Julius-Maximilians-
UNIVERSITÄT
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Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Advanced Algorithms

Winter term 2019/20

Lecture 3. 2D Linear Programming via sweep-lines and randomization

Source: **CG: A&A §4**

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Chair for Computer Science I

Maximizing Profit

You are the boss of a small company that produces two products, P_1 and P_2. If you produce x_1 units of P_1 and x_2 units of P_2, your profit in € is

$$G(x_1, x_2) = 300x_1 + 500x_2$$

Your production runs on three machines M_A, M_B, and M_C with the following capacities:

$$\begin{aligned} M_A: &\quad 4x_1 + 11x_2 \leq 880 \\ M_B: &\quad x_1 + x_2 \leq 150 \\ M_C: &\quad x_2 \leq 60 \end{aligned}$$

Which choice of (x_1, x_2) maximizes your profit?

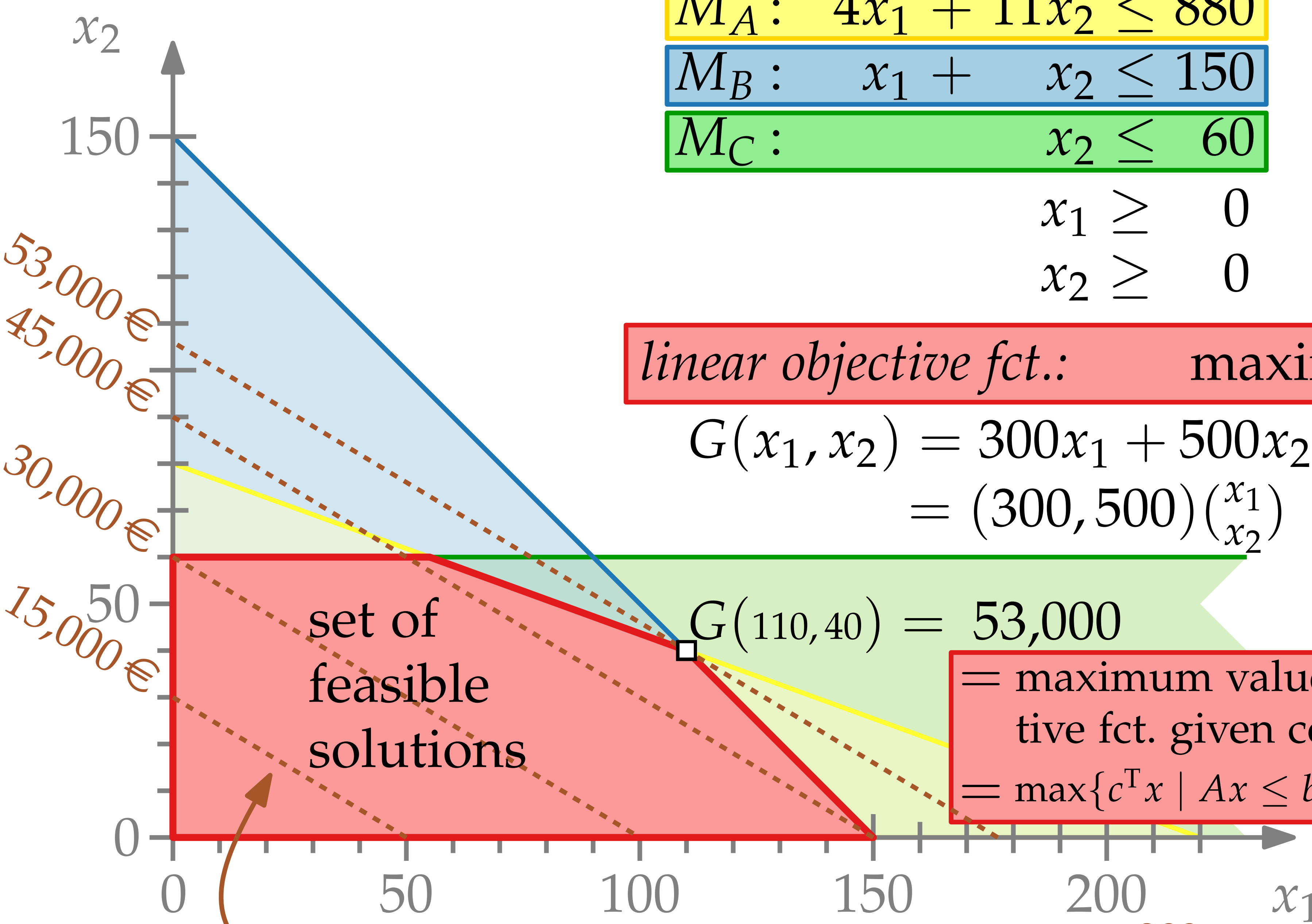
The Answer

linear constraints:

$$
\begin{aligned}
M_A: \quad 4x_1 + 11x_2 &\leq 880 \\
M_B: \quad x_1 + x_2 &\leq 150 \\
M_C: \quad x_2 &\leq 60 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{aligned}
\qquad
\begin{aligned}
Ax &\leq b \\
x &\geq 0
\end{aligned}
$$

linear objective fct.: maximize $c^{\mathrm{T}}x$

$$
\begin{aligned}
G(x_1, x_2) &= 300x_1 + 500x_2 \\
&= (300, 500)\binom{x_1}{x_2}
\end{aligned}
$$



$G(110, 40) = 53{,}000$

$=$ maximum value of objective fct. given constraints

$= \max\{c^{\mathrm{T}}x \mid Ax \leq b, x \geq 0\}$

„iso-profit line" (orthogonal to $\binom{300}{500}$)

Definition and Known Algorithms

Given a set H of n halfspaces in $\mathbb{R}^d$ and a direction c, find a point $x \in \bigcap H$ such that cx is maximum (or minimum).

Many algorithms known, e.g.:

– Simplex [Dantzig '47]

– Ellipsoid method [Khatchiyan '79]

– Inner-point method [Karmakar' 84]

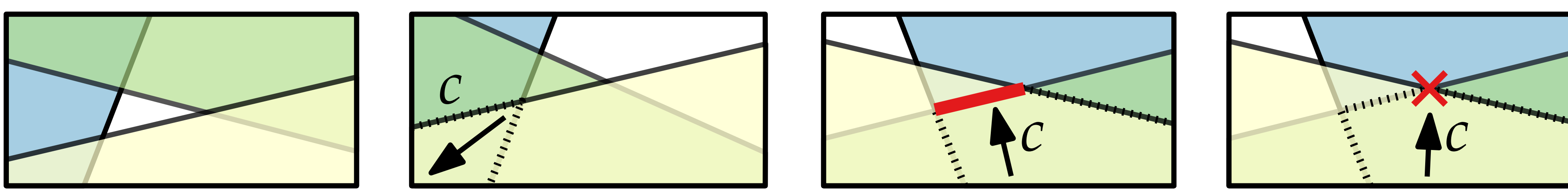
Good for instances where n *and* d are large.

We consider $d = 2$.

VERY important problem, e.g., in Operations Research.

["Book" application: casting]

$\bigcap H$ bounded.



$\bigcap H = \emptyset$ $\bigcap H$ unbnd. in dir. c set of optima: segment vs. point

First Approach

- compute $\bigcap H$ iteratively
- walk $\partial\left(\bigcap H\right)$, find vertex x w/ cx maximum, $O(n)$ time

IntersectHalfplanes(H)

Let $H = (h_1, \ldots, h_n)$
$C \leftarrow h_1$
foreach i from 2 to n **do**
  $C \leftarrow C \cap h_i$
return C

How??

$C :=$ chain of line segments $(s_1, \ldots, s_t)$

Walk around C to find $s_j, s_{j'} \in C$ intersecting h_i

Update C

Running time: $T_{\mathrm{IH}}(n) = n \cdot O(n)$

Total Time: $O(n^2)$:(

Exercise: Compute $C \cap h_i$ faster.

Second Approach

- compute $\bigcap H$ via divide and conquer
- walk $\partial(\bigcap H)$, find vertex x w/ cx maximum, $O(n)$ time

IntersectHalfplanes(H)

if $|H| = 1$ **then**

  $C \leftarrow h$, where $\{h\} = H$

else

  split H into sets H_1 and H_2 with $|H_1|, |H_2| \approx |H|/2$

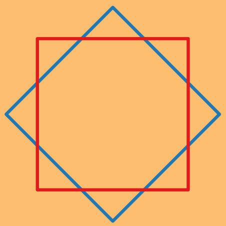
  $C_1 \leftarrow$ IntersectHalfplanes(H_1)

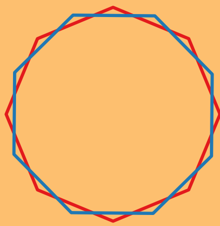
  $C_2 \leftarrow$ IntersectHalfplanes(H_2)

  $C \leftarrow$ IntersectConvexRegions(C_1, C_2)

return C

How complex can the new region be?

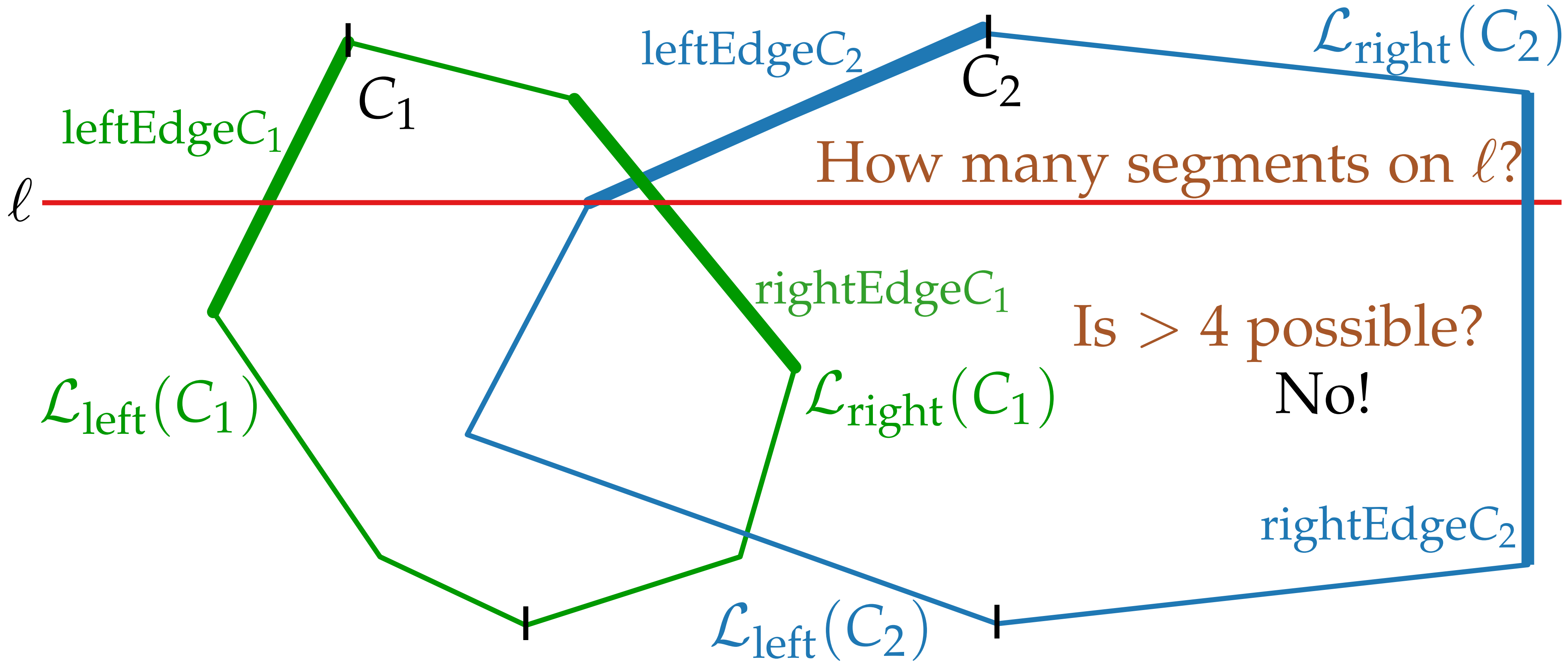




How??

Running time: $T_{\mathrm{IH}}(n) = 2T_{\mathrm{IH}}(n/2) + T_{\mathrm{ICR}}(n)$

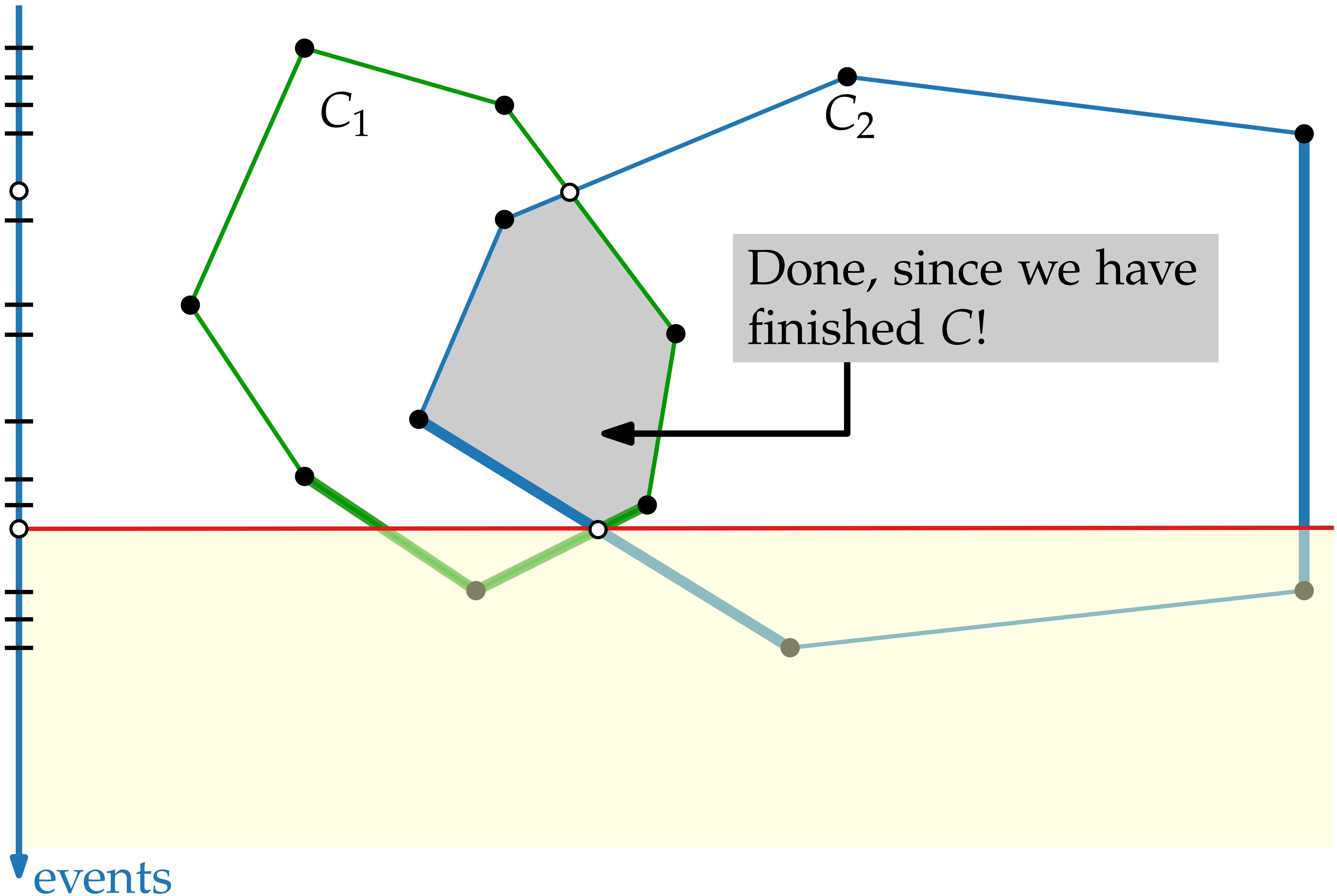
Intersecting Convex Regions



How does this help us?
$\rightsquigarrow$ sweep-line algorithm!

Theorem. The intersection of two convex polygonal regions can be computed in linear time.

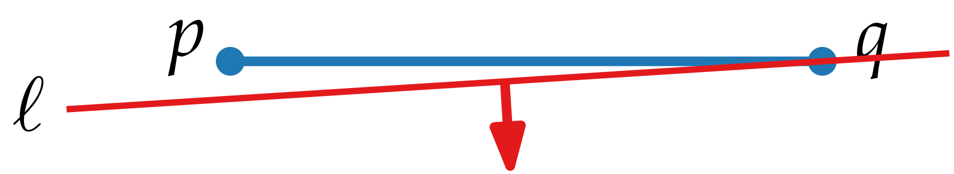
Sweep-Line Algorithm



Data Structures

1) event (-point) queue $\mathcal{Q}$

$$p \prec q \quad \Leftrightarrow_{\text{def.}} \quad y_p > y_q \quad \text{or} \quad (y_p = y_q \text{ and } x_p < x_q)$$

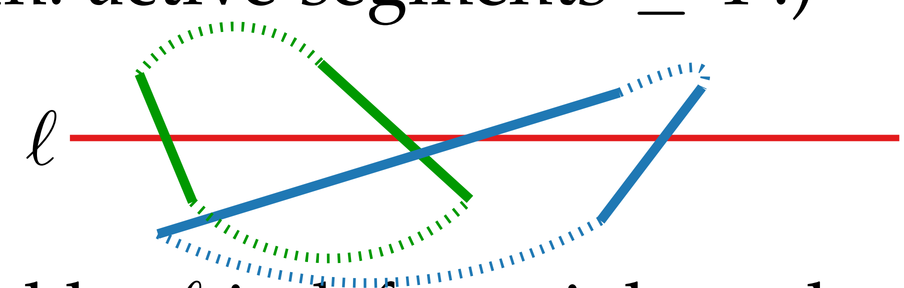


Store event pts in *sorted order* acc. to $\prec$

nextEvent() : either, next point (by $\prec$), or the intersection pt. of two active segments (below the sweep-line)

... runtime? $O(1)$, since num. active segments ≤ 4 :)

2) (sweep-line) status $\mathcal{T}$



Store the segments intersected by ℓ in left-to-right order.

Also, maintain the new convex hull.

Second Approach: Halfplane Intersection

Theorem. The intersection of two convex polygonal regions can be computed in linear time.

IntersectHalfplanes(H)

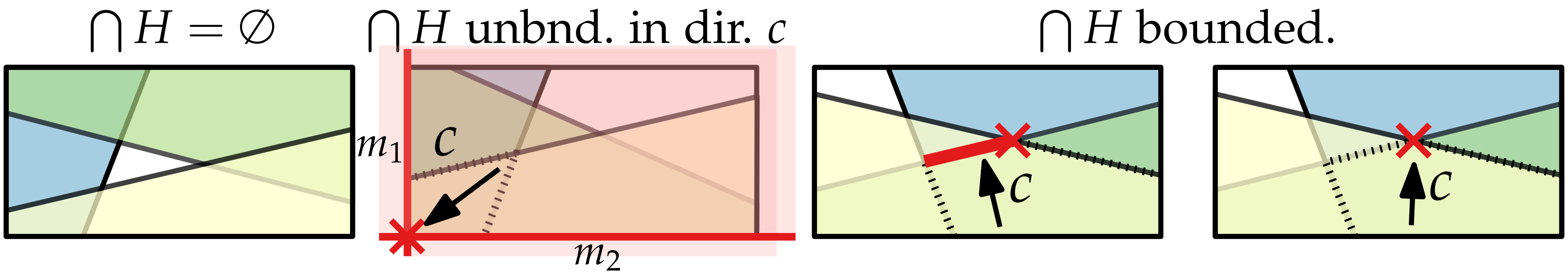
if $|H| = 1$ **then** $C \leftarrow h$, where $\{h\} = H$
else
  split H into sets H_1 and H_2 with $|H_1|, |H_2| \approx |H|/2$
  $C_1 \leftarrow$ IntersectHalfplanes(H_1)
  $C_2 \leftarrow$ IntersectHalfplanes(H_2)
  $C \leftarrow$ IntersectConvexRegions(C_1, C_2)
return C

Running time: $T_{\mathrm{IH}}(n) = 2T_{\mathrm{IH}}(n/2) + T_{\mathrm{ICR}}(n)$

Corollary. The intersection of n half planes can be computed in $O(n \log n)$ time.

Can we do better?

A Small Trick: Make Solution Unique



- Add two bounding halfplanes m_1 and m_2

$$m_1 = \begin{cases} x \leq M & \text{if } c_x > 0, \\ x \geq M & \text{otherwise,} \end{cases}$$ for some sufficiently large M

$$m_2 = \begin{cases} y \leq M & \text{if } c_y > 0, \\ y \geq M & \text{otherwise.} \end{cases}$$

Idea: M based on obj.fct. c.
see §4.5 of CG: A&A for more on unbounded LPs.

- Take the lexicographically largest solution.

$\Rightarrow$ Set of solutions is either empty or a uniquely defined pt.

Incremental Approach

Idea: Don't compute $\bigcap H$, but just *one* (optimal) point!

Randomized

$\text{2DBoundedLP}(H, c, m_1, m_2)$

compute random permutation of H

$H_0 = \{m_1, m_2\}$

$v_0 \leftarrow$ corner of $m_1 \cap m_2$

for $i \leftarrow 1$ **to** n **do**

if $v_{i-1} \in h_i$ **then**

$v_i \leftarrow v_{i-1}$ $O(1)$

else

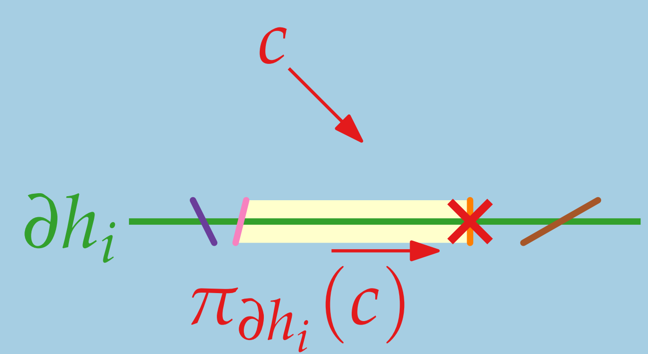
$v_i \leftarrow \text{1DBoundedLP}(\pi_{\partial h_i}(H_{i-1}), \pi_{\partial h_i}(c))$ $O(i)$

if $v_i = \text{nil}$ **then**

return nil

$H_i = H_{i-1} \cup \{h_i\}$ $O(1)$

return v_n



w-c running time:

$T(n) = \sum_{i=1}^{n} O(i) =$

$= O(n^2)$:-(

Result

Theorem. The 2D bounded LP problem can be solved in $O(n)$ expected time.

Proof. Let $X_i = \begin{cases} 1 & \text{if } v_{i-1} \notin h_i, \\ 0 & \text{else.} \end{cases}$ (indicator random variable).

Then the expected running time is

$$\begin{aligned} \mathbf{E}[T_{2d}(n)] &= \mathbf{E}[\textstyle\sum_{i=1}^{n}(1 - X_i) \cdot O(1) + X_i \cdot O(i)] \\ &= O(n) + \textstyle\sum \mathbf{E}[X_i] \cdot O(i) \\ &= O(n) + \textstyle\sum \mathbf{Pr}[X_i = 1] \cdot O(i) \quad = O(n). \end{aligned}$$

We fix the i random halfplanes in H_i.

$\mathbf{Pr}[X_i = 1]$ = probability that the optimal solution changes when h_i is added to H_{i-1}.

= probability that the optimal solution changes when h_i is removed from H_i.

Proof technique:
Backward analysis!

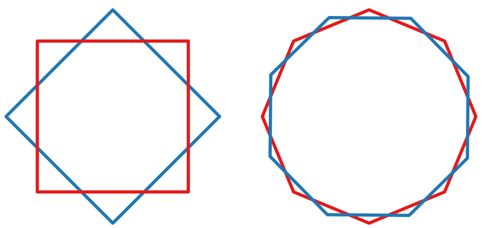
$\leq 2/i$. This is independent of the choice of H_i. □

Alt. for Intersecting Convex Regions

CG: A & A §2

Use sweep-line alg. for map overlay (line-segment intersections) !

Running time $T_{\mathrm{MO}}(n) = O((n + I)\log n)$,



where $I = \#$ intersection points.

here: $I \le n$ → $O(n \log n)$ for ICR

Running time $T_{\mathrm{IH}}(n) = 2T_{\mathrm{IH}}(n/2) + T_{\mathrm{ICR}}(n)$

$$\le 2T_{\mathrm{IH}}(n/2) + O(n \log n)$$

$$\in O(n \log^2 n)$$

As this is more general, it is unsurprisingly worse ... *

⇝ Better to use specialized algorithm for intersecting *convex* regions/polygons

* it can happen sometimes that general algorithms give optimal runtimes for special cases