

# Computational Geometry

## Convex Hulls in 3D or Mixing More Things

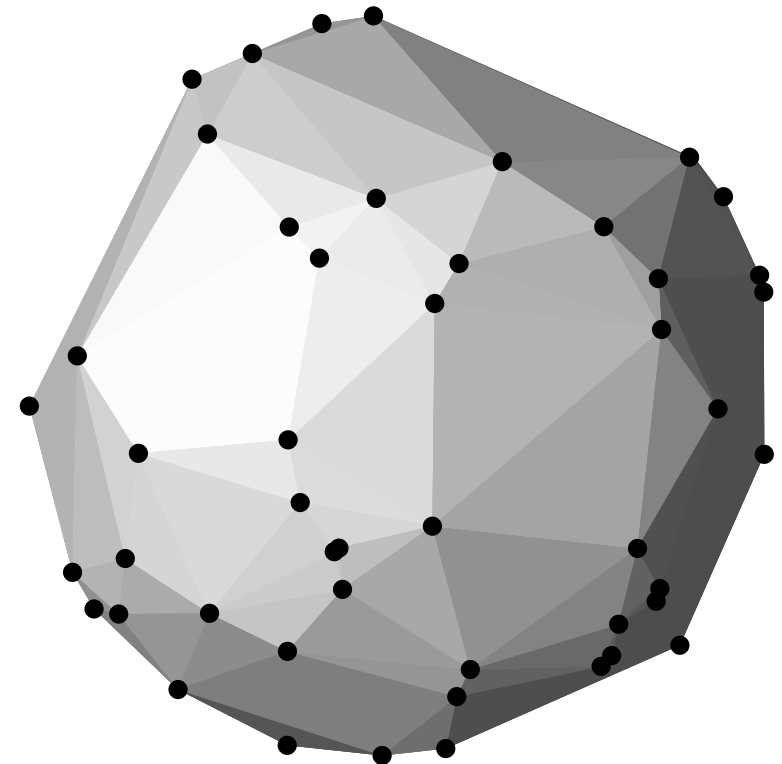
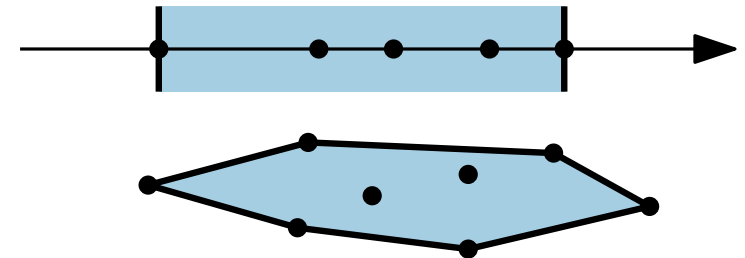
### Lecture #9

# Complexity of the Convex Hull

Given set  $S$  of  $n$  points in  $\mathbb{R}^d$ , what is max. #edges on  $\partial\text{CH}(S)$ ?

dim	w-c complexity of $\text{CH}(S)$
1	$2 \in \Theta(1)$
2	$n \in \Theta(n)$
3	$3n - 6 \in \Theta(n)$
$d$	$\Theta(n^{\lfloor d/2 \rfloor})$

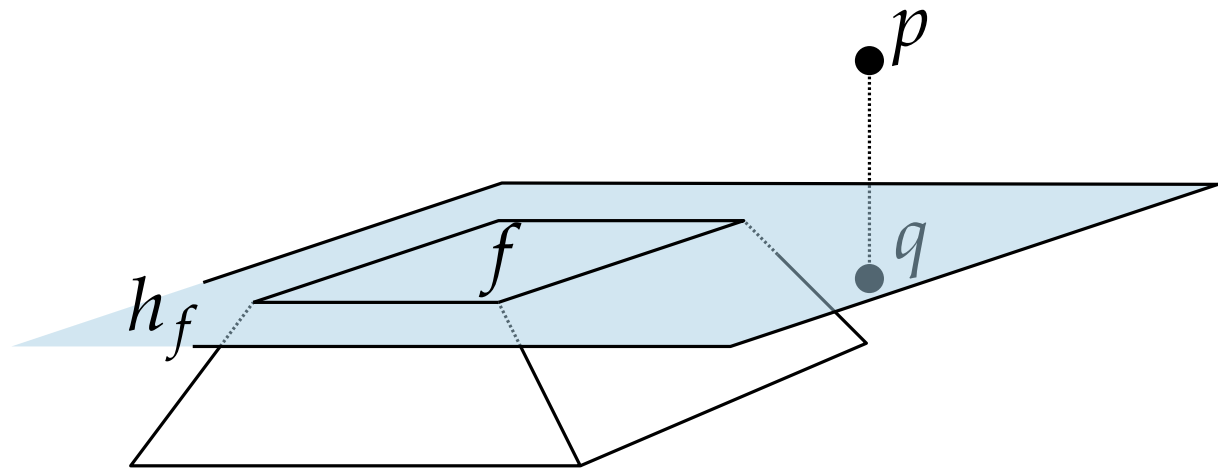
*Upper Bound Theorem*



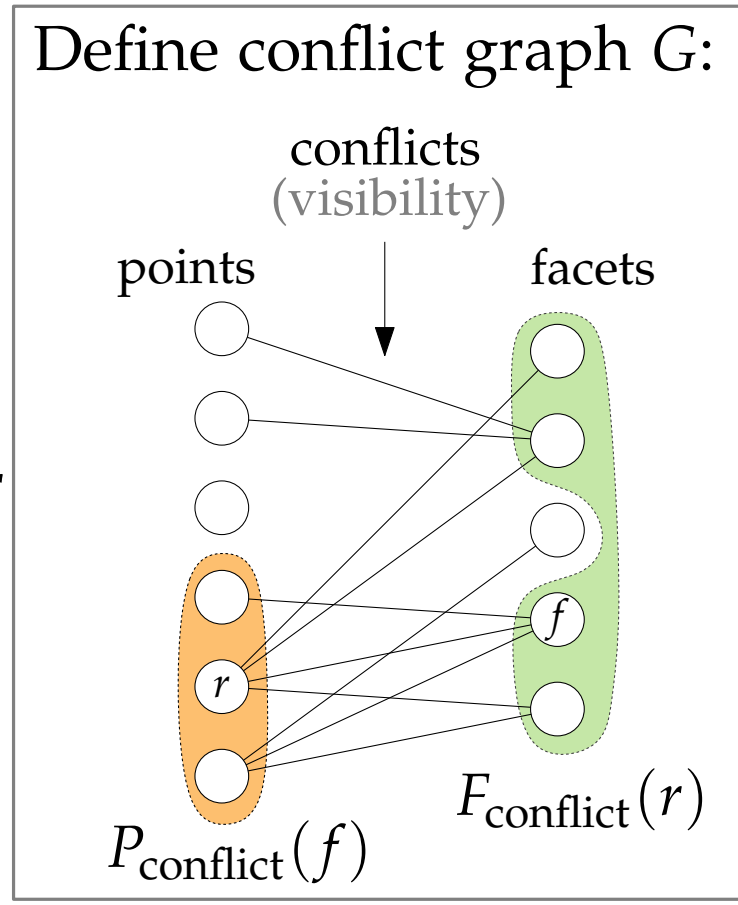
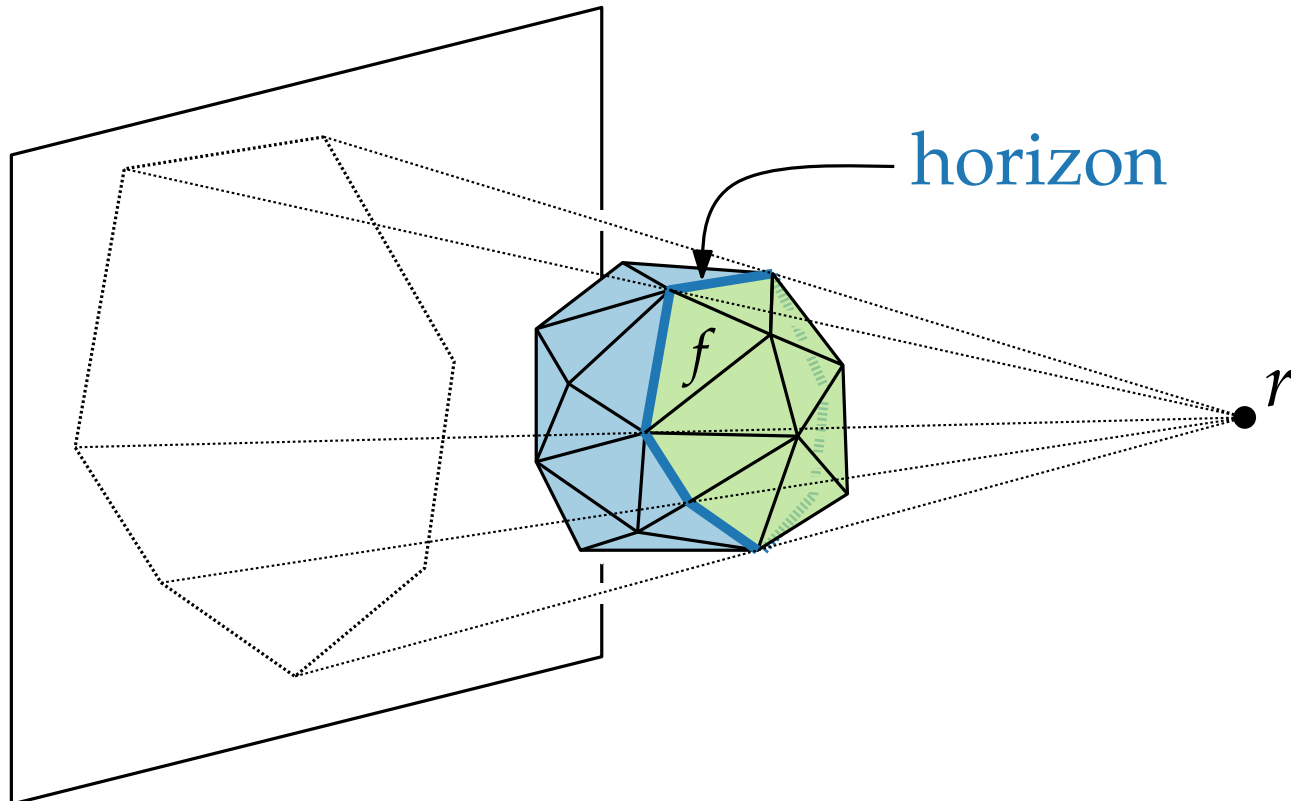
## Construction

randomized-incremental!

# Visibility



Face  $f$  is *visible* from  $p$  but not from  $q$ .



# Rand3DConvexHull( $P \subset \mathbb{R}^3$ )

pick non-coplanar set  $P' = \{p_1, \dots, p_4\} \subseteq P$

$C \leftarrow \text{CH}(P')$

compute rand. perm.  $(p_5, \dots, p_n)$  of  $P \setminus P'$

initialize conflict graph  $G$

**for**  $r = 5$  **to**  $n$  **do**

**if**  $F_{\text{conflict}}(p_r) \neq \emptyset$  **then**  $\{p_r \notin C\}$

delete all facets in  $F_{\text{conflict}}(p_r)$  from  $C$

$\mathcal{L} \leftarrow$  list of horizon edges visible from  $p_r$

**foreach**  $e \in \mathcal{L}$  **do**

$f \leftarrow C.\text{create\_facet}(e, p_r)$ ; create vtx for  $f$  in  $G$

$(f_1, f_2) \leftarrow \text{previously\_incident}_C(e)$

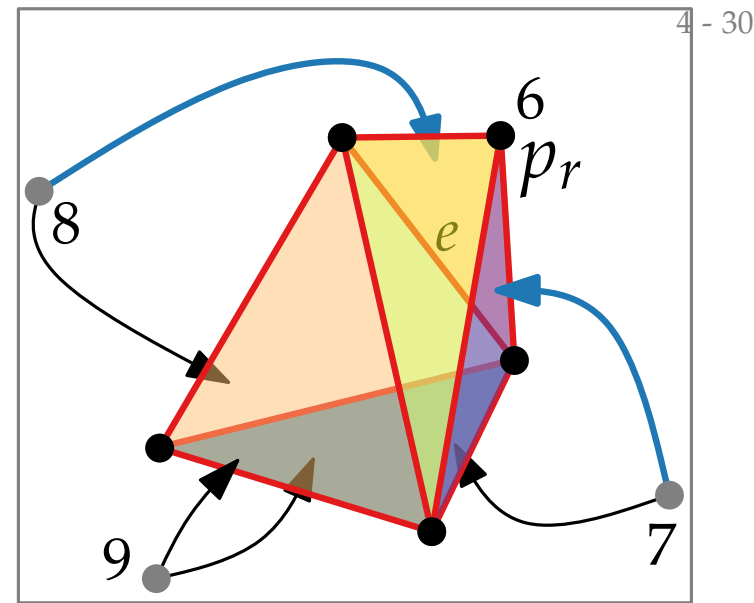
$P(e) \leftarrow P_{\text{conflict}}(f_1) \cup P_{\text{conflict}}(f_2)$

**foreach**  $p \in P(e)$  **do**

$\quad$  **if**  $f$  is visible from  $p$  **then** add edge  $(p, f)$  to  $G$

delete vtx  $\{p_r\} \cup F_{\text{conflict}}(p_r)$  from  $G$

**return**  $C$



Worst-case running time:  $O(n^3)$

# Analysis

**Idea:** Bound expected *structural change*, that is, the total #facets created by the algorithm.

**Lemma.** The expected #facets created is at most  $6n - 20$ .

*Proof.*  $E[\text{\#facets created}] =$   
 $= 4 + \sum_{r=5}^n E[\underbrace{\text{\#facets incident to } p_r \text{ in CH}(P_r)}_{\text{\#edges}}] \leq 6n - 20$

For  $r > 4$ :

$$\begin{aligned}
 E[\text{deg}(p_r, \text{CH}(P_r))] &= \frac{1}{r-4} \sum_{i=5}^r \text{deg}(p_i, \text{CH}(P_r)) \\
 &\leq \frac{1}{r-4} [(\underbrace{\sum_{i=1}^r \text{deg}(p_i)}_{2 \cdot \text{\# edges of CH}(P_r)} - 12)] \\
 &\leq \frac{1}{r-4} [2 \cdot (3r - 6) - 12] \leq 6
 \end{aligned}$$

# Running Time

**Theorem:** The convex hull of a set of  $n$  pts in  $\mathbb{R}^3$  can be computed in  $O(n \log n)$  expected time.

$O(n)$  time

```

Rand3DConvexHull( $P \subset \mathbb{R}^3$ )
{
  pick set  $P' = \{p_1, \dots, p_4\} \subseteq P$  of 4 non-coplanar pts
   $C \leftarrow \text{CH}(P')$ 
  compute a random permutation  $(p_5, \dots, p_n)$  of  $P \setminus P'$ 
  initialize conflict graph  $G$ 
  for  $r = 5$  to  $n$  do
    if  $F_{\text{conflict}}(p_r) \neq \emptyset$  then
      delete all facets in  $F_{\text{conflict}}(p_r)$  from  $C$ 
       $\mathcal{L} \leftarrow$  list of horizon edges visible from  $p_r$ 
      foreach  $e \in \mathcal{L}$  do
         $f \leftarrow C.\text{create\_facet}(e, p_r)$ ; create vtx for  $f$  in  $G$ 
         $(f_1, f_2) \leftarrow$  previously\_incident $_C(e)$ 
         $P(e) \leftarrow P_{\text{conflict}}(f_1) \cup P_{\text{conflict}}(f_2)$ 
        foreach  $p \in P(e)$  do
          if  $f$  visible from  $p$  then add edge  $(p, f)$  to  $G$ 
      delete vtx  $\{p_r\} \cup F_{\text{conflict}}(p_r)$  from  $G$ 
  return  $C$ 

```

using *configuration spaces*, Section 9.5 [Comp. Geom A&A]

Stage  $r$  of for-loop (w/o outer foreach loop)

takes time  $O(|F_{\text{conflict}}(p_r)|) = O(\#\text{facets del. when adding } p_r)$

This part of for-loop in total:

$E[\#\text{facets deleted}] = \leq E[\#\text{facets created}] = O(n)$ .

Lemma

Outer foreach-loop:

– in stage  $r$ :  $O(\sum_{e \in \mathcal{L}} |P(e)|)$

– in total:

$$O\left(\sum_{e \text{ on horizon at some time}} |P(e)|\right) = \cancel{O(n^2)} O(n \log n)$$

# Running times – expected vs. worst case

**Theorem:** The convex hull of a set of  $n$  pts in  $\mathbb{R}^3$  can be computed in  $O(n \log n)$  expected time.

**Exercise:** Give a simple deterministic algorithm that computes the convex hull in  $O(n^2)$  (worst-case) time.

# Convex Hulls and Half-Space Intersections Plane

Define duality  $\star$  between pts and (non-vertical) lines:

For  $p = (p_x, p_y)$ , define the line  $p^\star : y = p_x x - p_y$ .



For  $l : y = mx + b$ , define  $l^\star$  to be the pt  $q$  with  $q^\star = l$ , that is,  $l^\star = (m, -b)$ .

**Observe:** Let  $p \in \mathbb{R}^2$  and let  $l$  be a non-vertical line.

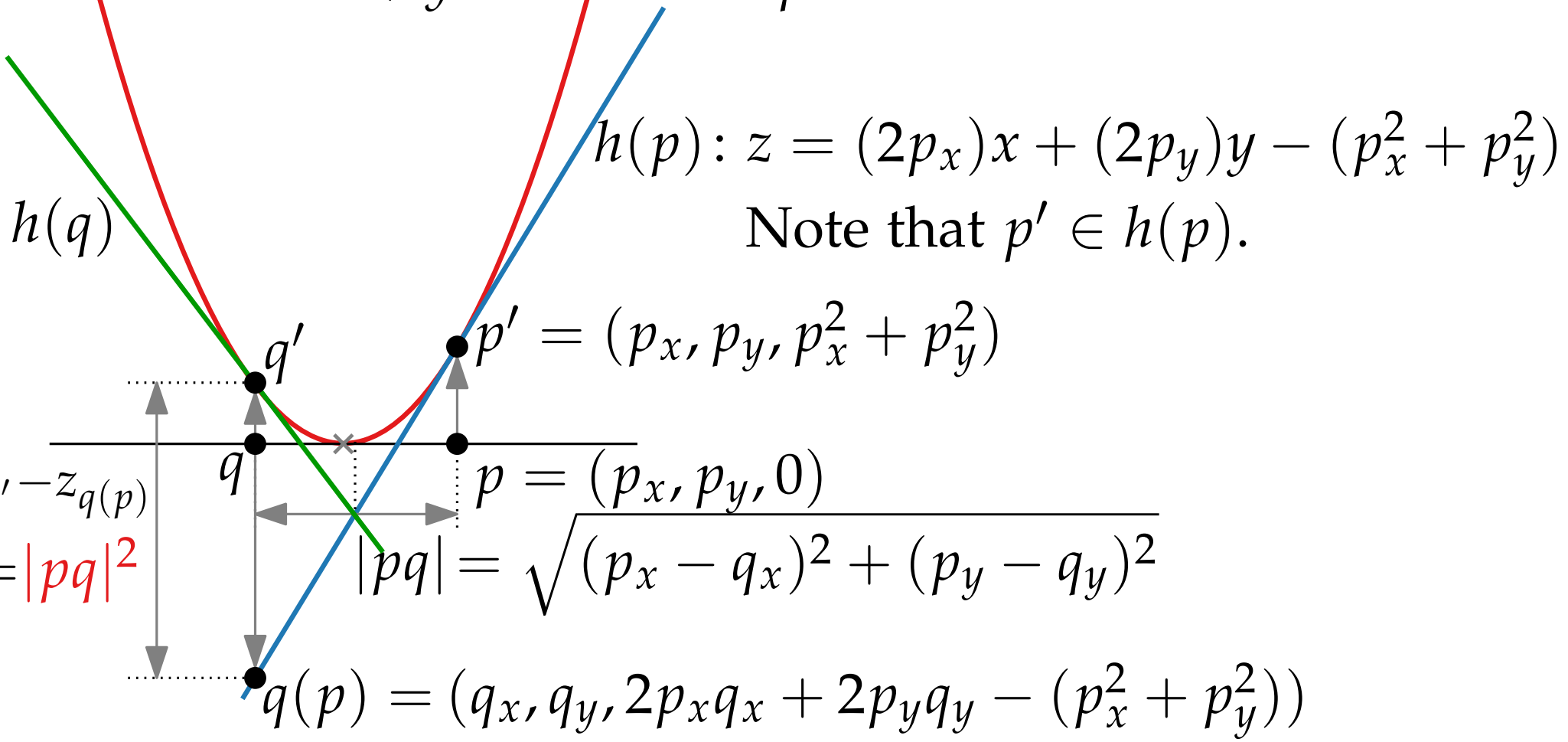
$\star$  is incidence-preserving:  $p \in l \Leftrightarrow l^\star \in p^\star$

$\star$  is order-preserving:  $p$  above  $l \Leftrightarrow l^\star$  above  $p^\star$



# Voronoi Diagrams Revisited

Let  $U: z = x^2 + y^2$  be the *unit paraboloid* in  $\mathbb{R}^3$ .



$\Rightarrow h(p)$  and  $U$  encode dist. betw.  $p$  and any other pt in  $z=0$ .

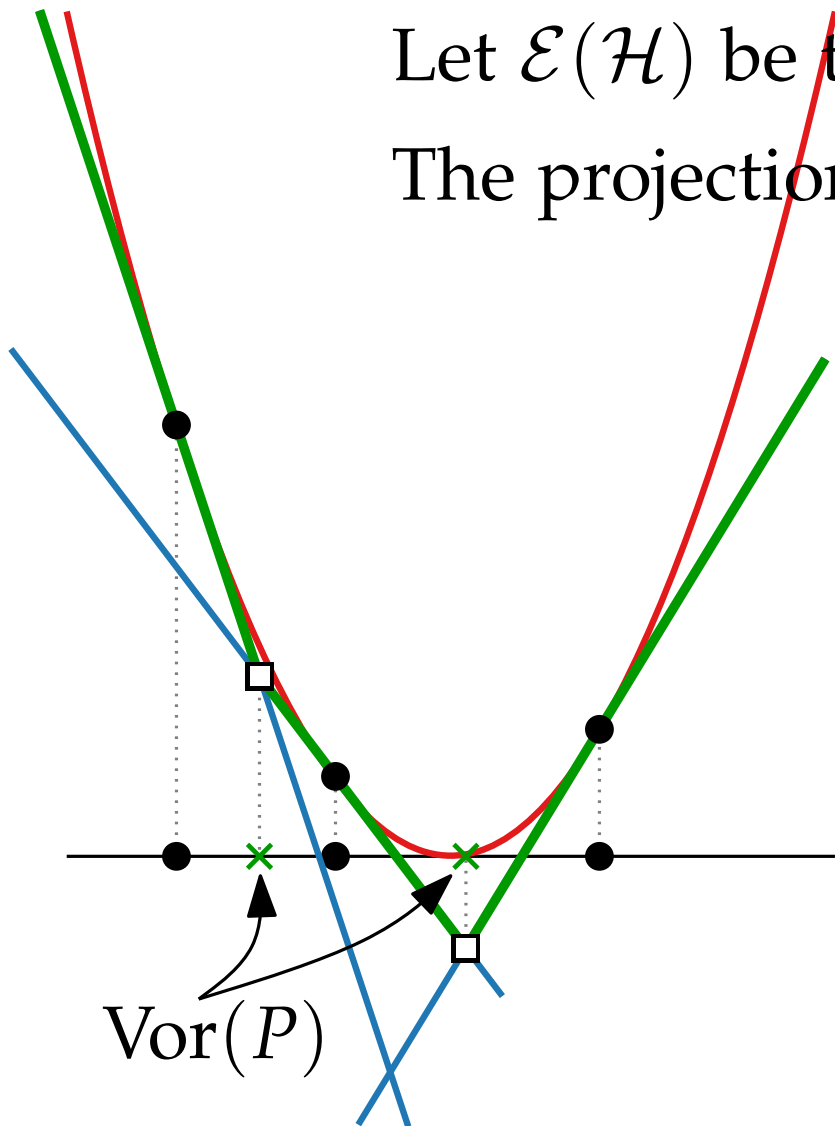
$\Rightarrow h(p) \cap U = \{p'\} \Rightarrow h(p)$  is tangent to  $U$  (in  $p'$ )

# The Upper Envelope Strikes Back

**Theorem:** Let  $P \subset \mathbb{R}^2 \times \{0\}$  and  $\mathcal{H} = \{h(p) \mid p \in P\}$ .

Let  $\mathcal{E}(\mathcal{H})$  be the upper envelope of  $\mathcal{H}$ .

The projection of  $\mathcal{E}(\mathcal{H})$  on  $z = 0$  is  $\text{Vor}(P)$ .



can compute  $\text{Vor}(P)$  in  $\mathbb{R}^2$   
via upper envelope in  $\mathbb{R}^3$



exercise 11.10

[Comp. Geom A&A]

upper envelope in  $\mathbb{R}^3$  is in  
one-to-one correspondence to  
lower convex hull of pt set  $\mathcal{H}^*$



use algorithm `Rand3DConvexHull!`