Algorithms for Graph Visualization

Summer Semester 2019
Lecture #8

Planar Orientations
Tessellations and Visibility Representations

Ref: [GD: AVG, Ch. 4]
Topological Numbering

Let $G = (V, E)$ be a directed graph w/ edge weights $w : E \rightarrow \mathbb{N}$

- **Topological numbering** of $G$:
  mapping $\mu : V \rightarrow \mathbb{N}$ with $\mu(u) < \mu(v)$ for every edge $(u, v)$

- **Topological sort** of $G$:
  topological numbering where $\mu(V) = \{1, \ldots, n\}$

- **Weighted topological numbering** of $(G, w)$:
  topol. numb. with $\mu(u) + w(u, v) \leq \mu(v)$ for every edge $(u, v)$
  *optimal* when: $\max_{v \in V} \mu(v) - \min_{v \in V} \mu(v)$ is minimized.

- Can be calculated in $O(n + m)$ time.  
  Exercise!
**st-graphs**

**st-graph**: a directed *acyclic* graph $G = (V, E)$ with exactly one source and exactly one sink.

- $G$ numbered topologically: each path traverses nodes in increasing order.
- For any vertex $v$, there is a directed $(s, t)$-path containing $v$.

**Planar st-graph**: an st-graph with a planar embedding such that $s$ and $t$ are on the outer face.
Planar \( st \)-graphs

- Normally drawn upwards planar.
- Two outer faces \( s^*/t^* \) left/right.
- For each \( e = (u, v) \in E \):
  - \( \text{orig}(e) = u \)
  - \( \text{dest}(e) = v \)
  - \( \text{left}(e) \in F \): face left of \( e \)
  - \( \text{right}(e) \in F \): face right of \( e \).
- \( G^* = (V^* = F, E^*) \):
  - \( e \in E \Rightarrow (\text{left}(e), \text{right}(e)) \in E^* \)
- Multigraph
- \( st^* \)-graph
Properties of Planar $st$-Graphs

**Lemma** $1$ \ Every face $f$ of $G$ consists of two paths from its source $\text{orig}(f)$ to its sink $\text{dest}(f)$.

**Lemma** $2$ \ At each vertex $v \in V$, the incoming/outgoing edges each form an interval, and these intervals are separated by the faces $\text{left}(v)/\text{right}(v)$.

Statements imply the same in the dual.
Properties of Planar $st$-graphs

**Lemma 3** For faces $f$ and $g$ exactly one of the following is true:
- There is a path from $\text{dest}(f)$ to $\text{orig}(g)$ in $G$.
- There is a path from $\text{dest}(g)$ to $\text{orig}(f)$ in $G$.
- There is a path from $f$ to $g$ in $G^\ast$.
- There is a path from $g$ to $f$ in $G^\ast$.

For $v \in V$: let $\text{orig}(v) = \text{dest}(v) = v$.
For $f \in F$: let $\text{left}(f) = \text{right}(f) = f$.

**Lemma 4** For objects $o_1, o_2 \in V \cup E \cup F$ exactly one of the following is true:
- There is a path from $\text{dest}(o_1)$ to $\text{orig}(o_2)$ in $G$.
- There is a path from $\text{dest}(o_2)$ to $\text{orig}(o_1)$ in $G$.
- There is a path from $\text{right}(o_1)$ to $\text{left}(o_2)$ in $G^\ast$.
- There is a path from $\text{right}(o_2)$ to $\text{left}(o_1)$ in $G^\ast$.

**Proof:** Exercise!
Tessellation / Tiling

- Tiles: axis-parallel rectangles

- can be unbounded, or degenerate (line segment/point)

- $\theta_1, \theta_2$ horizontally/vertically adjacent $\iff$ common vertical/horizontal boundary

- we write $\theta = [x_1(\theta), x_2(\theta)] \times [y_1(\theta), y_2(\theta)]$
Tessellation / Tiling

Def. A tessellation θ of a planar \( st \)-graph \( G \) places each object \( o \in V \cup E \cup F \) onto a tile \( \theta(o) \), so that:

- \( o_1 \neq o_2 \Rightarrow \text{int}(\theta(o_1)) \cap \text{int}(\theta(o_2)) = \emptyset \)
- \( \bigcup_{o \in V \cup E \cup F} \theta(o) \) is a rectangle.
- \( \theta(o_1) \) and \( \theta(o_2) \) horizontally adjacent \( \iff \)
  \( o_1 = \text{left}(o_2) \) or \( o_1 = \text{right}(o_2) \) or
  \( o_2 = \text{left}(o_1) \) or \( o_2 = \text{right}(o_1) \)
- \( \theta(o_1) \) and \( \theta(o_2) \) vertically adjacent \( \iff \)
  \( o_1 = \text{orig}(o_2) \) or \( o_1 = \text{dest}(o_2) \) or
  \( o_2 = \text{orig}(o_1) \) or \( o_2 = \text{dest}(o_1) \)
Tessellation Algorithm (for a Planar $st$-Graph $G$)

- Compute the dual $G^*$.
- Compute topological numberings $X$ of $G^*$ and $Y$ of $G$.
- For each object $o \in V \cup E \cup F$, set
  $$\theta(o) = [X(\text{left}(o)), X(\text{right}(o))] \times [Y(\text{orig}(o)), Y(\text{dest}(o))].$$
Tessellation Algorithm (for a Planar \(st\)-Graph \(G\))

- Compute the dual \(G^*\).
- Compute topological numberings \(X\) of \(G^*\) and \(Y\) of \(G\).
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Correctness:

- Lemma 4 guarantees disjointness.
- Neighbourhood conditions follow from the coordinate mapping.

Runtime: $O(n)$
Size Conditions

Minimum height/width $h, w : E \rightarrow \mathbb{R}_{\geq 0}$ for each edge tile.

- Compute optimal *weighted* topological numberings $Y$ of $G = (V, E; h)$ and $X$ of $G^* = (F, E^*; w)$.

- Vertex/Face tiles: modify $G$ to $G'$

  ![Diagram](image)

- Now each object of $G$ corresponds to an edge in $G'$.

**Thm:** A minimum area tessellation of a planar $st$-graph $G$ with minimum height/width $h, w : V \cup E \cup F \rightarrow \mathbb{R}_{\geq 0}$ can be computed in $O(n)$ time.
Visibility Representations

**Def.** A visibility representation $\Gamma$ of a planar $st$-graph $G$ has

- vertex $v$ as a horizontal segment $\Gamma(v)$
- and edge $(u, v)$ as a vertical segment $\Gamma(u, v)$

such that

- vertex segments are pairwise disjoint,
- edge segments are pairwise disjoint, and
- the edge segment $\Gamma(u, v)$ starts from the top of the vertex segment $\Gamma(u)$, ends on the bottom of vertex segment $\Gamma(v)$, and does not intersect other vertex segments.
Computing a Visibility Representation

- Use the tessellation: vertices are degenerate (i.e., line segments); faces are not degenerate

edge segments
Algorithm Visibility(planar st-graph $G$)

- Compute the dual $G^*$.  
- Compute optimal weighted topological numberings $Y$ of $G$ and $X$ of $G^*$ with *unit weights*.  
- For each vertex $v \in V$, set 
  \[ \Gamma(v) = [X(\text{left}(v)), X(\text{right}(v)) - 1] \times \{Y(v)\} . \]
- For each edge $e \in E$, set 
  \[ \Gamma(e) = \{X(\text{left}(e))\} \times [Y(\text{orig}(e)), Y(\text{dest}(e))] . \]
Algorithm Visibility(planar st-graph G)

- Compute the dual $G^*$.
- Compute optimal weighted topological numberings $Y$ of $G$ and $X$ of $G^*$ with unit weights.
- For each vertex $v \in V$, set
  $\Gamma(v) = [X(\text{left}(v)), X(\text{right}(v)) - 1] \times \{Y(v)\}$.
- For each edge $e \in E$, set
  $\Gamma(e) = \{X(\text{left}(e))\} \times [Y(\text{orig}(e)), Y(\text{dest}(e))]$.

**Thm:** In $O(n)$ time, the Visibility algorithm generates a visibility representation with integer coordinates and area $O(n^2)$. 