Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Lehrstuhl für
INFORMATIK I
Effiziente Algorithmen und
wissensbasierte Systeme

Institut für Informatik

Advanced Algorithms

Winter term 2019/20

Lecture 5. Online Algorithms

(based on lecture notes of Sabine Storandt)

Johannes Zink

Chair for Computer Science I

Introduction

Winter is about to begin . . .

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Is it worth buying new skis? Or should we rather rent them?

We don't know the weather (much) in advance.

Ski-Rental Problem

- Every day when there is "good" weather, you go skiing (it's a *good* day).

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(When to) buy skis? – We don't know T!

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Strategy I: buy on the first good day

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- Imagine this was the only good day the whole winter.

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- So Strategy I is M times worse than the optimal strategy.

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- Imagine there are many good days ($T > M$).

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$\Rightarrow$ Strategy III is deterministic and 2-competitive.

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costs for deterministic startegy

costs for optimal startegy

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not better than the det. Strategy **III**

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$\Rightarrow$ Strategy IV (with $\alpha = \frac{\sqrt{5}-1}{2} \approx 0.62$) is 1.81-competitive, randomized, and better than any deterministic strategy.

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With a more sophisticated probability distribution for the time we buy skis, we can even get a competitive ratio of $\frac{e}{e-1} \approx 1.58$.

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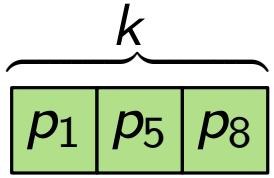

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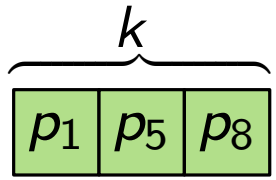

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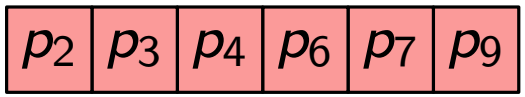

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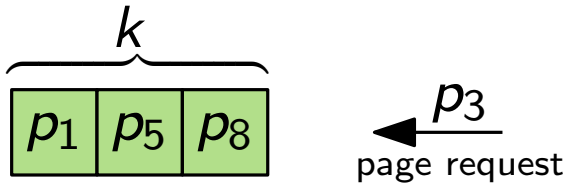

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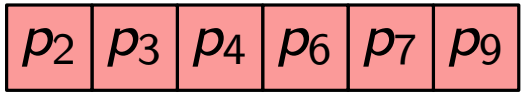

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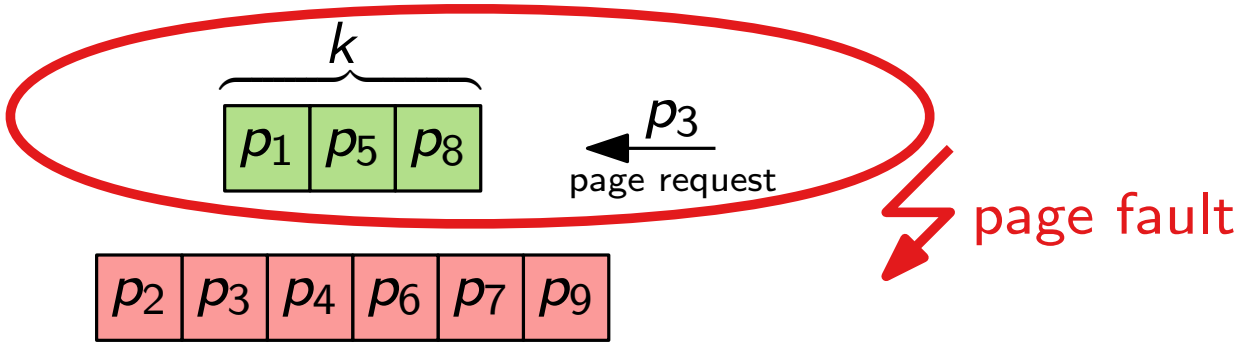

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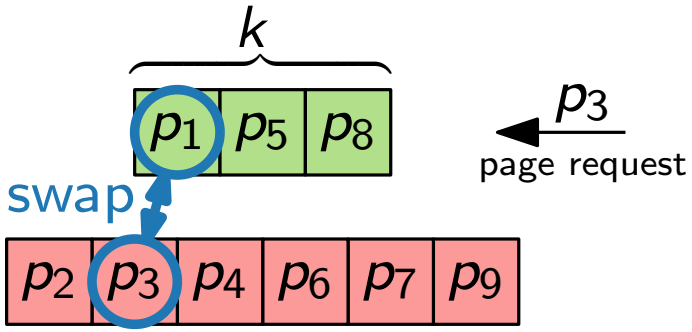

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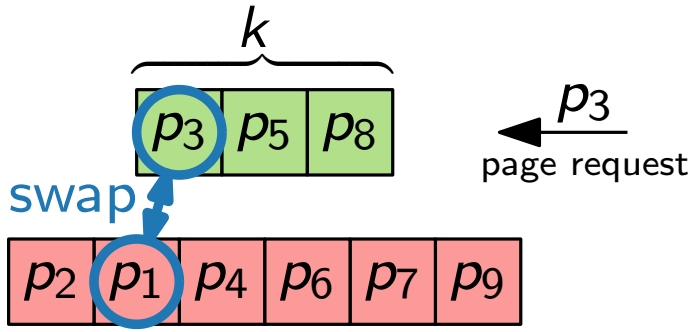

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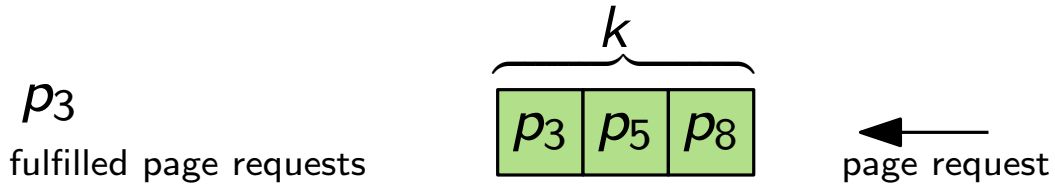

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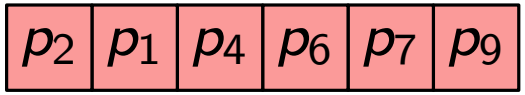

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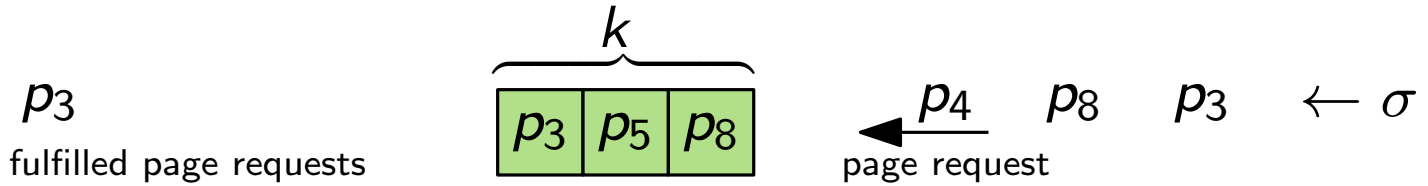

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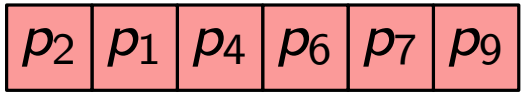

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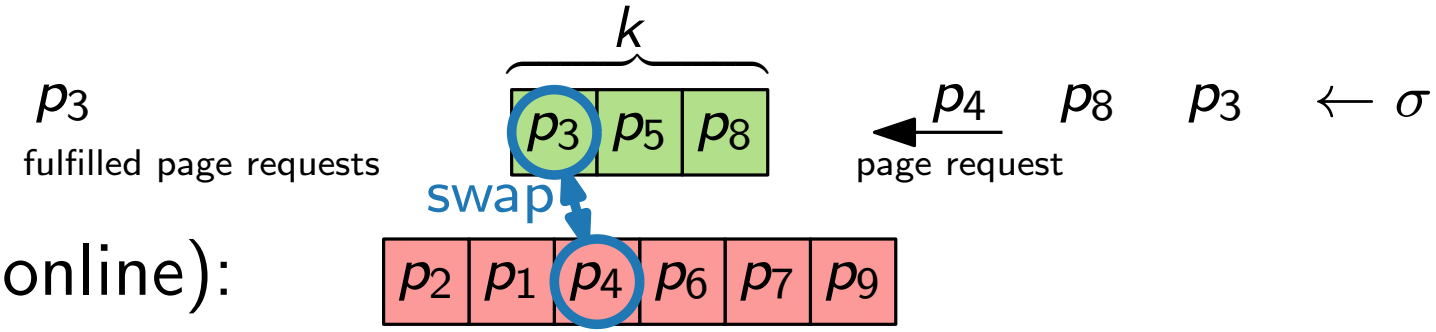

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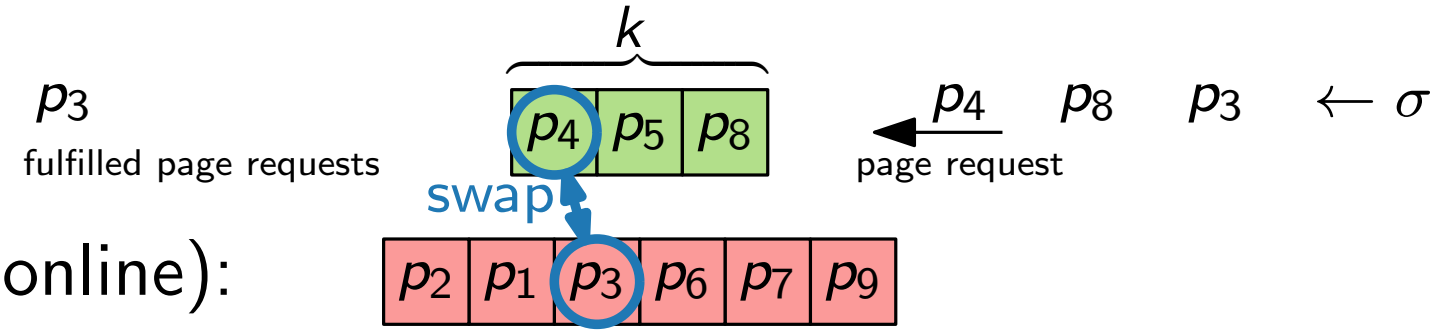

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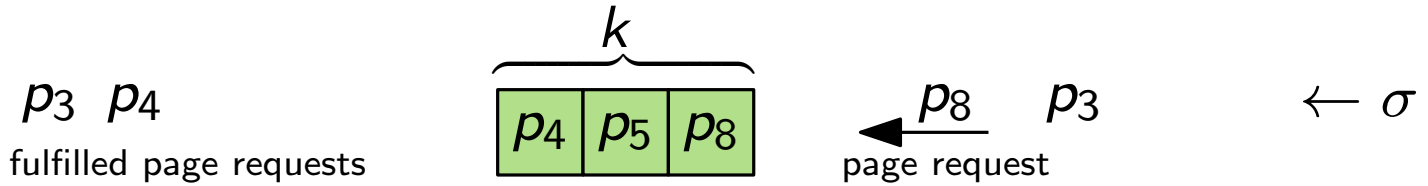

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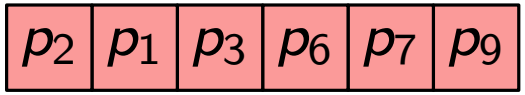

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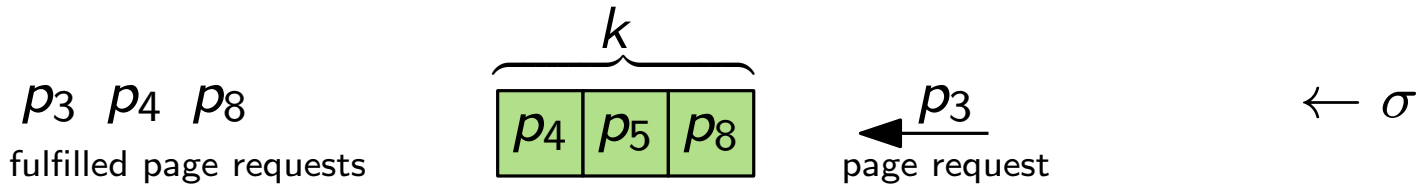

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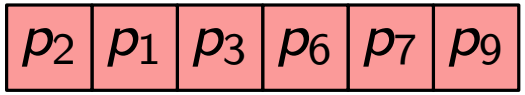

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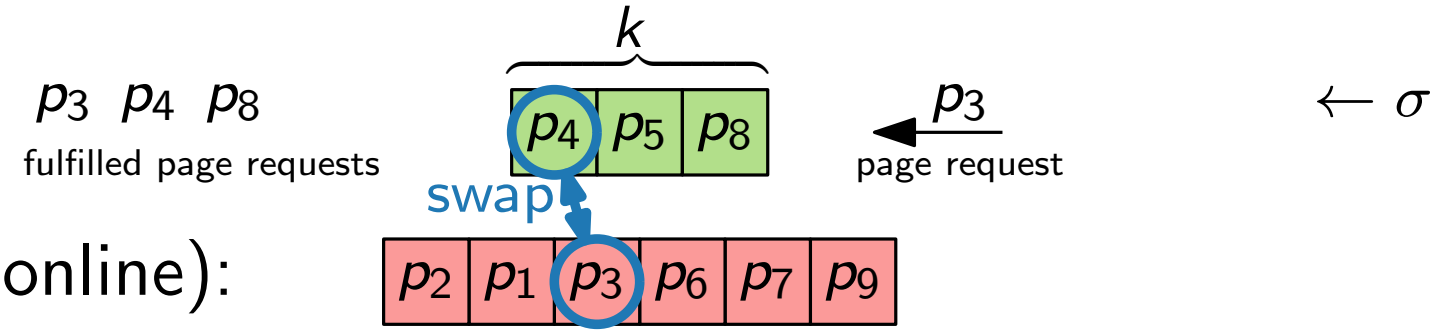

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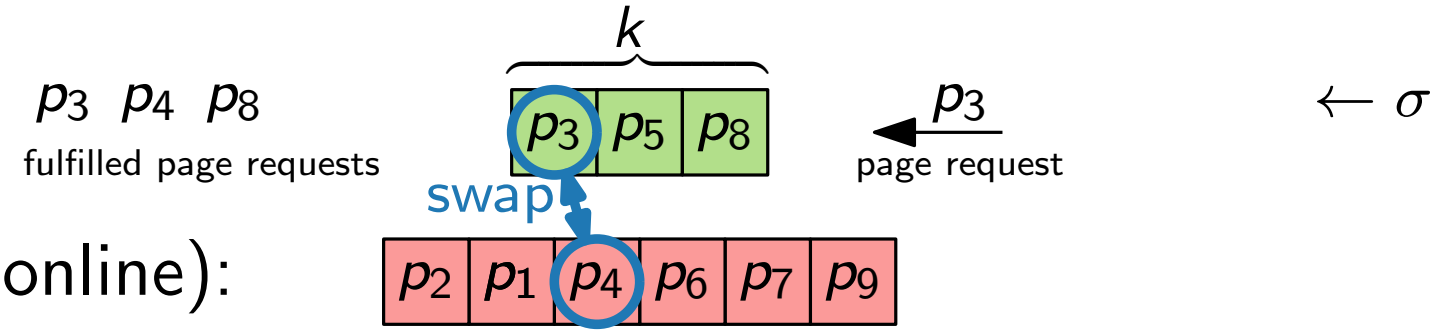

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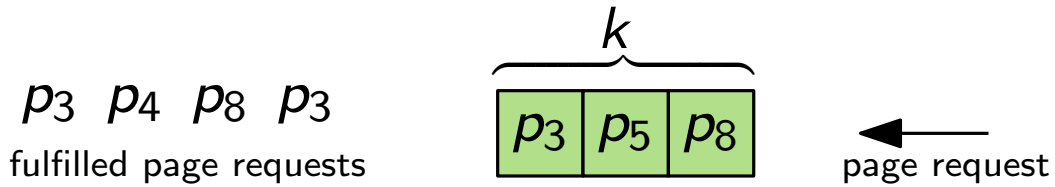

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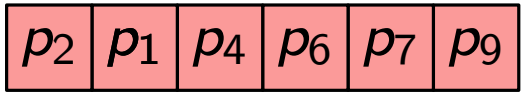

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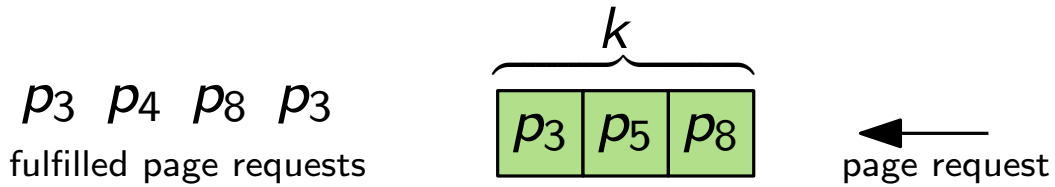

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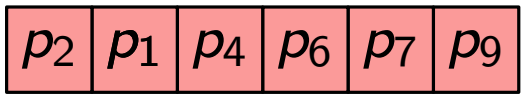

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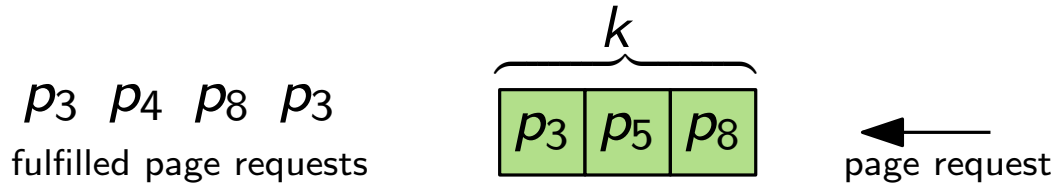

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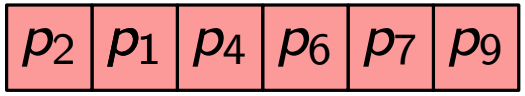

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- Then the next $k-1$ requested pages are in the cache already and the next page fault of MIN occurs on the $(i+k)$-th page of σ^* the earliest. Until then, the det. strategy has k faults.

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Theorem: LRU & FIFO are k-compet. No det. strategy is better.

Proof (only for LRU, FIFO similar):

- Remains to prove: No det. strategy is better than k-compet.
- Let there be $k+1$ pages in the memory system.
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Remark: $H_k = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}$ is the k-th harmonic number and for $k \geq 2$ it holds that $H_k < \ln(k) + 1$.

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- Since the first $d_{\text{begin}} = 0$, MIN has at least $\frac{c}{2}$ faults per phase.

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$\Rightarrow$ **Randomization helps!**

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