Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Computational Geometry

Orthogonal Range Queries
or
Fast Access to Data Bases

Lecture #5

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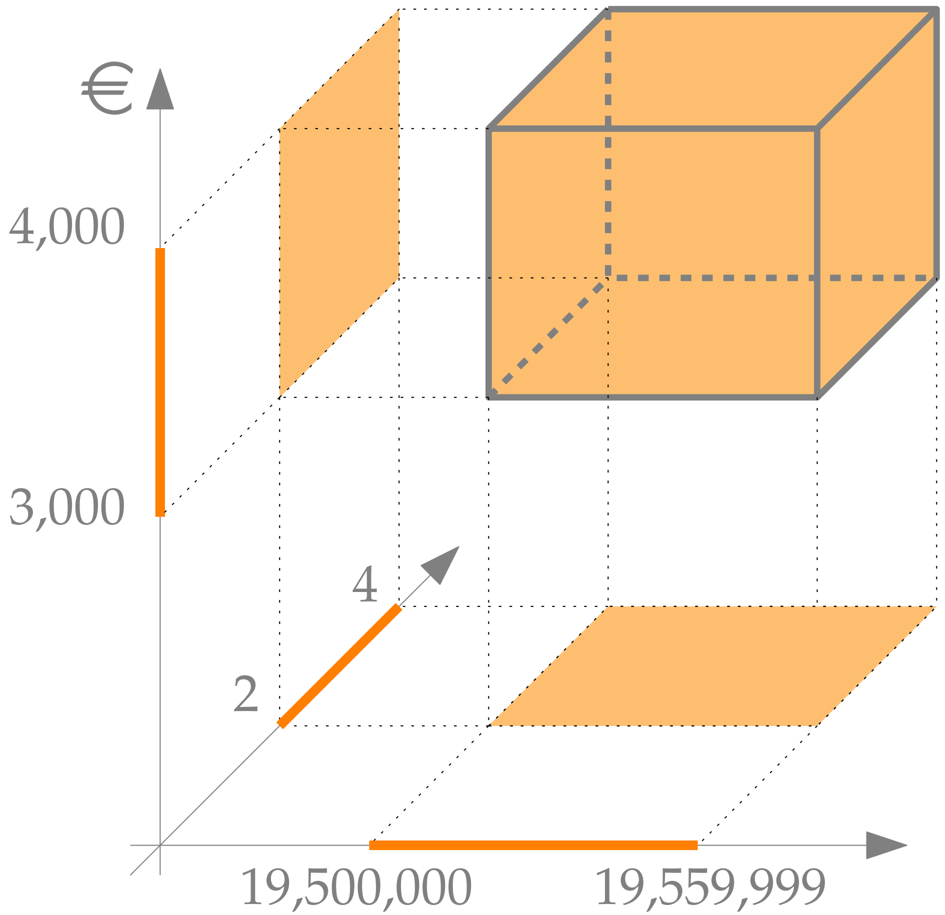
Winter Semester 2018/19

Orthogonal Range Queries

Example: Personnel management in a company



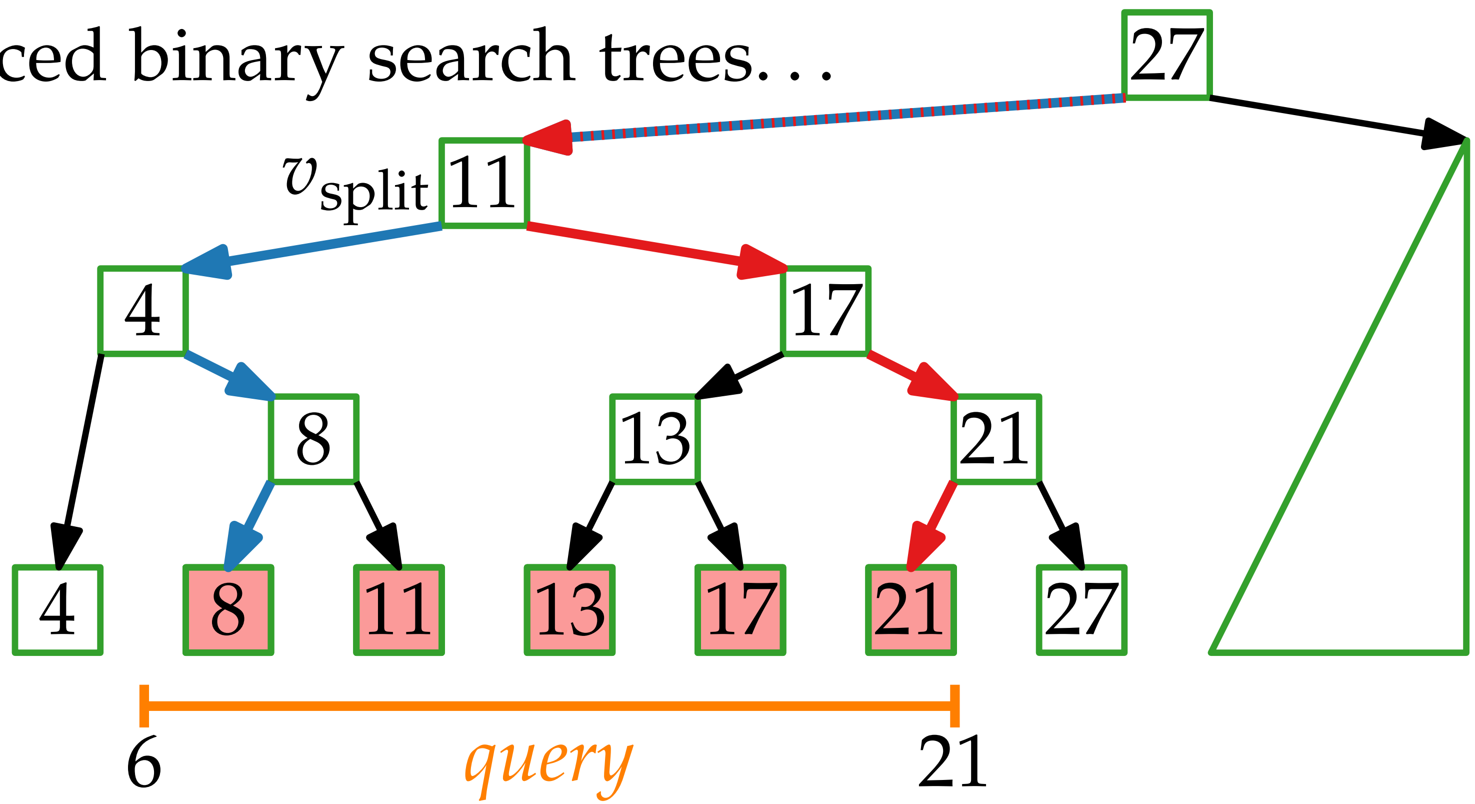
Typical queries for data bases!

1D Range Searching

Task: Preprocess a finite set $P \subset \mathbb{R}$ such that for any interval $[x, x']$ the set $P \cap [x, x']$ can be reported quickly. [2 min]

Solution: balanced binary search trees. . .

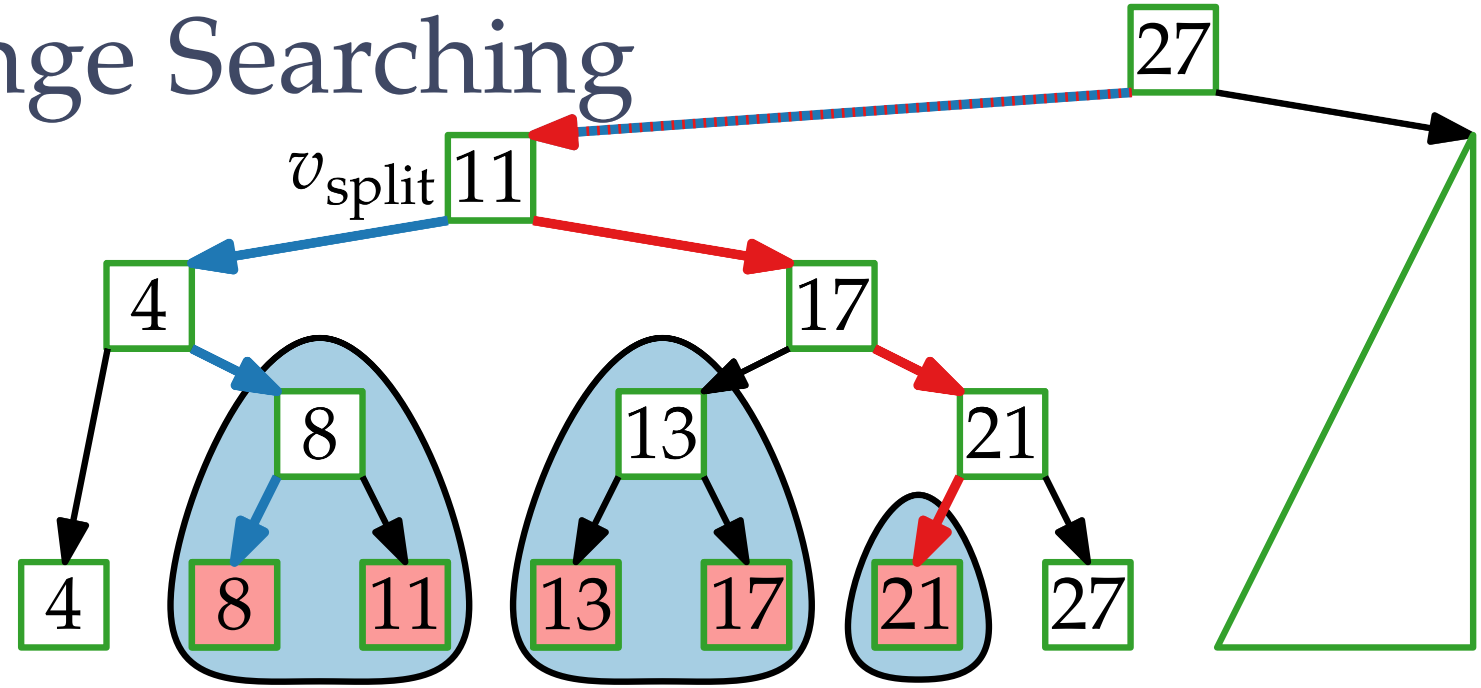
1. Search $x = 6$.
2. Search $x' = 21$.
3. Return all leaves 'inbetween".



Small changes: – keys only in leaves

– inner nodes store max. of their left subtrees

1D Range Searching



Observe: The result of a query is the disjoint union of at most $2h$ *canonical subsets* , where

– $h \in O(\log n)$ is the tree height,

– a canonical subset is an interval that contains all points stored in a subtree.

Theorem. A set of n real numbers can be preprocessed in $O(n \log n)$ time and $O(n)$ space such that 1D range queries take $\boxed{O(k + \log n)}$ time, where $k = |\text{output}|$.

output sensitive!

Extensions to 2D

Think... [3 min]

Task: Preprocess a finite set $P \subset \mathbb{R}^2$ such that for any range query $R = [x, x'] \times [y, y']$ the set $P \cap R$ can be reported quickly.

Solutions:

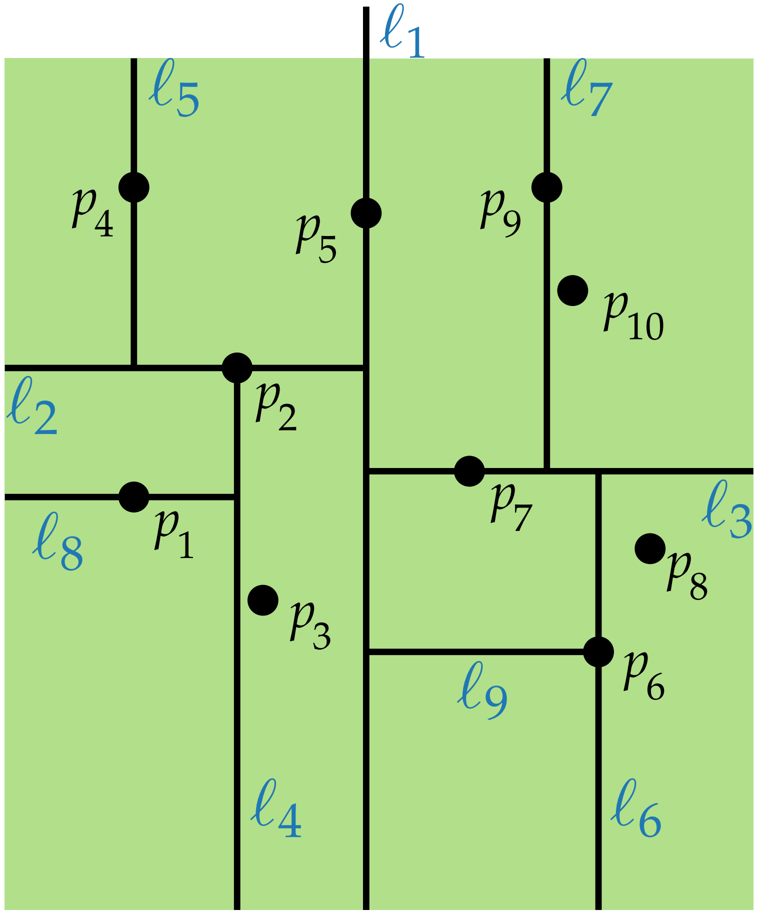
- *one* tree; query path alternates between x- and y-coord. } *kd-tree*
- first-level tree for x-coordinates; many second-level trees for y-coord. } *range tree*

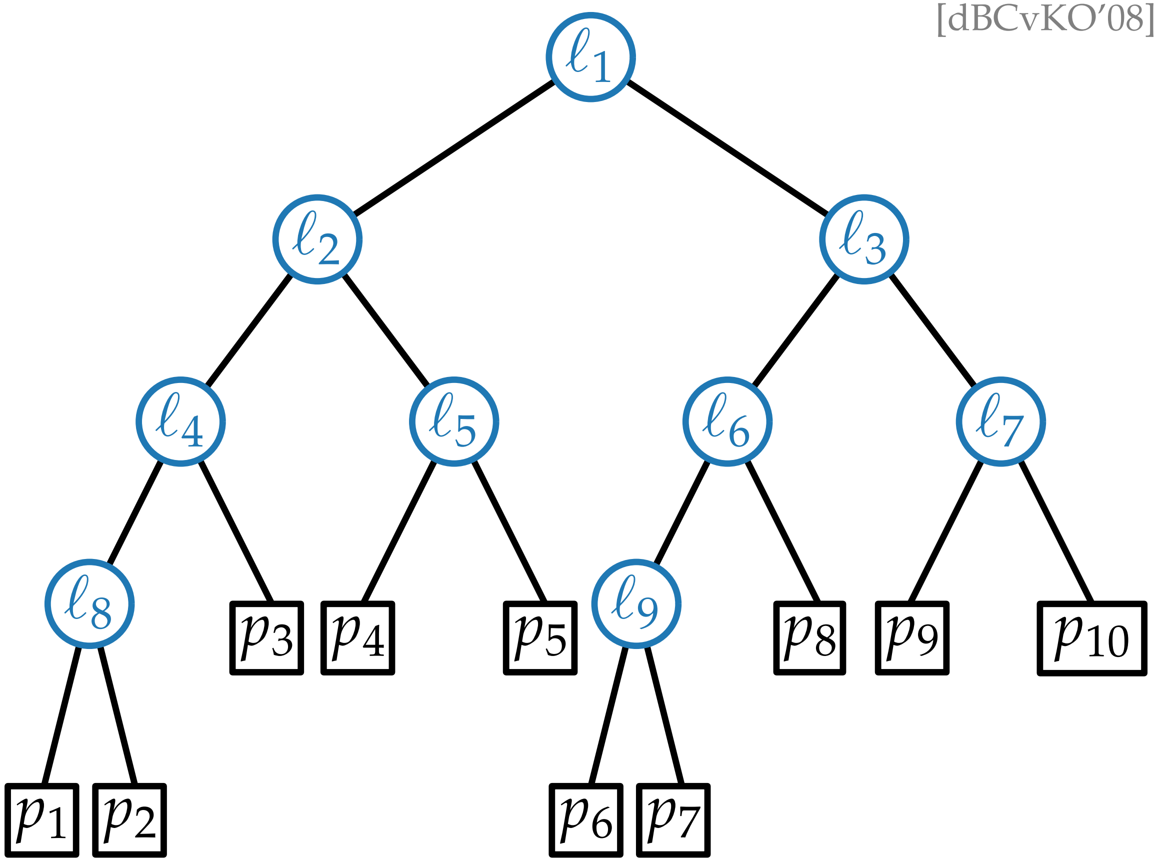
Assume: *General position!*

Here: no two points have the same x- or y-coordinate.

Kd-Trees: Example

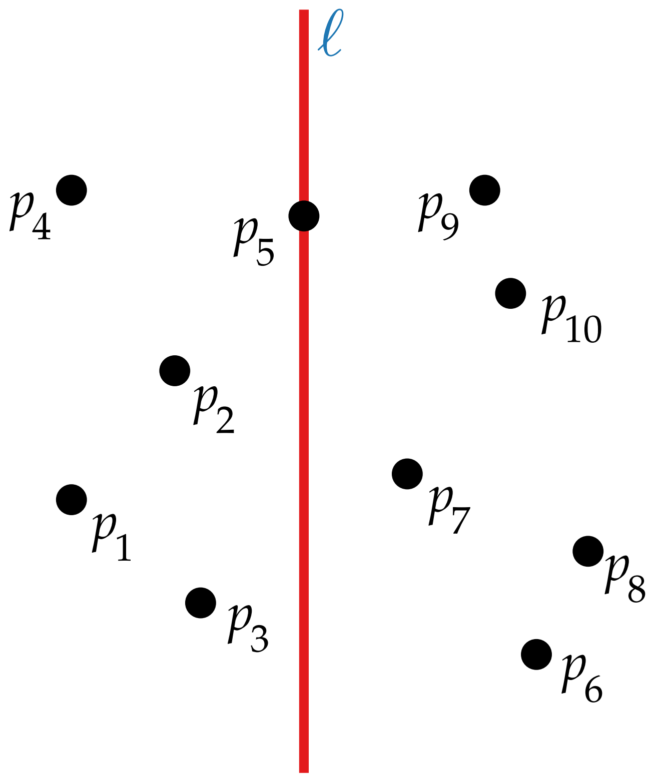
[dBCvKO'08]

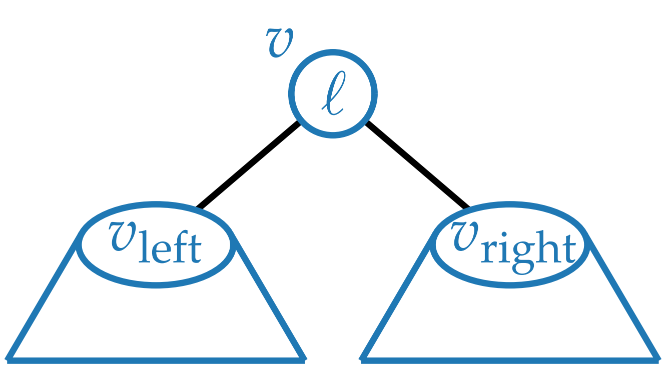




- Split any region that contains more than one point.
- Horizontal split lines/segm. belong to the region below.
  Vertical left.

Kd-Trees: Construction





Pseudo-code:

BuildKdTree(points P, int *depth*)
  if $|P| = 1$ **then**
    return (leaf storing the pt in P)
  else
    if *depth* is even **then**
      split P with the vertical line
        $\ell : x = x_{\text{median}(P)}$ into
        P_1 (pts left of or on ℓ) and
        $P_2 = P \setminus P_1$
    else
      split P horizontally...
    $v_{\text{left}} \leftarrow \text{BuildKdTree}(P_1, depth + 1)$
    $v_{\text{right}} \leftarrow \text{BuildKdTree}(P_2, depth + 1)$
    create a node v storing ℓ
    make v_{left} and v_{right} the children of v
    return (v)

Kd-Trees: Analysis

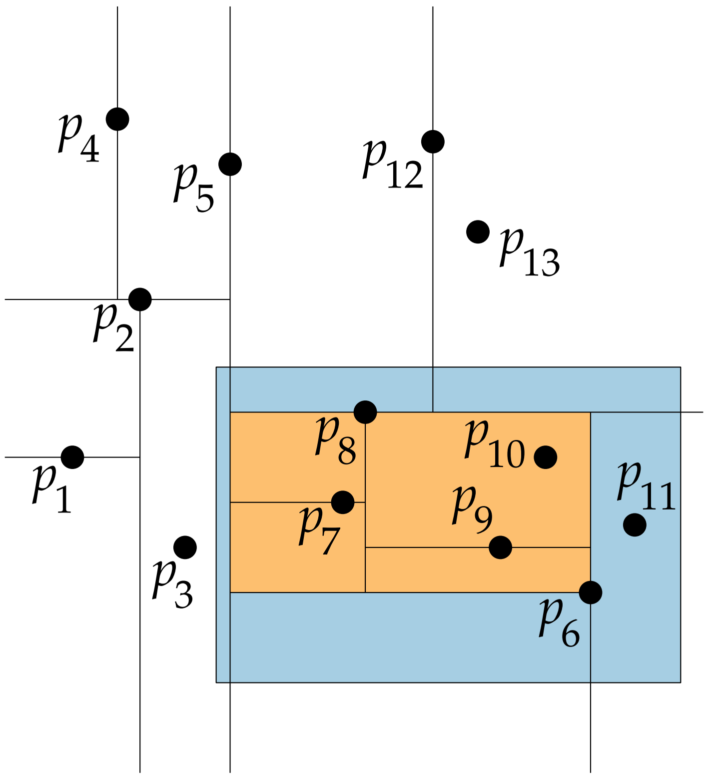
Construction time?

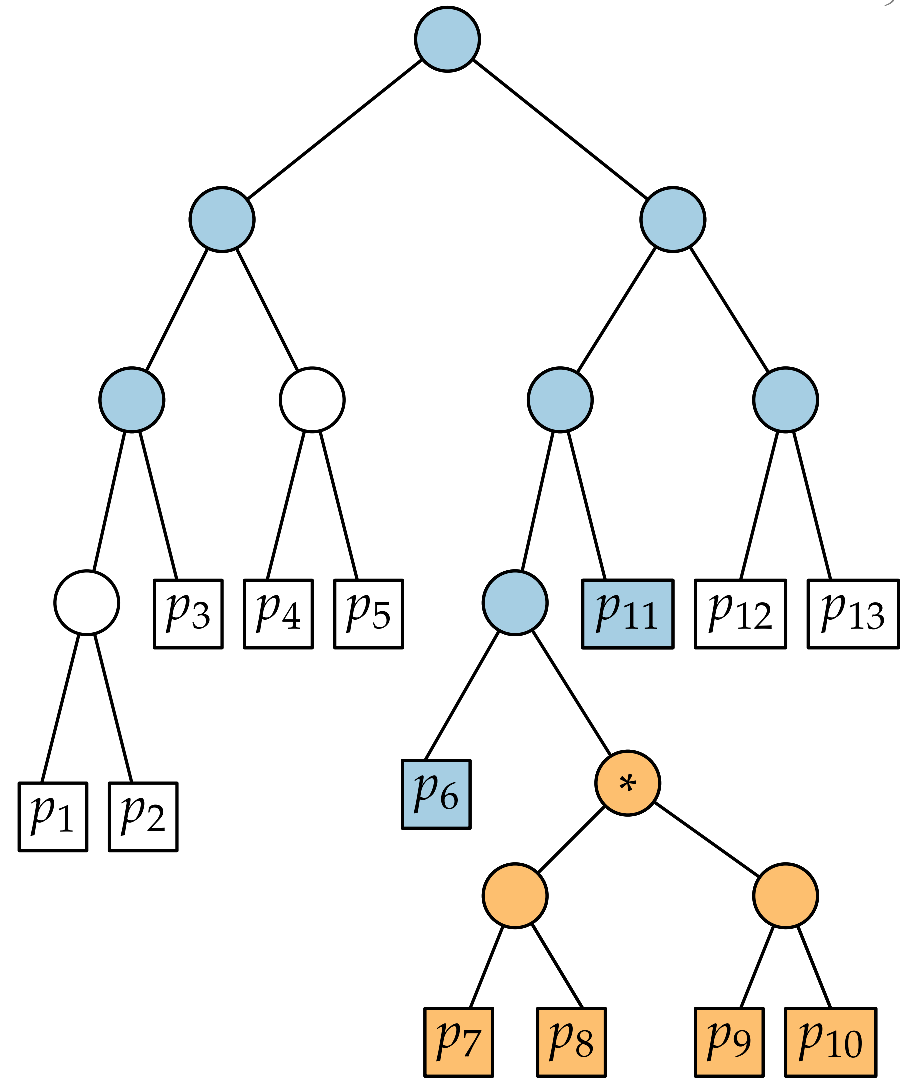
see Mergesort!

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ O(n) + 2T(\lceil n/2 \rceil) & \text{else.} \end{cases} = O(n \log n)$$

Lemma: A kd-tree for a set of n pts in the plane takes $O(n \log n)$ time to construct and uses $O(n)$ storage.

Kd-Trees: Querying





Lemma. Querying a kd-tree for n pts in the plane with an axis-parallel rectangle R takes $O(k + \sqrt{n}\,)$ time, where $k = |\text{output}|$.

in $\mathbb{R}^d$

$\boxed{O(k + n^{1-1/d})}$

Idea: $O(\sqrt{n})$ regions of the kd-tree intersect a vertical/horizontal line.

Extensions to 2D

Task: Preprocess a finite set $P \subset \mathbb{R}^2$ such that for any range query $R = [x, x'] \times [y, y']$ the set $P \cap R$ can be reported quickly.

Solutions:

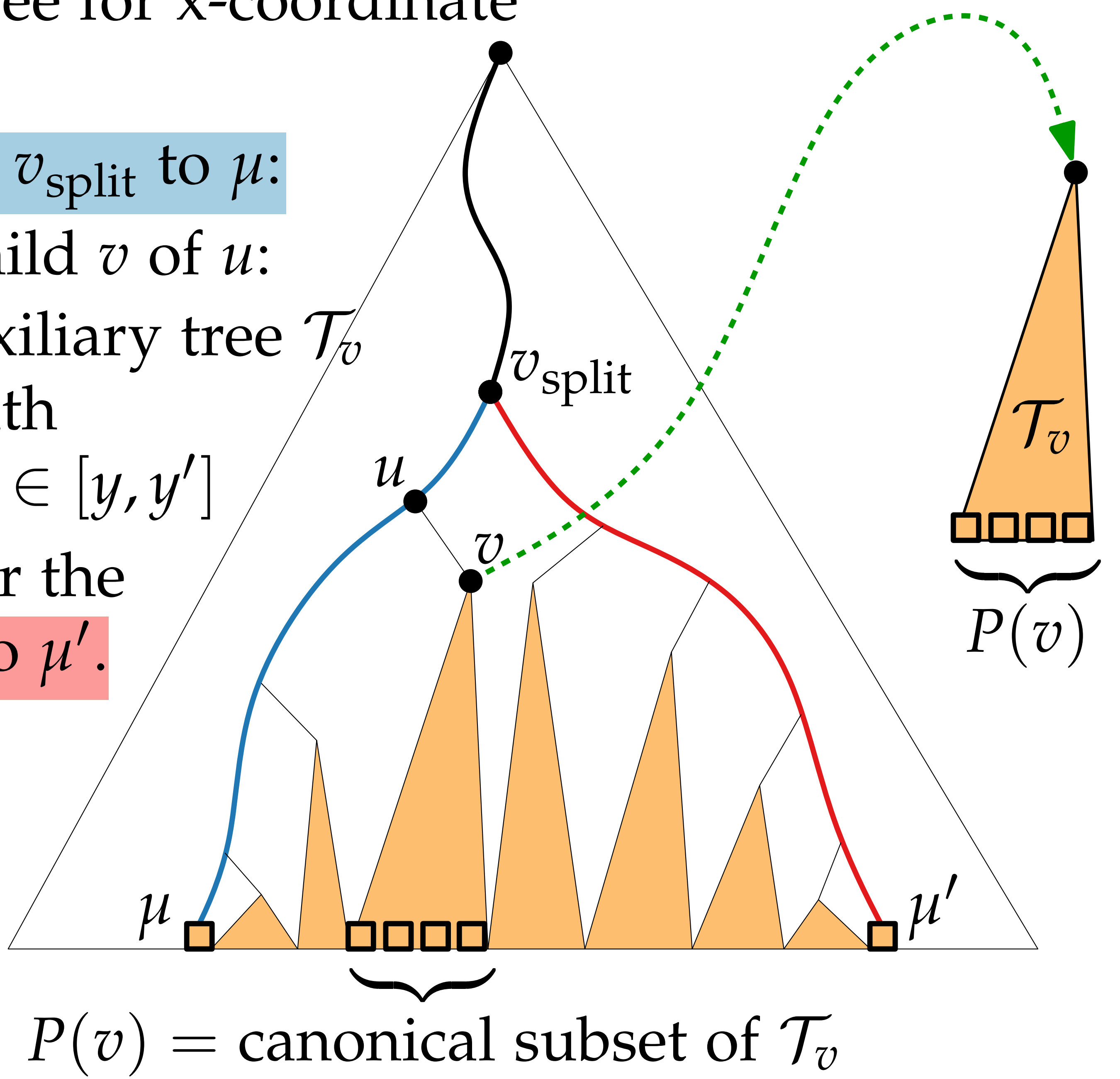
- *one* tree;
  query path alternates between x- and y-coord. } *kd-tree*
- first-level tree for x-coordinates;
  many second-level trees for y-coord. } *range tree*

Assume: *General position!*

Here: no two points have the same x- or y-coordinate.

Range Trees: Query Algorithm

1. Search in main tree for x-coordinate
2. For each node u on the path from v_{split} to μ:
   For the right child v of u:
   Search in auxiliary tree $\mathcal{T}_v$ for points with y-coordinate $\in [y, y']$
3. Symmetrically for the path from v_{split} to μ'.



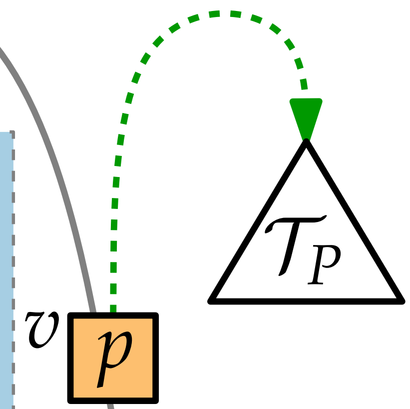
Range Trees: Construction

Build2DRangeTree(point[ ] P)

construct 2nd-level tree $\mathcal{T}_P$ on P (y-order)

if $P = \{p\}$ **then**

create leaf v:



else

x_{mid} = median x-coordinate of P

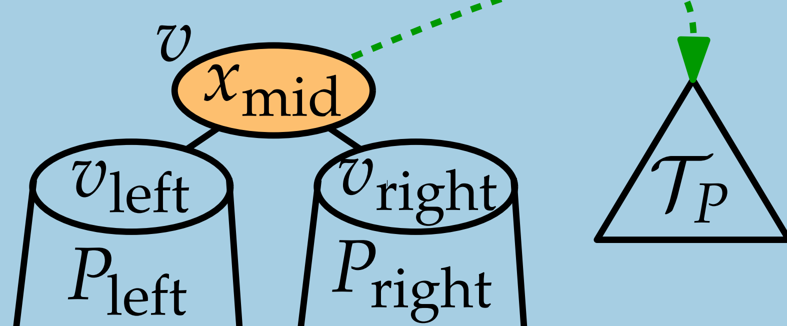
P_{left} = pts in P with x-coordinate $\leq x_{\text{mid}}$

P_{right} = $>$

v_{left} = Build2DRangeTree(P_{left})

v_{right} = Build2DRangeTree(P_{right})

create node v:



return v

Running time?

$O(n \log n)$:-(

Better:
Pre-sort once,
then build tree
bottom-up
in linear time.

$\Downarrow$

Total
construction
time $O(n \log n)$

Range Trees: Space Consumption

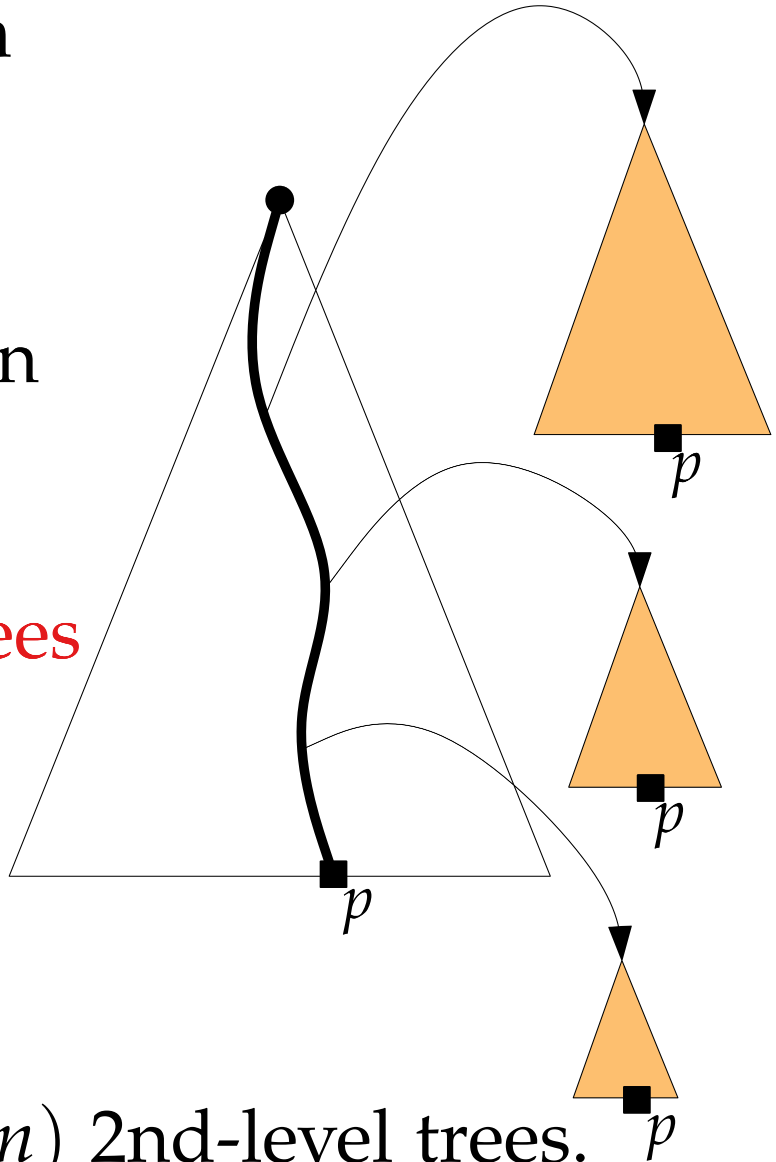
Each node v of the 1st-level tree has a pointer to a 2nd-level tree $\mathcal{T}_v$ with $|\mathcal{T}_v| = \Theta(|P(v)|)$.

Q: What's the *total* space consumption of all 2nd-level trees?

What's your guess:

- $\Theta(n^2)$,
- $\Theta(n \log n)$,
- $\Theta(n \log^2 n)$, or
- $\Theta(n)$?

How many trees will contain a given point?



A: Each $p \in P$ is stored in $h = \Theta(\log n)$ 2nd-level trees.

$\Rightarrow \Theta(n \log n)$ space

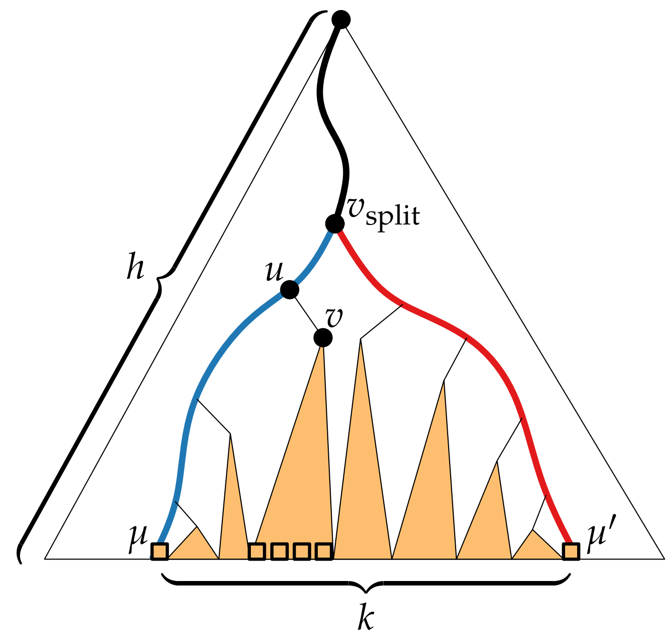
Range Trees: Query time

$$T(n,k) = \sum_{u \in \text{paths to } \mu \text{ and } \mu'} O(k_u + \log n)$$

$$= O(\textstyle\sum_u k_u) + O(\textstyle\sum_u \log n)$$

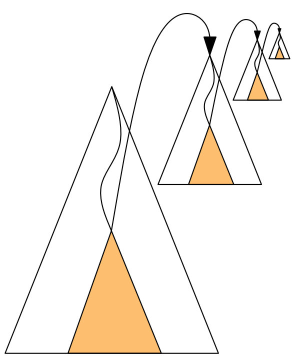
$$= O(k) \qquad + 2h \cdot O(\log n)$$

$$= O(k + \log^2 n)$$



$\mathbb{R}^d$? $O(n \log^{d-1} n)$ storage and construction time

$O(k + \log^d n)$ query time



See Chapter 5.4 in Comp. Geom A&A

Comparison

	kd-tree	range tree
construction time	$O(n \log n)$	$O(n \log n)$
storage	$O(n)$	$O(n \log n)$
query time	$O(k + \sqrt{n})$	$O(k + \log^2 n)$

Note: *trade-off* between space and query time

General Sets of Points

Idea: use *composite numbers* $(a|b)$ with lex order

$$p = (x, y) \longrightarrow \hat{p} = \big((x|y),\ (y|x)\big) \longrightarrow \textit{unique}\ \text{coordin.}$$

$$\text{range } R = [x, x'] \times [y, y']$$

$$\downarrow$$

$$\hat{R} = [(x|-\infty), (x'|+\infty)] \times [(y|-\infty), (y'|+\infty)]$$

Show: $p \in R \;\Leftrightarrow\; \hat{p} \in \hat{R}$

This removes our assumption about the input points being in general position.

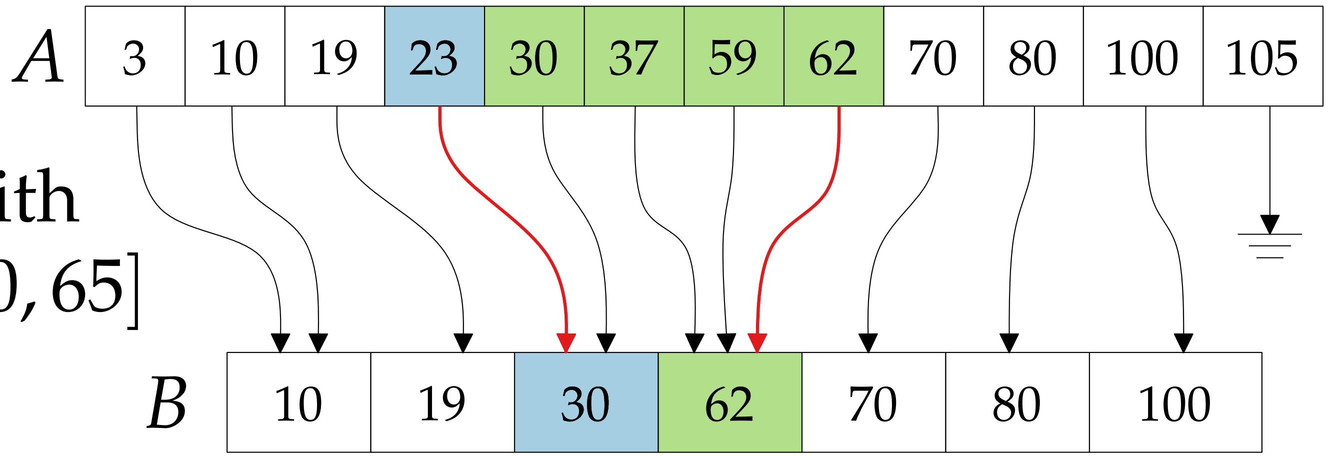
We can use kd-trees and range trees for *any* set of points; no matter how many points have the same x- or y-coord.

Fractional Cascading

Task 1: Given sets $B \subset A \subset \mathbb{N}$ stored in sorted order in arrays $A[1..n]$ and $B[1..m]$, support 1d range queries in the multiset $A \cup B$ in $k + 1 \cdot \log n$ time!

We allow $n \log m$ bits extra space.

Task 2: Assuming that task 1 can be solved, speed up 2d range queries: $O(k + \log^2 n) \to O(k + \log n)$ time!

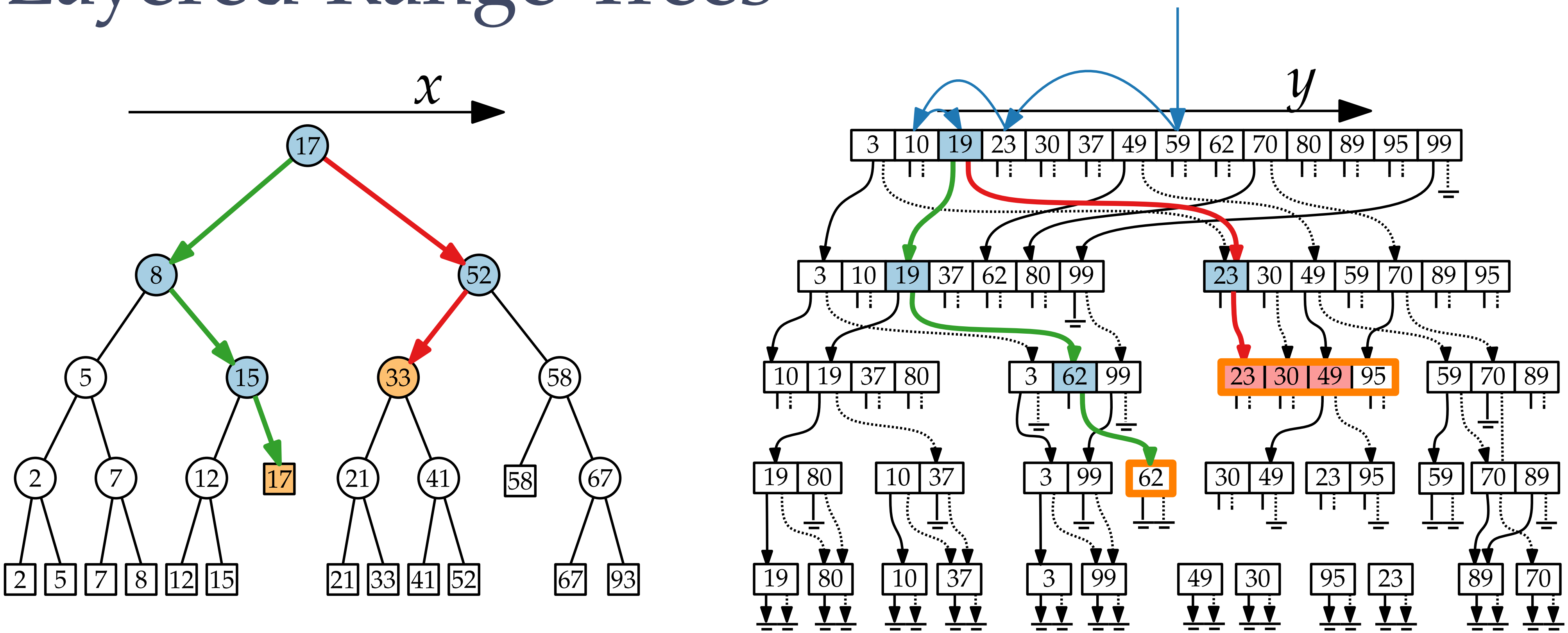


query with range $[20, 65]$

link $a \in A$ with smallest $b \geq a$ in B

[dBCvKO'08]

Layered Range Trees



$[16, 20] \times [53, 60] \;\to\; (21, 49), (33, 30), (52, 23)$

Theorem: Let $d \geq 2$ and let P be a set of n pts in $\mathbb{R}^d$. Given $O(n \log^{d-1} n)$ preprocessing time & storage, d-dim range queries on P can be answered in $O(k + \log^{d-1} n)$ time.