Algorithms for Graph Visualization

Summer Semester 2019
Lecture #7

Hierarchical Drawings

References:
Drawing Graphs: Eds. K. & W. (Ch. 5)
Graph Drawing: D. E. T. & T. (Ch. 9)
(based on slides from Marcus Krug, KIT)
Example
Example
Example

E-Mail-Graph between groups in Computer Science, KIT
Hierarchical Drawing

Problem statement:

- **Input:** directed graph $D = (V, A)$
- **Output:** Drawing of $D$ which *closely* reproduces the hierarchical properties of $D$. 
Hierarchical Drawing

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Desireable Properties

- vertices occur on (few) horizontal lines
- edges directed upwards
- edge crossings minimized
- edges upward, straight, and short as possible
- vertices evenly spaced
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- vertices occur on (few) horizontal lines
- edges directed upwards
- edge crossings minimized
- edges upward, straight, and short as possible
- vertices evenly spaced

Criteria can be contradictory!
Classical Approach

[Sugiyama, Tagawa, Toda '81]

Input

```plaintext
   3
  / \
 4   2
 /   /
1   6   7
```

```plaintext
5
```
Classical Approach [Sugiyama, Tagawa, Toda ’81]

Input → Cycle breaking

Diagram showing cycle breaking in a graph.
Classical Approach

[Sugiyama, Tagawa, Toda ’81]
Classical Approach

[Sugiyama, Tagawa, Toda ’81]

Input → Cycle breaking → Leveling

Crossing minimization
Classical Approach

[Sugiyama, Tagawa, Toda ’81]

Input → Cycle breaking → Leveling

Cycle breaking

Leveling

Input

Cycle breaking

Leveling

Crossing minimization

Vertex positioning
Classical Approach

[Sugiyama, Tagawa, Toda '81]

Input → Cycle breaking → Leveling

Crossing minimization → Vertex positioning → Edge drawing
Step 1: Cycle Breaking

Approach

- Find minimum set $A^*$ of edges which are not upwards.
- Remove $A^*$ and insert reversed edges.
Step 1: Cycle Breaking

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- Remove $A^*$ and insert reversed edges.

Problem Minimum Feedback Arc Set (FAS):

- Input: directed graph $D = (V, A)$
- Output: min. set $A^* \subseteq A$, so that $D - A^*$ acyclic
Step 1: Cycle Breaking

Approach
- Find minimum set $A^\star$ of edges which are not upwards.
- Remove $A^\star$ and insert reversed edges.

Problem Minimum Feedback Arc Set (FAS):
- Input: directed graph $D = (V, A)$
- Output: min. set $A^\star \subseteq A$, so that $D - A^\star$ acyclic

... NP-hard :-(

Step 1: Cycle Breaking

Approach
- Find minimum set $A^*$ of edges which are not upwards.
- Remove $A^*$ and insert reversed edges.

Problem Minimum Feedback Arc Set (FAS):
- Input: directed graph $D = (V, A)$
- Output: min. set $A^* \subseteq A$, so that $D - A^* + A^*_r$ acyclic

$\ldots$ NP-hard :-(
Greedy-Heuristic for FAS

GreedyMakeAcyclic(Digraph $D = (V, A)$)

$A' \leftarrow \emptyset$ (these will be the edges we keep)

return $(V, A')$
Greedy-Heuristic for FAS

GreedyMakeAcyclic(Digraph \( D = (V, A) \))

\[ A' \leftarrow \emptyset \] (these will be the edges we keep)

\[ \text{foreach } v \in V \text{ do} \]

\[ \text{return } (V, A') \]
Greedy-Heuristic for FAS

GreedyMakeAcyclic(Digraph $D = (V, A)$)

$A' \leftarrow \emptyset$ (these will be the edges we keep)

\begin{verbatim}
foreach $v \in V$ do
    if outdeg($v$) $>$ indeg($v$) then
        $A' \leftarrow A' \cup \text{out}(v)$
    else
        $A' \leftarrow A' \cup \text{in}(v)$

$A \leftarrow A \setminus (\text{out}(v) \cup \text{in}(v))$

return $(V, A')$
\end{verbatim}
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return $(V, A')$

• Timing:

$$\{vw | vw \in A\} = \{uv | uv \in A\}$$

(these will be the edges we keep)
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• Timing: $O(V + A)$
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return $(V, A')$

• Timing: $O(V + A)$

• Quality guarantee: $|A'| \geq \{vw | vw \in A\}$

(these will be the edges we keep)
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foreach $v \in V$ do

if $\text{outdeg}(v) > \text{indeg}(v)$ then

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end if

$A \leftarrow A \setminus (\text{out}(v) \cup \text{in}(v))$

end foreach

return $(V, A')$

- Timing: $O(V + A)$
- Quality guarantee: $|A'| \geq |A|/2$
Improved Greedy-Heuristic for FAS

- Each iteration of `foreach`, first look for sources and sinks. If there are none, pick $v$ such that $|\text{outdeg}(v) - \text{indeg}(v)|$ is maximized.
Improved Greedy-Heuristic for FAS

• Each iteration of **foreach**, first look for sources and sinks. If there are none, pick $v$ such that $|\text{outdeg}(v) - \text{indeg}(v)|$ is maximized.

• Timing: ?

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- Quality guarantee: \( |A'| \geq |A|/2 + |V|/6 \)
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Step 2: Leveling
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Problem

- **Input:** acyclic, directed graph $D = (V, A)$
- **Output:**
Step 2: Leveling

Problem

- **Input:** acyclic, directed graph $D = (V, A)$
- **Output:** Mapping $y : V \rightarrow \{1, \ldots, |V|\}$, so that for every $uv \in A$, $y(u) < y(v)$. 
Step 2: Leveling

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- **Input:** acyclic, directed graph $D = (V, A)$
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**Objective:** minimize . . .
Step 2: Leveling

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Objective: minimize . . .

- Number of layers, i.e. $|y(V)|$
- Length of the longest edge, i.e. $\max_{uv \in A} y(v) - y(u)$
- Total edge length, i.e. number of dummy vertices
Step 2: Leveling

Problem

- **Input:** acyclic, directed graph $D = (V, A)$
- **Output:** Mapping $y : V \rightarrow \{1, \ldots, \lvert V \rvert \}$, so that for every $uv \in A$, $y(u) < y(v)$.

Objective: **minimize** . . .

- **Number of layers**, i.e. $\lvert y(V) \rvert$
- **Length of the longest edge**, i.e. $\max_{uv \in A} y(v) - y(u)$
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Algorithm to Minimize the Number of Layers
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- for each source \( q \)
  set \( y(q) := 1 \)
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  set $y(q) := 1$
- for each non-source $v$
  set $y(v) := \max\{y(u) \mid uv \in A\} + 1$
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**Obs.** $y(v)$ is...
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Length of the longest path from a source to \( v \) plus 1.
...also optimal with respect to the number of layers!
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Length of the longest path from a source to \( v \) plus 1.
...also optimal with respect to the number of layers!

**Question:** Can we do this in linear time?
Linear time implementation

- for each source $q$
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Linear time implementation

ComputeLayering(AcyclicDigraph $D = (V, A)$)

$y = \text{new} \int[1..|V|]$ // all == 0

for each source $q \in V$

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Linear time implementation

ComputeLayering(AcyclicDigraph $D = (V, A)$)

\[
y = \text{new int}[1..|V|] \quad // \text{all } \equiv 0
\]

\textbf{foreach} source $q \in V$ \textbf{do}
\[
\_ \quad y(q) \leftarrow 1
\]

\textbf{foreach} non-source $v \in V$ \textbf{do}
\[
\_ \quad \text{ComputeYRec}(D, v, y)
\]

\textbf{return} $y$

\textbf{ComputeYRec}(AcyclicDigraph $D = (V, A)$, Vertex $v$, int[] $y$)

\textbf{if} $y(v) == 0$ \textbf{then}
\[
\_ \quad y(v) \leftarrow
\]

\textbf{return} $y(v)$
Linear time implementation

ComputeLayering(AcyclicDigraph \( D = (V, A) \))

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y = \text{new} \ int[1..|V|] \quad \text{// all} = 0
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\[
\text{foreach source } q \in V \ \text{do}
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\]

\text{return } y

ComputeYRec(AcyclicDigraph \( D = (V, A), \text{Vertex } v, \text{int}[] y \))

\[
\text{if } y(v) = 0 \ \text{then}
\]
\[
\quad y(v) \leftarrow \max \{ \text{ComputeYRec}(D, u, y) \mid uv \in A \} + 1
\]

\text{return } y(v)
Our Example
Our Example
Our Example

Looks good .... right?
Our Example

Looks good .... right?
Our Example

Looks good .... right?
Our Example

Looks good .... right?

The drawing can be very wide :-(
Goal: Narrower layer assignment.

Problem: Leveling with a given width.

- Input: acyclic, digraph $D = (V, A)$, width $W > 0$
- Output: Partition the vertex set into a minimum number of layers such that each layer contains at most $W$ elements.
Goal: Narrower layer assignment.

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Problem: Precedence-Constrained Multi-Processor Scheduling

- **Input:** $n$ jobs with unit (1) processing time, $W$ identical machines, and a partial ordering $<$ on the jobs.
- **Output:** Schedule respecting $<$ and having minimum processing time.
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- **NP-hard,** $(2 - \frac{2}{W})$-Approx., no $(\frac{4}{3} - \varepsilon)$-Approx. $(W \geq 3)$. 
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Approximating PCMPS

**Input:** Precedence graph (divided into layers of arbitrary width)
Approximating PCMPS

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Approximating PCMPS

**Input:** Precedence graph (divided into layers of arbitrary width)

Number of Machines is $W = 2$. 

Graph representation:

```
1 -- 2
<p>| |</p>
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
</table>
3 -- 4
```

```
5 -- 6
<p>| |</p>
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
</table>
7 -- 8
```

```
A -- D
<p>| |</p>
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
</table>
B -- C
```

```
E -- F
<p>| |</p>
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
</table>
G --
```
Approximating PCMPS

**Input:** Precedence graph (divided into layers of arbitrary width)

Number of Machines is $W = 2$.

**Output:** Schedule
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Output: Schedule

<table>
<thead>
<tr>
<th>$M_1$</th>
<th>$M_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>1 2 3 4 5 6 7 8 9 10</td>
</tr>
</tbody>
</table>
Approximating PCMPS

**Input:** Precedence graph (divided into layers of arbitrary width)

![Precedence Graph](image)

Number of Machines is $W = 2$.

**Output:** Schedule

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Input: Precedence graph (divided into layers of arbitrary width)

Output: Schedule

\[
\begin{array}{c|ccc}
M_1 & 1 & 2 & 4 \\
M_2 & - & 3 & - \\
t & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
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Approximating PCMPS

**Input:** Precedence graph (divided into layers of arbitrary width)

```
    1 -> 3 -> 5
      \   /   \
      4   2   6
        \____/    \
          7
```

Number of Machines is $W = 2$.

**Output:** Schedule

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![Precedence graph](image)

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Approximating PCMPS

**Input:** Precedence graph (divided into layers of arbitrary width)

![Precedence graph](image)

Number of Machines is $W = 2$.

**Output:** Schedule

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<th>$M_1$</th>
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**Question:** Good approximation factor?
Algorithm

• jobs stored in a list $L$
  (in any order, e.g., topologically sorted).

• for each time $t = 1, 2, \ldots$ schedule $\leq W$ available jobs.

• a job in $L$ is *available* when all its predecessors have been scheduled.

• As long as there are free machines and available jobs, take the first available job and assign it to a free machine.
Analysis for $W = 2$

Precedence graph $G <$

Schedule

\[
\begin{array}{c|cccccccccc}
M_1 & 1 & 2 & 4 & 5 & 6 & 8 & A & C & E & G \\
M_2 & - & 3 & - & - & 7 & 9 & B & D & F & - \\
t & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}
\]

„The art of the lower bound“
Analysis for $W = 2$

Precedence graph $G_{<}$

Schedule

<table>
<thead>
<tr>
<th>$M_1$</th>
<th>1 2 4 5 6 8 A C E G</th>
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<tr>
<td>$M_2$</td>
<td>- 3 -- 7 9 B D F --</td>
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$t$ | 1 2 3 4 5 6 7 8 9 10 |

„The art of the lower bound“

$\text{OPT} \geq$
Analysis for $W = 2$

```
Analysis for $W = 2$

Precedence graph $G_<$

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„The art of the lower bound“

$\text{OPT} \geq \lceil n/2 \rceil$
```
Analysis for $W = 2$

Precedence graph $G_<$

```
1  2  5  6  8  C  F  G
3  4  7  9  A  D  E
```

Schedule

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„The art of the lower bound“

$\text{OPT} \geq \lceil n/2 \rceil$ and $\text{OPT} \geq \lceil n/2 \rceil$
Analysis for $W = 2$

Precedence graph $G_<$

Schedule

```
 M_1 | 1 2 4 5 6 8 A C E G  
 M_2 | 3 - - 7 9 B D F -  
 t   | 1 2 3 4 5 6 7 8 9 10
```

"The art of the lower bound"

$$\text{OPT} \geq \lceil n/2 \rceil \quad \text{and} \quad \text{OPT} \geq \ell := \text{Number of layers of } G_<$$
Analysis for $W = 2$

Precedence graph $G_<$

```
1 -> 2, 3, 4
2 -> 5
3 -> 6, 7
4 -> 8
5 -> 9
6, 7 -> 8
8 -> C
9 -> D
C -> E
D -> F
E -> G
F -> A
G -> B
```

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$t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10

„The art of the lower bound“

$\text{OPT} \geq \lceil n/2 \rceil \quad \text{and} \quad \text{OPT} \geq \ell := \text{Number of layers of } G_<$

Goal: measure the quality of our algorithm using the lower bound(s).
Analysis for $W = 2$

Precedence graph $G_<$

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„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

Goal: measure the quality of our algorithm using the lower bound(s).

Bound $\text{ALG} \leq$
Analysis for $W = 2$

Precedence graph $G_<$

```
1 -> 3 -> 5
2 -> 6
4

6 -> 8 -> C
7 -> A

9 -> D

E -> G
```

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„The art of the lower bound“

$$\text{OPT} \geq \left\lceil \frac{n}{2} \right\rceil \quad \text{and} \quad \text{OPT} \geq \ell := \text{Number of layers of } G_<$$

**Goal:** measure the quality of our algorithm using the lower bound(s).

**Bound** $\text{ALG} \leq$
Analysis for $W = 2$

**Precedence graph** $G_<$

```
1 → 2 → 5 → 8 → E → F → G
3 → 6 → 9 → C → D
4
```

**Schedule**

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$$\text{OPT} \geq \left\lceil \frac{n}{2} \right\rceil \quad \text{and} \quad \text{OPT} \geq \ell := \text{Number of layers of } G_<$$

**Goal:** measure the quality of our algorithm using the lower bound(s).

**Bound** $\text{ALG} \leq$

- insertion of pauses ($-$) in the schedule (except the last) maps to layers of $G_<$
Analysis for $W = 2$

```
Precedence graph $G <$

1 → 2 → 3 → 5 → 6 → 9 → C → D → E
4 → 7 → A → B → G

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"The art of the lower bound"

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell : = \text{Number of layers of } G <$

**Goal:** measure the quality of our algorithm using the lower bound(s).

**Bound** $\text{ALG} \leq \lceil \frac{n+\ell}{2} \rceil$

insertion of pauses (−) in the schedule (except the last) maps to layers of $G <$
Analysis for $W = 2$

```
Precedence graph $G_<$
1 -> 2
1 -> 3
1 -> 4
2 -> 5
3 -> 6
4 -> 7
6 -> 8
6 -> 9
8 -> C
9 -> C

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$\text{OPT} \geq \lceil n/2 \rceil$ and $\text{OPT} \geq \ell := \text{Number of layers of } G_<$

**Goal:** measure the quality of our algorithm using the lower bound(s).

**Bound** $\text{ALG} \leq \lceil \frac{n+\ell}{2} \rceil \approx$

insertion of pauses (−) in the schedule (except the last) maps to layers of $G_<$
Analysis for $W = 2$

Precedence graph $G_<$

Schedule

„The art of the lower bound“

$\text{OPT} \geq \lceil n/2 \rceil$ and $\text{OPT} \geq \ell := \text{Number of layers of } G_<$

**Goal:** measure the quality of our algorithm using the lower bound(s).

**Bound** $\text{ALG} \leq \lceil \frac{n+\ell}{2} \rceil \approx \lceil n/2 \rceil + \ell/2$

insertion of pauses (−) in the schedule (except the last) maps to layers of $G_<$
Analysis for $W = 2$

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‟The art of the lower bound‟

$\text{OPT} \geq \lceil n/2 \rceil$ and $\text{OPT} \geq \ell$ := Number of layers of $G_{<}$

Goal: measure the quality of our algorithm using the lower bound(s).

Bound $\text{ALG} \leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx \lceil n/2 \rceil + \ell/2 \leq$

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"The art of the lower bound"

$\text{OPT} \geq \lceil n/2 \rceil$ and $\text{OPT} \geq \ell := \text{Number of layers of } G_<$

Goal: measure the quality of our algorithm using the lower bound(s).

Bound $\text{ALG} \leq \lceil \frac{n+\ell}{2} \rceil \approx \lceil n/2 \rceil + \ell/2 \leq 3/2 \cdot \text{OPT}$

insertion of pauses (–) in the schedule (except the last) maps to layers of $G_<$
Analysis for $W = 2$

Precedence graph $G_<$

Schedule

Optimal bound on $\text{OPT}$

Goal: measure the quality of our algorithm using the lower bound(s).

Bound $\text{ALG} \leq \left\lceil \frac{n + \ell}{2} \right\rceil \approx \left\lceil \frac{n}{2} \right\rceil + \frac{\ell}{2} \leq \frac{3}{2} \cdot \text{OPT}$

"The art of the lower bound"

$\text{OPT} \geq \lceil n/2 \rceil$ and $\text{OPT} \geq \ell := \text{Number of layers of } G_<$

Insertion of pauses (−) in the schedule (except the last) maps to layers of $G_<$

Gen. $\leq (2 - 1/W) \cdot \text{OPT}$
Step 3: Crossing minimization
Step 3: Crossing minimization

Problem:

- Input:
- Output:
Step 3: Crossing minimization

Problem:

- Input: Graph $G$, layering $y: V \rightarrow \{1, \ldots, |V|\}$
- Output:
Step 3: Crossing minimization

Problem:

- **Input:** Graph $G$, layering $y : V \rightarrow \{1, \ldots, |V|\}$
- **Output:** (Re-)ordering of vertices in each layer so that the number of crossings in minimized.
Step 3: Crossing minimization

Problem:

- **Input:** Graph $G$, layering $y: V \rightarrow \{1, \ldots, |V|\}$
- **Output:** (Re-)ordering of vertices in each layer so that the number of crossings in minimized.

- NP-hard, even for 2 layers
- hardly any approaches optimize over multiple layers :-(

[Garey & Johnson '83]
Iterative crossing reduction – idea
Iterative crossing reduction – idea

- add dummy-vertices for edges connecting *far* layers.
- consider adjacent layers \((L_1, L_2), (L_2, L_3), \ldots\)
  bottom-to-top.
- minimize crossings by permuting \(L_{i+1}\) while keeping \(L_i\)
  fixed.
Iterative crossing reduction – idea

- add dummy-vertices for edges connecting *far* layers.
- consider adjacent layers \((L_1, L_2), (L_2, L_3), \ldots\) bottom-to-top.
- minimize crossings by permuting \(L_{i+1}\) while keeping \(L_i\) fixed.

**Obs.** The number of crossings only depends on permutations of adjacent layers.
Iterative crossing reduction – Algorithm

(1) choose a random permutation of $L_1$. 
Iterative crossing reduction – Algorithm

1. choose a random permutation of $L_1$.
2. iteratively consider adjacent layers $L_i$ und $L_{i+1}$.
3. minimize crossings by permuting $L_{i+1}$ and keeping $L_i$ fixed.
4. repeat steps 2–3 in the reverse order (starting from $L_h$).
5. repeat steps 2–4 until no further improvement is achieved.
6. repeat steps 1–5 with different starting permutations.
Iterative crossing reduction – Algorithm

(1) choose a random permutation of $L_1$.

(2) iteratively consider adjacent layers $L_i$ und $L_{i+1}$.

(3) minimize crossings by permuting $L_{i+1}$ and keeping ($L_i$ fixed).

(4) repeat steps (2)–(3) in the reverse order (starting from $L_h$).

(5) repeat steps (2)–(4) until no further improvement is achieved.

(6) repeat steps (1)–(5) with different starting permutations.
Iterative crossing reduction – Algorithm

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Iterative crossing reduction – Algorithm

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Iterative crossing reduction – Algorithm

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(4) repeat steps (2)–(3) in the reverse order (starting from $L_h$).

(5) repeat steps (2)–(4) until no further improvement is achieved.

(6) repeat steps (1)–(5) with different starting permutations.
One-sided Crossing Minimization

Problem

• Input:

• Output:
One-sided Crossing Minimization

Problem

- **Input:** bipartite graph $G = (L_1 \cup L_2, E)$, permutation $\pi_1$ on $L_1$
- **Output:**

![Graph](image-url)
One-sided Crossing Minimization

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- Input: bipartite graph $G = (L_1 \cup L_2, E)$, permutation $\pi_1$ on $L_1$
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One-sided crossing minimization is NP-hard.

\[ \text{Abb. aus [Kaufmann und Wagner: Drawing Graphs]} \]
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Algorithms

- barycenter heuristic
- median heuristic
- Greedy-Switch
- ILP

...
Barycentre Heuristic

- Intuition: few intersections occur when vertices are close to their neighbours

[Sugiyama et al. '81]
Barycentre Heuristic

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- The barycentre of $u$ is the average $x$-coordinate of the neighbours of $u$ in layer $L_1$ 

$$x_2(u) := \text{bary}(u) := \frac{1}{\deg(u)} \sum_{v \in N(u)} x_1(v)$$
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• vertices with the same barycentre of are offset by a small \( \delta \)
• linear runtime
• relatively good results
• optimal if no crossings are required
• \( O(\sqrt{n}) \)-approximation factor
Barycentre Heuristic

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Worst case?

\([\text{Sugiyama et al. '81}]\)
Barycentre Heuristic

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- optimal if no crossings are required $\leftarrow$ exercise!
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Worst case?

\[u \rightarrow v\]

\[k^2 - 1\]

\[k - 1\]
Barycentre Heuristic

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- linear runtime
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- optimal if no crossings are required \(\Rightarrow\) exercise!
- \(O(\sqrt{n})\)-approximation factor
Median heuristic

- $\{v_1, \ldots, v_k\} := N(u)$ with $\pi_1(v_1) < \pi_1(v_2) < \cdots < \pi_1(v_k)$

- $x_2(u) := \text{med}(u) := \begin{cases} 0 & \text{when } N(u) = \emptyset \\ \pi_1(v_{\lceil k/2 \rceil}) & \text{otherwise} \end{cases}$

- move vertices $u$ und $v$ by small $\delta$, when $x_2(u) = x_2(v)$
Median heuristic

- \{v_1, \ldots, v_k\} := N(u) with \pi_1(v_1) < \pi_1(v_2) < \cdots < \pi_1(v_k)
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proof in [DETT]

[Eades & Wormald '94]
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proof in [DETT]

\[
2k(k + 1) + k^2 \quad \text{vs.} \quad (k + 1)^2
\]
Greedy-Switch heuristic

- iteratively swap each adjacent node as long as crossings decrease.
- runtime $O(L^2)$ per iteration; at most $|L^2|$ iterations
- suitable as post-processing for other heuristics
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Greedy-Switch heuristic

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Worst case?

![Diagram](image-url)
Greedy-Switch heuristic

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Worst case?

![Diagram showing a worst-case scenario](image-url)
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![Diagram of worst case scenario]
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Worst case?

$\approx k^2/4$  $\approx 2k$
Integer Linear Program

- Constant $c_{ij} := \text{num. of crossings between edges incident to } v_i \text{ or } v_j \text{ when } \pi_2(v_i) < \pi_2(v_j)$

\[ \text{Diagram: } v_i \quad v_j \]
Integer Linear Program

- Constant $c_{ij} := \text{num. of crossings between edges incident to } v_i \text{ or } v_j \text{ when } \pi_2(v_i) < \pi_2(v_j)$

- Variable $x_{ij}$ for each $1 \leq i < j \leq n_2 := |L_2|$

$$x_{ij} = \begin{cases} 
1 & \text{when } \pi_2(v_i) < \pi_2(v_j) \\
0 & \text{otherwise}
\end{cases}$$
Integer Linear Program

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\]

- The number of crossings of a permutations $\pi_2$

\[
\text{cross}(\pi_2) = \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} (c_{ij} - c_{ji})x_{ij} + \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} c_{ji}
\]
ILP (cont.)

- Minimize the number of crossings:

\[
\text{minimize } \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} (c_{ij} - c_{ji})x_{ij}
\]
ILP (cont.)

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  \]

- Constraints:
  \[
  0 \leq x_{ij} + x_{jk} - x_{ik} \leq 1 \quad \text{for } 1 \leq i < j < k \leq n_2
  \]
ILP (cont.)

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i.e., if \( x_{ij} = 1 \) and \( x_{jk} = 1 \), then \( x_{ik} = 1 \)
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\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{array}
\]
ILP (cont.)

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(Transitivity)
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(Transitivity)

Solution can be found via Branch-and-Bound on small degree graphs relatively quickly.
Our Example – iterations
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Step 4: Vertex positioning

Goal: paths should be close to straight, nodes evenly spaced
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Goal: paths should be close to straight, nodes evenly spaced

Exact: Quadratic Program (QP)

Heuristic: iterative approach
Quadratic Program

- Consider the path $p_e = (v_1, \ldots, v_k)$ of an edge $e = v_1v_k$ with dummy vertices: $v_2, \ldots, v_{k-1}$
Quadratic Program

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- $x$-coordinate of $v_i$ according to the line $v_1v_k$ (with equal spacing):

$$x(v_i) = x(v_1) + \frac{i - 1}{k-1} (x(v_k) - x(v_1))$$
Quadratic Program

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  \[ x(v_i) = x(v_1) + \frac{i - 1}{k - 1} (x(v_k) - x(v_1)) \]

- Define the deviation from the line
  \[ \text{dev}(p_e) := \sum_{i=2}^{k-1} \left( x(v_i) - \overline{x(v_i)} \right)^2 \]
Quadratic Program

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QP (cont.)

- Objective function:

\[
\min \sum_{e \in E} \text{dev}(p_e)
\]
QP (cont.)

- Objective function:

\[
\min \sum_{e \in E} \text{dev}(p_e)
\]

- Constraints: for all vertices \(v, w\) in the same layer with \(w\) right of \(v\)

\[
x(w) - x(v) \geq \rho(w, v)
\]
QP (cont.)

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- Problem: QP and potentially exponential width
Iterative Heuristic

• compute an Initial-Layout
Iterative Heuristic

- compute an Initial-Layout
- apply the following steps as long as improvements can be made.
Iterative Heuristic

- compute an Initial-Layout
- apply the following steps as long as improvements can be made.
  1. vertex positioning,
  2. edge straightening,
  3. compactifying the layout width.
Our Example
Our Example
Step 5: drawing the edges
Step 5 – drawing the edges (vertices w/ +ve area)
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Step 5 – drawing the edges (vertices w/ +ve area)

All figs. from [Kaufmann und Wagner: Drawing Graphs]
Our Example
Our Example
Our Example
Classical Approach

[Sugiyama, Tagawa, Toda ’81]

Input → Cycle breaking → Leveling

- Crossing minimization
- Vertex positioning
- Edge drawing