Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Approximationsalgorithmen

8. Vorlesung: Scheduling Jobs on Parallel Machines via Parametrized Pruning

– Folien von J. Spoerhase –

Referenzen: [Vazirani, §17] und [Williamson & Shmoys, §2.3]

[Lenstra, Shmoys, Tardos, *Mathematical Programming*, 1990]

Alexander Wolff Lehrstuhl für Informatik I WS 2018/19

Scheduling on Parallel Machines

Given: A set J of **jobs**, a set M of **machines**, and for each $j \in J$ and $i \in M$ the **processing time** $p_{ij} \in \mathbb{N}^+$ of j on i.

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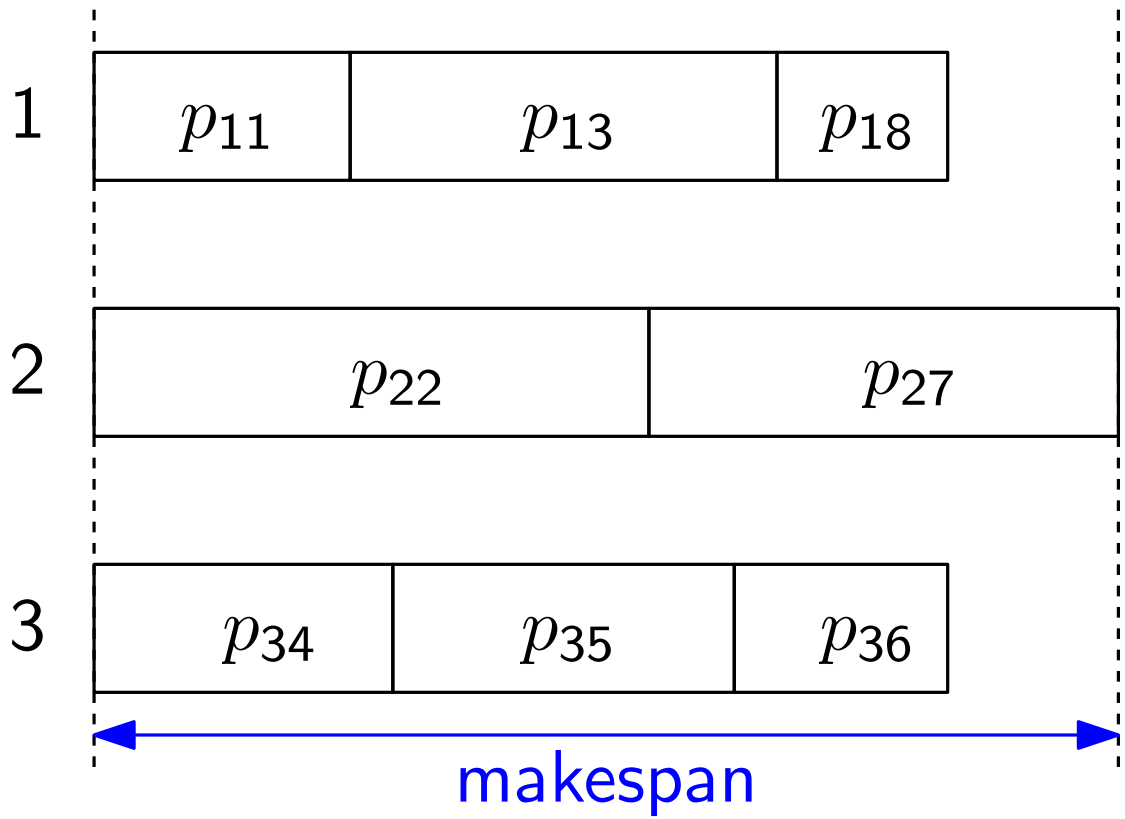

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Minimize t

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\begin{aligned}
\text{Minimize} \quad & t \\
\text{subject to} \quad & \sum_{i \in M} x_{ij} = 1, && j \in J \\
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$\Rightarrow \mathsf{OPT} = m$ and $\mathsf{OPT}_{\mathsf{frac}} = 1$.

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Define the "pruned" relaxation LP(T):

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LP(T) has no objective function; we just need to decide whether a feasible solution exists.

But why does this LP solve the problem with the integrality gap?

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Use binary search to find the smallest T so that LP(T) has a solution. Let T^* be this value of T.

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Any extreme-point solution to LP(T) must set at least $n-m$ jobs integrally.

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Why is this useful ....?

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Thm. This algorithm is a 2-approximation (assuming that we have an F-perfect matching).

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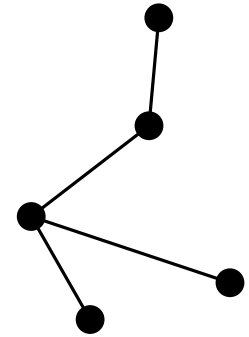

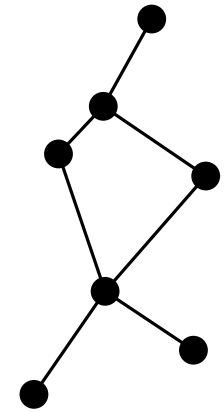

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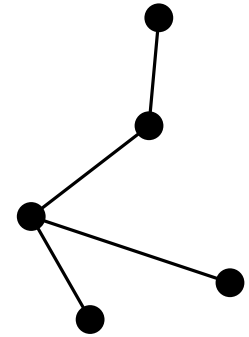

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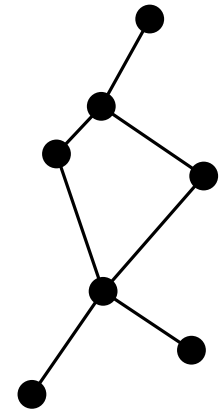

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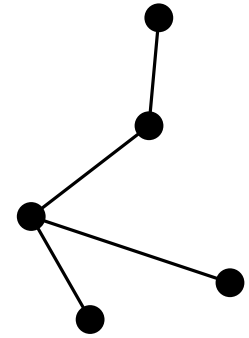

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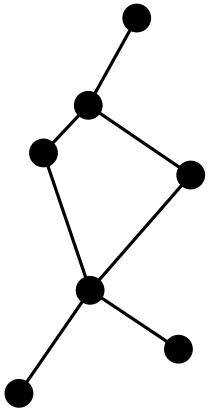

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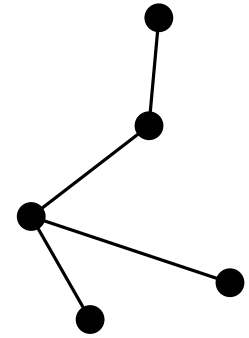

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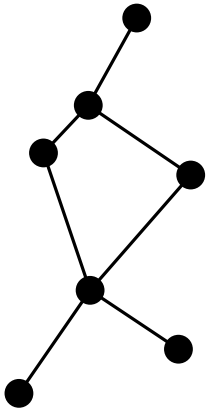

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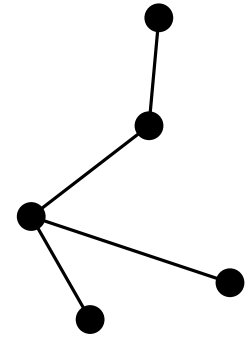

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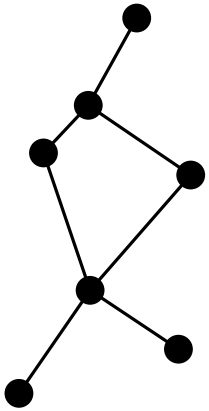

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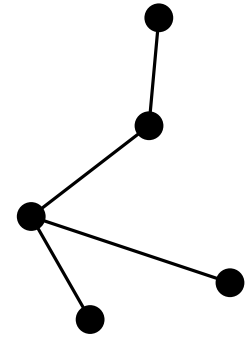

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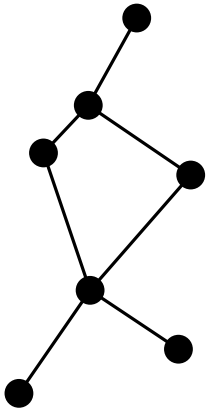

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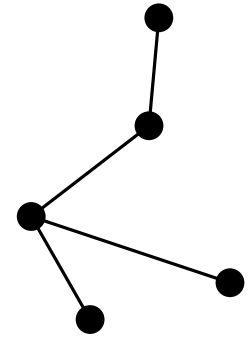

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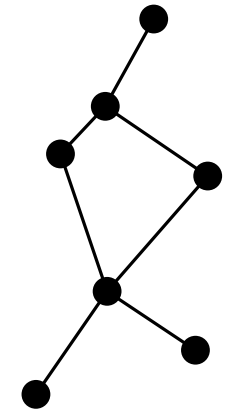

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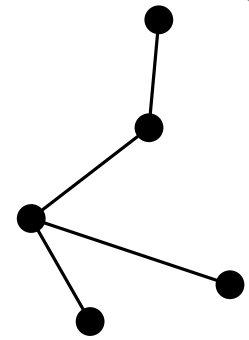

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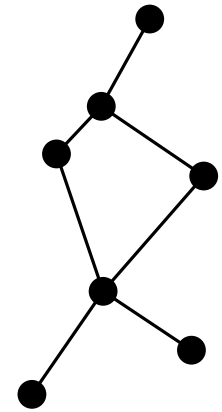

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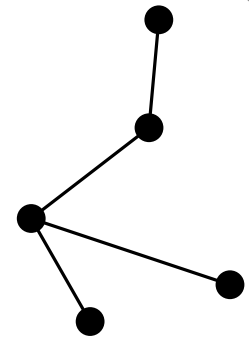
Lem. 3. The bipartite graph $G = (J \cup M, E)$ is a pseudo-forest.

Extreme-point solutions have $\leq n + m$ non-zero variables (L1).
Each component of G corresponds to extreme-point solution.

Lem. 4. The graph H has an F-perfect matching.

H is a pseudo-forest, too. The leaves in H are machines.

Pseudo-Trees and -Forests



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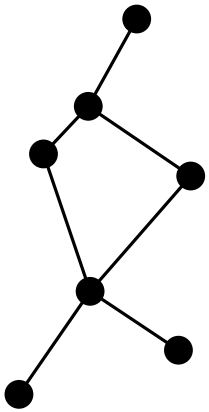

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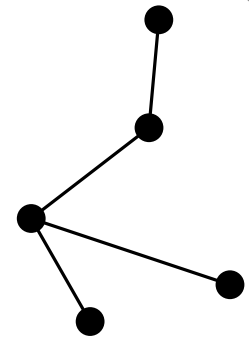

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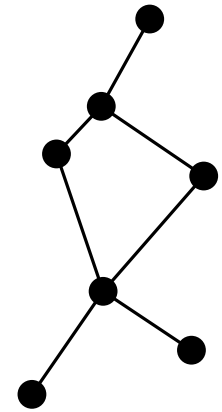

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After picking leaf edges exhaustively, only even cycles remain.

Scheduling on Parallel Machines

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Algorithm:

- LP(T) has no feasible solutions for any $T < m$.
- Extreme-pt. solution: Assign $1/m$ of j_1 and $m - 1$ other jobs to each machine. $\Rightarrow$ Makespan $2m - 1$.

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