Advanced Algorithms

Winter term 2019/20

Lecture 5. Online Algorithms
(based on lecture notes of Sabine Storandt)

Johannes Zink
Introduction

Winter is about to begin . . .
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. . . this means the ski season is back!
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Is it worth buying new skis? Or should we rather rent them?
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But what if there is not always enough snow?

Is it worth buying new skis? Or should we rather rent them?

We don’t know the weather (much) in advance.
Ski-Rental Problem

• Every day when there is “good” weather, you go skiing (it’s a good day).
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• Every day in the morning, you know if today is a good day.
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- Renting skis for 1 day costs 1 [Euro].
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- In the end, there will have been $T$ good days.
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• In the end, there will have been $T$ good days.

(When to) buy skis? – We don’t know $T$!
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Renting costs 1/day
Buying costs $M$
$T$ good days
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**Strategy I: buy on the first good day**
Ski-Rental Problem

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- Imagine this was the only good day the whole winter.
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Strategy II: never buy, always rent
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Strategy II: never buy, always rent

- Imagine there are many good days ($T > M$).
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- So Strategy II is $T/M$ times worse than the optimal strategy.
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**Strategy III: buy after \( M \) good days**
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- Observation: the optimal solution pays min(M, T)
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- Observation: the optimal solution pays $\min(M, T)$
- If $T \leq M$, the competitive ratio is 1.
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- Observation: the optimal solution pays $\min(M, T)$
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$\Rightarrow$ Strategy III is deterministic and 2-competitive.
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**Theorem:** No deterministic strategy is better than 2-competitive.
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  \[ \frac{E[c_{\text{Strategy IV}}]}{c_{\text{OPT}}} = \frac{\frac{1}{2} \cdot 2M + \frac{1}{2} \cdot (1+\alpha)M}{M} = \frac{3+\alpha}{2} \]
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- Case $T = \alpha M$: $\frac{E[c_{Strategy IV}]}{c_{OPT}} = \frac{\frac{1}{2} \cdot \alpha M + \frac{1}{2} \cdot (1+\alpha)M}{\alpha M} = 1 + \frac{1}{2\alpha}$
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- The w. c. ratio is minimum if $\frac{3+\alpha}{2} = 1 + \frac{1}{2\alpha} \Rightarrow \alpha = \frac{\sqrt{5}-1}{2}$
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⇒ Strategy IV (with $\alpha = \frac{\sqrt{5}-1}{2} \approx 0.62$) is 1.81-competitive, randomized, and better than any deterministic strategy.
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⇒ Strategy IV (with $\alpha = \frac{\sqrt{5}-1}{2} \approx 0.62$) is 1.81-competitive, randomized, and better than any deterministic strategy.

With a more sophisticated probability distribution for the time we buy skis, we can even get a competitive ratio of \[ \frac{e}{e-1} \approx 1.58. \]
Online vs. Offline Algorithms
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- No full information available initially (online problem)
Online vs. Offline Algorithms

**Online Algorithm**

- No full information available initially (*online problem*)
- Decisions are made with incomplete information.
Online vs. Offline Algorithms

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Online vs. Offline Algorithms

<table>
<thead>
<tr>
<th>Online Algorithm</th>
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- The objective value of the returned solution divided by the obj. v. of an optimal [offline] solution is the *competitive ratio*. 
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- No full information available initially (*online problem*)
- Decisions are made with incomplete information.
- The algorithm may get more informations over time or by exploring the problem instance.
- The objective value of the returned solution divided by the obj. v. of an optimal [offline] solution is the *competitive ratio*.

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Objective value:

- Minimize the number of page faults while fulfilling $\sigma$. 
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**Theorem:** LRU & FIFO are $k$-competitive.
No deterministic strategy is better.
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- Clearly, MIN also faults in $P_0$; consider $P_i$ with $i \geq 1$.
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- Similarly, if LRU has a page fault on $p$ in $P_i$, there had to be $k$ distinct page requests in between.
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- Remains to prove: No det. strategy is better than $k$-compet.
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- For every det. strategy there’s a worst-case page sequence $\sigma^*$ always requesting the page that is currently not in the cache.
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- Then the next \( k - 1 \) requested pages are in the cache already and the next page fault of MIN occurs on the \((i + k)\)-th page of \( \sigma^* \) the earliest. Until then, the det. strategy has \( k \) faults.
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$\Rightarrow$ The comp. ratio cannot be better than

$$\frac{|\sigma^*|}{\left\lceil \frac{|\sigma^*|}{k} \right\rceil} \sim \infty \Rightarrow k.$$
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Randomized strategy MARKING:
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Theorem: Marking is $2H_k$-competitive.
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**Remark:** $H_k = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}$ is the $k$-th harmonic number and for $k \geq 2$ it holds that $H_k < \ln(k) + 1$. 

Paging

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**Proof:**
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**Proof:** We consider phase $P_i$. Definitions:

- $S_{\text{MARK}}$ ($S_{\text{MIN}}$): set of pages in the cache of Marking (MIN)
- $d_{\text{begin}}$: $|S_{\text{MIN}} - S_{\text{MARK}}|$ at the beginning of $P_i$
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- A page is *stale* if it is unmarked, but was marked in $P_{i-1}$.
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- MIN has at least $\max(c - d_{\text{begin}}, d_{\text{end}}) \geq \frac{1}{2}(c - d_{\text{begin}} + d_{\text{end}})$
  $$= \frac{c}{2} - \frac{d_{\text{begin}}}{2} + \frac{d_{\text{end}}}{2}$$ faults. Over all phases, all $\frac{d_{\text{begin}}}{2}$ and $\frac{d_{\text{end}}}{2}$ cancel out, except for the first $\frac{d_{\text{begin}}}{2}$ and the last $\frac{d_{\text{end}}}{2}$.
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- Since the first $d_{\text{begin}} = 0$, MIN has at least $\frac{c}{2}$ faults per phase.
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**Proof**: 
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- Moreover, in this phase, there are $s = k - c \leq k - 1$ requests to stale pages; for requests $j = 1, \ldots, s$ to stale pages we compute the expected number of faults $E[F_j]$. 
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$$E[F_j] = \frac{s(j) - c(j)}{s(j)} \cdot 0 + \frac{c(j)}{s(j)} \cdot 1 \leq \frac{c}{k + 1 - j}$$
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• So the competitive ratio of Marking is $\frac{c+c(H_k-1)}{c/2} = 2H_k$. 
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• Moreover, in this phase, there are $s = k - c \leq k - 1$ requests to stale pages; for requests $j = 1, \ldots, s$ to stale pages we compute the expected number of faults $E[F_j]$.

• $c(j)$: # clean pages requested in this phase so far

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• $E[F_j] = \frac{s(j) - c(j)}{s(j)} \cdot 0 + \frac{c(j)}{s(j)} \cdot 1 \leq \frac{c}{k+1-j}$

• $\sum_{j=1}^{s} E[F_j] \leq \sum_{j=1}^{s} \frac{c}{k+1-j} \leq \sum_{j=2}^{k} \frac{c}{j} = c \cdot (H_k - 1)$

• So the competitive ratio of Marking is $\frac{c + c(H_k - 1)}{c/2} = 2H_k$. □
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