Computational Geometry

Convex Partition
or
Oblivious Routing

Thomas van Dijk

Winter Semester 2019/20
Oblivious routing
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Result:
Deterministic oblivious routing in triangulations.
Oblivious routing

Result:
Deterministic oblivious routing in triangulations.

Result:
Randomized oblivious routing convex partitions.
Minimum Convex Partition of Points Sets

(Often: assume no colinear points.)

Point set $P$
Minimum Convex Partition of Points Sets

(Often: assume no colinear points.)

Convex hull $CH(P)$
Minimum Convex Partition of Points Sets
(Often: assume no colinear points.)

Convex hull $CH(P)$

Points in faces are not allowed.
Minimum Convex Partition of Points Sets

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Convex hull \( CH(P) \)

Points in faces are not allowed.
Minimum Convex Partition of Points Sets

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A triangulation of $P$: 
Minimum Convex Partition of Points Sets

(Often: assume no colinear points.)

A triangulation of $P$: 8 faces
Minimum Convex Partition of Points Sets

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A triangulation of $P$: 8 faces
Minimum Convex Partition of Points Sets

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A minimal convex partition: 5 faces
Minimum Convex Partition of Points Sets

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Minimum number of faces?
Minimum Convex Partition of Points Sets
(Often: assume no colinear points.)

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Minimum Convex Partition of Points Sets

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Minimum number of faces? 4 faces
Minimum Convex Partition of Points Sets

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Minimum number of faces? 4 faces
Known results

Not as much as we would like...
Known results

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... no polynomial-time algorithm known.
Known results

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... no polynomial-time algorithm known.

... NP-hardness unknown.
Known results

Not as much as we would like...
  ... no polynomial-time algorithm known.
  ... NP-hardness unknown.
  ... it is fixed parameter tractable ...
Known results

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... by an unconvincing parameter.

(number of interior points)
Known results

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\( O(n^{3h+3}) \) time, where \( h \) is \# nested convex hulls.
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\[(\text{n}^{3h+3})\text{ time, where } h \text{ is } \# \text{ nested convex hulls.}\]

Integer Linear Program (ILP)
Known results

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(time, where $h$ is # nested convex hulls)

**Exact:** $O(n^{3h+3})$ time, where $h$ is # nested convex hulls.

Integer Linear Program (ILP)

**Approx:** Factor 3 in $O(n \log n)$ time.
Known results

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**Exact:** \(O(n^{3h+3})\) time, where \(h\) is \# nested convex hulls.

**Integer Linear Program (ILP)**

**Approx:** Factor 3 in \(O(n \log n)\) time.

Factor \(\frac{30}{11}\) in \(O(n^2)\) time.
Known results

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\(O(n^{3h+3})\) time, where \(h\) is \# nested convex hulls.

Integer Linear Program (ILP)

Approx: Factor 3 in \(O(n \log n)\) time.

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Integer Linear Programming (ILP)

Variables $x \in \mathbb{Z}^n$
Integer Linear Programming (ILP)

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Objective function  Maximize \( c^T x = \sum_i c_i x_i \)
Integer Linear Programming (ILP)

**Variables** \( x \in \mathbb{Z}^n \)

**Objective function** Maximize \( c^T x = \sum_i c_i x_i \)

**Constraints** \( Ax \leq b \)
Integer Linear Programming (ILP)

**Max Independent Set**

**Variables**

\[ x \in \mathbb{Z}^n \]

**Objective function**

Maximize \[ c^\top x = \sum_i c_i x_i \]

**Constraints**

\[ Ax \leq b \]

Given a graph \( G = (V, E) \), pick a maximum cardinality set \( S \subseteq V \) such that no two vertices are adjacent.
Integer Linear Programming (ILP)

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**Constraints**  \( Ax \leq b \)

Given a graph \( G = (V, E) \), pick a maximum cardinality set \( S \subseteq V \) such that no two vertices are adjacent.
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Max Independent Set

Variables \( x_v \in \{0, 1\} \quad \forall v \in V \)

Objective function Maximize \( c^\top x = \sum_i c_i x_i \)

Constraints \( Ax \leq b \)

Given a graph \( G = (V, E) \), pick a maximum cardinality set \( S \subseteq V \) such that no two vertices are adjacent.
Integer Linear Programming (ILP)

\[ x_v \in \mathbb{Z}, \quad 0 \leq x_v \leq 1 \]

**Variables**

\[ x_v \in \{0, 1\} \quad \forall v \in V \]

**Objective function**

Maximize \( c^\top x = \sum_i c_i x_i \)

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**Max Independent Set**

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Maximize \( 1_n^T x = \sum_{v \in V} x_v \)

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Given a graph \( G = (V, E) \), pick a maximum cardinality set \( S \subseteq V \) such that no two vertices are adjacent.
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**Max Independent Set**

**Variables**
$x_v \in \{0, 1\} \quad \forall v \in V$

**Objective function**
Maximize $1^T_n x = \sum_{v \in V} x_v$

**Constraints**
$x_u + x_v \leq 1 \quad \forall \{i, j\} \in E$

Given a graph $G = (V, E)$, pick a maximum cardinality set $S \subseteq V$ such that no two vertices are adjacent.
Integer Linear Programming (ILP)

Binary variable $x_{ij}$ meaning: do we select $\{i, j\}$ as an edge?
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Minimize \( \sum_{\{i,j\} \in P^2} x_{ij} \) subject to:
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Minimize $\sum_{\{i,j\} \in P^2} x_{ij}$ subject to:

$x_{ij} = 1 \quad \forall \overline{ij}$ on $CH(P)$
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$\sum_{k\in \text{Behind}(i,j)} x_{ik} \geq 1 \quad \forall (i, j) \in P^2$, where $i$ is interior
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$\sum_{k \in \text{Behind}(i,j)} x_{ik} \geq 1 \quad \forall (i, j) \in P^2$, where $i$ is interior

$x_{ij} + x_{kl} \leq 1 \quad \forall \overline{ij}$ and $\overline{kl}$ that cross
Integer Linear Programming (ILP)

Binary variable \(x_{ij}\) meaning: do we select \(\{i, j\}\) as an edge?

Minimize \(\sum_{\{i,j\} \in P^2} x_{ij}\) subject to:

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x_{ij} = 1 \quad \forall \overline{ij} \text{ on } CH(P)
\]

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\sum_{k \in \text{Behind}(i,j)} x_{ik} \geq 1 \quad \forall (i, j) \in P^2, \text{ where } i \text{ is interior}
\]

\[
x_{ij} + x_{kl} \leq 1 \quad \forall \overline{ij} \text{ and } \overline{kl} \text{ that cross}
\]

\[
\sum_{j \in P} x_{ij} \geq 3 \quad \forall i \in P, \text{ where } i \text{ is interior}
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3-Approximation

**Theorem** A 3-approximation of a minimum convex partition of $P$ can be computed in $O(n \log n)$ time. (This assumes no colinear points.)
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**Lemma**  There is a convex partition $E$ of $P$ such that $|R(E)| \leq \frac{3}{2}k + \frac{3}{2}$. (This assumes no colinear points.)
A lower bound

Let \( k \geq 3 \) be number of interior points; \( n - k \) outer points.
A lowerbound

Let $k \geq 3$ be number of interior points; $n - k$ outer points. Let $CH_{in}(P) := \text{convex hull of inner points}$. 
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Type b
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Let $k \geq 3$ be number of interior points; $n - k$ outer points.
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Claim: Type $a$ vertex needs at least one edge to $CH(P)$. 
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Claim: Type \( a \) vertex needs at least one edge to \( CH(P) \).
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Claim: Type $a$ vertex needs at least one edge to $CH(P)$.

Claim: Type $b$ vertex needs at least two edges to $CH(P)$. 
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Degree sum:
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**Degree sum:** $2(n - k)$

Every outer point has degree two on the convex hull.
A lowerbound

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Claim: Type $b$ vertex needs at least two edges to $CH(P)$. 

Degree sum: $2(n - k) + 3k$

Every outer point has degree two on the convex hull. 

Every interior point has degree at least 3.
A lowerbound

Let \( k \geq 3 \) be number of interior points; \( n - k \) outer points. Let \( CH_{in}(P) := \) convex hull of inner points.

Claim: Type \( a \) vertex needs at least one edge to \( CH(P) \).
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Degree sum: \( 2(n - k) + 3k \)

Every outer point has degree two on the convex hull.
Every interior point has degree at least 3. (No colinear points.)
A lowerbound

Let $k \geq 3$ be number of interior points; $n - k$ outer points. Let $CH_{in}(P) :=$ convex hull of inner points.

Claim: Type $a$ vertex needs at least one edge to $CH(P)$. Claim: Type $b$ vertex needs at least two edges to $CH(P)$.

**Degree sum:** \[2(n - k) + 3k + a + 2b\]

Every outer point has degree two on the convex hull. Every interior point has degree at least 3. (No colinear points.)

There are $a + 2b$ edges arriving at $CH$ from interior.
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\( \# \) edges \( \geq n + \frac{k}{2} + \frac{a}{2} + b \)
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Every outer point has degree two on the convex hull.

Every interior point has degree at least 3. (No colinear points.)

There are $a + 2b$ edges arriving at $CH$ from interior.

$\# \text{ edges} \geq n + \frac{k}{2} + \frac{a}{2} + b \quad \Rightarrow \quad \# \text{ faces} \geq \frac{k}{2} + \frac{a}{2} + b + 1$
Lemma There is a convex partition $E$ of $P$ such that $|R(E)| \leq \frac{3}{2}k + \frac{3}{2}$. 
An upperbound

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Induction on $k$: $k = 0$
An upperbound

**Lemma** There is a convex partition $E$ of $P$ such that

$$|R(E)| \leq \frac{3}{2}k + \frac{3}{2}.$$

**Induction on** $k$: $k = 0 \implies |R(E)| \leq \frac{3}{2}$
An upperbound

Lemma There is a convex partition $E$ of $P$ such that
$$|R(E)| \leq \frac{3}{2} k + \frac{3}{2}.$$  

Induction on $k$:  

$k = 1$
An upperbound

Lemma There is a convex partition $E$ of $P$ such that
$$|R(E)| \leq \frac{3}{2} k + \frac{3}{2}.$$ 

Induction on $k$: $k = 1 \implies |R(E)| \leq 3$
An upperbound

**Lemma**  There is a convex partition $E$ of $P$ such that $|R(E)| \leq \frac{3}{2}k + \frac{3}{2}$.

**Induction on $k$:**  $k = 1 \implies |R(E)| \leq 3$

Triangulate CH arbitrarily.
An upperbound

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Lemma There is a convex partition $E$ of $P$ such that $|R(E)| \leq \frac{3}{2}k + \frac{3}{2}$.

Induction on $k$: $k \geq 2$
An upperbound

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**Induction on $k$:** $k \geq 2$
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**Induction on** $k$:  \[ k \geq 2 \]

Ind. hyp. holds for $Q$
**An upperbound**

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Chain $C$ with $\ell \geq 2$ points.
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$Q$ has partition with $\frac{3}{2} (k - \ell) + \frac{3}{2}$ faces.
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New faces:
An upper bound

**Lemma** There is a convex partition $E$ of $P$ such that
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**Induction on** $k$: $k \geq 2$

$Q$ has partition with $\frac{3}{2}(k - \ell) + \frac{3}{2}$ faces.

New faces: $\ell + 1$. 

Chain $C$ with $\ell \geq 2$ points.
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New faces:  

$\ell + 1$.

Total:  

$$\frac{3}{2}k + \frac{3}{2} - \frac{1}{2}\ell + 1.$$
**An upperbound**

**Lemma** There is a convex partition $E$ of $P$ such that $|R(E)| \leq \frac{3}{2}k + \frac{3}{2}$.

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New faces: $\ell + 1$.

Total: $\frac{3}{2}k + \frac{3}{2} - \frac{1}{2}\ell + 1$.

Note! $\ell \geq 2$
Putting it together

**Theorem**  A 3-approximation of a minimum convex partition of $P$ can be computed in $O(n \log n)$ time.  (This assumes no colinear points.)

**Proof:**
Putting it together

**Theorem**  A 3-approximation of a minimum convex partition of $P$ can be computed in $O(n \log n)$ time.  (This assumes no colinear points.)

**Proof:**  At the excercise sheet!