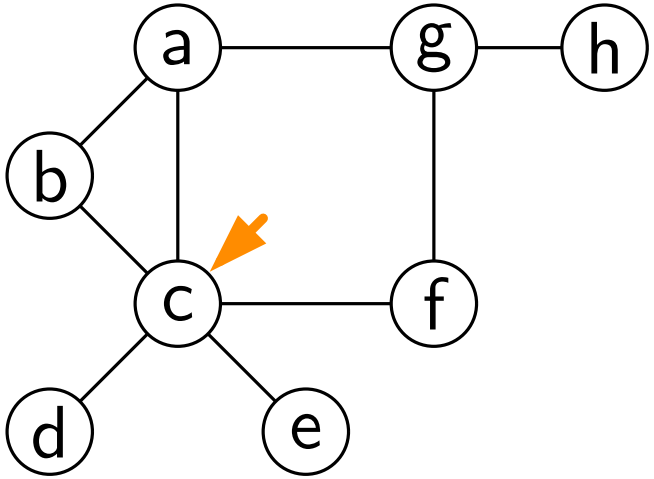

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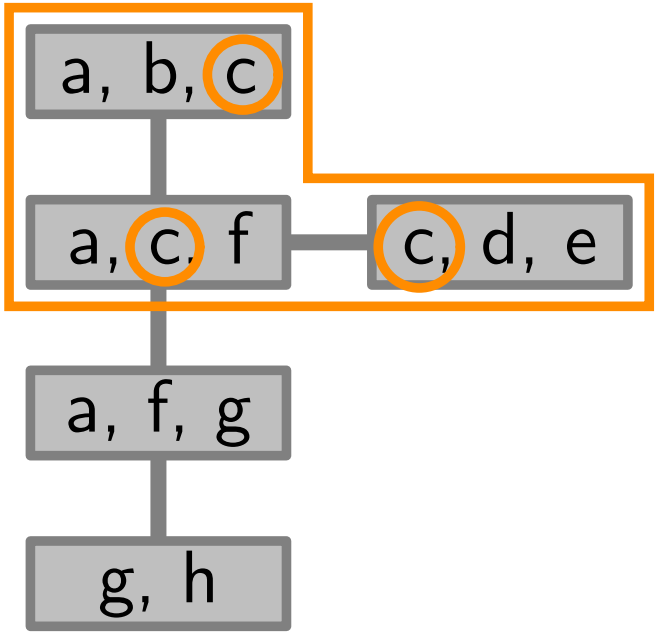

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2 each edge is contained in at least one bag

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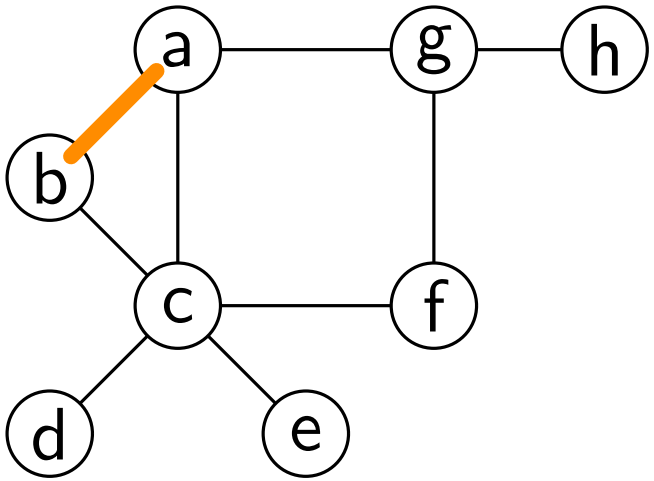

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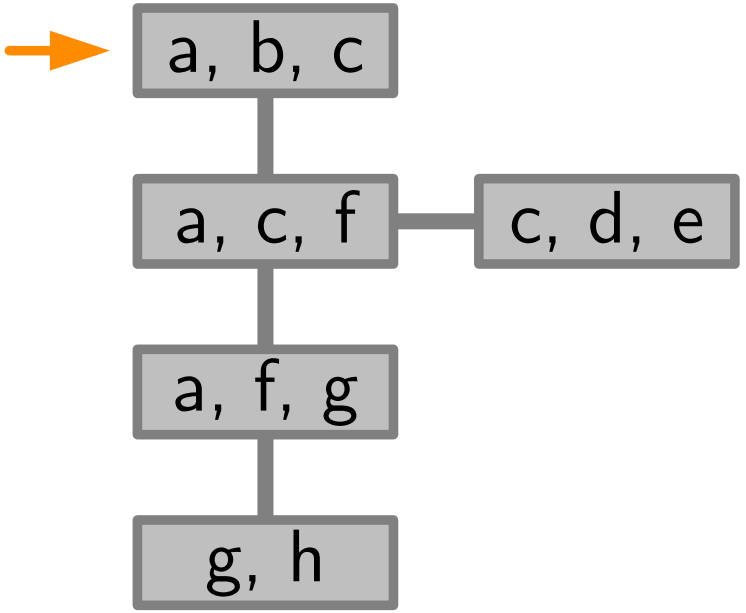

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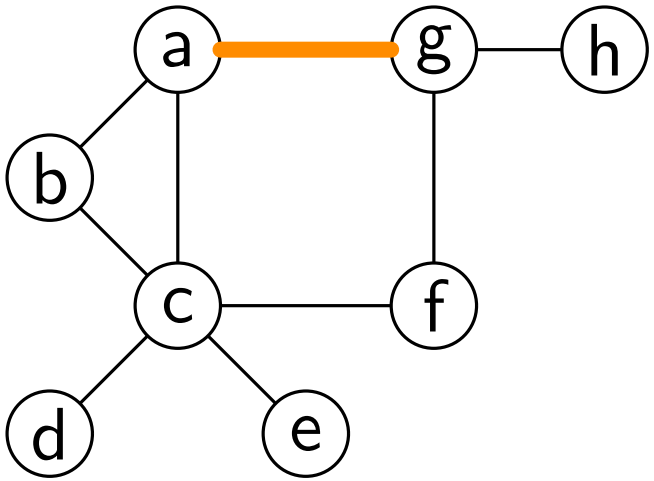

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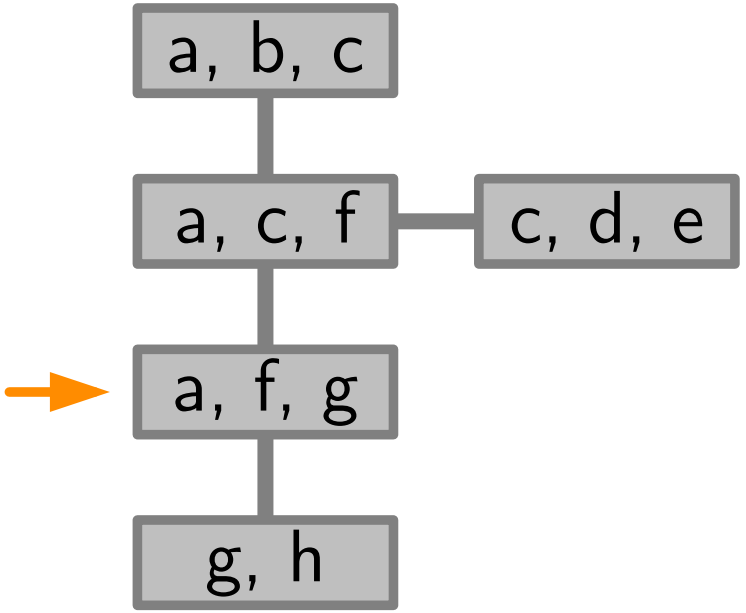

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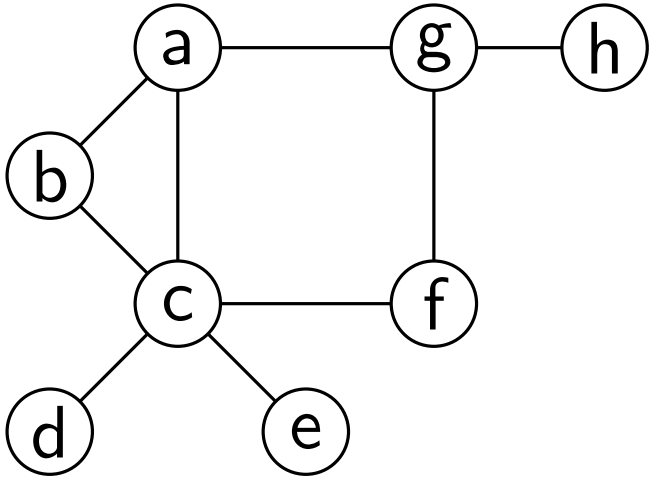

Tree Decomposition:

a, b, c,

d, e, f,

g, h

Example: Tree Decomposition

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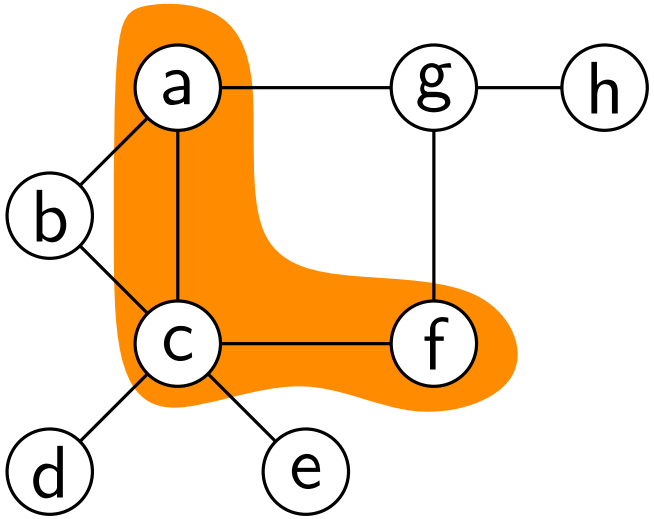

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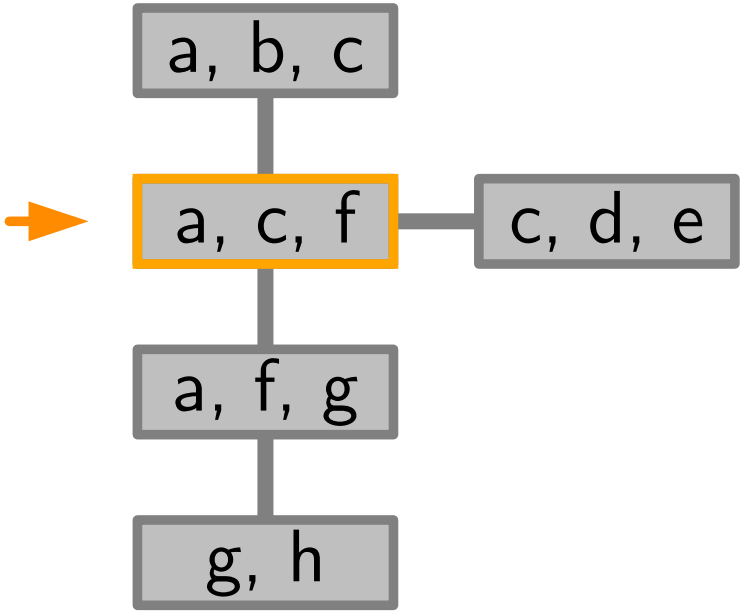

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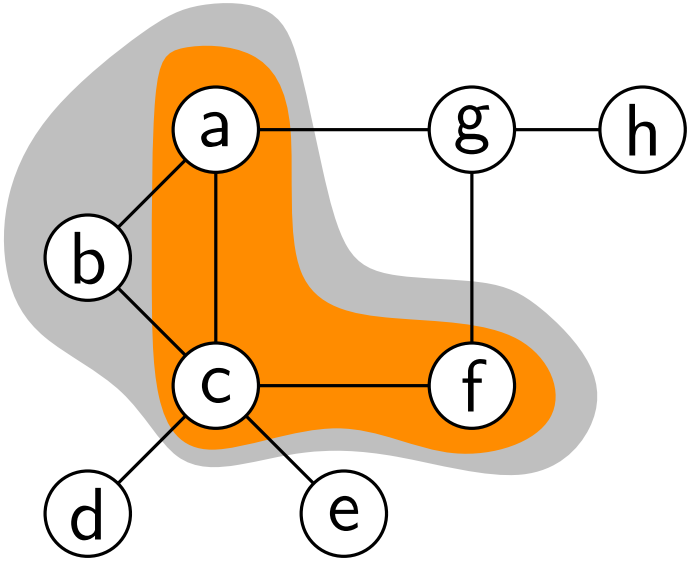

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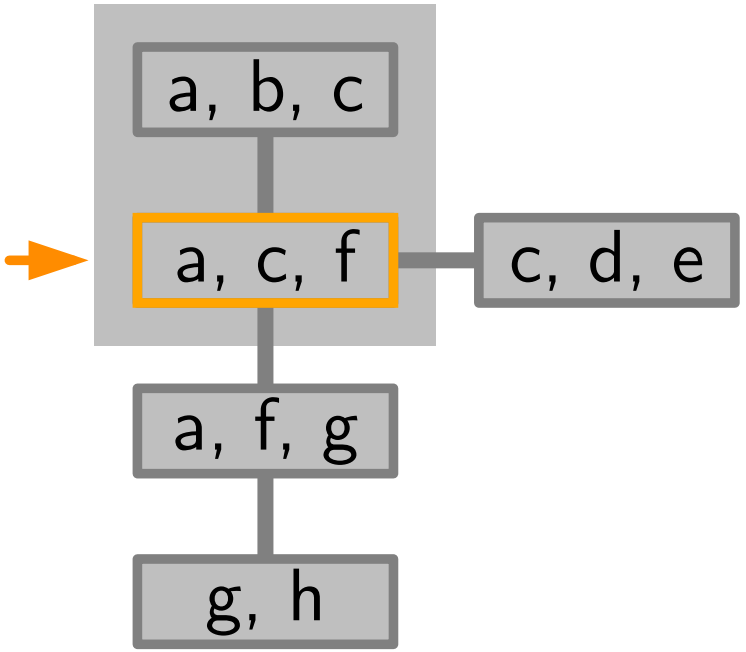

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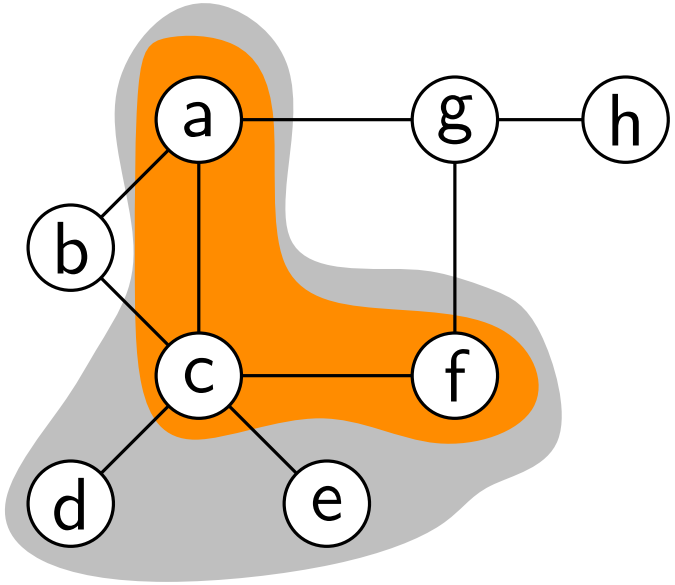

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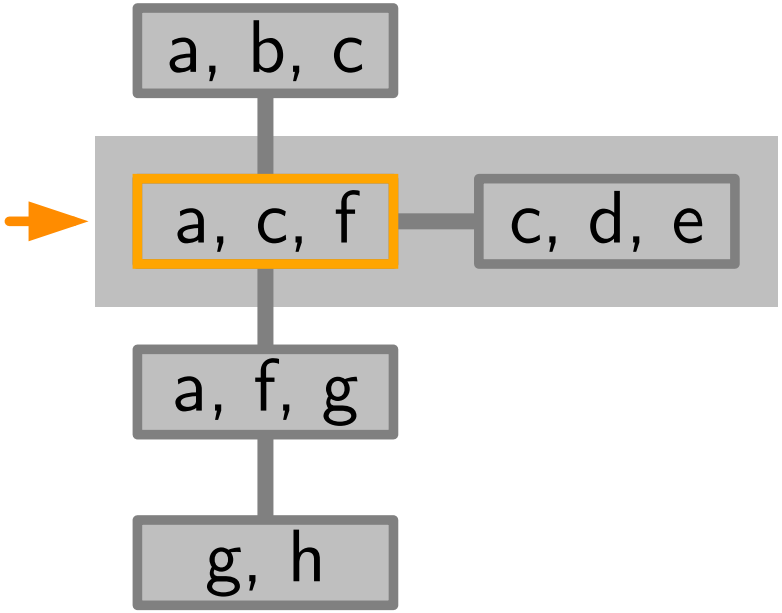

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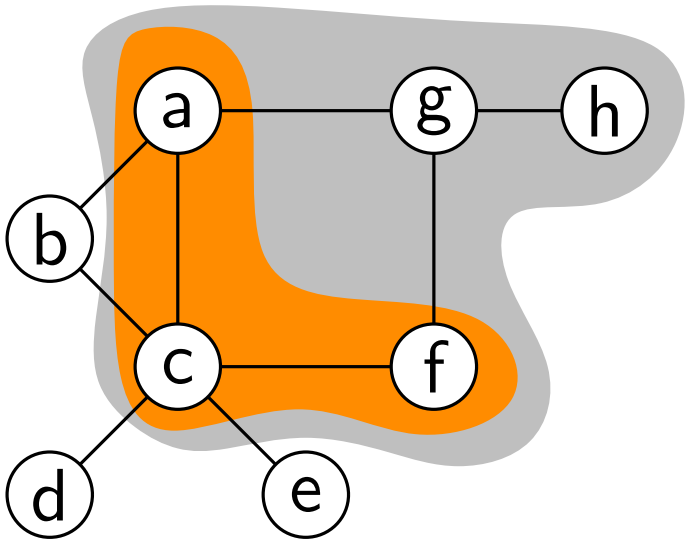

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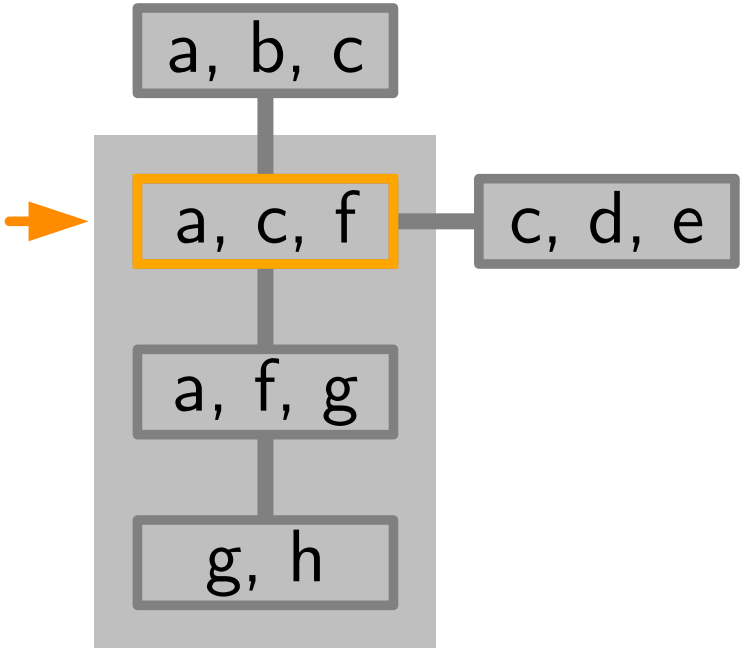

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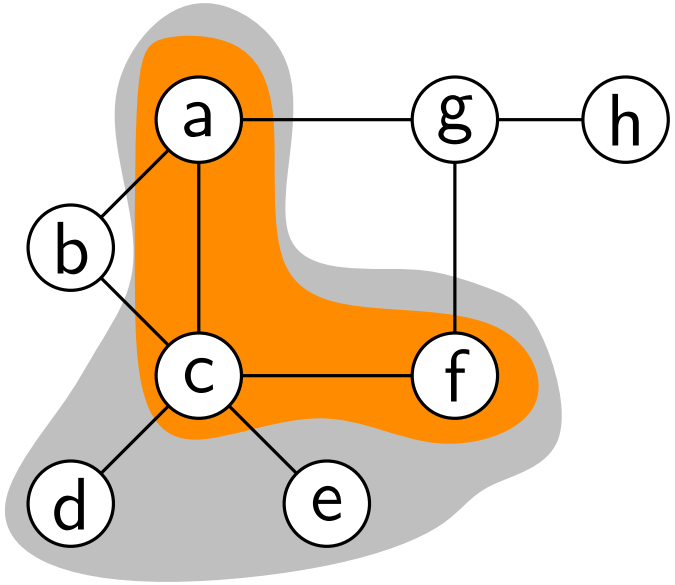

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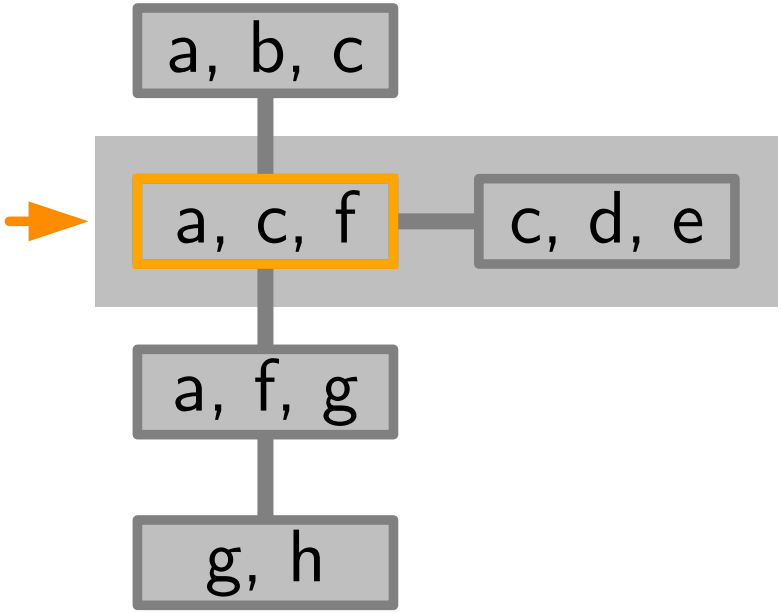

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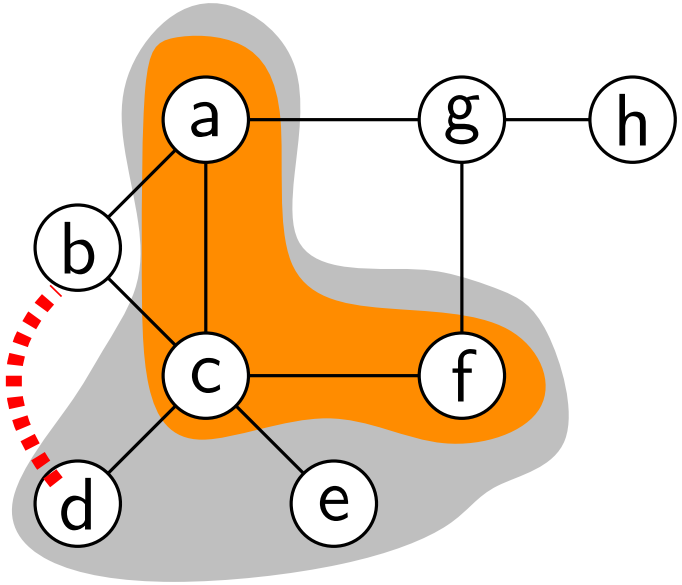

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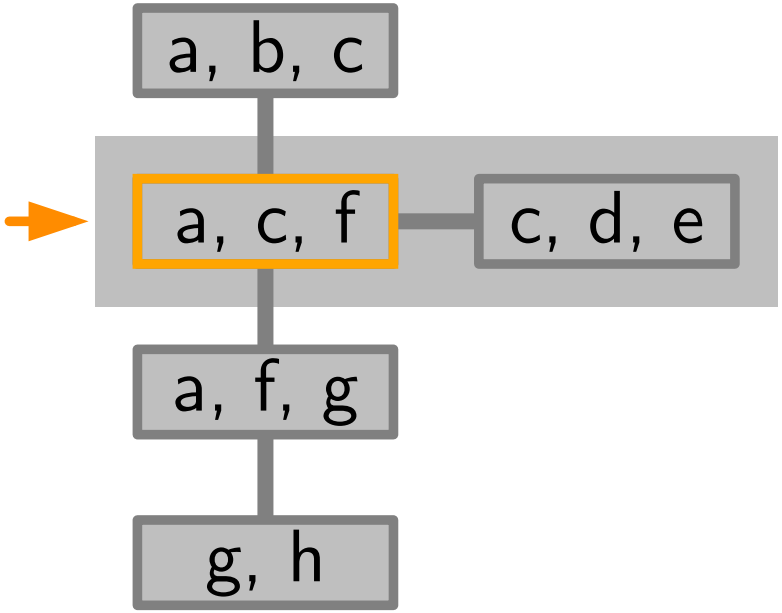

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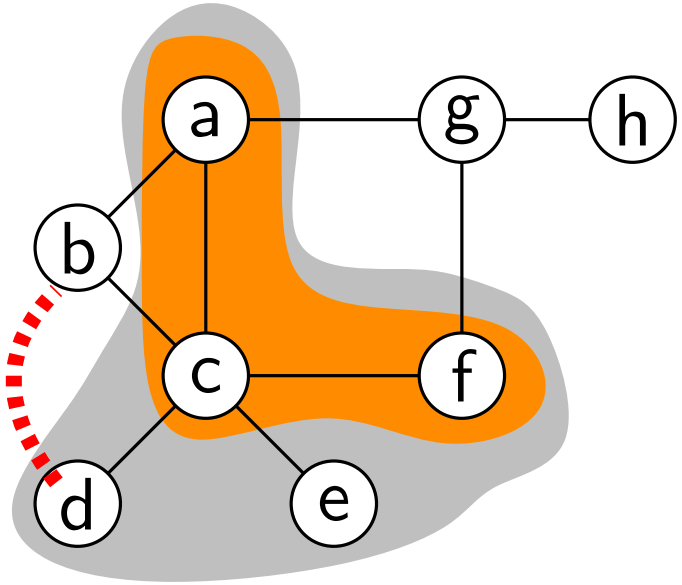

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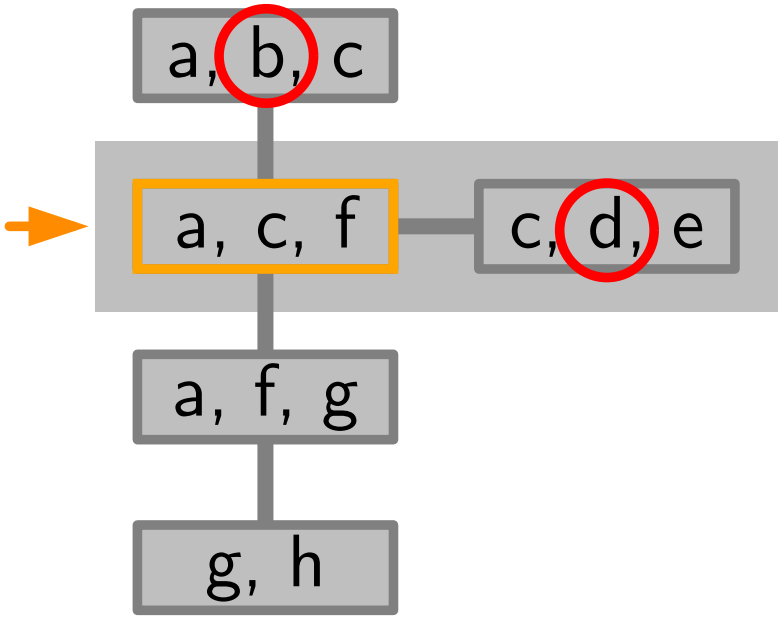

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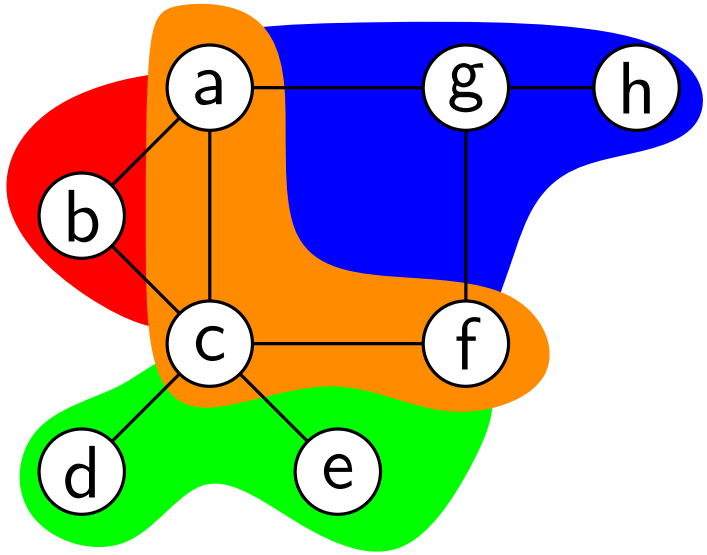

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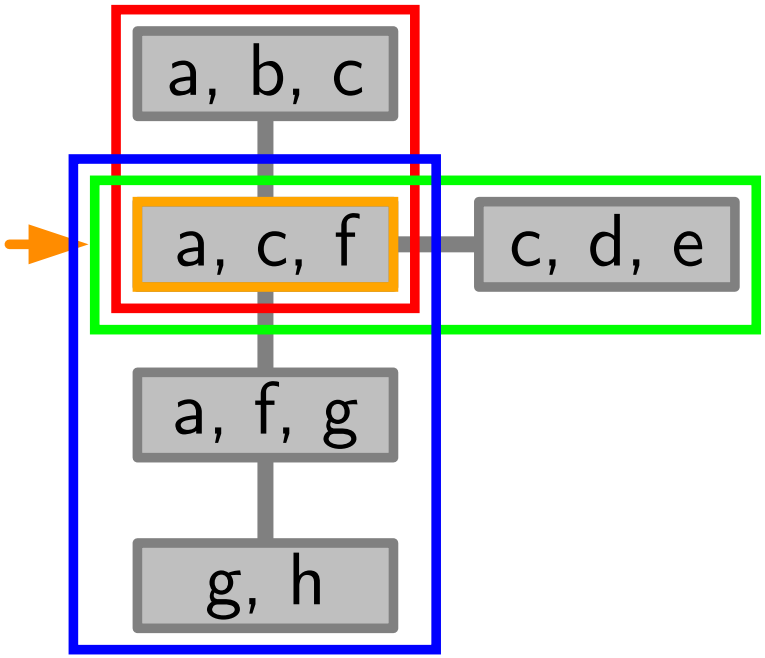

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Thm: There is a tree decomposition of width $\mathrm{tw}(G)$ where $|P|$ is polynomial in n, i.e., the tree has polynomial size in n

Parameterized Problems

Given: Instance of size n and parameter k

Def. Problem is FPT when solvable in $O(f(k) \cdot poly(n))$ time.

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Ex.: k-Vertex Cover

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Given: Instance of size n and parameter k

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See **PA §13.3**

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"natural parameterization"

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Independent Set (treewidth)

Given: Graph G, number k
Parameter: $\mathrm{tw}(G)$
Question: Does G have an independent set of size $\geq k$?

Parameterized Problems

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Channel Assignment (treewidth)	

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Given: Graph $G = (V, E)$, number k

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actually fixed-parameter linear: $O(f(k)n)$
algorithm is constructive (provides an optimal tree decomp.)

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How can we make "fixed-treewidth-tractable" algorithms?

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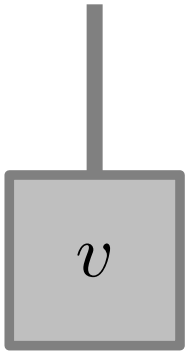
- **Leaf:** the bag is a leaf and contains only one vertex



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- **Leaf:** the bag is a leaf and contains only one vertex
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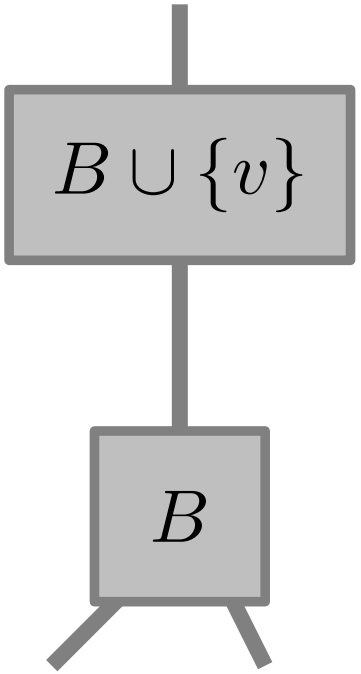
  The bag has exactly one child and contains the child's vertices and exactly one new vertex.



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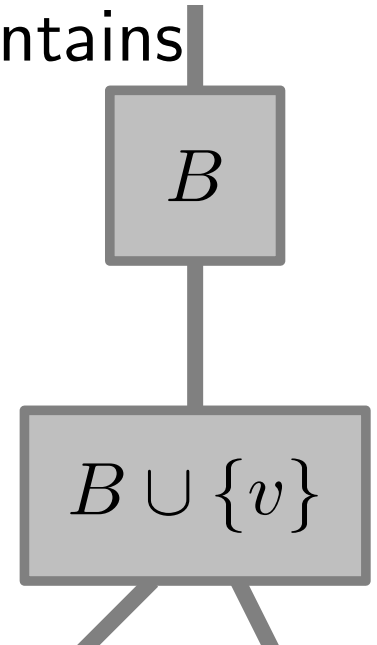
- **Forget:**
  The bag has exactly one child and contains one vertex fewer than the child.



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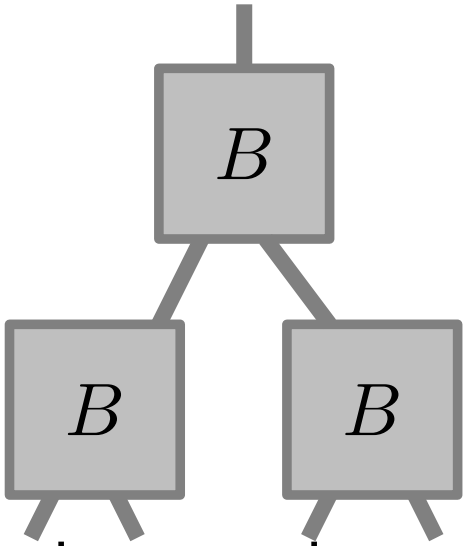
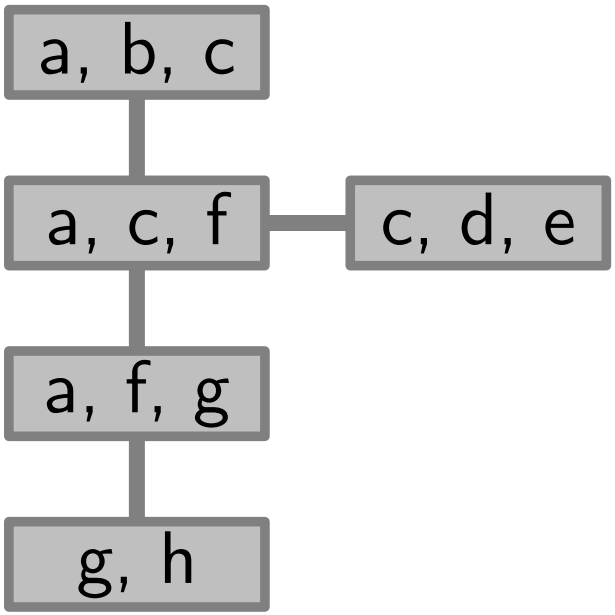
- **Join:**
  The bag has exactly two children and these three nodes have exactly the same vertices



item #1: nice tree decompositoins

Thm: For each tree decomposition, there is a nice tree decomposition of the same width and polynomially many more bags. This can also be constructed in polynomial time.



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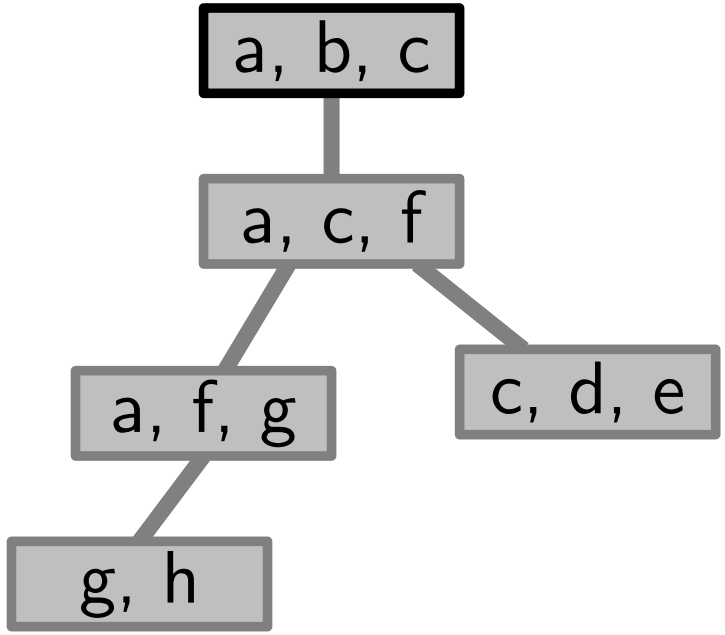

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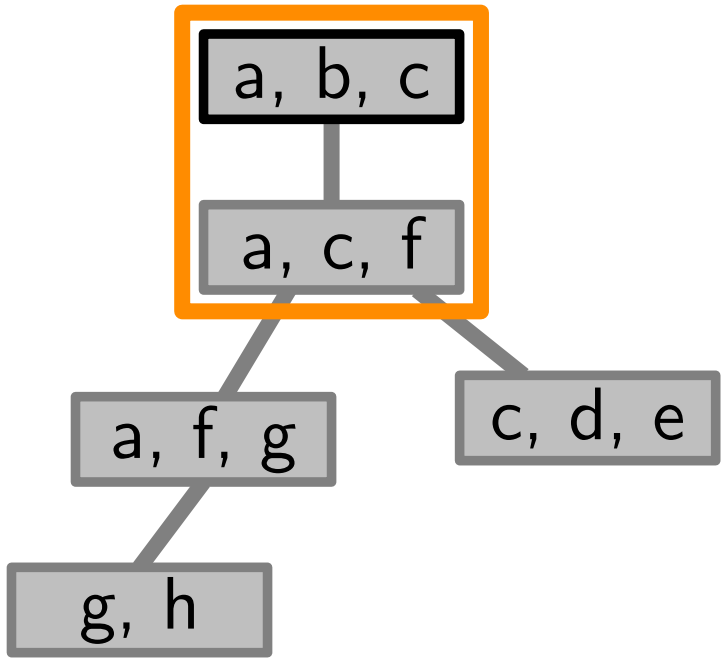

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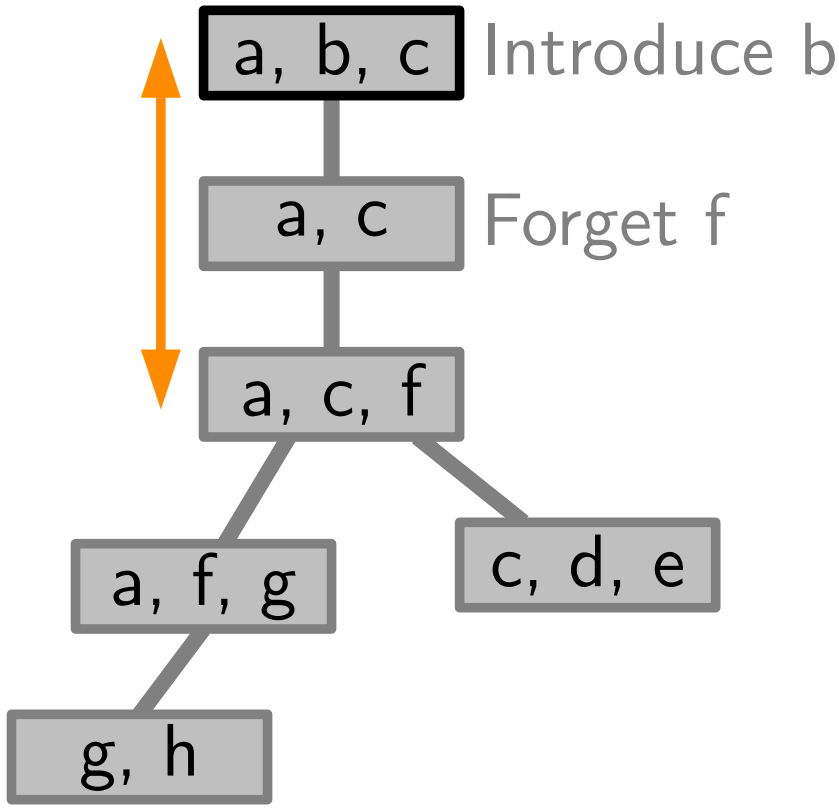

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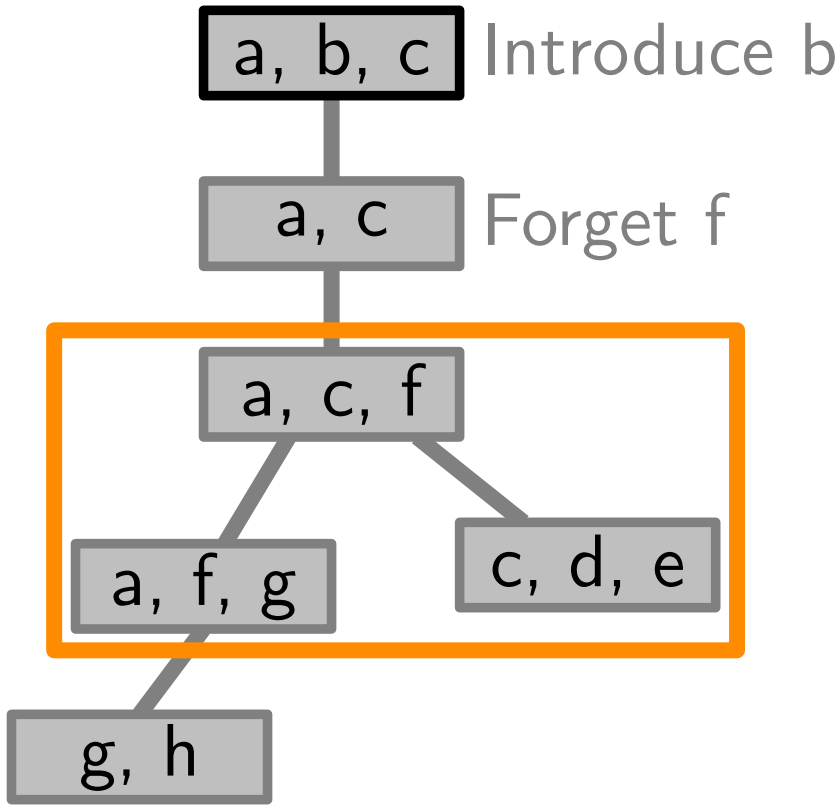

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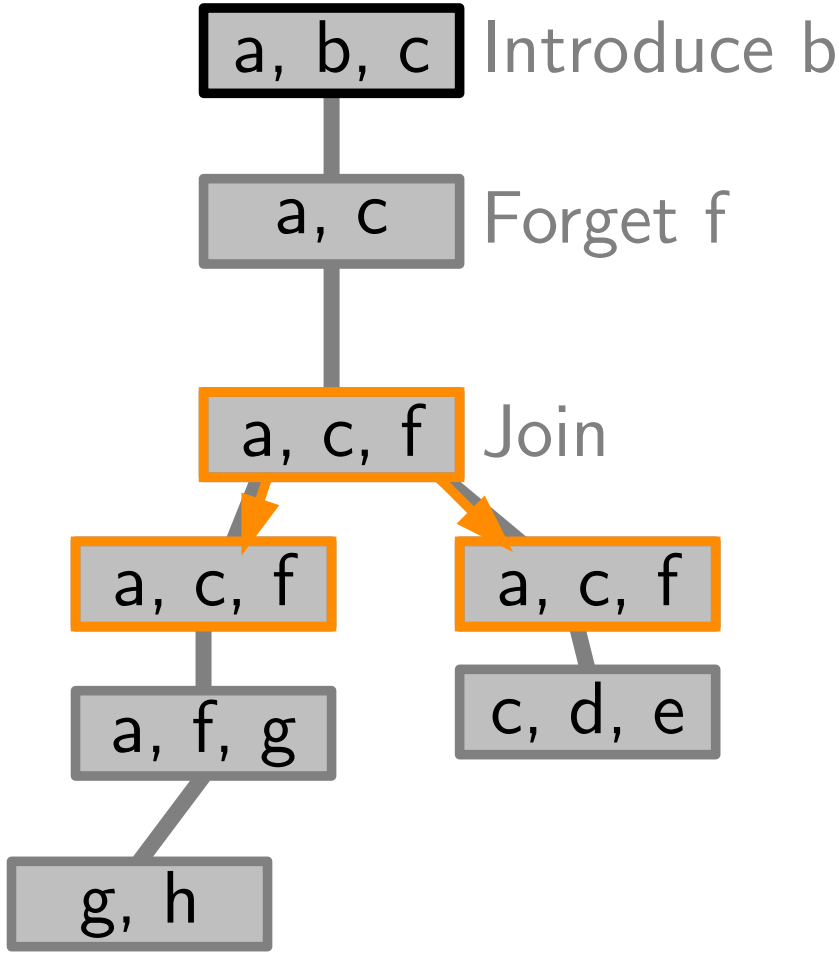

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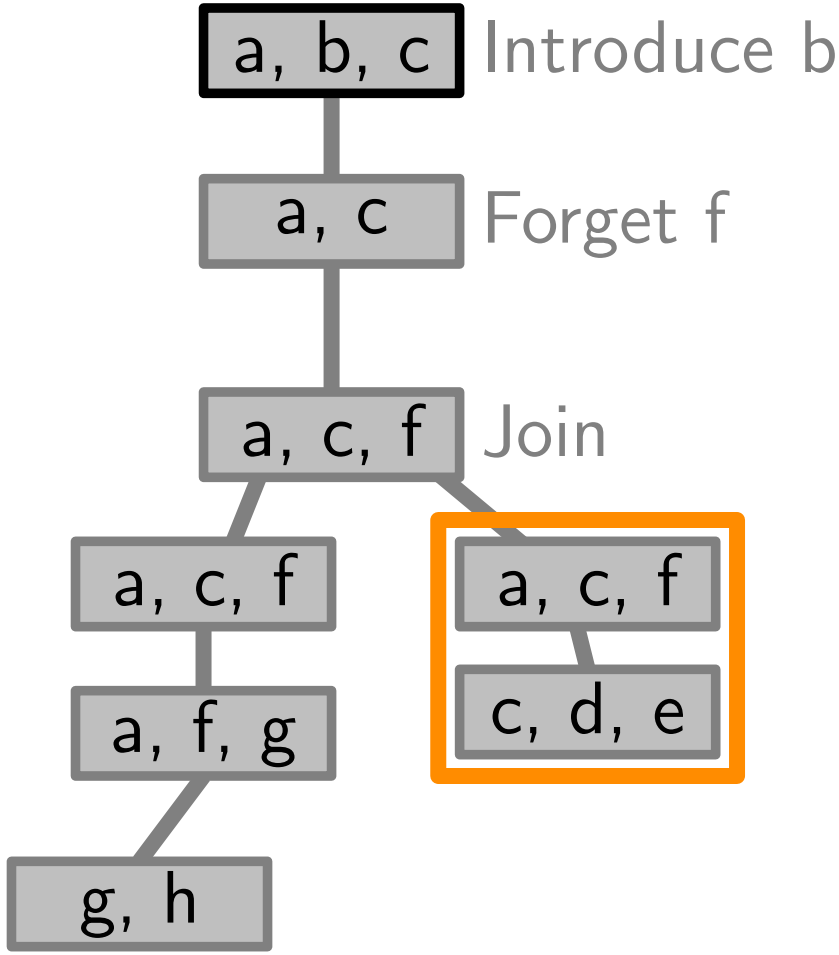

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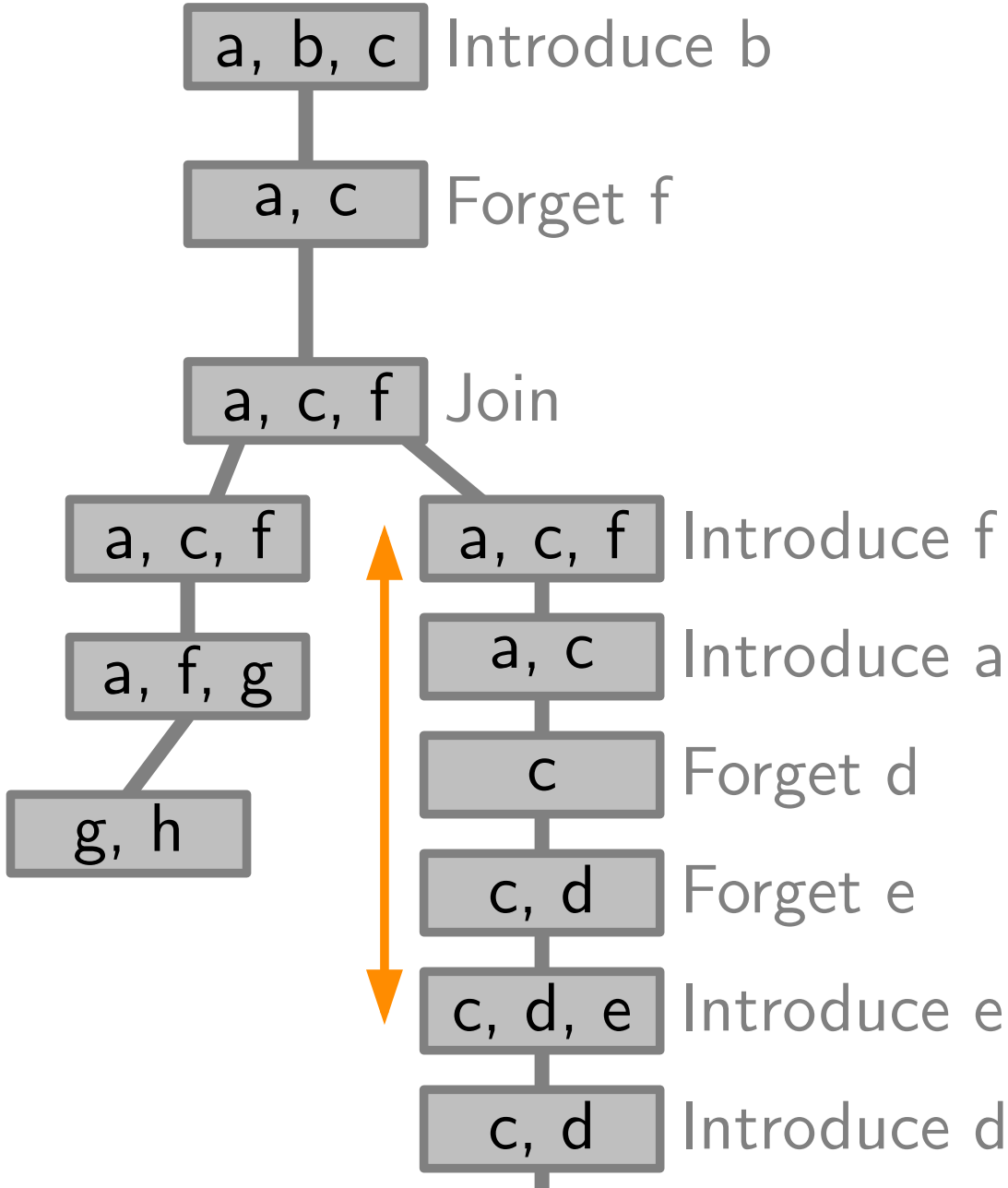

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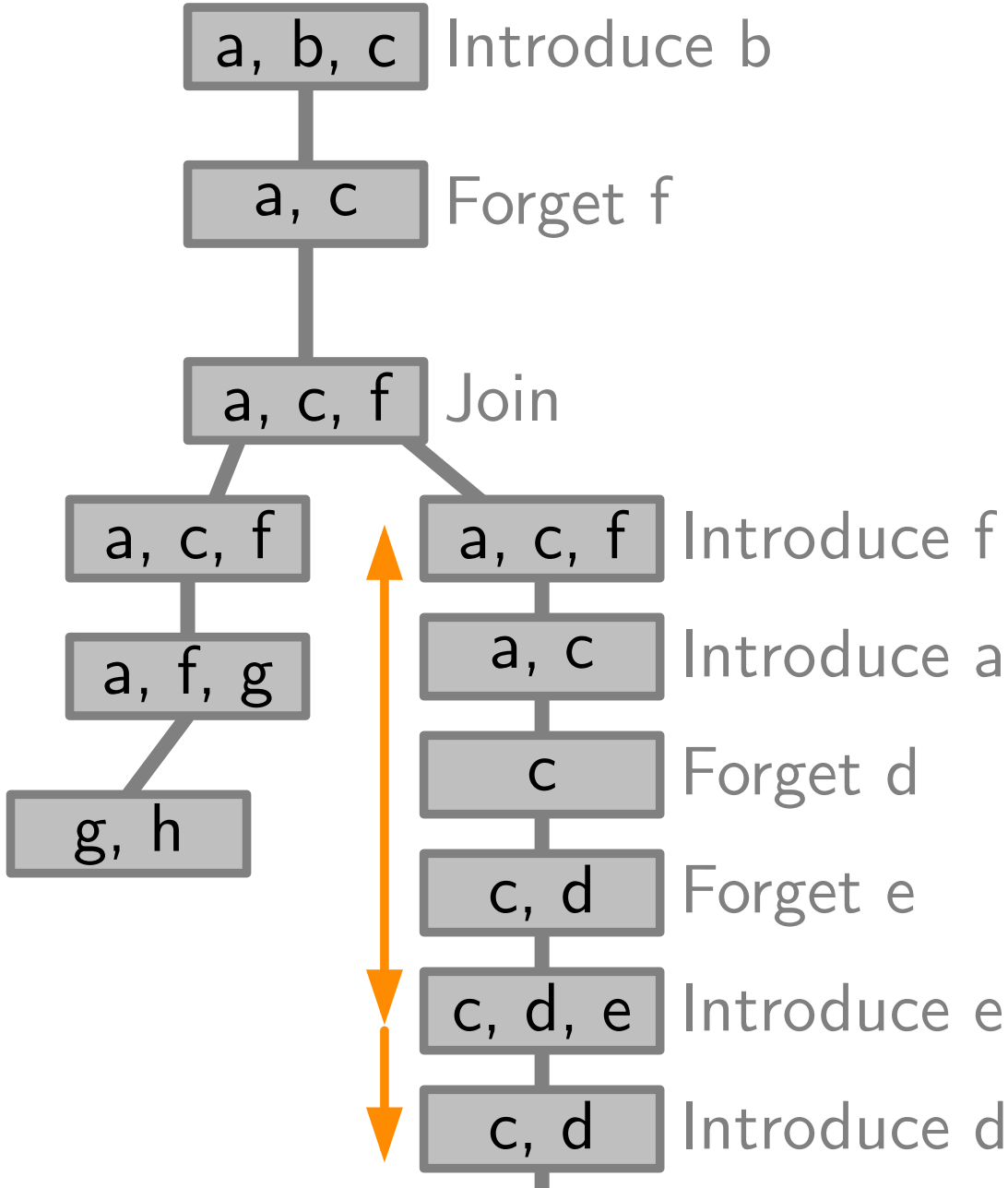

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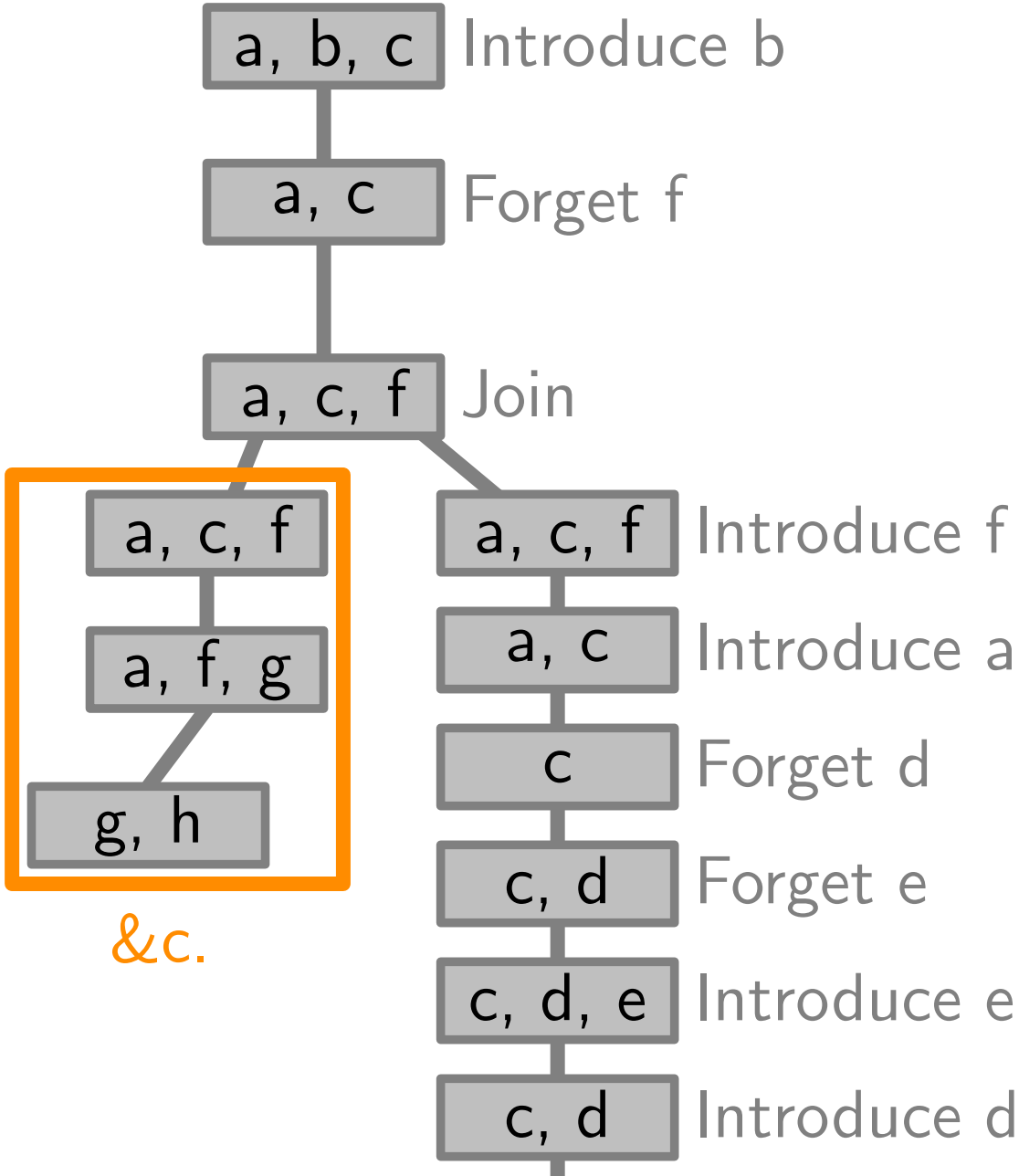

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Cor: For FPT-Algorithms it suffices to use nice tree decomp.

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Cor: For FPT-Algorithms it suffices to use nice tree decomp.

Strategy: Build a recurrence for each type of bag, and use dynamic programming.

Indep.Set on Nice Tree Decomp.

Let $G(i) :=$ Graph induced by the vertices in the subtree at i

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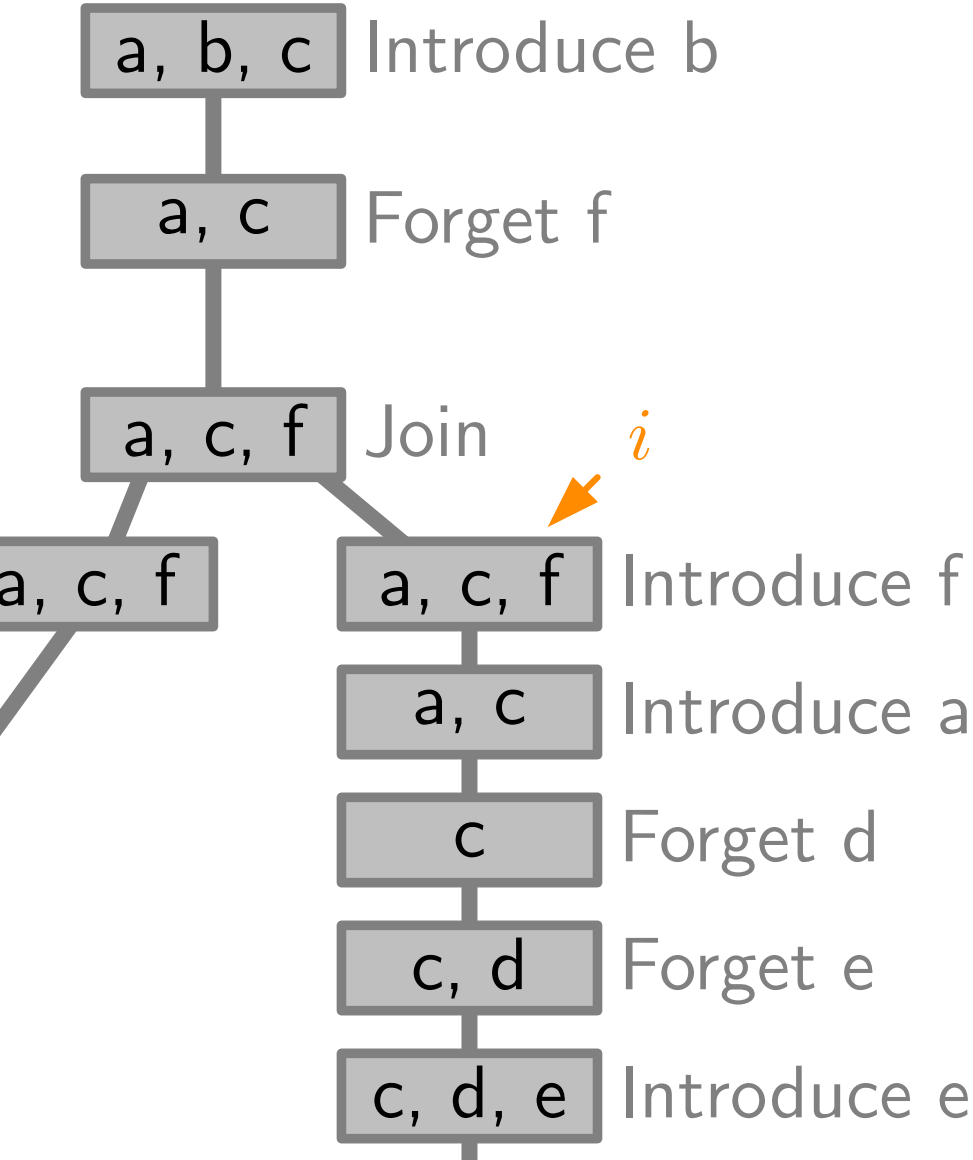

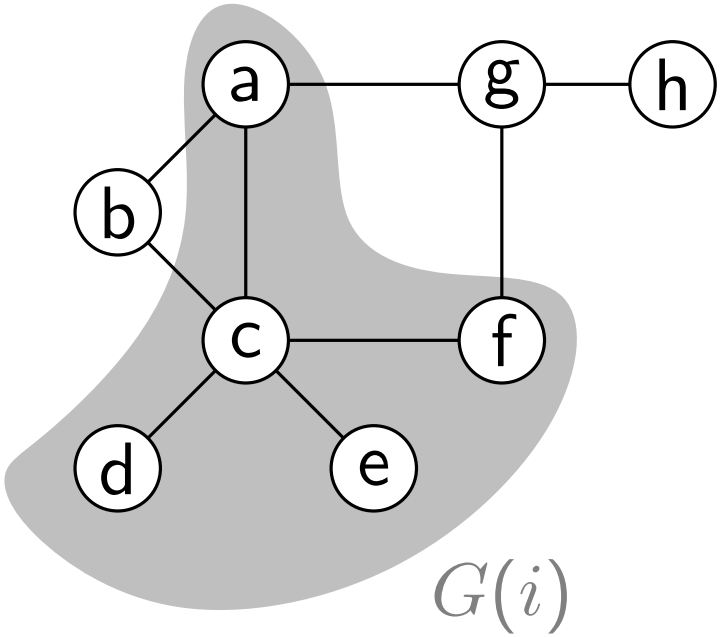

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Let $X_i = \{v\}$.

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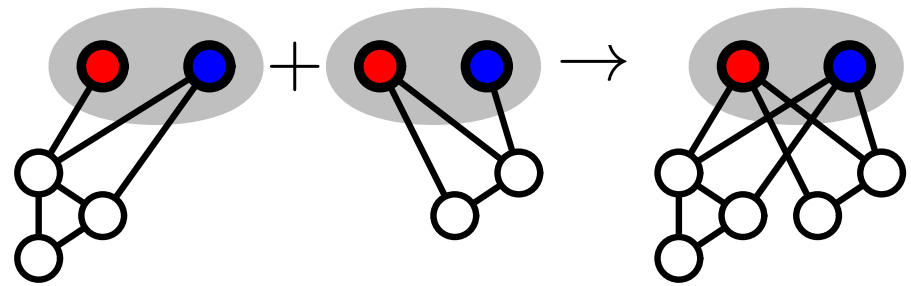
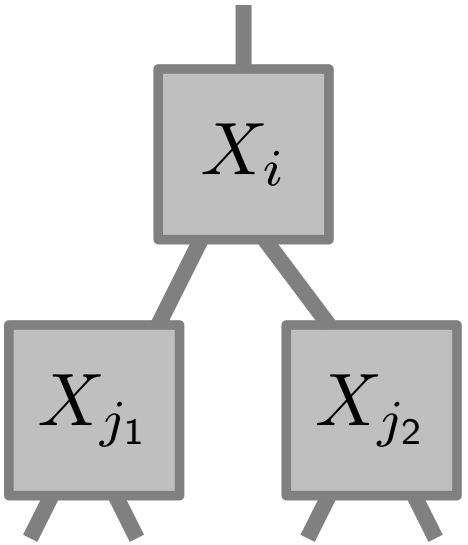


When i is a **Join** ...

with children j_1 and j_2

$R(i, S) =$



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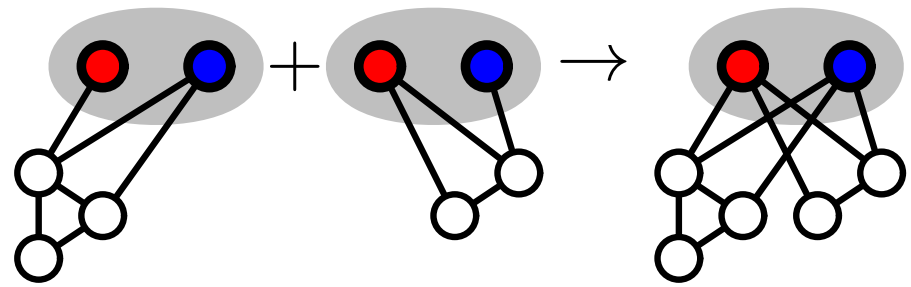

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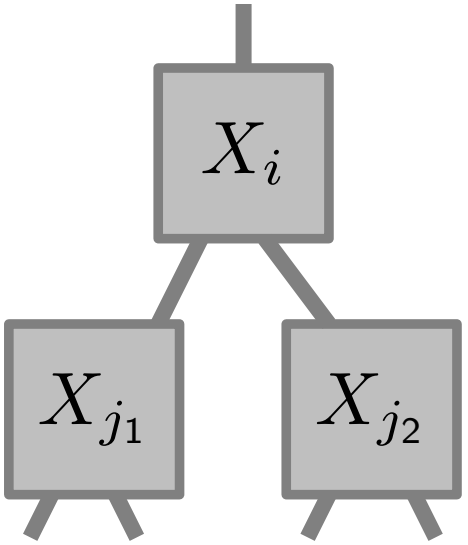
$$R(i, S) = R(j_1, S) + R(j_2, S)$$



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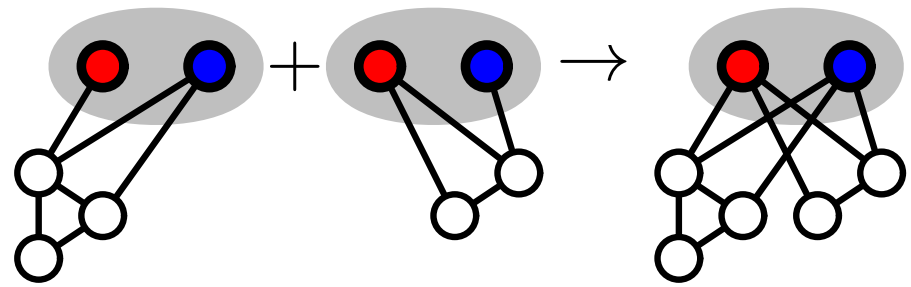

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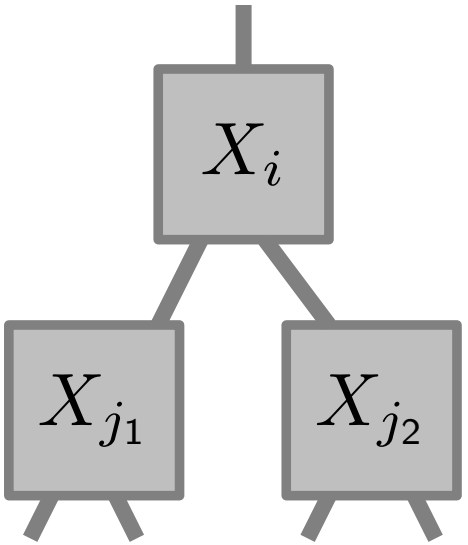

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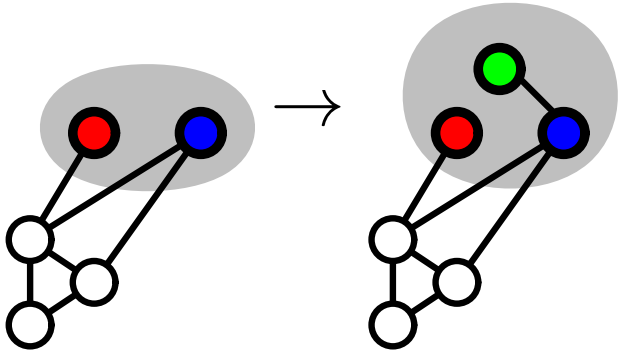
When i is an **Introduce** ...
with child j and $X_i = X_j \cup \{v\}$

For each $S \subseteq X_j$: $R(i, S) =$



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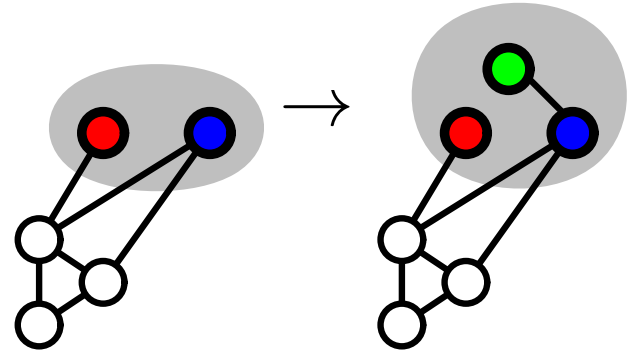
with child j and $X_i = X_j \cup \{v\}$

For each $S \subseteq X_j$: $R(i, S) = \quad R(j, S)$



Indep.Set on Nice Tree Decomp.

Let $G(i) :=$ Graph induced by the vertices in the subtree at i

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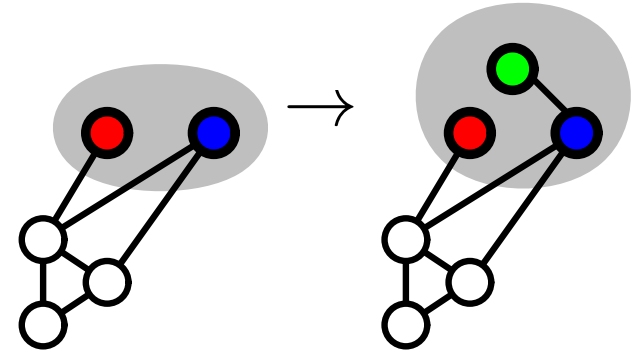

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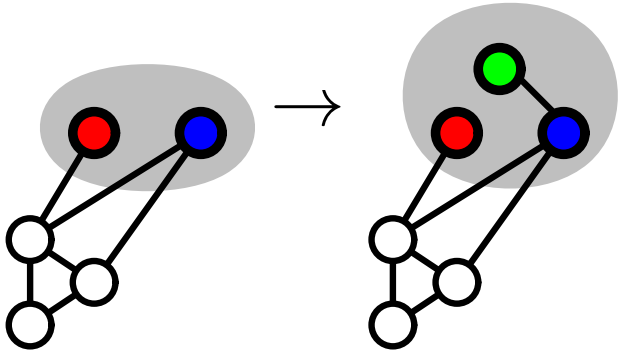

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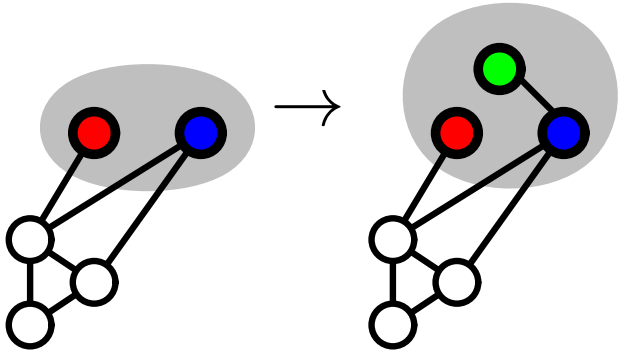

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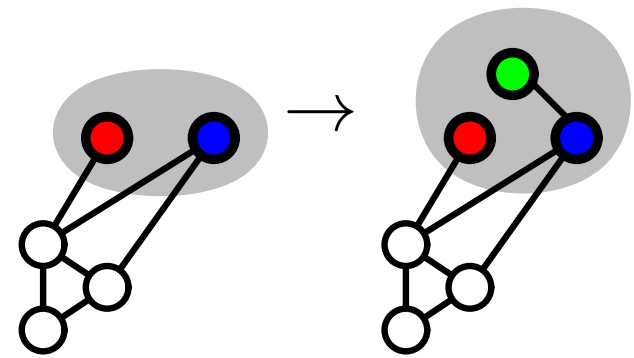

For each $S \subseteq X_j$: $R(i, S) = \quad R(j, S)$
and if v has neighbors in S, $R(i, S \cup \{v\}) = -\infty$
if v has no neighbors in S, $R(i, S \cup \{v\}) = R(j, S) + \omega(v)$

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When i is a **Forget** ...
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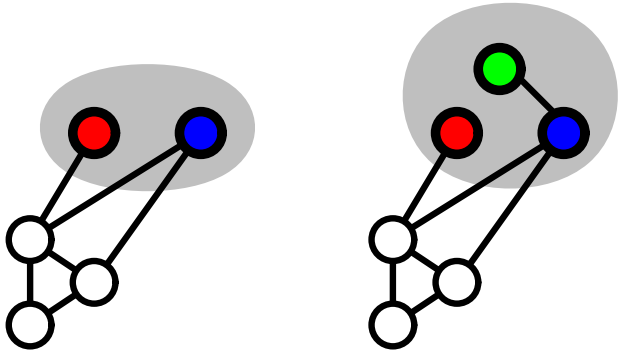
$R(i, S) =$



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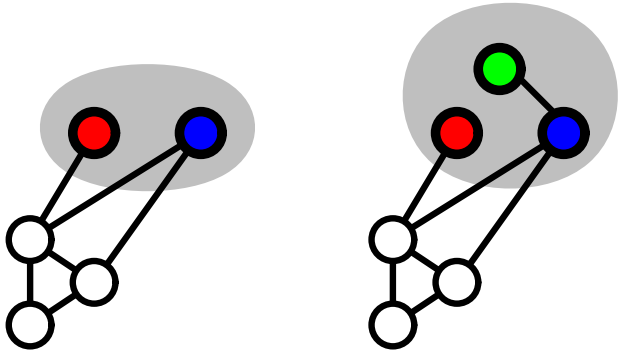

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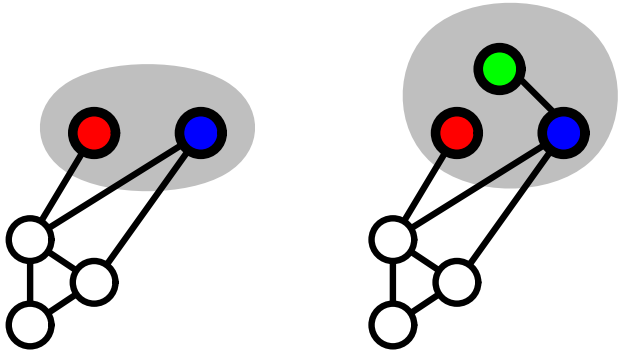

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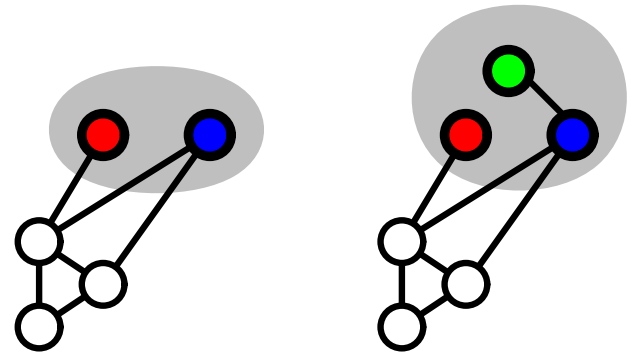
$R(i, S) = \max\{\ R(j, S)\ , R(j, S \cup \{v\})\ \}$



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Algo.: Compute $R(i, S)$ for all i and corresponding S

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Runtime: ?

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Thm: The independent set problem is FPT parameterized by treewidth.